$\mathbb{A}^1$-ALGEBRAIC TOPOLOGY (AFTER F. MOREL)

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Abstract. These notes are based on lectures delivered at the University of Zurich in the academic year 2022/23 on F. Morel's $\mathbb{A}^1$-algebraic topology. A condensed version of these lectures was also presented at the Graduate Summer School on Motivic Homotopy Theory at PCMI in July 2024. The original goal of the lectures was to understand F. Morel's fundamental theorem asserting that strongly invariant sheaves of abelian groups are strictly invariant. The proof of this result is difficult and intricate. In these notes, we reorganize and simplify Morel's original arguments in order to make the proof more accessible and transparent. In addition, we include a proof of F. Morel's identification of Milnor–Witt K-theory with certain unstable homotopy groups of motivic spheres. In particular, we develop Milnor–Witt K-theory from scratch, constructing residue and norm homomorphisms and verifying their standard compatibilities.

Contents

Key words and phrases. $\mathbb{A}^1$-algebraic topology, strictly invariant sheaves, Milnor–Witt K-theory, motivic spaces.

1. INTRODUCTION

These notes are intended as an introduction to motivic homotopy theory, with a particular emphasis on unstable phenomena. Our primary reference is F. Morel's book *$\mathbb{A}^1$-Algebraic Topology over a Field* [Mor12]. In this section, we give an informal overview of the types of results covered in these notes.

1.1. **The basic setup.**

We work over a ground field k, which we fix once and for all. Unfortunately, for the main results, we have to assume that k is perfect. We denote by Sm_k the category of smooth k-varieties. The main objects of study in $\mathbb{A}^1$-algebraic topology are the so-called motivic spaces, so we start by defining them.

Definition 1.1.1. A family of étale morphisms of qcqs schemes $(e_i : X_i \to X)_i$ is called a Nisnevich cover if for every point $x \in X$ there exist an index i_0 and a point $x' \in X_{i_0}$ over x such that the induced morphism $\kappa(x) \to \kappa(x')$ is an isomorphism. This defines the Nisnevich topology on schemes.

Remark 1.1.2. The Nisnevich topology is finer than the Zariski topology but coarser than the étale topology. It is the natural topology to use in motivic homotopy theory. The main historic reason is that algebraic K-theory satisfies Nisnevich descent [Nis89] but not étale descent. So, unless otherwise stated, the word "sheaf" will always mean "sheaf in the Nisnevich topology".

Definition 1.1.3. A k-space is a Nisnevich sheaf of Kan complexes on Sm_k. (Here we follow the ∞-categorical terminology and call "Nisnevich sheaf" a presheaf admitting Nisnevich descent.) The ∞-category of k-spaces is denoted by $\mathrm{Spc}(k)$.

Example 1.1.4.

(1) If X is a smooth k-variety, then X defines a sheaf of sets on Sm_k by the Yoneda embedding, and hence a discrete k-space. This gives a fully faithful embedding $\mathrm{Sm}_k \hookrightarrow \mathrm{Spc}(k)$.

(2) If A is a Kan complex, then we may consider the constant sheaf on Sm_k associated to A, namely the one given by $X \mapsto A^{\pi_0(X)}$. (To prove that $X \mapsto A^{\pi_0(X)}$ admits Nisnevich descent, it suffices to verify the Brown–Gersten property [MV99, §3.1, page 100, Definition 1.13 & Proposition 1.16] which follows immediately from the fact that a dense open in a connected smooth k-variety is also connected.) This gives a fully faithful embedding $\mathcal{S} \hookrightarrow \mathrm{Spc}(k)$ from the ∞-category $\mathcal{S}$ of Kan complexes. If no confusion can arise, we also write A for the constant sheaf associated to the Kan complex A.

Definition 1.1.5.

(1) Let $\mathcal{X}$ be a presheaf of Kan complexes on Sm_k. We say that $\mathcal{X}$ is $\mathbb{A}^1$-invariant if, for every $U \in \mathrm{Sm}_k$, the obvious map $\mathcal{X}(U) \to \mathcal{X}(\mathbb{A}^1 \times U)$ is an equivalence.

(2) A motivic space is an $\mathbb{A}^1$-invariant k-space. Explicitly, a presheaf of Kan complexes $\mathcal{X}$ is a motivic space if it admits Nisnevich descent and is $\mathbb{A}^1$-invariant. We denote by $\mathcal{H}(k) \subset \mathrm{Spc}(k)$ the full subcategory spanned by motivic spaces; $\mathcal{H}(k)$ is known as the Morel–Voevodsky ∞-category.

Example 1.1.6. The constant sheaf on Sm_k associated to a Kan complex is $\mathbb{A}^1$-invariant, and hence determines a motivic space. Thus, we have a fully faithful embedding $\mathcal{S} \hookrightarrow \mathcal{H}(k)$.

Remark 1.1.7. It is difficult to give explicit interesting examples of motivic spaces. Instead, we have a very inexplicit procedure of turning every k-space into a motivic one in a "minimal way". This is the so-called $\mathbb{A}^1$-localisation functor

$$\mathrm{L}_{\mathbb{A}^1} : \mathrm{Spc}(k) \to \mathcal{H}(k)$$

which is left adjoint to the obvious inclusion. (See [MV99, §2.3, page 93, Lemma 3.20] and Subsection 2.4.1 below. This functor is not left exact, but preserves finite direct products.) A map of k-spaces is said to be an $\mathbb{A}^1$-equivalence if it is sent to an equivalence by $\mathrm{L}_{\mathbb{A}^1}$. We often use the symbol "$\simeq_{\mathbb{A}^1}$" to designate an $\mathbb{A}^1$-equivalence.

Definition 1.1.8. Let $\mathcal{X}$ be a k-space. We set

$$\pi_0^{\mathbb{A}^1}(\mathcal{X}) := \pi_0(\mathrm{L}_{\mathbb{A}^1}\mathcal{X}).$$

Explicitly, $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is the Nisnevich sheafification of the presheaf $U \mapsto \pi_0\Gamma(U; \mathrm{L}_{\mathbb{A}^1}\mathcal{X})$.

Definition 1.1.9. Let $\mathcal{X}$ be a pointed k-space. We set, for $n \geqslant 0$,

$$\pi_n^{\mathbb{A}^1}(\mathcal{X}) := \pi_n(\mathrm{L}_{\mathbb{A}^1}\mathcal{X}).$$

Explicitly, $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is the Nisnevich sheafification of the presheaf $U \mapsto \pi_n\Gamma(U; \mathrm{L}_{\mathbb{A}^1}\mathcal{X})$. This is a sheaf of pointed sets for $n = 0$, a sheaf of groups for $n = 1$, and a sheaf of abelian groups for $n \geqslant 2$.

1.2. **The results we cover in these notes.** Given a pointed motivic space $\mathcal{X}$, one would like to:

- compute the sheaves $\pi_n^{\mathbb{A}^1}(\mathcal{X})$;
- understand how the space $\mathcal{X}$ is "built" from the $\pi_n^{\mathbb{A}^1}(\mathcal{X})$'s via the machinery of Postnikov towers and obstruction theory.

Even in classical homotopy theory, these are outstanding and notoriously difficult problems in general (e.g., for the spheres), so we are not going to do this for motivic spaces. Instead, we will

(1) determine the general structure of the $\pi_n^{\mathbb{A}^1}(\mathcal{X})$'s;
(2) compute $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ for some simple motivic spaces and in some limited range;
(3) construct Postnikov towers in motivic spaces.

We now give more details. Below, we use the word "sheaf" as a shorthand for "Nisnevich sheaf on Sm_k". Also, sheaf cohomology is always taken with respect to the Nisnevich topology.

Definition 1.2.1.

(1) A sheaf of sets F is said to be $\mathbb{A}^1$-invariant if

$$F(U) \to F(\mathbb{A}^1 \times U)$$

is a bijection for every $U \in \mathrm{Sm}_k$.

(2) A sheaf of groups F is said to be strongly $\mathbb{A}^1$-invariant if

$$\mathrm{H}^i(U;F) \to \mathrm{H}^i(\mathbb{A}^1_U;F)$$

is an isomorphism for $i \in \{0,1\}$ and every $U \in \mathrm{Sm}_k$.

(3) A sheaf of abelian groups F is said to be n-strongly $\mathbb{A}^1$-invariant if

$$\mathrm{H}^i(U;F) \to \mathrm{H}^i(\mathbb{A}^1_U;F)$$

is an isomorphism for $0 \leqslant i \leqslant n$ and every $U \in \mathrm{Sm}_k$. It is said to be strictly $\mathbb{A}^1$-invariant if it is n-strongly $\mathbb{A}^1$-invariant for every n.

One of the first nontrivial results we prove in these notes is the following.

Proposition 1.2.2. *Let $\mathcal{X}$ be a pointed k-space. Then $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant and, for $n \geqslant 2$, $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is $n-1$-strongly $\mathbb{A}^1$-invariant.*

Proposition 1.2.2 is an easy consequence of the unstable $\mathbb{A}^1$-connectivity theorem which we prove in Section 2. It is thus available over any ground field. Assuming that k is perfect, we have much better results. Indeed, one of the main theorems that we prove in these notes asserts that the notions of n-strong $\mathbb{A}^1$-invariance, for $n \geqslant 1$, are all equivalent.

Theorem 1.2.3 (Morel)**.** *Assume that k is perfect. If a sheaf of abelian groups is strongly $\mathbb{A}^1$-invariant, then it is strictly $\mathbb{A}^1$-invariant.*

The proof of Theorem 1.2.3 is given in Section 3. This theorem immediately yields that $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is strictly $\mathbb{A}^1$-invariant for all $n \geqslant 2$. With some work, one can also improve upon Proposition 1.2.2 when $n = 1$. In total, we obtain the following.

Theorem 1.2.4 (Morel)**.** *Assume that k is perfect. Let $\mathcal{X}$ be a pointed k-space. Then $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant and, for $n \geqslant 2$, $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is strictly $\mathbb{A}^1$-invariant.*

Remark 1.2.5. If F is a sheaf of groups, then F is strongly $\mathbb{A}^1$-invariant if and only if the associated Nisnevich local classifying space $\mathrm{B}(F)$ is motivic. Similarly, if F is a sheaf of abelian groups, then F is n-strongly $\mathbb{A}^1$-invariant if and only if the associated Nisnevich local Eilenberg–Mac Lane space $\mathrm{K}(F,n)$ is motivic. Thus the previous results readily imply the following.

Corollary 1.2.6 (Morel)**.** *Assume that k is perfect. Let $\mathcal{X}$ be a connected pointed motivic space. Then the Postnikov tower*

$$\cdots \tau_{\leqslant n}\mathcal{X} \to \cdots \to \tau_{\leqslant 1}\mathcal{X} \to \tau_{\leqslant 0}\mathcal{X}$$

in the ∞-topos $\mathrm{Spc}(k) = \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k)$ consists of motivic spaces. Thus we have a good notion of Postnikov towers in $\mathcal{H}(k)$.

Another important structural result on strictly $\mathbb{A}^1$-invariant sheaves of abelian groups is purity.

Theorem 1.2.7 (Morel)**.** *Assume that k is perfect. Let X be the localisation of a smooth k-variety at a point x of codimension d. Then, if F is a strictly $\mathbb{A}^1$-invariant sheaf, we have*

$$\mathrm{H}^i_x(X;F) = 0 \qquad \textit{for} \qquad i \neq d.$$

Theorem 1.2.7 implies the existence of resolutions by Cousin complexes for strictly $\mathbb{A}^1$-invariant sheaves. It will be obtained as a byproduct of the proof of Theorem 1.2.3. As for computations of homotopy sheaves, we offer the following.

Theorem 1.2.8 (Morel)**.** *For $n \geqslant 2$, there is a canonical isomorphism:*

$$\pi^{\mathbb{A}^1}_{n-1}(\mathbb{A}^n \smallsetminus o) \simeq \mathrm{K}^{\mathrm{MW}}_n,$$

where the right-hand side is the n-th sheaf of unramified Milnor–Witt K-theory.

Milnor–Witt K-theory is explicitly defined by generators and relations. The theory is developed from scratch in Section 4. In particular, we construct residue and norm homomorphisms by adapting the classical approach of Milnor, Bass–Tate and Kato for Milnor K-theory. For simplicity, we stay away of the characteristic 2 case which requires special arguments. Theorem 1.2.8 is proved in Section 5.

