COUNTEREXAMPLES TO F. MOREL'S CONJECTURE ON $\boldsymbol{\pi}_0^{\mathbb{A}^1}$

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Abstract. We exhibit counterexamples to F. Morel's conjecture on the $\mathbb{A}^1$-invariance of the sheaves of connected components of $\mathbb{A}^1$-local spaces.

For a scheme S, we denote by $\mathcal{S}pc(S)$ the ∞-category $\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_S)$ of Nisnevich sheaves on smooth S-schemes. An object of $\mathcal{S}pc(S)$ is called an S-space. The Morel–Voevodsky ∞-category $\mathcal{H}(S)$ is the full sub-∞-category of $\mathcal{S}pc(S)$ consisting of $\mathbb{A}^1$-local S-spaces. Recall that an S-space $\mathcal{X}$ is $\mathbb{A}^1$-local if, for every $U \in \mathrm{Sm}_S$, the map $\mathrm{pr}_1^* : \mathcal{X}(U) \to \mathcal{X}(U \times \mathbb{A}^1)$ is an equivalence (in the ∞-category of spaces). The obvious inclusion admits a left adjoint $\mathrm{L}_{\mathbb{A}^1} : \mathcal{S}pc(S) \to \mathcal{H}(S)$.

Notation 1. Let $\boldsymbol{\pi}_0$ be the 0-th truncation functor in the ∞-topos $\mathcal{S}pc(S)$ and, for $i \geq 1$, let $\boldsymbol{\pi}_i$ be the composition of $\boldsymbol{\pi}_0$ with the i-th loop space functor. For an S-space $\mathcal{X}$, we set $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\mathcal{X}) = \boldsymbol{\pi}_0(\mathrm{L}_{\mathbb{A}^1}(\mathcal{X}))$ and, if $\mathcal{X}$ is pointed, we set $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathcal{X}) = \boldsymbol{\pi}_i(\mathrm{L}_{\mathbb{A}^1}(\mathcal{X}))$.

Now, assume that S is the spectrum of a perfect field k. In his monograph [Mor12], F. Morel proved that the sheaves $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathcal{X})$ are $\mathbb{A}^1$-invariant in the strongest possible sense for every pointed k-space $\mathcal{X}$ and every integer $i \geq 1$. (See [Mor12, Definition 1.7 & Theorem 1.9] for a precise statement.) The case $i = 0$ was left open and, in [Mor12, Conjecture 1.12], F. Morel expressed the hope that $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\mathcal{X})$ is also $\mathbb{A}^1$-invariant for every k-space $\mathcal{X}$.

We will exhibit counterexamples to F. Morel's conjecture. Curiously, our counterexample is based on an old counterexample to a different conjecture of F. Morel, namely his $\mathbb{A}^1$-connectivity conjecture over a general base, which we disproved in [Ayo06].

Definition 2. Let X be a smooth k-scheme. We say that X is $\mathbb{A}^1$-discrete if, for any extension K/k, every k-morphism $\mathbb{A}^1_K \to X$ factors as the structural projection $\mathbb{A}^1_K \to \mathrm{Spec}(K)$ followed by a K-point $\mathrm{Spec}(K) \to X$ of the scheme X.

We have the following well known fact.

Lemma 3. *Let X be a smooth k-scheme. Assume that X is proper and $\mathbb{A}^1$-discrete. Then, for a dense open immersion $j : V \to U$ of smooth k-schemes, composition with j gives a bijection*

$$\hom(U, X) \simeq \hom(V, X).$$

Proof. See [Deb01, Corollary 1.44]. □

We now give a general construction of $\mathbb{A}^1$-local k-spaces.

Construction 4. Let X be a smooth k-scheme and let $\mathcal{M} \in \mathcal{H}(X)$ be an $\mathbb{A}^1$-local X-space. We denote by $\Phi_X(M)$ the presheaf on Sm_k given informally by

$$U \in (\mathrm{Sm}_k)^{\mathrm{op}} \;\mapsto \coprod_{s:U\to X} \Gamma(U; s^*\mathcal{M})$$

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where, for every morphism $s : U \to X$, we write $s^* : \mathcal{H}(X) \to \mathcal{H}(U)$ for the pullback of $\mathbb{A}^1$-local spaces. More precisely, the functor Φ_X is given by the following composition

$$\mathcal{H}(X) \overset{(\star)}{\simeq} \mathrm{Sect}^{\mathrm{cocart}}\left(\int_{s:U\to X\in((\mathrm{Sm}_k)_{/X})^{\mathrm{op}}} \mathcal{H}(U) \middle/ ((\mathrm{Sm}_k)_{/X})^{\mathrm{op}}\right) \xrightarrow{\Gamma} \mathrm{Psh}((\mathrm{Sm}_k)_{/X}) \xrightarrow{\mathrm{ff}_{X,\sharp}} \mathrm{Psh}(\mathrm{Sm}_k)$$

where $\mathrm{ff}_{X,\sharp}$ is the left Kan extension along the forgetful functor $\mathrm{ff}_X : (\mathrm{Sm}_k)_{/X} \to \mathrm{Sm}_k$ and $(\star)$ is the obvious equivalence between $\mathcal{H}(X)$ and the ∞-category of cocartesian sections of the cocartesian fibration classified by $(U \to X) \mapsto \mathcal{H}(U)$.

