THE SLICE FILTRATION ON DM(k) DOES NOT PRESERVE GEOMETRIC MOTIVES

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In this appendix we give an unconditional argument for the following (un)-property of the slice filtration on $\mathbf{DM}(k)$:

Proposition 0.1 — *The slice filtration on* $\mathbf{DM}(k)$ *does not preserve geometric motives.*

Recall that the slice filtration is a sequence of transformations:

$$\nu^{\geq n} \to \nu^{\geq n-1} \to \cdots \to \mathrm{id}$$

where $\nu^{\geq n}(M) = \tau(M(-n))(n)$ with $\tau : \mathbf{DM}(k) \to \mathbf{DM}_{\mathrm{eff}}(k)$ the right adjoint to the full embedding $\mathbf{DM}_{\mathrm{eff}}(k) \subset \mathbf{DM}(k)$. When M is effective (e.g.. the motive $\mathrm{M}(X)$ of a smooth projective variety X) we have $\tau(M(-n)) = \underline{\mathsf{Hom}}_{\mathrm{eff}}(\mathbb{Z}(n), M)$ where $\underline{\mathsf{Hom}}_{\mathrm{eff}}$ stands for the internal hom in $\mathbf{DM}_{\mathrm{eff}}(k)$. We will prove the following:

Proposition 0.2 — *Assume that k is big enough. There exists a smooth projective k-variety X such that* $\underline{\mathsf{Hom}}_{\mathrm{eff}}(\mathbb{Z}(1), \mathrm{M}(X))$ *is not a geometric motive.*

We will implicitly assume k algebraically closed and work with rational coefficients.

1. Compacity in $\mathbf{DM}_{\mathrm{eff}}(k)$

Recall the following classical notions (see [Neeman. *Triangulated categories*]):

Definition 1.1 — *Let $\mathcal{T}$ be a triangulated category with arbitrary infinite sums. An object $U \in \mathcal{T}$ is called* compact *if the functor* $\hom_{\mathcal{T}}(U, -) : \mathcal{T} \to \mathcal{A}b$ *commutes with sums. The category $\mathcal{T}$ is* compactly generated, *if there exists a set $\underline{G}$ of compact objects in $\mathcal{T}$ such that the family of triangulated functors* $\hom_{\mathcal{T}}(U[n], -)$, *where $U \in \underline{G}$ and $n \in \mathbb{Z}$, is conservative (that is detects isomorphisms).*

If $\mathcal{T}$ is compactly generated by $\underline{G}$ then the subcategory $\mathcal{T}_{\mathrm{comp}}$ of compact objects is the pseudo-abelian envelop of the triangulated sub-category of $\mathcal{T}$ generated by $\underline{G}$.

Let $(A_n)_{n\in\mathbb{N}}$ be an inductive system in $\mathcal{T}$. Its *homotopy colimit* is the cone of:

$$(\mathrm{id} - s) : \oplus_{n\in\mathbb{N}} A_n \to \oplus_{n\in\mathbb{N}} A_n$$

where s is the composition $A_{n_0} \to A_{n_0+1} \to \oplus_{n\in\mathbb{N}} A_n$ on the factor A_{n_0}. It is denoted by $hocolim_{n\in\mathbb{N}} A_n$. We have the following lemma:

Lemma 1.2 — *If $U \in \mathcal{T}$ is compact, then* $\hom_{\mathcal{T}}(U, -)$ *commutes with $\mathbb{N}$-indexed homotopy colimits.*

The following proposition is well-known. It follows immediately from the commutation of Nisnevic hyper-cohomology with infinite sums of complexes:

Proposition 1.3 — *The category* $\mathbf{DM}_{\mathrm{eff}}(k)$ *is compactly generated by the set of* $\mathrm{M}(X)$ *with X in a set representing isomorphism classes of smooth k-varieties. Moreover the sub-category* $\mathbf{DM}^{\mathrm{gm}}_{\mathrm{eff}}(k)$ *is the sub-category of compact objects of* $\mathbf{DM}_{\mathrm{eff}}(k)$.