1.3. **Notation and conventions.** We work over a ground field k which we assume perfect starting from Section 3. By "k-variety" we mean a separated finite type k-scheme. We denote by Sch_k the category of k-varieties and by Sm_k its subcategory spanned by the smooth ones. We often consider presheaves on Sm_k (or Sch_k) which we tacitly left Kan extend to pro-objects in Sm_k (or Sch_k). A prominent class of such pro-objects are the "essentially smooth k-schemes", a convenient terminology introduced by F. Morel. These are the k-schemes that can be written as a limit of a cofiltered inverse system of affine étale schemes over a given smooth k-variety. Examples of essentially smooth k-schemes include Zariski and henselian localisations at points of smooth k-varieties. The dimension of a scheme is always taken to be its Krull dimension.

Often, we work with the Nisnevich topology which we abbreviate by "nis". Occasionally, we also consider the Zariski topology which we abbreviate by "zar". Unless stated otherwise, the word "sheaf" means "sheaf for the Nisnevich topology". Similarly, unless stated otherwise, sheaf cohomology is considered with respect to the Nisnevich topology. In particular, we write "$\mathrm{H}^n(X; -)$" and "$\mathrm{R}\Gamma(X; -)$" instead of "$\mathrm{H}^n_{\mathrm{nis}}(X; -)$" and "$\mathrm{R}\Gamma_{\mathrm{nis}}(X; -)$", except in a few places where both the Zariski and Nisnevich topologies are considered.

Although not strictly necessary for these notes, we follow the general trend and use the language of higher category theory following the book of Lurie [Lur09]. In particular, we denote by $\mathcal{S}$ the ∞-category of Kan complexes (aka., spaces, homotopy types, anima). Given a category C, we denote by $\mathrm{Psh}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$ the ∞-category of presheaves of Kan complexes on C and, if C is endowed with a topology τ, we denote by $\mathrm{Shv}_\tau(C) \subset \mathrm{Psh}(C)$ the full subcategory spanned by the τ-sheaves, i.e., those sheaves admitting τ-descent. The obvious inclusion admits an exact left adjoint $\mathrm{L}_\tau : \mathrm{Psh}(C) \to \mathrm{Shv}_\tau(C)$ called the sheafification functor.

We set $\mathrm{Spc}(k) := \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k)$ and call its objects k-spaces. The ∞-category $\mathrm{Spc}(k)$ is an ∞-topos. It admits internal notions of truncations, homotopy sheaves, classifying spaces, Eilenberg–Mac Lane spaces, etc., and we use the usual notations pertaining to these notions, namely $\tau_{\leq n}$, $\tau_{\geq n}$, $\pi_n(-)$, $\mathrm{B}(-)$, $\mathrm{K}(-, n)$, etc. Given a k-space $\mathcal{X}$, we denote by $\mathcal{X}_+$ the pointed k-space obtained by freely adjoining a base point to $\mathcal{X}$. Given two pointed k-spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $[\mathcal{X}, \mathcal{Y}]_{\mathrm{nis}}$ for the set $\pi_0 \mathrm{Map}_{\mathrm{Spc}(k)_*}(\mathcal{X}, \mathcal{Y})$.

We denote by $\mathcal{H}(k)$ the Morel–Voevodsky ∞-category. Objects of $\mathcal{H}(k)$ are called motivic spaces. We have inclusion functors $\mathcal{H}(k) \hookrightarrow \mathrm{Spc}(k) \hookrightarrow \mathrm{Psh}(\mathrm{Sm}_k)$ with left adjoints

$$\mathrm{Psh}(\mathrm{Sm}_k) \xrightarrow{\mathrm{L}_{\mathrm{nis}}} \mathrm{Spc}(k) \xrightarrow{\mathrm{L}_{\mathbb{A}^1}} \mathcal{H}(k).$$

We also write $\mathrm{L}_{\mathrm{mot}} : \mathrm{Psh}(\mathrm{Sm}_k) \to \mathcal{H}(k)$ for the composite functor $\mathrm{L}_{\mathbb{A}^1} \circ \mathrm{L}_{\mathrm{nis}}$. Given a presheaf of pointed Kan complexes $\mathcal{X}$, we set $\pi^{\mathbb{A}^1}_n(\mathcal{X}) := \pi_n(\mathrm{L}_{\mathrm{mot}}\mathcal{X})$. If $\mathcal{X}$ is a pointed k-space, then $\pi^{\mathbb{A}^1}_n(\mathcal{X}) =$

$\pi_n(\mathrm{L}_{\mathbb{A}^1}\mathcal{X})$. If $\mathcal{X}$ is a motivic space, then $\pi_n^{\mathbb{A}^1}(\mathcal{X}) = \pi_n(\mathcal{X})$. Also, given two pointed k-spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $[\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1}$ for $[\mathrm{L}_{\mathbb{A}^1}\mathcal{X}, \mathrm{L}_{\mathbb{A}^1}\mathcal{Y}]_{\mathrm{nis}} \simeq [\mathcal{X}, \mathrm{L}_{\mathbb{A}^1}\mathcal{Y}]_{\mathrm{nis}}$.

2. The unstable $\mathbb{A}^1$-connectivity theorem

In this section, we prove the unstable $\mathbb{A}^1$-connectivity theorem of F. Morel (see [Mor12, Theorem 6.38]). As a consequence, we obtain that $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is $n-1$-strongly invariant for every pointed k-space $\mathcal{X}$ and every integer $n \geqslant 1$. In [Mor12], the unstable $\mathbb{A}^1$-connectivity theorem is obtained as a consequence of the weak $\mathbb{A}^1$-Hurewicz theorems [Mor12, Theorems 6.35 & 6.37]. In particular, this requires prior knowledge that $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant and that $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is strictly $\mathbb{A}^1$-invariant for $n \geq 2$, which is the content of Theorem 1.2.4. In comparison, our proof is much more direct and elementary, and in particular does not require the ground field to be perfect. In fact, it can be considered as a "destabilisation" of F. Morel's proof of the stable $\mathbb{A}^1$-connectivity theorem [Mor05b, Theorem 6.1.8].

2.1. **Connectivity in algebraic topology.** We start by reviewing the classical notions of connectivity and truncatedness in algebraic topology.

Definition 2.1.1. Let $n \geqslant 0$ be an integer. We say that a Kan complex X is n-connective if it is nonempty and if for every point $x \in X$, we have $\pi_i(X, x) = 0$ for $0 \leqslant i \leqslant n-1$. (If $n = 0$, the last condition is vacuous, so that 0-connectivity is equivalent to nonemptiness.) For convenience, we also declare that every Kan complex is -1-connective.

Definition 2.1.2. Let $n \geqslant 0$ be an integer. We say that a Kan complex X is n-truncated if for every point $x \in X$, we have $\pi_i(X, x) = 0$ for $i \geqslant n+1$. (In particular, a 0-truncated Kan complex is just a discrete set.) We say that X is -1-truncated if it is empty or contractible, and we say that X is -2-truncated if it is contractible.

Here is a basic fact from topology.

Lemma 2.1.3. *Let X be a Kan complex and $n \geqslant -2$ an integer. Then there exists a unique map $X \to \tau_{\leqslant n}X$ with n-truncated codomain and $n+1$-connective fibres.*

Remark 2.1.4.

(1) If $n = -2$, then $\tau_{\leqslant -2}X = *$.
(2) If $n = -1$, then

$$\tau_{\leqslant -1}X = \begin{cases} * & \text{if} \quad X \neq \emptyset, \\ \emptyset & \text{if} \quad X = \emptyset. \end{cases}$$

(3) In general, the map $\pi_0(X) \to \pi_0(\tau_{\leqslant n}X)$ is surjective and, for $x \in X$, the induced map $\pi_i(X, x) \to \pi_i(\tau_{\leqslant n}X, x)$ is an isomorphism if $0 \leqslant i \leqslant n$ and is the projection to a singleton if $i \geqslant n+1$.

Remark 2.1.5. Given a Kan complex X, we obtain a tower of Kan complexes

$$X \to \cdots \to \tau_{\leqslant n+1}X \to \tau_{\leqslant n}X \to \cdots \to \tau_{\leqslant -2}X$$

called the Postnikov tower. Moreover, we have an equivalence

$$X \xrightarrow{\sim} \lim_n \tau_{\leqslant n}X.$$

Furthermore, the fibres of $\tau_{\leqslant n+1}X \to \tau_{\leqslant n}X$ are Eilenberg–Mac Lane spaces for $n \geqslant 0$. (See for example [GJ99, Chapter VI, §2].)

2.2. **Local connectivity.** Fix a site (C, τ). We are mainly interested in the smooth Nisnevich site $(\mathrm{Sm}_k, \mathrm{nis})$ and the small Nisnevich site $(\mathrm{\acute{E}t}_X, \mathrm{nis})$ of a scheme X.

Definition 2.2.1. Let $\mathcal{X}$ be a presheaf of Kan complexes on C. We say that $\mathcal{X}$ is locally n-connective (with $n \geqslant 0$) if the following conditions are satisfied.

1. For every $U \in C$, there is a τ-cover $(U_\alpha \to U)_\alpha$ such that $\mathcal{X}(U_\alpha)$ is non-empty for all α.
2. For every $U \in C$ and every $x \in \mathcal{X}(U)$, the sheafification of
$$V \in C_{/U} \mapsto \pi_i(\mathcal{X}(V), x)$$
is zero for all $0 \leq i \leq n - 1$.

We also declare that every presheaf of Kan complexes is locally -1-connective.

Remark 2.2.2.

1. The first condition in Definition 2.2.1 is equivalent to asking that the map of presheaves $\pi_0(\mathcal{X}) \to *$ induces an epimorphism after sheafification.
2. If (C, τ) has enough points, then $\mathcal{X}$ is locally n-connective if and only if for enough points ξ of (C, τ), the Kan complexes $\xi^*(\mathcal{X})$ are n-connective.
3. Taking for τ the chaotic topology yields the notion of n-connective presheaves of Kan complexes.

Definition 2.2.3. Let $\mathcal{X}$ be a presheaf of Kan complexes on C. We say that $\mathcal{X}$ is locally n-truncated (with $n \geqslant 0$) if, for every $U \in C$ and every $x \in \mathcal{X}(U)$, the sheafification of
$$V \in C_{/U} \mapsto \pi_i(\mathcal{X}(V), x)$$
is zero for $i \geqslant n+1$. (In particular, $\mathcal{X}$ is 0-truncated if and only if its sheafification $\mathrm{L}_\tau \mathcal{X}$ is discrete.) We say that $\mathcal{X}$ is -1-truncated if $\mathrm{L}_\tau \mathcal{X}$ is a subsheaf of the final sheaf. We say that $\mathcal{X}$ is -2-truncated if $\mathrm{L}_\tau \mathcal{X}$ is the final sheaf.

Remark 2.2.4.

1. If (C, τ) has enough points, then $\mathcal{X}$ is locally n-truncated if and only if for enough points ξ of (C, τ), the Kan complexes $\xi^*(\mathcal{X})$ are n-truncated.
2. Taking for τ the chaotic topology yields the notion of n-truncated presheaves of Kan complexes.

Remark 2.2.5. The notions of local connectivity and local truncatedness for a presheaf of Kan complexes $\mathcal{X}$ depend only on the sheafification of $\mathcal{X}$. Moreover, when restricted to sheaves on C, these notions coincide with the intrinsic notions of connectivity and truncatedness in the ∞-topos $\mathrm{Shv}_\tau(C)$; see [Lur09, Definitions 5.5.6.1 & 6.5.1.10]. Thus, when working exclusively with sheaves, we shall use the expressions "n-connective" and "n-truncated" instead of "locally n-connective" and "locally n-truncated".

Lemma 2.2.6. *Let $\mathcal{X}$ be a sheaf of Kan complexes on C and $n \geqslant -2$ an integer. There exists a unique map $\mathcal{X} \to \tau_{\leqslant n}\mathcal{X}$ in the ∞-topos* $\mathrm{Shv}_\tau(C)$ *with n-truncated codomain and $n+1$-connective fibres.*

Proof. This follows immediately from [Lur09, Definition 6.5.1.10]. □

Remark 2.2.7. As in Remark 2.1.5, given a sheaf of Kan complexes $\mathcal{X}$ on C, we have a functorial tower
$$\mathcal{X} \to \cdots \to \tau_{\leqslant n}\mathcal{X} \to \cdots \to \tau_{\leqslant -2}\mathcal{X}$$

in $\mathrm{Shv}_\tau(C)$, called the Postnikov tower. However, in general, the natural map

$$\mathcal{X} \to \lim_n \tau_{\leqslant n}\mathcal{X}$$

is not always an equivalence in the ∞-category $\mathrm{Shv}_\tau(C)$. Fortunately, for the Nisnevich topology, this convergence issue does not arise.

