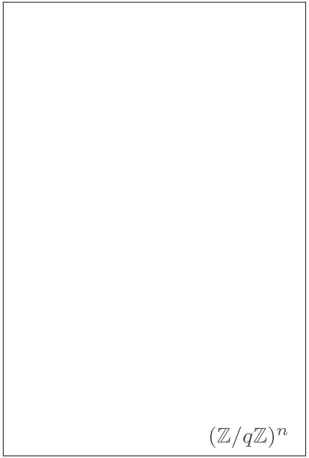
Bounds on the Minimum Lee Distance

KN Institutskolloquium

Jessica Bariffi
joint work with Violetta Weger (TUM)

23.10.2023

Motivation



Work over a q-ary alphabet $\mathbb{Z}/q\mathbb{Z}$, where $q = p^s$ a prime power.

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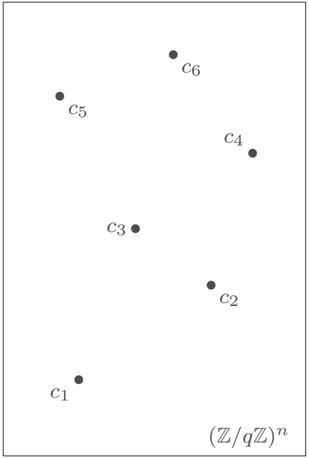

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Code: $\mathcal{C} \subseteq (\mathbb{Z}/q\mathbb{Z})^n$ k-dimensional submodule

- length n
- type k ($\mathbb{Z}/q\mathbb{Z}$-dimension)
- codewords $c_i \in \mathcal{C}$ for $i = 1, \ldots, q^k$

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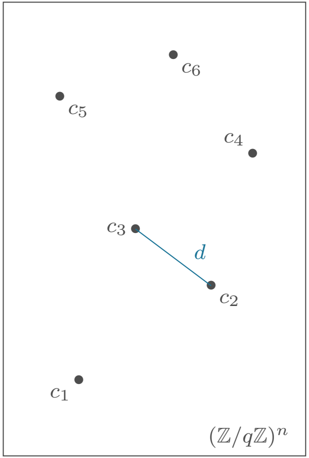

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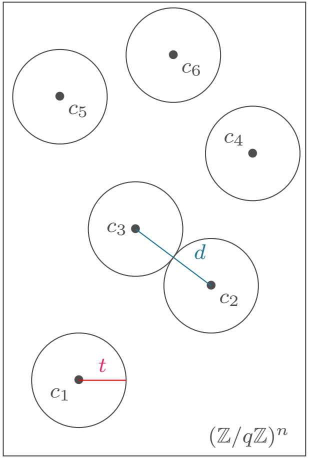

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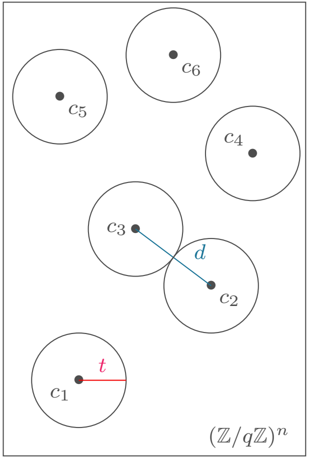

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One of the main tasks: find tight bounds on the minimum distance

Outline

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1. Singleton-like Bounds for Codes over Rings
2. Generalized Weights in the Hamming Metric
3. Lee Supports and Generalized Weights
4. Generalized Lee Distances

Ring-Linear Codes

Consider a code $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ as a $\mathbb{Z}/p^s\mathbb{Z}$-submodule of $(\mathbb{Z}/p^s\mathbb{Z})^n$.

- $\mathcal{C} \cong (\mathbb{Z}/p^s\mathbb{Z})^{k_0} \times (\mathbb{Z}/p^{s-1}\mathbb{Z})^{k_1} \times \ldots \times (\mathbb{Z}/p\mathbb{Z})^{k_{s-1}}$
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- Generator matrix in systematic form

$$G_{\mathsf{sys}} = \begin{pmatrix} \mathbb{I}_{k_0} & A_{1,2} & A_{1,3} & \cdots & A_{1,s} & A_{1,s+1} \\ 0 & p\mathbb{I}_{k_1} & pA_{2,3} & \cdots & pA_{2,s} & pA_{2,s+1} \\ 0 & 0 & p^2\mathbb{I}_{k_2} & \cdots & p^2A_{3,s} & p^2A_{3,s+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p^{s-1}\mathbb{I}_{k_{s-1}} & p^{s-1}A_{s,s+1} \end{pmatrix}$$