2. Finite generation in $\mathbf{HI}(k)$

Recall that $\mathbf{DM}_{\text{eff}}(k)$ admits a natural t-structure whose heart $\mathbf{HI}(k)$ is the category of homotopy invariant Nisnevic sheaves with transfers. For an object $M \in \mathbf{DM}_{\text{eff}}(k)$ we denote $h_i(M)$ the truncation with respect to this t-structure. Recall that $h_i(M)$ is simply the i-th homology sheaf of the complex M. We will also write $h_i(X)$ for $h_i(\mathrm{M}(X))$ when X is a smooth k-variety. We make the following definition:

Definition 2.1 — *A sheaf $F \in \mathbf{HI}(k)$ is called* finitely generated *if there exists a smooth variety X and a surjection $h_0(X) \twoheadrightarrow F$.*

It is clear that the property of being finitely generated is stable by quotients. It is also stable by extensions. Indeed, let $F \subset G$ in $\mathbf{HI}(k)$ such that F and G/F are finitely generated and chose surjections $a: \ h_0(X) \twoheadrightarrow F$ and $b: \ h_0(Y) \twoheadrightarrow G/F$. There exists a Nisnevic cover $U \to Y$ such that $b_{|U}$ lifts to $b': \ h_0(U) \longrightarrow G$. We get in this way a surjection $a \coprod b': \ h_0(X \coprod U) \twoheadrightarrow G$.

Assuming that k is countable we say that a sheaf F is countable if for any smooth k-variety X the set $F(X)$ is countable. Note the following technical lemma:

Lemma 2.2 — *Let F be a sheaf in $\mathbf{HI}(k)$ which is countable. There exists a chain $(S_n)_{n\in\mathbb{N}}$ of finitely generated sub-sheaves of F such that $F = \cup_{n\in\mathbb{N}} S_n$.*

Proof. Consider the set $\mathcal{S}$ whose elements are the finitely generated sub-sheaves of F. This set is countable as every finitely generated sub-sheaf of F is the image of a map $a : h_0(X) \to F$ with X a smooth k-variety and $a \in F(X)$. Fix a bijection $b : \mathbb{N} \xrightarrow{\sim} \mathcal{S}$ and denote $S_n = \sum_{i=0}^{n} b(i)$. We clearly have that $F = \cup_{n\in\mathbb{N}} S_n$. □

As a corollary we have the following:

Proposition 2.3 — *Let F be a countable sheaf in $\mathbf{HI}(k)$. Suppose that $\hom_{\mathbf{HI}(k)}(F, -)$ commutes with $\mathbb{N}$-indexed colimits. Then F is finitely generated.*

Proof. By lemma 2.2 we can write $F = colim_{n\in\mathbb{N}}(S_n)$ with S_n finitely generated sub-sheaves of F. Using $\hom(F, F) = colim_{n\in\mathbb{N}} \hom(F, S_n)$ one can find $n_0 \in \mathbb{N}$ such that the identity of F factors trough the inclusion $S_{n_0} \subset F$. This implies that $F = S_{n_0}$. □

Remark 2.4 — By working a little bit more, one shows under the hypothesis of 2.3 that F is finitely presented in the sense that there exists an exact sequence:

$$h_0(X_2) \longrightarrow h_0(X_1) \longrightarrow F \longrightarrow 0$$

with X_1 and X_2 two smooth k-varieties.

3. conclusion

Using propositions 1.3 and 2.3 we can prove the following:

Theorem 3.1 — *Let M be a geometric motive in $\mathbf{DM}_{\text{eff}}(k)$. Suppose that $h_i(M) = 0$ for $i < 0$. Then $h_0(M)$ is finitely generated*[1].

[1] In fact $h_0(M)$ is even finitely presented (see 2.4)

Proof. The motive M being geometric, it is defined over a finitely generated field (in particular a countable one). Hence, we may assume our base field k countable. It follows that the sheaves $h_i(M)$ are countable. This can be proved by reducing to the case $M = \mathrm{M}(X)$ with X a smooth k-variety and using Voevodsky's identification $\mathrm{M}(X) = C_*\mathbb{Z}_{tr}(X)$ with C_* the Suslin-Voevodsky complex.