Proposition 2.2.8. *In* $\mathrm{Spc}(k) := \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_k)$*, Postnikov towers converge, i.e., given a* k*-space* $\mathcal{X}$*, the natural map*

$$\mathcal{X} \to \lim_n \tau_{\leqslant n}\mathcal{X}$$

is an equivalence in $\mathrm{Spc}(k)$.

Proof. This follows from [MV99, §2.1, page 60, Theorem 1.37] and the usual bound on the Nisnevich cohomological dimension. For a modern account, see [Lur09, Proposition 7.2.1.10]. □

2.3. Connectivity in $\mathbb{A}^1$-algebraic topology.

Definition 2.3.1. Let $\mathcal{X}$ be a k-space. We say that $\mathcal{X}$ is $\mathbb{A}^1$-n-connective (with $n \geqslant -1$) if $\mathrm{L}_{\mathbb{A}^1}\mathcal{X}$ is n-connective as a k-space.

Remark 2.3.2.

(1) Every k-space is $\mathbb{A}^1$-(-1)-connective.

(2) A k-space $\mathcal{X}$ is $\mathbb{A}^1$-0-connective if and only if the map $\pi_0^{\mathbb{A}^1}(\mathcal{X}) \to *$ is an epimorphism of Nisnevich sheaves. It is $\mathbb{A}^1$-1-connective if and only if $\pi_0^{\mathbb{A}^1}(\mathcal{X}) \simeq *$.

(3) Assume that $n \geqslant 1$. A k-space $\mathcal{X}$ is $\mathbb{A}^1$-n-connective if and only if $\pi_0^{\mathbb{A}^1}(\mathcal{X}) \simeq *$ and, for every $x \in \mathrm{L}_{\mathbb{A}^1}\mathcal{X}(k)$, we have $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x) = *$ for $0 \leqslant i \leqslant n-1$.

We can now state the main result of this section.

Theorem 2.3.3 (F. Morel's unstable connectivity theorem)**.** *Let* $\mathcal{X}$ *be a* k*-space. If* $\mathcal{X}$ *is* n*-connective then* $\mathcal{X}$ *is also* $\mathbb{A}^1$*-*n*-connective.*

Here is an equivalent way of stating this theorem.

Theorem 2.3.4 (F. Morel's unstable connectivity theorem)**.** *The* $\mathbb{A}^1$*-localisation functor*

$$\mathrm{L}_{\mathbb{A}^1} : \mathrm{Spc}(k) \to \mathrm{Spc}(k)$$

preserves n*-connective objects for every* $n \geqslant -1$.

Before going into the proof, we give some applications.

Corollary 2.3.5. *Colimits in* $\mathrm{Spc}(k)$ *preserve* $\mathbb{A}^1$*-*n*-connective objects. Equivalently, colimits in* $\mathcal{H}(k)$ *preserve* n*-connective objects.*

Proof. This follows immediately from Theorem 2.3.3 using that $\mathrm{L}_{\mathbb{A}^1} : \mathrm{Spc}(k) \to \mathcal{H}(k)$ commutes with colimits and that colimits in $\mathrm{Spc}(k)$ preserve n-connective objects. □

Corollary 2.3.6. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be two pointed* k*-spaces. Assume that* $\mathcal{X}$ *is* $\mathbb{A}^1$*-*m*-connective and that* $\mathcal{Y}$ *is* $\mathbb{A}^1$*-*n*-connective. Then* $\mathcal{X} \wedge \mathcal{Y}$ *is* $\mathbb{A}^1$*-*$m+n$*-connective.*

Proof. The functor $\mathrm{L}_{\mathbb{A}^1} : \mathrm{Spc}(k) \to \mathrm{Spc}(k)$ commutes with finite products. (For instance, this can be seen using Lemma 2.4.3 below.) It follows that the map

$$\mathrm{L}_{\mathbb{A}^1}(\mathcal{X} \wedge \mathcal{Y}) \to \mathrm{L}_{\mathbb{A}^1}(\mathrm{L}_{\mathbb{A}^1}\mathcal{X} \wedge \mathrm{L}_{\mathbb{A}^1}\mathcal{X})$$

is an equivalence. The desired result follows now from Theorem 2.3.3. □

Definition 2.3.7. Let X be a pointed Kan complex. The n-connective cover of X, for $n \geqslant -1$, is defined to be the fibre of the canonical map $X \to \tau_{\leqslant n-1}X$. (The case $n = -1$ is redundant.) It is denoted by $\tau_{\geqslant n}X$. This generalises readily to sheaves of pointed Kan complexes on a site (C, τ).

Corollary 2.3.8. *Let $\mathcal{X}$ be a pointed motivic space. Then, for every $n \geqslant -1$, the n-connective cover $\tau_{\geqslant n}X$ of X, taken in the ∞-topos* $\mathrm{Spc}(k)$*, is a motivic space.*

Proof. Indeed, since $\mathcal{X}$ is motivic, the obvious map $\tau_{\geqslant n}\mathcal{X} \to \mathcal{X}$ factors through the $\mathbb{A}^1$-localisation of $\tau_{\geqslant n}\mathcal{X}$ as follows:

$$\tau_{\geqslant n}\mathcal{X} \to \mathrm{L}_{\mathbb{A}^1}(\tau_{\geqslant n}\mathcal{X}) \to \mathcal{X}.$$

By the $\mathbb{A}^1$-connectivity theorem, $\mathrm{L}_{\mathbb{A}^1}(\tau_{\geqslant n}\mathcal{X})$ is still n-connective. Thus, the second map factors through $\tau_{\geqslant n}\mathcal{X}$. This shows that $\tau_{\geqslant n}\mathcal{X}$ is a direct summand of $\mathrm{L}_{\mathbb{A}^1}(\tau_{\geqslant n}\mathcal{X})$, and hence is motivic as needed. □

Remark 2.3.9. Given a motivic space $\mathcal{X}$, it is natural to wonder if the truncations $\tau_{\leqslant n}\mathcal{X}$, for $n \geqslant -2$, are also motivic. The case $n = -2$ is obviously true. The case $n = -1$ is also true since any subsheaf of the final sheaf is $\mathbb{A}^1$-invariant. The case $n = 0$ asks if the Nisnevich sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant. This was a conjecture of F. Morel which turned out to be untrue as we show in Subsection 2.5. In some sense, the $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is the only obstruction. In general, we will see that the $\tau_{\leqslant n}\mathcal{X}$'s are all motivic provided that $\mathcal{X}$ is $\mathbb{A}^1$-connected (i.e. has trivial $\pi_0^{\mathbb{A}^1}(\mathcal{X})$).

For later use, we record the following result.

Corollary 2.3.10. *Let $\mathcal{X}$ be a pointed k-space. Then, for $n \geqslant 1$, the sheaf $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is $n-1$-strongly $\mathbb{A}^1$-invariant.*

Proof. A sheaf of abelian groups F on Sm_k is m-strongly $\mathbb{A}^1$-invariant if and only if the associated Eilenberg–Mac Lane space $\mathrm{K}(F, m)$ is motivic. (This is an easy fact which we establish in Proposition 3.1.3. We stress that the Eilenberg–Mac Lane space $\mathrm{K}(F, m)$ is taken in the ∞-topos $\mathrm{Spc}(k)$.) Without loss of generality, we may assume that $\mathcal{X}$ is motivic. For $n > 1$, we have a fibre sequence

$$\mathrm{K}(\pi_n(\mathcal{X}), n-1) \to \tau_{\geqslant n+1}\mathcal{X} \to \tau_{\geqslant n}\mathcal{X}$$

in $\mathrm{Spc}(k)$. Since $\tau_{\geqslant n}\mathcal{X}$ and $\tau_{\geqslant n+1}\mathcal{X}$ are motivic by Corollary 2.3.8, the k-space $\mathrm{K}(\pi_n(\mathcal{X}), n-1)$ is also motivic. (Indeed, the property of being motivic is preserved under limits.) □

Remark 2.3.11. The previous corollary is only a preliminary result. Later, assuming k perfect, we will see that $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is 1-strongly invariant and that $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is strictly $\mathbb{A}^1$-invariant for $n \geqslant 2$.

2.4. **Proof of the $\mathbb{A}^1$-connectivity theorem.** Let $\mathcal{X}$ be a pointed k-space which is n-connective. We need to show that $\mathrm{L}_{\mathbb{A}^1}\mathcal{X}$ is also n-connective. We show that F. Morel's argument in the stable setting [Mor05b] can be adapted to the unstable setting. In particular, as in loc. cit., our proof consists of two steps.

- Step 1: generic n-connectivity. In this step we show that the Kan complexes $\mathrm{L}_{\mathbb{A}^1}\mathcal{X}(K)$ are n-connective for all essentially smooth extensions K/k.
- Step 2: unramifiedness. In this step, we show that the sheaves $\pi_i^{\mathbb{A}^1}(\mathcal{X})$ are unramified, i.e., for every smooth k-variety X and every dense open $U \subset X$, the map

  $$\pi_i^{\mathbb{A}^1}(\mathcal{X})(X) \to \pi_i^{\mathbb{A}^1}(\mathcal{X})(U)$$

  is injective.

The first step relies on an explicit description of the motivic localisation endofunctor. The second step relies on Gabber's presentation lemma.

2.4.1. *Explicit motivic localisation.* Recall that

$$\mathrm{L}_{\mathrm{mot}} : \mathrm{Psh}(\mathrm{Sm}_k) \to \mathrm{Psh}(\mathrm{Sm}_k)$$

is the endofunctor enforcing Nisnevich descent and $\mathbb{A}^1$-invariance. We would like to give an explicit model for this functor. In fact, we will describe $\mathrm{L}_{\mathrm{mot}}$ in terms of $\mathrm{L}_{\mathrm{nis}}$, the functor enforcing Nisnevich descent. (The latter can be described using a Godement resolution.) Recall that we have a canonical cosimplicial scheme $\Delta^\bullet$ given in degree n by

$$\Delta^n = \mathrm{Spec}(k[t_0, \dots, t_n]/t_0 + \dots + t_n - 1)$$

which is isomorphic to the n-dimensional affine space $\mathbb{A}^n$.

Construction 2.4.1. Let $\mathcal{X}$ be a presheaf of Kan complexes on Sm_k. Consider the cosimplicial presheaf of Kan complexes $\mathcal{X}^{\Delta^\bullet}$ given by

$$[n] \in \mathbf{\Delta} \mapsto \mathcal{X}(- \times \Delta^n).$$

We then define

$$\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X}) := \operatorname*{colim}_{[n]\in\mathbf{\Delta}} \mathcal{X}^{\Delta^n}$$

to be the geometric realisation of $\mathcal{X}^{\Delta^\bullet}$. (Concretely, if we view $\mathcal{X}$ as a presheaf of simplicial sets, this is given by the diagonal of the presheaf of bisimplicial sets $\mathcal{X}_m(- \times \Delta^n)$ up to taking Kan replacements sectionwise.) The endofunctor $\mathcal{X} \mapsto \mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X})$ is usually called the Suslin–Voevodsky construction.

Lemma 2.4.2. *Let $\mathcal{X}$ be a presheaf of Kan complexes on* Sm_k*. Then* $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X})$ *is $\mathbb{A}^1$-invariant (see Definition 1.1.5). In fact,* $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X}) \simeq \mathrm{L}^{\mathrm{Psh}}_{\mathbb{A}^1}(\mathcal{X})$*, where*

$$\mathrm{L}^{\mathrm{Psh}}_{\mathbb{A}^1} : \mathrm{Psh}(\mathrm{Sm}_k) \to \mathrm{Psh}(\mathrm{Sm}_k)$$

is the endofunctor enforcing $\mathbb{A}^1$-invariance.