Remark 5. Denote by $p : X \to \mathrm{Spec}(k)$ the structural projection. It can be shown that $\mathrm{L}_{\mathbb{A}^1}(\Phi_X(\mathcal{M}))$ is equivalent to $p_\sharp(\mathcal{M})$ where $p_\sharp : \mathcal{H}(X) \to \mathcal{H}(k)$ is the left adjoint to the pullback functor p^*. We will not prove this here since we do not need it.

Proposition 6. *Keep the notations as in Construction 4. Assume that X is proper and $\mathbb{A}^1$-discrete. Then $\Phi_X(\mathcal{M})$ belongs to $\mathcal{H}(k)$, i.e., it has Nisnevich descent and is $\mathbb{A}^1$-invariant.*

Proof. We check the Brown–Gersten property for $\Phi_X(\mathcal{M})$. Clearly, the space $\Phi_X(\mathcal{M})(\emptyset)$ is contractible since $\mathcal{H}(\emptyset)$ is the final category. If $U = U_1 \coprod U_2$, a map $s : U \to X$ is the union of two maps $s_1 : U_1 \to X$ and $s_2 : U_2 \to X$, and we have

$$\Gamma(U; s^*\mathcal{M}) = \Gamma(U_1, s_1^*\mathcal{M}) \times \Gamma(U_2; s_2^*\mathcal{M}).$$

This yields an equivalence $\Phi_X(\mathcal{M})(U) \simeq \Phi_X(\mathcal{M})(U_1) \times \Phi_X(\mathcal{M})(U_2)$. Consider now a Nisnevich square of smooth k-schemes:

$$\begin{array}{ccc} U' & \xrightarrow{j'} & V' \\ \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e} \\ U & \xrightarrow{j} & V. \end{array}$$

We need to show that

$$\begin{array}{ccc} \coprod_{s':U'\to X} \Gamma(U'; s'^*\mathcal{M}) & \longleftarrow & \coprod_{t':V'\to X} \Gamma(V'; t'^*\mathcal{M}) \\ \uparrow & & \uparrow \\ \coprod_{s:U\to X} \Gamma(U; s^*\mathcal{M}) & \longleftarrow & \coprod_{t:V\to X} \Gamma(V; t^*\mathcal{M}) \end{array}$$

is cartesian in the ∞-category of spaces. Using what we just said, we may assume that V and V' are connected, and that j and j' have dense images. By Lemma 3, we have bijections $\hom(V, X) \simeq \hom(U, X)$ and $\hom(V', X) \simeq \hom(U', X)$. Thus, we may rewrite the above square as follows:

$$\begin{array}{ccc} \coprod_{t':V'\to X} \Gamma(U'; t'^*\mathcal{M}) & \longleftarrow & \coprod_{t':V'\to X} \Gamma(V'; t'^*\mathcal{M}) \\ \uparrow & & \uparrow \\ \coprod_{t:V\to X} \Gamma(U; t^*\mathcal{M}) & \longleftarrow & \coprod_{t:V\to X} \Gamma(V; t^*\mathcal{M}). \end{array}$$

The obvious map $\hom(V, X) \to \hom(V', X)$ is injective and the vertical arrows in the above square factor through the summands

$$\coprod_{t:V\to X} \Gamma(U'; t^*\mathcal{M}) \qquad \text{and} \qquad \coprod_{t:V\to X} \Gamma(V'; t^*\mathcal{M})$$

respectively. Thus, we are left to show that the square

$$\begin{array}{ccc} \Gamma(U'; t^*\mathcal{M}) & \longleftarrow & \Gamma(V'; t^*\mathcal{M}) \\ \uparrow & & \uparrow \\ \Gamma(U; t^*\mathcal{M}) & \longleftarrow & \Gamma(V; t^*\mathcal{M}) \end{array}$$

is cartesian for every $t : V \to X$. This is obvious, since $t^*\mathcal{M}$ belongs to $\mathcal{H}(V)$ by design.

It remains to see that $\Phi_X(\mathcal{M})$ is $\mathbb{A}^1$-local. Using that X is $\mathbb{A}^1$-discrete, we see that the map $\Phi_X(\mathcal{M})(U) \to \Phi_X(\mathcal{M})(U \times \mathbb{A}^1)$ is the coproduct over $s : U \to X$ of the maps

$$\Gamma(U; s^*\mathcal{M}) \to \Gamma(U \times \mathbb{A}^1; s^*\mathcal{M}).$$

These are equivalences since the $s^*\mathcal{M}$'s belong to $\mathcal{H}(U)$ by design. $\square$

Next, we describe the sheaves of connected components of the $\mathbb{A}^1$-local k-spaces we just constructed.