By 2.3 we need only to check that $\hom_{\mathbf{HI}(k)}(h_0(M), -)$ commutes with $\mathbb{N}$-colimits. Let $(A_n)_{n\in\mathbb{N}}$ be an inductive system and denote A its colimit. First, remark that A is also the homotopy colimit of $(A_n)_{n\in\mathbb{N}}$ in $\mathbf{DM}_{\mathrm{eff}}(k)$. Indeed, one has an exact triangle:

$$\oplus A_n \xrightarrow{\mathrm{id}-s} \oplus A_n \longrightarrow hocolim A_n \longrightarrow$$

It is easy to see that the morphism of sheaves $\mathrm{id} - s$ is injective. It follows that $hocolim_{n\in\mathbb{N}} A_n$ is the co-kernel of $\mathrm{id} - s$ which is canonically isomorphic to A.

Having this in mind, we can write:

$$\hom_{\mathbf{HI}(k)}(h_0(M), colim A_n) \stackrel{1}{=} \hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(h_0(M), hocolim A_n)$$
$$\stackrel{2}{=} \hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(M, hocolim A_n) \stackrel{3}{=} colim \hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(M, A_n)$$
$$\stackrel{4}{=} colim \hom_{\mathbf{HI}(k)}(h_0(M), A_n)$$

Equality (1) follows from the above discussion. Equalities (2) and (4) follow from the condition $h_i(M) = 0$ for $i < 0$. Equality (3) is the compactness of M. This proves the theorem. □

Let X be a smooth projective variety of dimension d. Using [Voevodsky, *triangulated category of motives.* Theorem 4.2.2 and Proposition 4.2.3] we have:

- the sheaf $h_i(\underline{\mathsf{Hom}}_{\mathrm{eff}}(\mathbb{Z}(1)[2], M(X)))$ is zero for $i < 0$,
- the sheaf $h_0(\underline{\mathsf{Hom}}_{\mathrm{eff}}(\mathbb{Z}(1)[2], M(X)))$ is canonically isomorphic to the Nisnevic sheaf $\mathrm{CH}^{d-1}_{/X}$ associated to the pre-sheaf: $U \rightsquigarrow \mathrm{CH}^{d-1}(U \times_k X)$.

To prove 0.2 it suffices by 3.1 to find a smooth projective variety X of dimension $d = 3$ such that $\mathrm{CH}^{d-1}_{/X}$ is not finitely generated. To do this, we will construct a quotient of $\mathrm{CH}^{d-1}_{/X}$ which is constant but not finitely generated.

Definition 3.2 — *Let U be a smooth k-scheme. A cycle $[Z] \in \mathrm{CH}^{d-1}(U \times_k X)$ is said to be U-algebraically equivalent to zero if there exist a smooth and connected U-scheme $V \to U$, a finite correspondence of degree zero $\sum_i n_i[T_i] \in Cor(V/U)$ (i.e. $n_i \in \mathbb{Z}$ and $t_i : T_i \to U$ are finite and surjective) and a cycle $[W] \in \mathrm{CH}^{d-1}(V \times_k X)$ such that $[Z]$ is rationally equivalent to $\sum_i n_i(t_i \times id_X)_*[W \cap (T_i \times X)]$.*

We denote $\mathrm{NS}^{d-1}(U \times_k X)_U$ the quotient of $\mathrm{CH}^{d-1}(U \times_k X)$ with respect to the U-algebraic equivalence. We let also $\mathrm{NS}^{d-1}_{/X}$ be the Nisnevic sheaf associated to the pre-sheaf $U \rightsquigarrow \mathrm{NS}^{d-1}(U \times_k X)_U$.

We have clearly a surjective morphism $\mathrm{CH}^{d-1}_{/X} \to \mathrm{NS}^{d-1}_{/X}$. The latter sheaf is constant (because our base field k is algebraically closed). Indeed, for any finitely generated extension $k \subset K$ we have $\mathrm{NS}^{d-1}_{/X}(K) = \mathrm{NS}^{d-1}(X \otimes_k K)$. It is a well-known fact that the Neron-Severi group is invariant by extensions of an algebraically closed field.

Now, it is easy to see that a constant sheaf is finitely generated if and only if its group of sections over k is a finite dimensional $\mathbb{Q}$-vector space (using that a map from $h_0(X)$ to a constant sheaf factors trough $\mathbb{Q}_{tr}(\pi_0(X))$ with $\pi_0(X)$ the set of connected components of the variety X). We are done since $\mathrm{NS}^2(X)$ is not finite dimensional for a generic quintic in $\mathbb{P}^4$.