Proof. This is a very standard argument. We need to show that $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant, i.e., that

$$\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X})(U) \to \mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_U)$$

is an equivalence for every $U \in \mathrm{Sm}_k$. Using the multiplication of $\mathbb{A}^1$, we reduce to showing that the maps

$$s_0^*,\ s_1^* : \mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_U) \to \mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X})(U),$$

induced by the zero and unit sections of $\mathbb{A}^1$, are homotopic. This follows from explicit cohomotopies for the morphisms of cosimplicial schemes

$$s_0,\ s_1 : \Delta^\bullet \to \mathbb{A}^1 \times \Delta^\bullet.$$

Moreover, the fact that $\mathcal{X} \to \mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X})$ is an $\mathbb{A}^1$-equivalence follows from the fact that each of the maps $\mathcal{X} \to \mathcal{X}^{\Delta^n}$ is an $\mathbb{A}^1$-equivalence which is verified by an explicit $\mathbb{A}^1$-homotopy. □

Lemma 2.4.3. *Let $\mathcal{X}$ be a presheaf of Kan complexes on* Sm_k*. There is an equivalence*

$$\begin{aligned} \mathrm{L}_{\mathrm{mot}}\mathcal{X} &\simeq (\mathrm{Sing}^{\mathbb{A}^1} \circ \mathrm{L}_{\mathrm{nis}})^{\circ\infty}(\mathcal{X}) \\ &= \operatorname*{colim}_{n\in\mathbb{N}} (\mathrm{Sing}^{\mathbb{A}^1} \circ \mathrm{L}_{\mathrm{nis}})^{\circ n}(\mathcal{X}). \end{aligned}$$

Proof. The natural transformations $\mathrm{id} \to \mathrm{L}_{\mathrm{nis}}$ and $\mathrm{id} \to \mathrm{Sing}^{\mathbb{A}^1}$ give rise to a sequence

$$\mathcal{X} \to \mathrm{L}_{\mathrm{nis}}\mathcal{X} \to \mathrm{Sing}^{\mathbb{A}^1}\mathrm{L}_{\mathrm{nis}}\mathcal{X} \to \mathrm{L}_{\mathrm{nis}}\mathrm{Sing}^{\mathbb{A}^1}\mathrm{L}_{\mathrm{nis}}\mathcal{X} \to \mathrm{Sing}^{\mathbb{A}^1}\mathrm{L}_{\mathrm{nis}}\mathrm{Sing}^{\mathbb{A}^1}\mathrm{L}_{\mathrm{nis}}\mathcal{X} \to \cdots .$$

Thus, we have an equivalence

$$\operatorname*{colim}_{n\in\mathbb{N}}\, (\mathrm{Sing}^{\mathbb{A}^1} \circ \mathrm{L}_{\mathrm{nis}})^{\circ n}(\mathcal{X}) \simeq \operatorname*{colim}_{n\in\mathbb{N}}\, (\mathrm{L}_{\mathrm{nis}} \circ \mathrm{Sing}^{\mathbb{A}^1})^{\circ n} \circ \mathrm{L}_{\mathrm{nis}}(\mathcal{X})$$

where the left-hand side is $\mathbb{A}^1$-invariant and the right-hand side has Nisnevich descent. Moreover, the natural maps from $\mathcal{X}$ to both sides of the above equivalence are countable compositions of $\mathbb{A}^1$-equivalences and Nisnevich local equivalences. This easily yields the desired result. □

Proposition 2.4.4. *Let $\mathcal{X}$ be a k-space. The morphism of Nisnevich sheaves*

$$\pi_0(\mathcal{X}) \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$$

is surjective.

Proof. For every presheaf of Kan complexes $\mathcal{Z}$ on Sm_k, the map $\mathcal{Z} \to \mathrm{L}_{\mathrm{nis}}(\mathcal{Z})$ induces an isomorphism on the sheafified π_0's. Thus, it is enough to show that the map $\mathcal{Z} \to \mathrm{Sing}^{\mathbb{A}^1}(\mathcal{Z})$ induces an epimorphism on the presheaf π_0's. Inspecting Construction 2.4.1, it suffices to prove the following general fact: if $X_\bullet$ is a simplicial object in Kan complexes, the induced map $\pi_0(X_0) \to \pi_0(|X_\bullet|)$ is surjective. This follows readily from the fact that, up to Kan replacement, $|X_\bullet|$ is given by the diagonal of $X_\bullet$ viewed as a bisimplicial set. □

Corollary 2.4.5. *Let $\mathcal{X}$ be a k-space. If $\mathcal{X}$ is connected, then $\mathrm{L}_{\mathbb{A}^1}(\mathcal{X})$ is also connected.*

This shows that the $\mathbb{A}^1$-connectivity theorem holds for 1-connective k-spaces. So in the sequel, we may assume that $n \geqslant 2$. We will need the following.

Lemma 2.4.6. *Let $X_\bullet$ be a simplicial Kan complex. Assume that there is an integer $n \geqslant -1$ such that X_m is $n-m$-connective for every m. Then $|X|$ is n-connective.*

Proof. The cases $n = -1$ and $n = 0$ are obvious. The case $n = 1$ is also clear. Indeed, in this case X_0 is connected and so is $|X|$. Thus, we assume that $n \geqslant 2$ in the sequel and we set $Y = |X|$.

Step 1. Here we show that $\pi_1(Y)$ is trivial, i.e., that Y is simply connected. We do this "by hand". Recall that there is a functor

$$\begin{array}{ccc} \mathcal{S} & \to & \mathcal{G}pd \\ S & \mapsto & \tau_{\leqslant 1}S \end{array}$$

from Kan complexes to groupoids, sending a Kan complex to its fundamental groupoid. This functor is a left adjoint, and hence commutes with colimit. Thus, it suffices to prove the following.

Claim. Let $G_\bullet$ be a simplicial groupoid such that G_1 is connected and G_0 is connected and simply connected (i.e., $G_0 \simeq *$). Then $\mathrm{colim}_{[n]\in\mathbf{\Delta}}G_n \simeq *$.

To prove the claim, consider a connected groupoid $H = [*/B]$ and view it as a constant simplicial groupoid. Then, fixing equivalences $G_0 \simeq *$ and $G_1 \simeq [*/A_1]$, we have:

$$\begin{aligned} \mathrm{Map}(G_\bullet, H) &= \lim\nolimits_{[n]\in\mathbf{\Delta}} \mathrm{Map}(G_n, H) \\ &\simeq \lim\Big(\mathrm{Map}(*, H) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathrm{Map}([*/A_1], H) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \cdots\Big) \\ &\simeq \lim\Big([*/B] \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; [\mathrm{Hom}(A_1, B)/B] \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \cdots\Big). \end{aligned}$$

(Note that $\mathrm{Map}([*/A_1],[*/B])$ is the groupoid of natural transformations from $[*/A_1]$ to $[*/B]$ which is easily identified with $[\mathrm{Hom}(A_1,B)/B]$.) Now, we have a fully faithful inclusion

$$[*/B] \hookrightarrow [\mathrm{Hom}(A_1,B)/B]$$

sending the base point of $[*/B]$ to the zero morphism from A_1 to B. Moreover, the first two maps in the above cosimplicial diagram are given precisely by this fully faithful embedding. This proves that the above limit is equivalent to $[*/B] = H$ as needed.

Step 2. We now treat the general case. Recall that we are assuming that $n \geqslant 2$. From Step 1, we know that $Y = |X|$ is simply connected. Thus, to show that Y is n-connective, we may use the Hurewicz theorem to reduce to showing that the reduced homology groups $\widetilde{\mathrm{H}}_i(Y)$ vanish for $i < n$. The reduced homology of Y can be computed as the homology of the complex $\widetilde{\mathbb{Z}}[Y] := \mathbb{Z}[Y]/\mathbb{Z}$ which is quasi-isomorphic to the total complex $\mathrm{Tot}(\widetilde{\mathbb{Z}}[X_\bullet])$ associated to the double complex $\widetilde{\mathbb{Z}}[X_\bullet] := \mathbb{Z}[X_\bullet]/\mathbb{Z}$. By assumption, for every $m \geq 0$, the complex $\widetilde{\mathbb{Z}}[X_m]$ is concentrated in homological degree $\geqslant n - m$. We conclude using a spectral sequence argument. □

We use the previous lemma to study the effect of the Suslin–Voevodsky construction on local connectivity.

Definition 2.4.7. Let $\mathcal{X}$ be a presheaf of Kan complexes on Sm_k. We say that $\mathcal{X}$ is weakly n-connective if for every local henselian essentially smooth k-scheme X of dimension d, the Kan complex $\mathcal{X}(X)$ is $n - d$-connective.

Remark 2.4.8. Note that a locally n-connective presheaf of Kan complexes on Sm_k is weakly n-connective. Note also that the endofunctor $\mathrm{L}_{\mathrm{nis}}$ preserves weakly n-connective presheaves. The notion of weak n-connectivity was introduced in [Ayo21] for a very similar purpose, namely for proving that the $\mathbb{P}^1$-localisation endofunctor preserves étale-local connectivity.

Proposition 2.4.9. *Let $\mathcal{X}$ be presheaf of Kan complexes on* Sm_k*. If $\mathcal{X}$ is weakly n-connective, then* $\mathrm{Sing}^{\mathbb{A}^1}\mathrm{L}_{\mathrm{nis}}(\mathcal{X})$ *is also weakly n-connective.*

Proof. By replacing $\mathcal{X}$ with $\mathrm{L}_{\mathrm{nis}}(\mathcal{X})$, we may assume that $\mathcal{X}$ has Nisnevich descent. Given a local henselian essentially smooth k-scheme X of dimension d, we need to show that $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X})(X)$ is $n - d$-connective. Since

$$\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{X})(X) = |\mathcal{X}(X \times \Delta^\bullet)|,$$

it is enough, by Lemma 2.4.6, to show that $\mathcal{X}(X \times \mathbb{A}^m)$ is $n-d-m$-connective. Thus, the proposition follows from the next lemma. □

Lemma 2.4.10. *Let $\mathcal{X}$ be a k-space. If $\mathcal{X}$ is weakly n-connective, then for every essentially smooth k-scheme X of dimension d, the Kan complex $\mathcal{X}(X)$ is $n - d$-connective.*

Proof. We know the lemma for X local henselian, and we want to extend it to general essentially smooth k-schemes X. We argue by induction on $d = \dim(X)$. Without loss of generality, we may assume that $n - d \geqslant 1$. Replacing $\mathcal{X}$ by an iterated loop space, we reduce to showing that $\mathcal{X}(X)$ is connected. Given $\alpha_1, \alpha_2 \in \pi_0(\mathcal{X}(X))$, we need to show that $\alpha_1 = \alpha_2$. This holds over the generic points of X. Thus, we can find a dense open $U \subset X$ such that $\alpha_1|_U = \alpha_2|_U$. Assume that U maximal with this property. We claim that $U = X$. If not, let ξ be a generic point of $X \smallsetminus U$, and let $Y = X^h_\xi$

be the henselisation of X at ξ. Since Y has dimension $\leqslant d$ and $\mathcal{X}$ is weakly n-connective, we have $\alpha_1|_Y = \alpha_2|_Y$. Let $V = U \cup \{\xi\}$ considered as a pro-open in X. There is a pro-Nisnevich square

$$\begin{tikzcd}
Y \smallsetminus \{\xi\} \arrow[r] \arrow[d] & Y \arrow[d] \\
U \arrow[r] & V
\end{tikzcd}$$

which, by Nisnevich descent for $\mathcal{X}$, gives rise to a cartesian square of Kan complexes

$$\begin{tikzcd}
\mathcal{X}(V) \arrow[r] \arrow[d] & \mathcal{X}(U) \arrow[d] \\
\mathcal{X}(Y) \arrow[r] & \mathcal{X}(Y \smallsetminus \{\xi\}).
\end{tikzcd}$$

Since $\mathcal{X}(Y \smallsetminus \{\xi\})$ is simply connected (by induction), and $\alpha_1|_U = \alpha_2|_U$ and $\alpha_1|_Y = \alpha_2|_Y$, we see that $\alpha_1|_V = \alpha_2|_V$. This equality has to be true on a open neighbourhood of V in X which contradicts the maximality of U and finishes the proof. □

Definition 2.4.11. Let $\mathcal{X}$ be a presheaf of Kan complexes on Sm_k. We say that $\mathcal{X}$ is generically n-connective if for every essentially smooth field extension K/k, the Kan complex $\mathcal{X}(K)$ is n-connective.

Clearly, weak n-connectivity implies generic n-connectivity. For the remainder of the proof, we only need to remember the following.

