TUM School of Computation, Information and Technology
Department of Electrical and Computer Engineering
Technical University of Munich

Information-theoretic Bounds on the Size of Spheres in the Lee Metric

Jessica Bariffi

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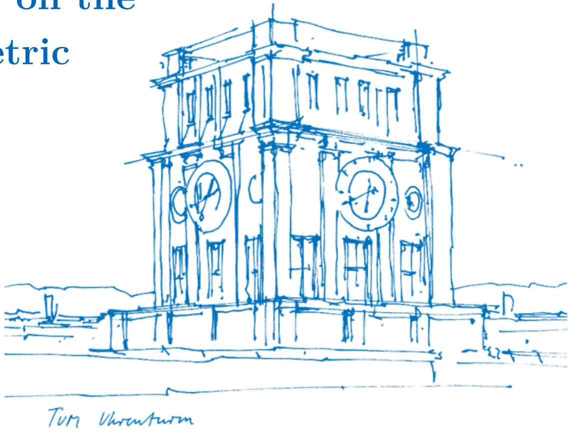

Motivation

- Crucial task in Coding Theory: understand limits in performance of error-correction

  $\Longrightarrow$ Sphere-packing bounds, Gilbert-Varshamov bound, ...

  $\Longrightarrow$ Derived using bounds on n-dimensional spheres in corresponding metrics

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Disclaimer

Method presented can be used for any additive metric.

Outline

1. Lee Metric and Spheres
2. Information-Theoretic Tools
3. Bounds in the Lee Metric

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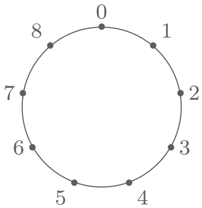
The Lee Metric



The *Lee weight* of an element $a \in \mathbb{Z}/q\mathbb{Z}$ defines the **minimum number of arcs** separating a from the origin 0.

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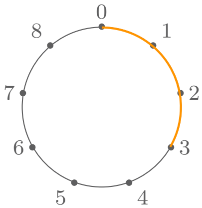

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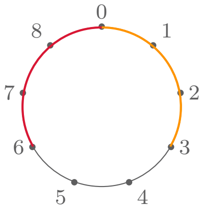

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$$\mathrm{wt}_{\mathsf{L}}(a) = \mathrm{wt}_{\mathsf{L}}(q-a)$$

$$\mathrm{wt}_{\mathsf{H}}(a) \leq \mathrm{wt}_{\mathsf{L}}(a) \leq \lfloor q/2 \rfloor$$

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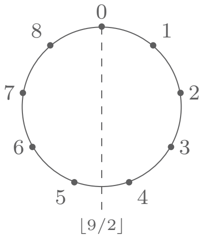

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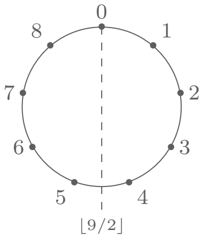

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Definition

For any integer $a \in \mathbb{Z}/q\mathbb{Z}$ and any vector $x, y \in (\mathbb{Z}/q\mathbb{Z})^n$ we define their *Lee weight* as

$$\mathrm{wt}_{\mathsf{L}}(a) := \min(a, |\, q-a\,|) \quad \text{and} \quad \mathrm{wt}_{\mathsf{L}}(x) := \sum_{i=1}^{n} \mathrm{wt}_{\mathsf{L}}(x_i)$$

The *Lee distance* between x and y is given by $\mathrm{d}_{\mathsf{L}}(x,y) := \mathrm{wt}_{\mathsf{L}}(x-y)$.

Spheres in the Lee Metric

n-dimensional Lee sphere of radius t	$\mathcal{S}_{t,q}^{(n)} := \{x \in (\mathbb{Z}/q\mathbb{Z})^n \mid \mathrm{wt}_{\mathsf{L}}(x) = t\}$
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Example Lee weight $t = 2$ in $\mathbb{Z}/5\mathbb{Z}$

$\longrightarrow$ vectors containing either 2 elements of Lee weight 1 or 1 element of Lee weight 2.

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Theorem - [Roth, ´06]

Whenever $t \leq q/2$, we have $\left|\mathcal{S}_{t,q}^{(n)}\right| = \sum_{i=0}^{n} 2^i \binom{n}{i}\binom{t}{i}$.

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- Generating functions
- Convolutions
- Counting integer partitions
- Typical sequences
- ...