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Example over $\mathbb{Z}/9\mathbb{Z} = \mathbb{Z}/3^2\mathbb{Z}$

- Subtype $(k_0, k_1) = (3, 0)$

$$G_{\mathsf{sys}} = \begin{pmatrix} 1 & 0 & 0 & 3 & 7 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 4 \end{pmatrix}$$

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- Subtype $(k_0, k_1) = (1, 2)$

$$G_{\mathsf{sys}} = \begin{pmatrix} 1 & 5 & 4 & 3 & 7 \\ 0 & 3 & 0 & 6 & 6 \\ 0 & 0 & 3 & 0 & 3 \end{pmatrix}$$

Hamming Weight

Counts 1 if a symbol is nonzero, and 0 else.

For $x \in (\mathbb{Z}/q\mathbb{Z})^n$:

$$\mathrm{wt}_{\mathsf{H}}(x) := |\{i = 1, \dots, n \mid x_i \neq 0\}|$$

Lee Weight

For $a \in \mathbb{Z}/q\mathbb{Z}$: $\mathrm{wt}_{\mathsf{L}}(a) = \min\{a, |q-a|\}$.

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Distance Measures

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Example

Consider $x = (0, 1, 2, 3, 4, 5, 6) \in (\mathbb{Z}/7\mathbb{Z})^7$

$$\mathrm{wt}_{\mathsf{H}}(x) = 0 + 1 + 1 + 1 + 1 + 1 + 1 = 6$$

$$\mathrm{wt}_{\mathsf{L}}(x) = 0 + 1 + 2 + 3 + 3 + 2 + 1 = 12$$

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The *Hamming*- and *Lee distance* between two tuples $x, y \in \mathbb{Z}/p^s\mathbb{Z}$ is given by weight of their difference, i.e.,

$$\mathrm{d}_{\mathsf{H}}(x, y) := \mathrm{wt}_{\mathsf{H}}(x - y) \quad \text{and} \quad \mathrm{d}_{\mathsf{L}}(x, y) := \mathrm{wt}_{\mathsf{L}}(x - y).$$

The Singleton Bound
Hamming Metric

Given a code $\mathcal{C} \subseteq (\mathbb{Z}/q\mathbb{Z})^n$ of dimension k and minimum distance $\mathrm{d}_{\mathsf{H}}(\mathcal{C}) = d$, then

$$d \leq n - k + 1.$$

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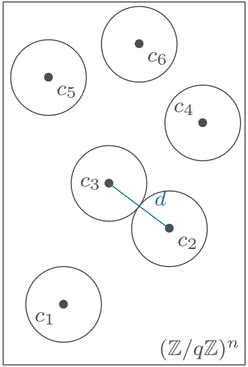

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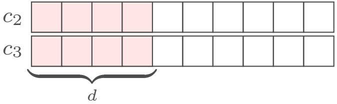

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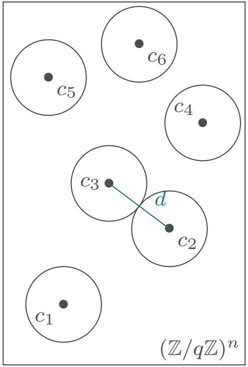

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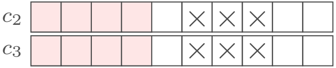

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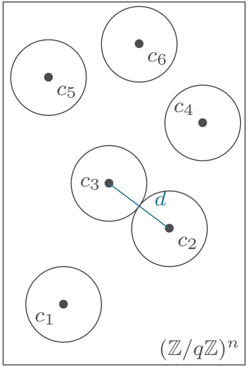

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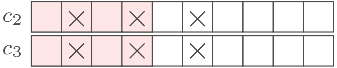

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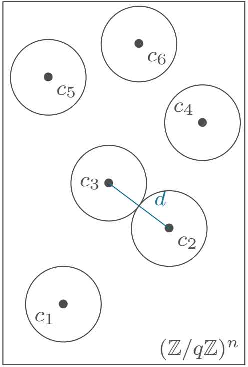

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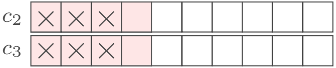

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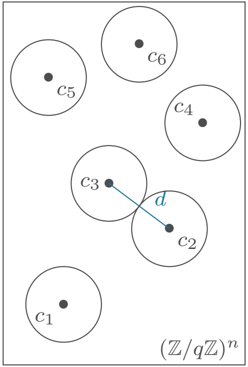