Corollary 2.4.12. *Let $\mathcal{X}$ be a k-space. If $\mathcal{X}$ is n-connective, then $\mathrm{L}_{\mathbb{A}^1}(\mathcal{X})$ is generically n-connective.*

Proof. If $\mathcal{X}$ is n-connective, then it is weakly n-connective. By Proposition 2.4.9, the endofunctor $\mathrm{Sing}^{\mathbb{A}^1} \circ \mathrm{L}_{\mathrm{nis}}$ preserves weak n-connectivity. By Lemma 2.4.3, the same is true for $\mathrm{L}_{\mathrm{mot}}$. This finishes the proof since weak n-connectivity implies generic n-connectivity. □

2.4.2. *Applying Gabber's lemma.* We will prove the following result.

Theorem 2.4.13. *Let $\mathcal{X}$ be a pointed motivic space. Let X be a local essentially smooth k-scheme and $\eta \in X$ its generic point. Then the maps $\pi_n(\mathcal{X}(X)) \to \pi_n(\mathcal{X}(\eta))$ have trivial kernel for all $n \geqslant 0$. In particular, they are injective for $n \geqslant 1$.*

The unstable $\mathbb{A}^1$-connectivity theorem follows easily from Corollary 2.4.12 and Theorem 2.4.13.

Proof of the unstable $\mathbb{A}^1$-connectivity theorem. Let $\mathcal{X}$ be a k-space which is n-connective for some $n \geqslant 2$. Then $\mathcal{X}(k)$ is nonempty, and fixing a base point we may assume that $\mathcal{X}$ is pointed. By Corollary 2.4.12, we have $\pi_i^{\mathbb{A}^1}(\mathcal{X})(\eta) = *$ for $0 \leqslant i \leqslant n-1$ and η a generic point of an essentially smooth k-scheme. By Theorem 2.4.13, it follows that $\pi_i^{\mathbb{A}^1}(\mathcal{X})(X) = *$ for $1 \leqslant i \leqslant n-1$ and X a local henselian essentially smooth k-scheme. The same is true for $i = 0$ by Corollary 2.4.5. Thus $\mathcal{X}$ is $\mathbb{A}^1$-n-connective as needed. □

The proof of Theorem 2.4.13 relies on the following geometric result.

Theorem 2.4.14 (Gabber's presentation lemma)**.** *Let X be an irreducible smooth k-variety of dimension $d \geqslant 1$, $x \in X$ a point and $Z \subset X$ a nowhere dense closed subset. Then there exists an open neighbourhood $U \subset X$ of x and an étale morphism $f : U \to \mathbb{A}^1_V$, with $V \subset \mathbb{A}^{d-1}$ open, such that:*

(1) the induced map $Z \cap U \to V$ is finite;

(2) the induced map $Z \cap U \to f(Z \cap U)$ is an isomorphism and $Z \cap U = f^{-1}(f(Z \cap U))$.

Comments on the proof. When k is infinite, this is proven in [Gab94]; see also [CTHK97, Theorem 3.1.1]. The case of finite fields was treated in [HK20, Theorem 1.1 & Remark 1.3]. $\square$

Proof of Theorem 2.4.13. Replacing $\mathcal{X}$ with its iterated loop spaces, we reduce to the case $n = 0$. It is then enough to show the following.

Claim. Let X be an irreducible smooth k-variety with generic point η, and let $x \in X$. Given a class $\alpha \in \pi_0(\mathcal{X}(X))$ such that $\alpha|_\eta = 0$, there is an open neighbourhood U of x such that $\alpha|_U = 0$. (Here, 0 denotes the base point of $\mathcal{X}$.)

By assumption, there is a nowhere dense closed subset $Z \subset X$ such that $\alpha|_{X \smallsetminus Z} = 0$. Thus, we can lift α to a class

$$\alpha' \in \pi_0 \mathrm{fib}\{\mathcal{X}(X) \to \mathcal{X}(X \smallsetminus Z)\} =: \pi_0 \mathcal{X}_Z(X).$$

Apply Gabber's presentation lemma to get an étale morphism $f : U \to \mathbb{A}^1_V$ from an open neighbourhood $U \subset X$ of x satisfying the properties (1) and (2) in Theorem 2.4.14. Shrinking V around the image of x, we may assume furthermore that there is a section $s : V \to \mathbb{A}^1_V$ which is disjoint from $f(Z \cap U)$. We have a Nisnevich square

$$\begin{tikzcd}
U \smallsetminus (Z \cap U) \arrow[r] \arrow[d] & U \arrow[d, "f"] \\
\mathbb{A}^1_V \smallsetminus f(Z \cap U) \arrow[r] & \mathbb{A}^1_V.
\end{tikzcd}$$

Since $\mathcal{X}$ has Nisnevich descent, we deduce a cartesian square

$$\begin{tikzcd}
\mathcal{X}(\mathbb{A}^1_V) \arrow[r] \arrow[d] & \mathcal{X}(\mathbb{A}^1_V \smallsetminus f(Z \cap U)) \arrow[d] \\
\mathcal{X}(U) \arrow[r] & \mathcal{X}(U \smallsetminus (Z \cap U)).
\end{tikzcd}$$

Taking the fibres of the horizontal arrows in this square, we obtain an equivalence

$$\mathcal{X}_{f(Z \cap U)}(\mathbb{A}^1_V) \simeq \mathcal{X}_{Z \cap U}(U).$$

Thus, $\alpha'|_U$ is the image of a unique element of

$$\alpha'' \in \pi_0 \mathrm{fib}\{\mathcal{X}(\mathbb{A}^1_V) \to \mathcal{X}(\mathbb{A}^1_V \smallsetminus f(Z \cap U))\} = \pi_0 \mathcal{X}_{f(Z \cap U)}(\mathbb{A}^1_V)$$

and it is enough to show that α'' maps to the base point in $\mathcal{X}(\mathbb{A}^1_V) \simeq \mathcal{X}(V)$. To do so, it suffices to show that the map

$$\mathcal{X}_{f(Z \cap U)}(\mathbb{A}^1_V) \to \mathcal{X}(\mathbb{A}^1_V)$$

is nullhomotopic. Since $s^* : \mathcal{X}(\mathbb{A}^1_V) \to \mathcal{X}(V)$ is an equivalence, it suffices to see that the composition of

$$\mathcal{X}_{f(Z \cap U)}(\mathbb{A}^1_V) \to \mathcal{X}(\mathbb{A}^1_V) \xrightarrow{s^*} \mathcal{X}(V)$$

is nullhomotopic, which is clear since the image of s is contained in $\mathbb{A}^1_V \smallsetminus f(Z \cap U)$. $\square$

2.5. **A counterexample to F. Morel's $\pi_0^{\mathbb{A}^1}$-conjecture.** We present here a counterexample to the following.

Conjecture 2.5.1 (F. Morel)**.** *Let $\mathcal{X}$ be a k-space. Then $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant.*

We will produce a motivic k-space $\mathcal{X}$ such that the sheaf $\pi_0(\mathcal{X}) = \pi_0^{\mathbb{A}^1}(\mathcal{X})$ is not $\mathbb{A}^1$-invariant. This is based on [Ayo06, Ayo23].

Definition 2.5.2. Let X be a k-variety. We say that X is $\mathbb{A}^1$-discrete if for every finitely generated extension K/k, any map $\mathbb{A}^1_K \to X$ is constant, i.e., factors as $\mathbb{A}^1_K \to \mathrm{Spec}(K) \to X$.

Example 2.5.3.

(1) $\mathbb{A}^1 \smallsetminus 0$ is $\mathbb{A}^1$-discrete.
(2) Abelian varieties are $\mathbb{A}^1$-discrete.
(3) Projective smooth curves of genus $\geqslant 1$ are $\mathbb{A}^1$-discrete.

We will use the following well-known fact.

Lemma 2.5.4. *Let X be a proper and $\mathbb{A}^1$-discrete k-variety. Then for every smooth k-variety U and every dense open $V \subset U$, the restriction map* $\mathrm{Hom}(U, X) \to \mathrm{Hom}(V, X)$ *is a bijection.*

We only sketch an argument when k has characteristic zero. A proof in general case can be found in [Deb01, Corollary 1.44].

Proof in characteristic zero. Using resolution of singularities, we can give a short proof. Any map $V \to X$ extends by properness to a map $U' \to X$, where $U' \to U$ is a blowup with centre disjoint from V. We may assume that $U' \to U$ is a sequence of blowups with smooth centres. By induction, we can assume that $U' = \mathrm{Bl}_Z(U)$, with $Z \subset U \smallsetminus V$ a smooth closed subvariety. In this case, the exceptional divisor $E \subset U'$ is a projective bundle over Z. Thus, by $\mathbb{A}^1$-discreteness, the map $E \to X$ factors through Z. It follows that $U' \to X$ factor through U as needed. □

Construction 2.5.5. Let X be a k-variety, and let $\mathcal{M}$ be a motivic space over X, i.e., an object of the Morel–Voevodsky ∞-category $\mathcal{H}(X) \subset \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_X)$ spanned by the $\mathbb{A}^1$-invariant sheaves. (Just replace "k" with "X" in Definition 1.1.5.) We define a presheaf of Kan complexes on Sm_k, denoted by $\Phi(\mathcal{M})$, by sending $U \in \mathrm{Sm}_k$ to

$$\coprod_{s: U \to X} (s^* \mathcal{M})(U)$$

where $s^* : \mathcal{H}(X) \to \mathcal{H}(U)$ is the inverse image functor. (Explicitly, $s^*\mathcal{M}$ is given by $\mathrm{L}_{\mathrm{mot}}(\underline{s}^*\mathcal{M})$ with $\underline{s}^*$ the left Kan extension functor along the base change functor $\mathrm{Sm}_X \to \mathrm{Sm}_U$.)

The key proposition is the following.

Proposition 2.5.6. *Let X be a proper $\mathbb{A}^1$-discrete k-variety, and let $\mathcal{M} \in \mathcal{H}(X)$ be a motivic space over X. Then $\Phi(\mathcal{M})$ is a motivic space over k.*

Proof. We need to check that $\Phi(\mathcal{M})$ has Nisnevich descent and is $\mathbb{A}^1$-invariant. Clearly, we have $\Phi(\mathcal{M})(\emptyset) = *$. Also, if $U = U_1 \coprod U_2$, then

$$\Phi(\mathcal{M})(U) = \Phi(\mathcal{M})(U_1) \times \Phi(\mathcal{M})(U_2).$$

Next, we fix a Nisnevich square

$$\begin{tikzcd}
V' \arrow[r, "j'"] \arrow[d, "e'"] & U' \arrow[d, "e"] \\
V \arrow[r, "j"] & U
\end{tikzcd}$$

with e étale and j an open immersion. We need to check that $\Phi(\mathcal{M})$ transforms this square into a homotopy cartesian one. By what we just said, we may assume that U and U' are connected, and that V is a dense open in U. We need to show that

$$\begin{tikzcd}
\coprod_{t':V'\to X} (t'^*\mathcal{M})(V') & \coprod_{s':U'\to X} (s'^*\mathcal{M})(U') \arrow[l] \\
\coprod_{t:V\to X} (t^*\mathcal{M})(V) \arrow[u] & \coprod_{s:U\to X} (s^*\mathcal{M})(U) \arrow[l] \arrow[u]
\end{tikzcd}$$

is cartesian. Since $\mathrm{Hom}(U, X) = \mathrm{Hom}(V, X)$ and $\mathrm{Hom}(U', X) = \mathrm{Hom}(V', X)$, we can rewrite this square as follows:

$$\begin{tikzcd}
\coprod_{s':U'\to X} (s'^*\mathcal{M})(V') & \coprod_{s':U'\to X} (s'^*\mathcal{M})(U') \arrow[l] \\
\coprod_{s:U\to X} (s^*\mathcal{M})(V) \arrow[u] & \coprod_{s:U\to X} (s^*\mathcal{M})(U). \arrow[l] \arrow[u]
\end{tikzcd}$$

Since $U' \to U$ is dominant, $\mathrm{Hom}(U, X) \to \mathrm{Hom}(U', X)$ is injective. It is thus enough to show that the coproduct, indexed by $s : U \to X$, of the squares

$$\begin{tikzcd}
(s^*\mathcal{M})(V') & (s^*\mathcal{M})(U') \arrow[l] \\
(s^*\mathcal{M})(V) \arrow[u] & (s^*\mathcal{M})(U) \arrow[l] \arrow[u]
\end{tikzcd}$$

is cartesian. Each of the above squares is cartesian since the presheaves $s^*\mathcal{M}$ have Nisnevich descent. We conclude using that coproducts preserve cartesian squares in the ∞-category of Kan complexes. It remains to show that $\Phi(\mathcal{M})$ is $\mathbb{A}^1$-invariant. Fix $U \in \mathrm{Sm}_k$ and consider the map

$$\Phi(\mathcal{M})(U) \to \Phi(\mathcal{M})(\mathbb{A}^1_U).$$

Since $\mathrm{Hom}(\mathbb{A}^1_U, X) = \mathrm{Hom}(U, X)$, we can write this map as

$$\coprod_{s:U\to X} s^*\mathcal{M}(U) \to \coprod_{s:U\to X} s^*\mathcal{M}(\mathbb{A}^1_U),$$

which is a coproduct of equivalences. This finishes the proof. □

We the same assumptions as in Proposition 2.5.6, we can also describe $\pi_0(\Phi(\mathcal{M})) = \pi_0^{\mathbb{A}^1}(\Phi(\mathcal{M}))$.