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Stay tuned for Hugo's talk right after this! ;)

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3. Bounds in the Lee Metric

Types and Spheres

- Finite alphabet $\mathcal{A}$ with additive weight function wt
- Maximum weight over $\mathcal{A}$: $\mu = \max_{a \in \mathcal{A}}(\mathrm{wt}(a))$

Definition: type

The *type* of any tuple $x \in \mathcal{A}^n$ is defined as the tuple $\theta(x) := (\theta_0(x), \ldots, \theta_{|\mathcal{A}|-1}(x))$, where

$$\theta_i(x) = \frac{1}{n} \left| \{k = 1, \ldots, n \mid x_k = i\} \right| .$$

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$\Longrightarrow$ Can recover weight from type: $\mathrm{wt}(x) = n \sum_{i=0}^{\mu} \theta_i(x)\, \mathrm{wt}(i)$

$\Longrightarrow$ If $\mathrm{wt}(x) = t$, the type induces an integer partition of t of parts of size at most μ.

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Sphere size via types

$$\left| \mathcal{S}_{t,\mathcal{A}}^{(n)} \right| = \sum_{\theta \in \Theta_t^{(n)}} \frac{n!}{(n\theta_0)! \cdot \ldots \cdot (n\theta_{|\mathcal{A}|-1})!} =: \sum_{\theta \in \Theta_t^{(n)}} \binom{n}{n\theta}$$

Bounds via Entropy

- Random variable X over finite alphabet $\mathcal{A}$
- P_X probability distribution of X: $P_X(a) = \mathbb{P}(X = a)$, $a \in \mathcal{A}$.
- Q another probability distribution over $\mathcal{A}$

Definition: Entropy and Kullback-Leibler Divergence

$$H(P_X) = \sum_{\substack{a \in \mathcal{A} \\ P_X(a) \neq 0}} P_X(a) \log_2(P_X(a))$$

$$\mathsf{D}(P_X \; || \; Q) = -\sum_{a \in \mathcal{A}} P_X(a) \log\left(\frac{P_X(a)}{Q(a)}\right)$$

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Theorem [Cover & Thomas]

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Sphere size: Bounded by sequence whose type maximizes the entropy.

Lee-Boltzmann Distribution

Example

$$\mathcal{S}_{2,5}^{(3)} = \Big\{ (1,\,1,\,0),\ldots,\,(1,\,4,\,0),\ldots,(4,\,4,\,0),\ldots,\;(2,\,0,\,0),\ldots,\,(3,\,0,\,0),\ldots \Big\}$$

Draw $a \in \mathcal{S}_{2,5}^{(3)}$ uniformly at random, then ...

- smaller Lee weights are more likely to occur in the vector a.
- some sequences are more likely $\longrightarrow$ typical sequence.

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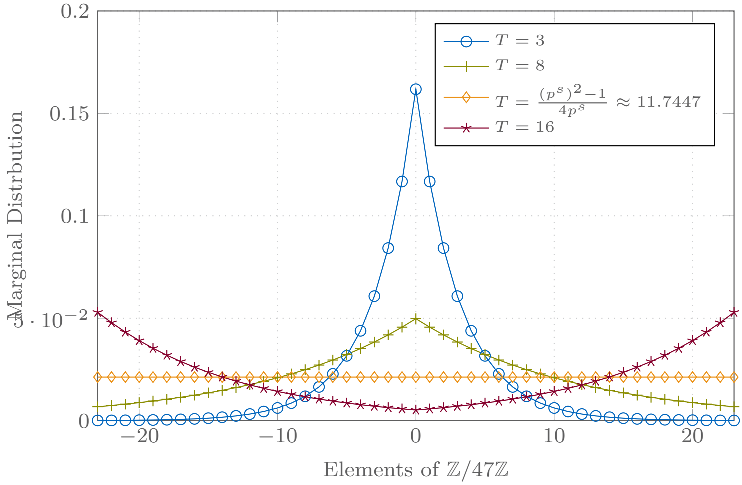
- Define $E := \left\{ P = (p_0, \ldots, p_{q-1}) \mid \sum_{i=0}^{q-1} p_i \,\mathrm{wt_L}(i) = \delta \right\} \longrightarrow$ distributions of tuples in $\mathcal{S}_{\delta n,q}^{(n)}$.

Theorem - [B., Bartz, Liva, Rosenthal - 21‘]

The distribution in E maximizing the entropy is given by $B_\delta = (B_\delta(0), \ldots, B_\delta(q-1))$, where

$$B_\delta(i) := \frac{1}{Z(\beta)} \exp\left(-\beta \,\mathrm{wt_L}(i)\right).$$

Lee-Boltzmann Distribution - Example over $\mathbb{Z}/47\mathbb{Z}$



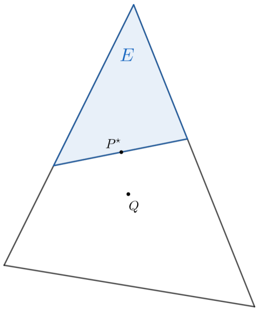
Conditional Limit Theorem

Conditional Limit Theorem - [Cover & Thomas]

Let E be a closed convex set of probability distributions over an alphabet $\mathcal{X}$ and let Q be a distribution over $\mathcal{X}$ but not in E. Let $X_1, \ldots, X_n$ be discrete random variables drawn i.i.d. $\sim Q$. Define $X^n = (X_1, \ldots, X_n)$ and let $P^\star = \arg\min_{P \in E} D(P \,||\, Q)$. Then

$$\mathbb{P}\left(X_1 = a \,|\, P_{X^n} \in E\right) \longrightarrow P^\star(a)$$

in probability as n grows large for any $a \in \mathcal{X}$.