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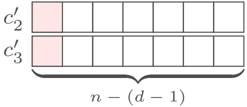

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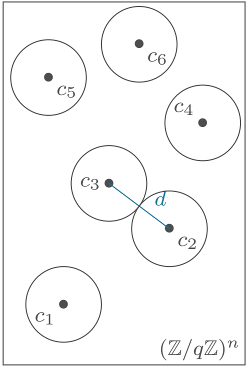

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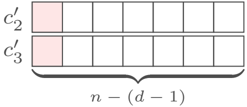

- for codewords of length $n-(d-1)$ we can only have at most $q^{n-(d-1)}$ distinct many.
  $\implies q^k \leq q^{n-(d-1)}$

Shiromoto (2000)
Let $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ be a linear code of type k and define the maximal Lee weight $M := \lfloor p^s/2 \rfloor$. Then

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Example Let $\mathcal{C} \subseteq (\mathbb{Z}/27\mathbb{Z})^5$ be the code generated by $G = \begin{pmatrix} 1 & 10 & 4 & 20 & 9 \\ 0 & 3 & 9 & 18 & 9 \end{pmatrix}$.

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Need new techniques for the Lee metric!

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Hamming Support of a Vector

Consider a finite integer residue ring $\mathbb{Z}/q\mathbb{Z}$ and a vector $x \in (\mathbb{Z}/q\mathbb{Z})^n$.

- Support of x as set of nonzero indices

$$\mathrm{supp}_{\mathsf{H}}(x) = \{i \in \{1, \ldots, n\} \mid x_i \neq 0\} .$$

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Let $x = (1, 4, 2, 0, 2, 0, 3)$ with $\mathrm{wt}_{\mathsf{H}}(x) = 5$

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Hamming Support of a Code

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Note the Hamming weight of a code coincides in all three point of views.

Generalized Hamming Weights

Let $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ be a linear code of of dimension k. Then for any $r \in \{1, \ldots, k\}$ the r-th generalized weight is given by

$$\mathrm{d}^r(\mathcal{C}) := \min\{\mathrm{wt}_{\mathsf{H}}(\mathcal{D}) \mid \mathcal{D} \leq \mathcal{C} \ \text{ and } \ \dim(\mathcal{D}) = r\}.$$

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$\mathrm{d}^1_\mathsf{H}(\mathcal{C})$: minimum weight of a subcode generated by one nonzero codeword

$\mathcal{D}_1 = \langle(1,1,0,0)\rangle$, $\mathcal{D}_2 = \langle(0,1,1,0)\rangle$ ... $\mathcal{D}_7 = \langle(1,1,1,1)\rangle$

$\mathrm{wt}_\mathsf{H}(\mathcal{D}_1) = 2$ $\mathrm{wt}_\mathsf{H}(\mathcal{D}_2) = 2$ ... $\mathrm{wt}_\mathsf{H}(\mathcal{D}_7) = 4$

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$\mathrm{d}^2_{\mathsf{H}}(\mathcal{C})$: minimum weight of a subcode generated by two independent codewords

$$\mathcal{D}_1 = \left\langle \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \right\rangle, \quad \mathcal{D}_2 = \left\langle \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \right\rangle, \dots,$$

$\mathrm{wt}_{\mathsf{H}}(\mathcal{D}_1) = 3 \qquad \mathrm{wt}_{\mathsf{H}}(\mathcal{D}_2) = 4, \ \dots$

Generalized Hamming Weights

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$$\implies \mathrm{d}^2_{\mathsf{H}}(\mathcal{C}) = 3$$

$\mathrm{d}^3_{\mathsf{H}}(\mathcal{C})$: minimum weight of a subcode generated by three independent codewords

$$\implies \mathrm{d}^3_{\mathsf{H}}(\mathcal{C}) = 4$$

Let $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ be a linear code of of dimension k. Then for any $r \in \{1, \ldots, k\}$ the r-th generalized weight is given by

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$$\mathrm{d}^1_{\mathsf{H}}(\mathcal{C}) = 2 \qquad \mathrm{d}^2_{\mathsf{H}}(\mathcal{C}) = 3 \qquad \mathrm{d}^3_{\mathsf{H}}(\mathcal{C}) = 4$$

Let $\mathcal{C}$ be a linear code of dimension k. Then we have

$$\mathrm{d}_{\mathsf{H}}(\mathcal{C}) = \mathrm{d}^1_{\mathsf{H}}(\mathcal{C}) < \mathrm{d}^2_{\mathsf{H}}(\mathcal{C}) < \ldots < \mathrm{d}^k_{\mathsf{H}}(\mathcal{C}) = \mathrm{wt}_{\mathsf{H}}(\mathcal{C}).$$

Note: Then we can derive

$$\mathrm{d}_{\mathsf{H}}(\mathcal{C}) \leq \mathrm{wt}_{\mathsf{H}}(\mathcal{C}) - (k-1) \overset{\text{non-degenerate}}{=} n - k + 1.$$

Outline

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- type (dimension) k
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- rank K
- ...