Proposition 2.5.7. *Let X be a proper $\mathbb{A}^1$-discrete k-variety and let $\mathcal{M} \in \mathcal{H}(X)$ be a motivic X-space. Then $\pi_0^{\mathbb{A}^1}(\Phi(\mathcal{M}))$ is the sheaf sending $U \in \mathrm{Sm}_k$ to*

$$\coprod_{s:U\to X} \pi_0^{\mathbb{A}^1}(s^*\mathcal{M})(U).$$

In particular, for $\Phi(\mathcal{M})$ to satisfy F. Morel's conjecture, it is necessary that $\pi_0^{\mathbb{A}^1}(\mathcal{M})$ is $\mathbb{A}^1$-invariant.

Proof. Let G be the presheaf of sets given as in the statement. By definition $\pi_0(\Phi(\mathcal{M}))$ is the sheafification of the presheaf F sending $U \in \mathrm{Sm}_k$ to

$$\coprod_{s:U\to X} \pi_0((s^*\mathcal{M})(U)).$$

We have an obvious morphism $F \to G$. We will show that G is a Nisnevich sheaf and that $F \to G$ induces isomorphisms on henselian local schemes.

The proof of the first property is identical to the proof of the previous proposition. For the second property, fix an essentially smooth henselian local scheme W. Then $F(W) \to G(W)$ can be identified with

$$\coprod_{s:W\to W} \pi_0((s^*\mathcal{M})(W)) \to \coprod_{s:W\to X} (\pi_0(s^*\mathcal{M}))(W)$$

which is obviously an isomorphism. □

Corollary 2.5.8. *To find a counterexample to F. Morel's conjecture, it is enough to find a smooth proper $\mathbb{A}^1$-discrete X and a motivic space $\mathcal{M} \in \mathcal{H}(X)$ such that $\pi_0(\mathcal{M})$ is not $\mathbb{A}^1$-invariant.*

To go further, we need to explain a construction of counterexamples to another conjecture of F. Morel, namely his $\mathbb{A}^1$-connectivity conjecture over nonzero dimensional bases. Recall that, for every integer $n \geqslant 0$, there is a sheaf $\mathrm{K}_n^{\mathrm{M}}$ on Sm_k, called the sheaf of unramified Milnor K-theory. It is strictly $\mathbb{A}^1$-invariant, and over a field extension F/k, we have:

$$\mathrm{K}_n^{\mathrm{M}}(F) = \overbrace{F^\times \otimes \cdots \otimes F^\times}^{n \text{ times}} / \langle \cdots \otimes (1-a) \otimes \cdots \otimes a \otimes \cdots \rangle.$$

This sheaf admits a Gersten resolution:

$$\mathrm{K}_n^{\mathrm{M}}(U) \simeq \left[\bigoplus_{x\in U^{(0)}} \mathrm{K}_n^{\mathrm{M}}(x) \to \bigoplus_{x\in U^{(1)}} \mathrm{K}_{n-1}^{\mathrm{M}}(x) \to \cdots \to \bigoplus_{x\in U^{(c)}} \mathrm{K}_{n-c}^{\mathrm{M}}(x) \to \cdots \right].$$

This is a Zariski local resolution which is termwise flabby, but which is functorial only for smooth morphisms.

Construction 2.5.9. Let X be a smooth threefold and $Y \to X$ a closed surface, possibly singular. We define a sheaf $\mathrm{K}_{Y,2}^{\mathrm{M}}$ on X by setting for $U \in \mathrm{Sm}_X$

$$\mathrm{K}_{Y,2}^{\mathrm{M}}(U) = \mathrm{fib}\left\{\mathrm{R}\Gamma(U; \mathrm{K}_2^{\mathrm{M}}) \to \mathrm{R}\Gamma(U \smallsetminus U \times_X Y; \mathrm{K}_2^{\mathrm{M}})\right\}[2].$$

By an easy inspection, we see that

$$\mathrm{K}_{Y,2}^{\mathrm{M}}(U) = \left[\bigoplus_{v\in V^{(0)}} \kappa(x)^\times \to \bigoplus_{x\in V^{(1)}} \mathbb{Z} \right]$$

where $V = U \times_X Y$ and where $\bigoplus_{x\in V^{(1)}} \mathbb{Z}$ is placed in degree zero.

Lemma 2.5.10. $\mathrm{K}^{\mathrm{M}}_{Y,2}$ *is a motivic complex on* Sm_X *which is connective. In particular, the associated Eilenberg–Mac Lane object* $\mathcal{M} := \mathrm{K}(\mathrm{K}^{\mathrm{M}}_{Y,2})$ *belongs to* $\mathcal{H}(X)$.

Now, by construction, $\pi_0(\mathcal{M})$ is the sheaf cl_Y associated to the presheaf

$$U \mapsto \mathrm{CL}(U \times_X Y)$$

where $\mathrm{CL}(W)$ is the group of Weil divisors modulo rational equivalence on a k-variety W. There are examples where this sheaf is not $\mathbb{A}^1$-invariant. This is the case when Y has an isolated singularity at a k-point $o \in Y(k)$ admitting a resolution $b : Y' \to Y$ such that $C = b^{-1}(o)$ is a cuspidal rational curve. In this case $\mathrm{Pic}(C) \simeq \mathbb{G}_{\mathrm{a}} \oplus \mathbb{Z}$, and we obtain a surjective map

$$\mathrm{cl}_Y \to o_*\mathrm{Pic}(C)$$

by taking a divisor $D \subset Y$ and intersecting its strict transform with C. For more details, see [Ayo06, Lemme 3.2]. By Proposition 2.5.7, the motivic space $\Phi(\mathcal{M})$ does not satisfy F. Morel's conjecture.

3. Strongly $\mathbb{A}^1$-invariant sheaves are strictly $\mathbb{A}^1$-invariant

We fix a perfect field k. The goal of this section is to prove the following result. (See Definition 1.2.1 for the notions of strong and strict $\mathbb{A}^1$-invariance.)

Theorem 3.0.1 (Morel)**.** *Let* M *be a strongly* $\mathbb{A}^1$*-invariant sheaf of abelian groups on* Sm_k*. Then* M *is strictly* $\mathbb{A}^1$*-invariant.*

Recall that strong $\mathbb{A}^1$-invariance is just 1-strong $\mathbb{A}^1$-invariance while strict $\mathbb{A}^1$-invariance is n-strong $\mathbb{A}^1$-invariance for all $n \geqslant 0$. Thus, to prove Theorem 3.0.1, it suffices to prove the implication

$$n\text{-strong } \mathbb{A}^1\text{-invariance} \Rightarrow n+1\text{-strong } \mathbb{A}^1\text{-invariance}$$

for all $n \geqslant 1$. However, as the proof is rather involved, we first treat the case $n = 1$ and then repeat the argument for $n \geqslant 2$. In fact, it turns out that the case $n \geqslant 2$ is slightly easier than the case $n = 1$. We start with some general preliminary results.

3.1. Easy consequences of n-strong $\mathbb{A}^1$-invariance.

We fix an integer $n \geqslant 1$. We denote by M a Nisnevich sheaf of abelian groups on Sm_k which we often assume to be n-strongly $\mathbb{A}^1$-invariant. We first recall the Eilenberg–Mac Lane construction.

Notation 3.1.1. If $F_\bullet$ is a complex of presheaves of abelian groups concentrated in nonnegative degrees, i.e., $F_i = 0$ for $i < 0$, we denote by $\mathrm{K}(F)$ the presheaf of pointed Kan complexes obtained from $F_\bullet$ using the Dold–Kan correspondence

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Ab}) \simeq \mathrm{Compl}_{\geqslant 0}(\mathrm{Ab}).$$

By construction, the homotopy presheaves of $\mathrm{K}(F)$ are the homology presheaves of $F_\bullet$.

Construction 3.1.2. Let M be a sheaf of abelian groups on Sm_k, and let $n \geqslant 0$ be an integer. The Eilenberg–Mac Lane space $\mathrm{K}(M, n)$ is the pointed k-space obtained by imposing Nisnevich descent to the presheaf of Kan complexes $\mathrm{K}(M[n])$. It can be constructed explicitly as follows. Choose a resolution in the abelian category of sheaves on Sm_k:

$$M[n] \simeq [I_n \to I_{n-1} \to \cdots \to I_1 \to J_0],$$

such that $I_1, \ldots, I_n$ are injective sheaves, and then set

$$\mathrm{K}(M, n) = \mathrm{K}([I_n \to \cdots \to I_1 \to J_0]).$$

In particular, for every smooth k-variety U, we have:

$$\pi_i\Gamma(U;\mathrm{K}(M,n)) = \begin{cases} 0 & \text{if } \ i \geqslant n+1, \\ \mathrm{H}^{n-i}(U,M) & \text{if } \ 0 \leqslant i \leqslant n. \end{cases}$$

This readily implies the following tautological characterisation of n-strong $\mathbb{A}^1$-invariance.

Proposition 3.1.3. *Let M be a sheaf of abelian groups on* Sm_k*, and let $n \geqslant 0$ be an integer. Then the following conditions are equivalent:*

(1) the sheaf M is n-strongly $\mathbb{A}^1$-invariant;
(2) the k-space $\mathrm{K}(M,n)$ *is motivic.*

The above proposition is useful because it gives access to the theory of motivic spaces in studying n-strongly $\mathbb{A}^1$-invariant sheaves. We will illustrate this in this subsection, but first we need to discuss Thom spaces and purity (following Morel–Voevodsky [MV99]).

Definition 3.1.4. Let X be a smooth k-scheme and let N be a vector bundle on X. The Thom space of N is the quotient

$$N/N\smallsetminus 0_X$$

in $\mathrm{Spc}(k)$, considered as a pointed k-space. When viewed as an object in $\mathcal{H}(k)$, i.e., after applying $\mathrm{L}_{\mathbb{A}^1}$, we denote this quotient by $\mathrm{Th}(N)$ and we keep calling it the Thom space of N.

Proposition 3.1.5. *Assume that the vector bundle N has rank d. Then* $\mathrm{Th}(N)$ *is d-connective.*

Proof. The statement is Zariski local on X. Indeed, assume that $X = X_1 \cup X_2$ with X_1 and X_2 open in X, and let $X_{12} = X_1 \cap X_2$. Denote by N_1, N_2 and N_{12} the restrictions of N to X_1, X_2 and X_{12}. Then $\mathrm{Th}(N)$ is the pushout in $\mathcal{H}(k)$ of the diagram

$$\begin{tikzcd} \mathrm{Th}(N_{12}) \arrow[r] \arrow[d] & \mathrm{Th}(N_1) \\ \mathrm{Th}(N_2). & \end{tikzcd}$$

Thus, the claim follows from Corollary 2.3.5. Therefore, it suffices to prove the result when N is free, i.e., $N \simeq \mathbb{A}^d_X$. In this case, we have an equivalence

$$\mathbb{A}^d_X/\mathbb{A}^d_X\smallsetminus 0_X \simeq \overbrace{(\mathbb{A}^1/\mathbb{A}^1\smallsetminus 0) \wedge \cdots \wedge (\mathbb{A}^1/\mathbb{A}^1\smallsetminus 0)}^{d \text{ times}} \wedge X_+$$

in $\mathrm{Spc}(k)$. Thus, by Corollary 2.3.6, it is enough to show that $\mathrm{Th}(\mathbb{A}^1)$ is 1-connective. But, the obvious morphism

$$\mathbb{A}^1/\mathbb{A}^1\smallsetminus 0 \to \Sigma\mathbb{G}_\mathrm{m},$$

induced by the projection $\mathbb{A}^1 \to \mathrm{Spec}(k)$, is an $\mathbb{A}^1$-equivalence. (Here and elsewhere, we denote by $\mathbb{G}_\mathrm{m}$ the k-space $\mathbb{A}^1\smallsetminus 0$ pointed by the unit section.) By the $\mathbb{A}^1$-connectivity theorem, $\Sigma\mathbb{G}_\mathrm{m}$ is $\mathbb{A}^1$-1-connective. This finishes the proof. □

Thom spaces are especially important because of the next result.