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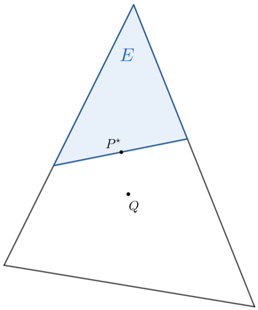

In our case

- $Q \sim \mathcal{U}(\mathbb{Z}/q\mathbb{Z})$
- E set of distributions of tuples in $\mathcal{S}_{t,q}^{(n)}$
- $P^\star = B_\delta$, for $\delta = t/n$

Marginal Distribution in the Lee Sphere

Lemma - Marginal Distribution in the Lee Sphere [BBLR, ´21]
Consider a random vector $A \in \mathcal{S}_{\delta n,q}^{(n)}$ and let $P(a)$ be the marginal distribution of an element of A. Then, for every $a \in \mathbb{Z}/q\mathbb{Z}$ we have

$$P(a) \longrightarrow B_\delta(a) := \frac{1}{Z(\beta)} \exp\left(-\beta \, \mathrm{wt}_\mathsf{L}(a)\right),$$

where β is the unique real solution to the Lee weight constraint $\delta = \sum_{i=0}^{q-1} \mathrm{wt}_\mathsf{L}(i)\mathbb{P}(X = i)$ and $Z(\beta)$ denotes the normalization constant

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Distribution Maximizing Entropy

Definition (Boltzmann-like Distribution)

For any $a \in \mathcal{A}$ and $0 < \delta < \mu$, we define the probability distribution

$$P_\beta(a) := \frac{q^{-\beta\,\mathrm{wt}(a)}}{Z(\beta)}$$

where β is the unique solution to the weight constraint $\mathbb{E}[\mathrm{wt}(a)] = \sum_{a\in\mathcal{A}} P_\beta(a)\,\mathrm{wt}(a) = \delta$ and $Z(\beta)$ is chosen s.t. $\sum_{a\in\mathcal{A}} P_\beta(a) = 1$, i.e. $Z(\beta) = \sum_{a\in\mathcal{A}} q^{-\beta\,\mathrm{wt}(a)}$.

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- one-to-one correspondence between β and δ
- Denote $H_\delta = H(P_\beta)$

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Theorem [Löliger, 1994]

For any $0 < \delta \leq \overline{w}$ and $n \in \mathbb{N}$ we have $\frac{1}{n} \log_q \left| \mathcal{B}_{\delta n}^{(n)} \right| \leq H_\delta$.

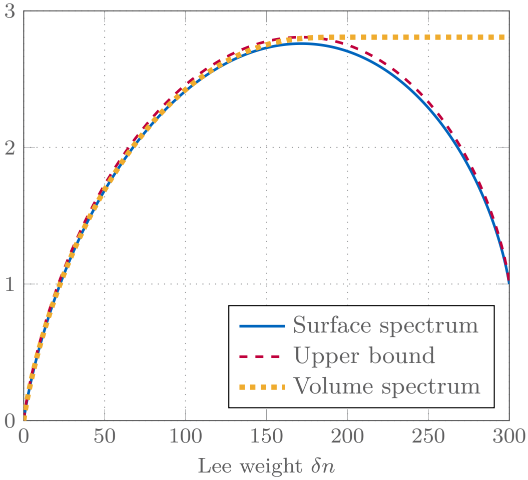
Extending Bounds for any Radius

Theorem - [BBLR, 22'] and [SCJB, 24']
For any $0 < \delta \leq \mu$ we have

$$\frac{1}{n} \log_q \left| \mathcal{B}_{\delta n}^{(n)} \right| \leq \begin{cases} H_\delta & \text{if } 0 < \delta \leq \overline{w} \\ \log_q(|\mathcal{A}|) & \text{if } \overline{w} < \delta \leq \mu \end{cases}.$$