For $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ of rank K define for all $r = 1, \dots K$

$$\mathrm{d}_{\mathsf{L}}^r(\mathcal{C}) = \min\left\{|\mathrm{supp}_{\mathsf{L}}(\mathcal{D})| \mid \mathcal{D} \leq \mathcal{C};\ \mathrm{rk}(\mathcal{D}) = r\right\}.$$

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- For a code $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ of rank K we want something like

$$\text{d}_\mathsf{L}(\mathcal{C}) = \text{d}_\mathsf{L}^1(\mathcal{C}) < \text{d}_\mathsf{L}^2(\mathcal{C}) < \ldots < \text{d}_\mathsf{L}^K(\mathcal{C}) = \text{wt}_\mathsf{L}(\mathcal{C}).$$

Meet-Support

$$\mathrm{supp}_{\mathsf{L},\mathsf{meet}}(\mathcal{C}) := \left(\min_{c\in\mathcal{C}}\{\max\{\mathrm{wt}_{\mathsf{L}}(c_1), 0\}\}, \dots, \min_{c\in\mathcal{C}}\{\max\{\mathrm{wt}_{\mathsf{L}}(c_n), 0\}\}\right)$$

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Problem Does not satisfy the desired properties!

Example Consider $\mathcal{C}=\langle(1,2)\rangle\subset(\mathbb{Z}/9\mathbb{Z})^2$ with minimum distance $\mathrm{d}_{\mathsf{L}}(\mathcal{C})=3$.

$$\mathcal{C}=\{(0,0),(1,2),(2,4),(3,6),(4,8),(5,1),(6,3),(7,5),(8,7)\}$$

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$$\mathrm{supp}_{\mathsf{L},\mathsf{meet}}(\mathcal{C}) := \left(\min_{c\in\mathcal{C}}\{\max\{\mathrm{wt}_{\mathsf{L}}(c_1),0\}\},\dots,\min_{c\in\mathcal{C}}\{\max\{\mathrm{wt}_{\mathsf{L}}(c_n),0\}\}\right)$$

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Hence, $\mathrm{supp}_{\mathsf{L},\mathsf{meet}}(\mathcal{C}) = (1,1) \Longrightarrow \mathrm{d}^1_{\mathsf{L},\mathsf{meet}}(\mathcal{C}) = 2 < \mathrm{d}_{\mathsf{L}}(\mathcal{C}) = 3.$ We wanted $\mathrm{d}_{\mathsf{L}}(\mathcal{C}) = \mathrm{d}^1_{\mathsf{L},\mathsf{meet}}(\mathcal{C})$.

Join Support

$$\mathrm{supp}_{\mathsf{L},\mathsf{join}}(\mathcal{C}) := (\max\{\mathrm{wt}_{\mathsf{L}}(c_1) \mid c \in \mathcal{C}\}, \ldots, \max\{\mathrm{wt}_{\mathsf{L}}(c_n) \mid c \in \mathcal{C}\})$$

Increasing Property

$$\mathrm{d}_{\mathsf{L}}(\mathcal{C}) \leq \mathrm{d}^1_{\mathsf{L},\mathsf{join}}(\mathcal{C}) < \mathrm{d}^2_{\mathsf{L},\mathsf{join}}(\mathcal{C}) < \cdots < \mathrm{d}^K_{\mathsf{L},\mathsf{join}}(\mathcal{C}) \leq \mathrm{wt}_{\mathsf{L},\mathsf{join}}(\mathcal{C})$$

Bound on the Minimum Distance

$$\mathrm{d}_{\mathsf{L}}(\mathcal{C}) \leq \left\lfloor \frac{p}{2} \right\rfloor p^{s-1}(n-K+1)$$

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Bound on the Minimum Distance

$$\mathrm{d}_{\mathsf{L}}(\mathcal{C}) \leq \left\lfloor \frac{p}{2} \right\rfloor p^{s-1}(n - K + 1)$$

- Most of the times better than Shiromoto's bound (at least less sparse).
- Can only be attained for $p = 3$.
- Have to go through all the codewords.