Theorem 3.1.6 (Morel–Voevodsky purity)**.** *Let X be a smooth k-variety and let $Y \subset X$ be a closed subvariety with Y smooth. Then, there is an $\mathbb{A}^1$-equivalence*

$$X/X\smallsetminus Y \simeq_{\mathbb{A}^1} \mathrm{Th}(\mathrm{N}_Y)$$

where N_Y *is the normal bundle of Y in X.*

Corollary 3.1.7. *Let X be a smooth k-variety and $U \subset X$ open. Assume that $X \smallsetminus U$ is smooth of codimension $\geqslant d$. Then X/U is $\mathbb{A}^1$-d-connective.*

Proof of the Morel–Voevodsky purity theorem. We split the proof in two steps.

Step 1. We first assume that there is a cartesian square

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & X \arrow[d, "e"] \\
\mathbb{A}^n \times \{0\} \arrow[r] & \mathbb{A}^n \times \mathbb{A}^d
\end{tikzcd}$$

with e étale. (Note that it is always possible to find such a square Zariski locally on X.) We then obtain a common étale neighbourhood of Y in X and in $\mathbb{A}^n \times Y$, namely

$$X' = X \times_{\mathbb{A}^n} Y \smallsetminus (Y \times_{\mathbb{A}^n} Y \smallsetminus Y).$$

This gives a chain of $\mathbb{A}^1$-equivalences

$$X/X \smallsetminus Y \simeq X'/X' \smallsetminus Y \simeq \mathbb{A}^n \times Y/(\mathbb{A}^n \smallsetminus 0) \times Y \simeq_{\mathbb{A}^1} \mathrm{Th}(\mathrm{N}_Y).$$

(Note that the first two equivalences belong to $\mathrm{Spc}(k)$.)

Step 2. We now treat the general case. This is done using the deformation to the normal cone. We only sketch the argument and refer the reader to [MV99, §3.2, page 115, Theorem 2.23] for more details. There is a commutative diagram with cartesian squares

$$\begin{tikzcd}
Y \times (\mathbb{A}^1 \smallsetminus 0) \arrow[r] \arrow[d] & Y \times \mathbb{A}^1 \arrow[d] & Y \arrow[l] \arrow[d] \\
X \times (\mathbb{A}^1 \smallsetminus 0) \arrow[r] \arrow[d] & D \arrow[d] & \mathrm{N}_Y \arrow[l] \arrow[d] \\
\mathbb{A}^1 \smallsetminus 0 \arrow[r] & \mathbb{A}^1 & 0. \arrow[l]
\end{tikzcd}$$

(Here D is the deformation to the normal cone of $Y \subset X$.) We deduce maps in $\mathrm{Spc}(k)$:

$$\frac{X \times \{1\}}{(X \smallsetminus Y) \times \{1\}} \xrightarrow{(1)} \frac{D}{D \smallsetminus (Y \times \mathbb{A}^1)} \xleftarrow{(2)} \frac{\mathrm{N}_Y}{\mathrm{N}_Y \smallsetminus 0_Y}.$$

We claim that (1) and (2) are $\mathbb{A}^1$-equivalences. To prove this, we can argue locally on X. Thus, we may assume that we have a cartesian square as in Step 1. This yields a commutative diagram:

$$\begin{tikzcd}
\frac{X \times \{1\}}{(X \smallsetminus Y) \times \{1\}} \arrow[r, "(1)"] \arrow[d] & \frac{D}{D \smallsetminus (Y \times \mathbb{A}^1)} \arrow[d] & \frac{\mathrm{N}_Y}{\mathrm{N}_Y \smallsetminus 0_Y} \arrow[l, "(2)"'] \arrow[d] \\
\frac{\mathbb{A}^d \times Y}{(\mathbb{A}^d \smallsetminus 0) \times Y} \arrow[r, "i_1"] & \frac{\mathbb{A}^d \times Y \times \mathbb{A}^1}{(\mathbb{A}^d \smallsetminus 0) \times Y \times \mathbb{A}^1} & \frac{\mathbb{A}^d \times Y}{(\mathbb{A}^d \smallsetminus 0) \times Y}, \arrow[l, "i_0"']
\end{tikzcd}$$

which is enough to conclude. □

Now, we return to n-strongly $\mathbb{A}^1$-invariant sheaves.

Theorem 3.1.8. *Let M be an n-strongly $\mathbb{A}^1$-invariant sheaf on Sm_k. Let X be smooth k-variety and $U \subset X$ open. Assume that $X \smallsetminus U$ has codimension $\geqslant d$. Then the map*

$$\mathrm{H}^i(X; M) \to \mathrm{H}^i(U; M)$$

is an isomorphism for $0 \leqslant i \leqslant \min(d-2, n-1)$ and is injective for $i = \min(d-1, n)$.

Proof. We can filter X by opens

$$U = U_0 \subset U_1 \subset \cdots \subset U_m = X$$

so that $U_s \smallsetminus U_{s-1}$ is smooth of codimension $\geqslant d$ for every $1 \leqslant s \leqslant m$. It is clearly enough to prove the theorem for each of the inclusions $U_{s-1} \subset U_s$. Thus, we may assume that $Y = X \smallsetminus U$ is smooth of codimension d. For all integers $i \geqslant 0$, we have bijections

$$\mathrm{H}^i(X; M) \simeq [X_+, \mathrm{K}(M, i)]_{\mathrm{nis}} \qquad \text{and} \qquad \mathrm{H}^i(U; M) \simeq [U_+, \mathrm{K}(M, i)]_{\mathrm{nis}}.$$

Therefore, it suffices to show that the maps

$$[X_+, \mathrm{K}(M, i)]_{\mathrm{nis}} \to [U_+, \mathrm{K}(M, i)]_{\mathrm{nis}}$$

have the required properties of the statement. The cofibre sequence

$$U_+ \to X_+ \to X/U$$

yields an exact sequence

$$[X/U, \mathrm{K}(M, i)]_{\mathrm{nis}} \to [X_+, \mathrm{K}(M, i)]_{\mathrm{nis}} \to [U_+, \mathrm{K}(M, i)]_{\mathrm{nis}} \to [X/U, \mathrm{K}(M, i+1)]_{\mathrm{nis}}.$$

It is thus enough to show that

$$[X/U, \mathrm{K}(M, i)]_{\mathrm{nis}} = 0 \qquad \text{for} \qquad i \leqslant \min(d-1, n).$$

For $0 \leqslant i \leqslant n$, the k-space $\mathrm{K}(M, i)$ is motivic, and so we are left to showing that

$$[X/U, \mathrm{K}(M, i)]_{\mathbb{A}^1} = 0 \qquad \text{for} \qquad i \leqslant \min(d-1, n).$$

This follows from Corollary 3.1.7 asserting that X/U is $\mathbb{A}^1$-d-connective. □

Remark 3.1.9. We note the following particular cases of Theorem 3.1.8.

(1) If M is $\mathbb{A}^1$-invariant (i.e., 0-strongly $\mathbb{A}^1$-invariant), the map $M(X) \to M(U)$ is injective for any dense open $U \subset X$.

(2) If M is 1-strongly $\mathbb{A}^1$-invariant, the map $M(X) \to M(U)$ is an isomorphism for any open $U \subset X$ such that $X \smallsetminus U$ has codimension $\geqslant 2$. Under the same conditions, the map

$$\mathrm{H}^1(X; M) \to \mathrm{H}^1(U; M)$$

is injective.

In fact, for later use, it will be more convenient to have a reformulation of Theorem 3.1.8 in terms of cohomology with support.

Notation 3.1.10. Let X be an essentially smooth k-scheme and let $Z \subset X$ be a closed subset. Given a sheaf of abelian groups F on $\mathrm{\acute{E}t}_X$, we set

$$\mathrm{H}^i_Z(X; F) = \mathrm{H}^i(\mathrm{fib}(\mathrm{R}\Gamma(X; F) \to \mathrm{R}\Gamma(X \smallsetminus Z; F)))$$

for $i \in \mathbb{Z}$. If F is a sheaf on Sm_k, the pro-cofiber sequence $(X \smallsetminus Z)_+ \to X_+ \to X/X \smallsetminus Z$ shows that

$$\mathrm{H}^i_Z(X; F) \simeq [X/X \smallsetminus \mathrm{Z}, \mathrm{K}(F, i)]_{\mathrm{nis}}.$$

Theorem 3.1.11. *Let X be an essentially smooth k-scheme, and $Z \subset X$ a closed subset of X of codimension $\geqslant d$. Let M be an n-strongly $\mathbb{A}^1$-invariant sheaf. Then*

$$\mathrm{H}^i_Z(X; M) = 0 \qquad \textit{for} \qquad 0 \leqslant i \leqslant \min(d-1, n).$$

Proof. If $T \subset Z \subset X$ are closed subsets of X, we have a long exact sequence in cohomology with support

$$\cdots \to \mathrm{H}^{i-1}_{Z-T}(X\smallsetminus T; M) \to \mathrm{H}^i_T(X; M) \to \mathrm{H}^i_Z(X; M) \to \mathrm{H}^i_{Z\smallsetminus T}(X\smallsetminus T; M) \to \mathrm{H}^{i+1}_T(X; M) \to \cdots .$$

Thus, to prove the theorem for Z, it is enough to prove it for $T \subset X$ and $Z\smallsetminus T \subset X\smallsetminus T$. By a simple induction, we may thus assume that Z is smooth. Now, by construction and Theorem 3.1.6, we have

$$\mathrm{H}^i_Z(X; M) \simeq [X/X\smallsetminus Z, \mathrm{K}(M, i)]_{\mathrm{nis}} \simeq [X/X\smallsetminus Z, \mathrm{K}(M, i)]_{\mathbb{A}^1} \simeq [\mathrm{Th}(\mathrm{N}_Z), \mathrm{K}(M, i)]_{\mathbb{A}^1}$$

for $i \leqslant n$. Moreover, $\mathrm{Th}(\mathrm{N}_Z)$ is $\mathbb{A}^1$-d-connective by Proposition 3.1.5, which yields the vanishing of this group when $i \leqslant d-1$. □

Remark 3.1.12. Keep the notations as in Theorem 3.1.11. Then, if M is 1-strongly $\mathbb{A}^1$-invariant, we have:

- $\mathrm{H}^0_Z(X; M) = 0$ if $d \geqslant 1$,
- $\mathrm{H}^1_Z(X; M) = 0$ if $d \geqslant 2$.

Proposition 3.1.13. *Let M be n-strongly $\mathbb{A}^1$-invariant and let X be an essentially smooth local k-scheme with closed point $x \in X$. If* $\dim(X) \leqslant n+1$, *then* $\mathrm{H}^i_x(X; M) = 0$ *except possibly for* $i = \dim(X)$.

Proof. That $\mathrm{H}^i_x(X; M) = 0$ for $i \leqslant \dim(X) - 1$ is a consequence of Theorem 3.1.11. It remains to see that $\mathrm{H}^i_x(X; M) = 0$ for $i \geqslant \dim(X) + 1$. For this, we use the short exact sequence

$$\mathrm{H}^{i-1}(X\smallsetminus x; M) \to \mathrm{H}^i_x(X; M) \to \mathrm{H}^i(X; M)$$

and the usual bound on the Nisnevich cohomological dimension of schemes (see for example [MV99, §3.1, page 98, Proposition 1.8]). □

Remark 3.1.14. In proving the implication n-strong $\mathbb{A}^1$-invariance ⇒ $n+1$-strong $\mathbb{A}^1$-invariance, a key step consists in understanding the local cohomology groups

$$\mathrm{H}^{n+1}_x(X; M)$$

where M is n-strongly $\mathbb{A}^1$-invariant, and X is an essentially smooth local k-scheme of dimension $n+1$ and with closed point $x \in X$.