Proposition 7. *Keep the notations as in Construction 4. Assume that X is proper and $\mathbb{A}^1$-discrete. Then the sheaf $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\Phi_X(\mathcal{M}))$ is given by*

$$U \mapsto \coprod_{s:U\to X} \Gamma(U; \boldsymbol{\pi}_0^{\mathbb{A}^1}(s^*\mathcal{M})).$$

In particular, $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\Phi_X(\mathcal{M}))$ is $\mathbb{A}^1$-invariant if and only if $\boldsymbol{\pi}_0^{\mathbb{A}^1}(s^*\mathcal{M})$ is $\mathbb{A}^1$-invariant for every morphism $s : U \to X$. (In particular, a necessary condition is that $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\mathcal{M})$ is $\mathbb{A}^1$-invariant.)

Proof. Since $\Phi_X(\mathcal{M})$ is $\mathbb{A}^1$-local, we have $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\Phi_X(\mathcal{M})) = \boldsymbol{\pi}_0(\Phi_X(\mathcal{M}))$. Thus, it is the sheafification of the ordinary presheaf of sets

$$U \mapsto \coprod_{s:U\to X} \pi_0\Gamma(U; s^*\mathcal{M}),$$

which we denote by F. Let G be the presheaf described in the statement. We will show that G is a Nisnevich sheaf and that the obvious map $F \to G$ induces isomorphisms on stalks. This will prove the first statement.

The proof that G is a Nisnevich sheaf is identical to the proof that $\Phi_X(\mathcal{M})$ has Nisnevich descent: we check that G takes a Nisnevich square to a cartesian square of sets, and this boils down to the property that $\boldsymbol{\pi}_0^{\mathbb{A}^1}(t^*\mathcal{M})$ is a Nisnevich sheaf for every $t : V \to X$, which is true by design. To prove that $F \to G$ induces an isomorphism on stalks, we fix a henselian essentially smooth k-scheme W. The map $F(W) \to G(W)$ is then the coproduct, over $r : W \to X$, of the maps $\pi_0\Gamma(W; r^*\mathcal{M}) \to \Gamma(W; \boldsymbol{\pi}_0(r^*\mathcal{M}))$, which are obviously isomorphisms.

For the last statement, using that X is $\mathbb{A}^1$-discrete, we see that $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\Phi_X(\mathcal{M}))$ is $\mathbb{A}^1$-invariant if and only if, for every $s : U \to X$, the map

$$\Gamma(U; \boldsymbol{\pi}_0^{\mathbb{A}^1}(s^*\mathcal{M})) \to \Gamma(U \times \mathbb{A}^1; \boldsymbol{\pi}_0^{\mathbb{A}^1}(s^*\mathcal{M}))$$

is an equivalence. Applying this property for a composite $q \circ s : V \to X$ with $q : V \to U$ a smooth morphism, we deduce immediately that the previous condition is equivalent to asking that $\boldsymbol{\pi}_0^{\mathbb{A}^1}(s^*\mathcal{M})$ is $\mathbb{A}^1$-invariant for every $s : U \to X$. □

It is now clear how to produce counterexamples to F. Morel's conjecture.

Construction 8. Let X be a smooth, proper and $\mathbb{A}^1$-discrete k-scheme. (For example, X can be an abelian variety or a product of curves of genera ≥ 1.) Let $\mathcal{M} \in \mathcal{H}(X)$ be an $\mathbb{A}^1$-local X-space such that $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\mathcal{M})$ is not $\mathbb{A}^1$-invariant. Then Proposition 7 insures that $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\Phi_X(M))$ is also not $\mathbb{A}^1$-invariant. An explicit example of such an $\mathcal{M}$ can be obtained as follows, assuming that X has dimension ≥ 3. Let $Y \subset X$ be a closed integral surface and $o \in Y(k)$ a rational point admitting a Zariski neighbourhood $N \subset Y$ which is also an étale neighbourhood of the singular point of the projective surface $S \subset \mathbb{P}^3$ defined by the equation $w(x^3 - y^2z) + F(x, y, z) = 0$, where F is a general homogeneous polynomial of degree 4. This is the surface used in [Ayo06] to produce a counterexample to Morel's connectivity conjecture. In particular, we have a complex of abelian groups $\mathcal{K}_{S,1}^{M,!}$ on Sm_S, concentrated in homological degrees 0 and 1 and sending an irreducible $T \in \mathrm{Sm}_S$ to the two-term Gersten complex

$$k(T)^\times \to \coprod_{x \in T^{(1)}} \mathbb{Z}.$$

We write also $\mathcal{K}_{S,1}^{M,!}$ for the associated Eilenberg–Mac Lane space which is an object of $\mathcal{H}(S)$. Letting $i : N \to X$ be the obvious inclusion and $e : N \to S$ the étale neighbourhood of the singular point of S, we set $\mathcal{M} = i_* e^* \mathcal{K}_{S,1}^{M,!}$. As was shown in [Ayo06], the sheaf $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\mathcal{M})$ restricted to a neighbourhood of o in X is not $\mathbb{A}^1$-invariant.

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