Column Support

Support on matrices $A = (a_1^\top, \dots, a_n^\top)$ where $a_i \in (\mathbb{Z}/p^s\mathbb{Z})^m$.

$$\mathrm{supp}_{\mathsf{col}}(A) := (\max\{\mathrm{supp}(a_1)\}, \dots, \max\{\mathrm{supp}(a_n)\})$$

$$\mathrm{supp}_{\mathrm{col}}(\mathcal{C}) := \min\{\mathrm{supp}_{\mathrm{col}}(G) \mid \langle G \rangle = \mathcal{C}\}$$

$$\mathrm{d}^r_{\mathsf{col}}(\mathcal{C}) := \min\{|\mathrm{supp}_{\mathrm{col}}(\mathcal{D})| \mid \mathcal{D} \leq \mathcal{C}, \mathrm{rk}(\mathcal{D}) = r\}, \text{ for } r = 1, \dots, K$$

NEW! Based on the generator matrices of a subcode of rank r.

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\mathrm{supp}_{\mathsf{col}}(A) &:= (\max\{\mathrm{supp}(a_1)\}, \ldots, \max\{\mathrm{supp}(a_n)\}) \\
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Column Support

Support on matrices $A = (a_1^\top, \dots, a_n^\top)$ where $a_i \in (\mathbb{Z}/p^s\mathbb{Z})^m$.

$$\mathrm{supp}_{\mathsf{col}}(A) := (\max\{\mathrm{supp}(a_1)\}, \dots, \max\{\mathrm{supp}(a_n)\})$$

$$\mathrm{supp}_{\mathrm{col}}(\mathcal{C}) := \min\{\mathrm{supp}_{\mathrm{col}}(G) \mid \langle G \rangle = \mathcal{C}\}$$

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Problem Still sparse and hard to control subcodes of smaller ranks

Outline

Filtration of Chain Rings

Consider a code $\mathcal{C} \subseteq (\mathbb{Z}/p^s\mathbb{Z})^n$ of rank K. Define for each $i = 0, \dots, s-1$ the i-th filtration of $\mathcal{C}$ as

$$\mathcal{C}_i := \mathcal{C} \cap \langle p^i \rangle.$$

The maximum Lee weight in each filtration is $M_i := \left\lfloor \frac{p^{s-i}}{2} \right\rfloor p^i$

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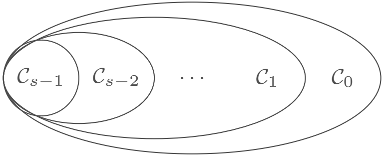

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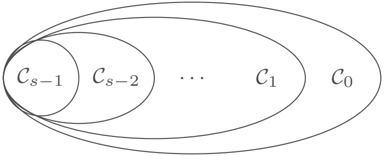

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Generalized Lee Distances

For every $r = 1, \dots, s$, define the r-th *generalized Lee distance* as

$$\mathrm{d}_{\mathsf{L}}^{r}(\mathcal{C}) = \mathrm{d}_{\mathsf{L}}(\mathcal{C}_{r-1}).$$

Example over $\mathbb{Z}/27\mathbb{Z}$

Consider a code $\mathcal{C} \subseteq (\mathbb{Z}/27\mathbb{Z})^4$ of subtype $(1, 1, 1)$ with generator matrix

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Defining more Parameters

Example Let $\mathcal{C} \subseteq (\mathbb{Z}/27\mathbb{Z})^5$ be the code of subtype $(1,1,0)$ and $\mathrm{d}_\mathsf{L}(\mathcal{C}) = 9$, generated by

$$G_0 = \begin{pmatrix} 1 & 1 & 4 & 20 & 9 \\ 0 & 3 & 9 & 18 & 9 \end{pmatrix}.$$

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ℓ_i : maximum prime power in each row i and
n': maximum number of appearances in each row of this prime number.

$$\mathrm{d_L}(\mathcal{C}) \leq \begin{cases} p^{s-\ell+\sigma} + (n - K - n')\, M_{s-\ell+\sigma} & \text{if } \ell \geq 1, \\ p^\sigma + (n-K)M_\sigma & \text{else.} \end{cases}$$

Examples

$\mathcal{C} = \langle G \rangle \subseteq (\mathbb{Z}/27\mathbb{Z})^5$ with $\mathrm{d}_\mathsf{L}(\mathcal{C}) = 9$

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Thank you for your attention!