Moreover, it holds

$$\frac{1}{n} \log_q \left| \mathcal{S}_{\delta n}^{(n)} \right| \leq H_\delta$$

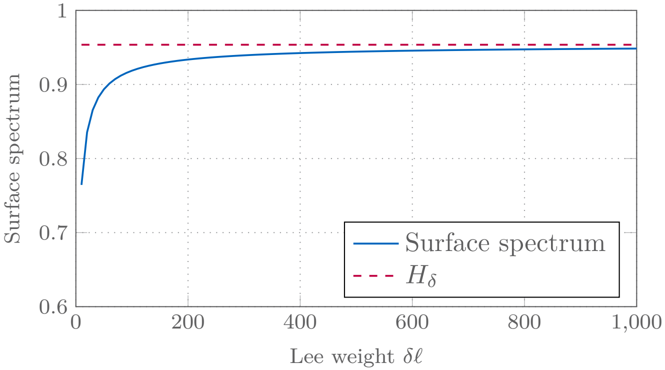


Asymptotically Tight

Theorem (Continuation)

The bounds provided are asymptotically tight, i.e.,

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \log_q \left| \mathcal{S}_{\delta n}^{(n)} \right| = H_\delta \quad \text{and} \quad \lim_{n \longrightarrow \infty} \frac{1}{n} \log_q \left| \mathcal{B}_{\delta n}^{(n)} \right| = \begin{cases} H_\delta & \text{if } 0 < \delta \leq \overline{w} \\ \log_q(|\mathsf{A}|) & \text{if } \overline{w} < \delta \leq \mu \end{cases}.$$



Ring-Linear Codes

Definition
A linear code $\mathcal{C} \subset (\mathbb{Z}/q\mathbb{Z})^n$ is a $\mathbb{Z}/q\mathbb{Z}$-submodule of $(\mathbb{Z}/q\mathbb{Z})^n$.

Parameters

- Blocklength n
- $\mathbb{Z}/q\mathbb{Z}$-dimension $k := \log_q(|\mathcal{C}|)$
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- **Memoryless Lee Channel**

  Transmit $x \in \mathcal{C}$

  Receive: $y = x + e \in (\mathbb{Z}/q\mathbb{Z})^n$ where $e_i \sim B_\delta$ for some δ

  $\implies$ Channel matching to the Lee metric under ML decoding.

Sphere-Packing Bound – Another Point of View

Theorem - [BBLR, '23]

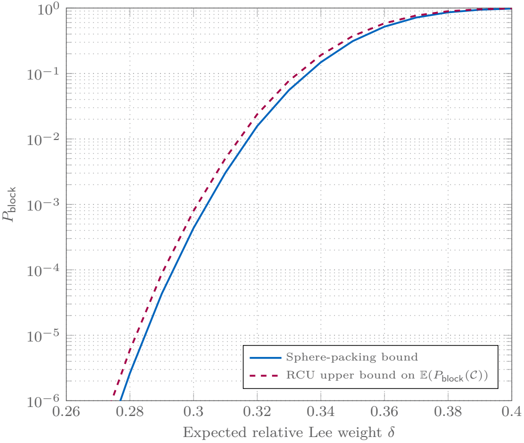
The block error probability of **any** code $\mathcal{C} \subseteq (\mathbb{Z}/q\mathbb{Z})^n$ of rate R over a memoryless Lee channel is lower bounded as

$$P_{\text{block}}(\mathcal{C}) > \frac{1}{Z(\beta)^n} \sum_{d=d_0+1}^{rn} \left| \mathcal{S}_{d,q}^{(n)} \right| \mathbb{E}\left(-\beta d\right) + \frac{1}{Z(\beta)^n} \left(\left| \mathcal{S}_{d_0,q}^{(n)} \right| - \xi \right) \mathbb{E}\left(-\beta d_0\right)$$

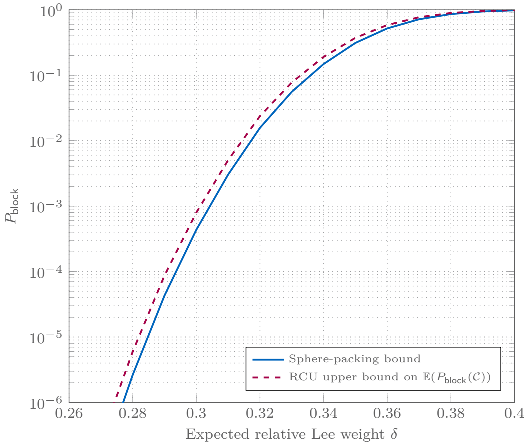
where d_0 and ξ are chosen so that

$$\sum_{d=0}^{d_0-1} \left| \mathcal{S}_{d,q}^{(n)} \right| + \xi = 2^{n(\log_2(q)-R)} \quad \text{and} \quad 0 < \xi \leq \left| \mathcal{S}_{d_0,q}^{(n)} \right|.$$

Sphere-Packing Bound - Comparison



Sphere-Packing Bound - Comparison



Thank you for your attention!