3.2. **Contraction.**

In this subsection, we discuss the notion of contraction of an n-strongly $\mathbb{A}^1$-invariant sheaf. This is based on the following observation.

Lemma 3.2.1. *Let F be an n-strongly $\mathbb{A}^1$-invariant sheaf on* Sm_k *and let X be a smooth k-variety. Then the sheaf $F^X = F(- \times X)$ is also n-strongly $\mathbb{A}^1$-invariant.*

Proof. Since F is n-strongly $\mathbb{A}^1$-invariant, the associated Eilenberg–Mac Lane k-space $\mathrm{K}(F, n)$ is motivic. Consider the pointed k-space $\mathrm{K}(F, n)^X$ sending a smooth k-variety U to the Kan complex $\mathrm{K}(F, n)(U \times X)$. Then $\mathrm{K}(F, n)^X$ is also motivic. By Corollary 2.3.8, its n-connective cover $\tau_{\geqslant n}\mathrm{K}(F, n)^X$ is motivic as well. Now, it is easy to see that $\tau_{\geqslant n}\mathrm{K}(F, n)^X \simeq \mathrm{K}(F^X, n)$, showing that $\mathrm{K}(F^X, n)$ is motivic. We conclude using Proposition 3.1.3. □

Notation 3.2.2. Given a presheaf of abelian groups F on Sm_k, we set

$$F_{-1} = \ker\left\{F^{\mathbb{A}^1\smallsetminus 0} \xrightarrow{1^*} F\right\}.$$

Clearly, this is a direct summand of $F^{\mathbb{A}^1\smallsetminus 0}$. In particular, by the previous lemma, if F is an n-strongly $\mathbb{A}^1$-invariant sheaf, so is F_{-1}.

Definition 3.2.3. Let F be a presheaf of abelian groups on Sm_k. The presheaf F_{-1} is called the contraction of F. We define the r-th contraction F_{-r} of F by induction: $F_{-r} = (F_{-r+1})_{-1}$.

Remark 3.2.4. Given a presheaf of pointed sets $\mathcal{X}$ on Sm_k, we can define $F^{\mathcal{X}}$ to be the kernel of the map $\underline{\mathrm{Hom}}(\mathcal{X}, F) \to F$ given by evaluation at the base point of $\mathcal{X}$. Then, the same argument used in the proof of Lemma 3.2.1 shows that, if F is an n-strongly $\mathbb{A}^1$-invariant sheaf, so is $F^{\mathcal{X}}$. Recall that we usually write $\mathbb{G}_{\mathrm{m}}$ for $\mathbb{A}^1\smallsetminus 0$ pointed by the unit section. We may then form the smash product $\mathbb{G}_{\mathrm{m}}^{\wedge r}$ which is the quotient of $(\mathbb{A}^1\smallsetminus 0)^{\times r}$ by the union of the hypersurfaces defined by the equations $t_i = 1$, for $1 \leqslant i \leqslant r$, where t_i is the coordinate at the i-th factor. Then, we have $F_{-r} = F^{\mathbb{G}_{\mathrm{m}}^{\wedge r}}$. Thus, if F is an n-strongly $\mathbb{A}^1$-invariant sheaf, so are its contractions. (Of course, this also follows by induction from Lemma 3.2.1.)

Proposition 3.2.5. *Let X be an essentially smooth k-scheme and let $Y \subset X$ be an essentially smooth closed subscheme of codimension d. Let M be an n-strongly $\mathbb{A}^1$-invariant sheaf. Assume that $n \geqslant d$. Also assume that we have fixed a trivialisation of the normal bundle $\mathrm{N}_Y \simeq \mathbb{A}^d_Y$. Then, there is an isomorphism*

$$\mathrm{H}^d_Y(X; M) \simeq M_{-d}(Y).$$

Proof. It is enough to treat the case where X is a smooth k-variety as our construction will be compatible with further étale localisation. Using that M is n-strongly invariant, we have a chain of bijections

$$\begin{aligned} \mathrm{H}^d_Y(X; M) &\simeq [X/X\smallsetminus Y, \mathrm{K}(M, d)]_{\mathrm{nis}} \\ &\simeq [X/X\smallsetminus Y, \mathrm{K}(M, d)]_{\mathbb{A}^1} \\ &\simeq [\mathrm{Th}(\mathrm{N}_Y), \mathrm{K}(M, d)]_{\mathbb{A}^1} \\ &\simeq [Y_+ \wedge (\mathbb{A}^1/\mathbb{A}^1\smallsetminus 0)^{\wedge d}, \mathrm{K}(M, d)]_{\mathbb{A}^1}. \end{aligned}$$

Now, recall that there is an $\mathbb{A}^1$-equivalence

$$\mathbb{A}^1/\mathbb{A}^1\smallsetminus 0 \simeq_{\mathbb{A}^1} \Sigma\mathbb{G}_{\mathrm{m}}.$$

So, we may continue the previous chain of bijections with the following lines:

$$\begin{aligned} &\simeq [Y_+ \wedge \Sigma^d\mathbb{G}_{\mathrm{m}}^{\wedge d}, \mathrm{K}(M, d)]_{\mathbb{A}^1} \\ &\simeq [Y_+ \wedge \Sigma^d\mathbb{G}_{\mathrm{m}}^{\wedge d}, \mathrm{K}(M, d)]_{\mathrm{nis}}. \end{aligned}$$

Using the adjunction (Σ^d, Ω^d) on $\mathrm{Spc}(k)$ and the fact that $\Omega^d(\mathrm{K}(M, d)) \simeq \mathrm{K}(M, 0)$, we may further continue the chain of bijections with the following lines

$$\begin{aligned} &\simeq [Y_+ \wedge \mathbb{G}_{\mathrm{m}}^{\wedge d}, \mathrm{K}(M, 0)]_{\mathrm{nis}} \\ &\simeq M(Y_+ \wedge \mathbb{G}_{\mathrm{m}}^{\wedge d}) \\ &\simeq M^{\mathbb{G}_{\mathrm{m}}^{\wedge d}}(Y) \\ &= M_{-d}(Y). \end{aligned}$$

This finishes the proof. □

To go further, we need to discuss the coniveau spectral sequence.

3.3. **Coniveau spectral sequence.** Let X be a noetherian scheme and F a Nisnevich sheaf on X. There is a spectral sequence for computing $\mathrm{H}^*(X; F)$, called the coniveau spectral sequence. It looks as follows:

$$\mathrm{E}_1^{p,q} = \bigoplus_{x \in X^{(p)}} \mathrm{H}_x^{p+q}(X; F) \Rightarrow \mathrm{H}^{p+q}(X; F),$$

where $X^{(p)}$ is the set of p-codimensional points of X, and $\mathrm{H}_x^n(X; F)$ is the local cohomology of X supported at x and valued in F. Recall that

$$\mathrm{H}_x^n(X; F) = \mathrm{H}^n(i^! F)_x$$

where $i : \overline{\{x\}} \to X$ is the closed immersion and $i^!$ is the right adjoint to i_*. An alternative way of defining these groups is as follows:

$$\mathrm{H}_x^n(X; F) = \mathrm{H}^n(\mathrm{fib}(\mathrm{R}\Gamma(X_x^h; F) \to \mathrm{R}\Gamma(X_x^h \setminus x; F)))$$

where X_x^h is the henselisation of X at x. Before constructing the coniveau spectral sequence, we will picture it under some assumptions.

Definition 3.3.1. Let F be a sheaf on Sm_k. We say that F is unramified if for every open $U \subset X$ of a smooth k-variety X we have:

1. $F(X) \to F(U)$ is injective if U is dense in X.
2. $F(X) \to F(U)$ is an isomorphism if $X \setminus U$ has codimension $\geqslant 2$.

By Remark 3.1.9, a 1-strongly $\mathbb{A}^1$-invariant sheaf is unramified.

Lemma 3.3.2. *Let F be a sheaf of abelian groups on Sm_k. Let X be a smooth k-variety and let $x \in X$ be a point of X.*

1. *If x is a generic point of X, then $\mathrm{H}_x^0(X; F) = F(x)$ and $\mathrm{H}_x^i(X; F) = 0$ for $i \geqslant 1$.*
2. *If x has codimension one in X, then $\mathrm{H}_x^i(X; F) = 0$ for $i \geqslant 2$. Moreover, if F is unramified, then $\mathrm{H}_x^0(X; F) = 0$ and*
$$\mathrm{H}_x^1(X; F) = \frac{F(\eta_{X_x^h})}{F(X_x^h)},$$
where X_x^h is the henselisation of X at x and $\eta_{X_x^h}$ is its generic point.
3. *If x has codimension $d \geqslant 2$ in X, then $\mathrm{H}_x^i(X, F) = 0$ for $i \geqslant d + 1$. Moreover, if F is unramified, then $\mathrm{H}_x^0(X; F) = \mathrm{H}_x^1(X; F) = 0$ and*
$$\mathrm{H}_x^i(X; F) = \mathrm{H}^{i-1}(X_x^h \setminus x; F)$$
for $2 \leqslant i \leqslant d$.

Proof. We have a long exact sequence

$$0 \to \mathrm{H}_x^0(X; F) \to \mathrm{H}^0(X_x^h; F) \to \mathrm{H}^0(X_x^h \setminus x; F) \to \mathrm{H}_x^1(X; F) \to \mathrm{H}^1(X_x^h; F) \to \mathrm{H}^1(X_x^h \setminus x; F) \to$$

$$\cdots \to \mathrm{H}_x^n(X; F) \to \mathrm{H}^n(X_x^h; F) \to \mathrm{H}^n(X_x^h \setminus x; F) \to \mathrm{H}_x^{n+1}(X; F) \to \cdots .$$

If x is a generic point, then $X_x^h = x$ and $X_x^h \setminus \{x\} = \emptyset$. This proves (1). If x has codimension one, $X_x^h \setminus x$ is the spectrum of a field, and thus has no higher cohomology. This proves (2). If x has codimension $d \geq 2$, then $X_x^h \setminus x$ has dimension $d - 1$, and its cohomology vanishes in degree $\geq d$. This proves (3). □

Remark 3.3.3. Here is a picture of the E_1-page of the coniveau spectral sequence when the sheaf F is unramified:

$$\begin{array}{c|cccccccc}
q\uparrow & & & & & & & \\
\hline
& & & & & & & \rightarrow & p \\
& H^0_x & H^1_x & H^2_x & H^3_x & H^4_x & H^5_x & & \\
& 0 & 0 & 0 & H^2_x & H^3_x & H^4_x & & \\
& 0 & 0 & 0 & 0 & H^2_x & H^3_x & & \\
& 0 & 0 & 0 & 0 & 0 & H^2_x & & \\
& 0 & 0 & 0 & 0 & 0 & 0 & &
\end{array}$$

where we write "H^n_x" as a shorthand for "$H^n_x(X; F)$".

Remark 3.3.4. For a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups F, the coniveau spectral sequence will be shown to be identically zero except at the horizontal line ($q = 0$). This implies that $H^*(X; F)$ is the cohomology of the complex

$$E_1^{0,0} \to E_1^{1,0} \to \cdots \to E_1^{p,0} \to \cdots .$$

This complex makes sense for any sheaf of groups on $\acute{E}t_X$.

Definition 3.3.5. Let X be a noetherian scheme and let F be a sheaf of abelian groups on $\acute{E}t_X$. The line ($q = 0$) of the coniveau spectral sequence for F is called the Cousin complex and it is denoted by $C^\bullet(X; F)$. Note that, for $p \in \mathbb{N}$, we have

$$C^p(X; F) = \bigoplus_{x \in X^{(p)}} H^p_x(X; F).$$

Remark 3.3.6. The proof of the implication 1-strong $\mathbb{A}^1$-invariance $\Rightarrow$ 2-strong $\mathbb{A}^1$-invariance is based on the study of the Cousin complex of a 1-strongly $\mathbb{A}^1$-invariant sheaf in dimension 2.

We end this subsection with a construction of the coniveau spectral sequence. It is obtained by filtering F in the ∞-category of sheaves on $\acute{E}t_X$.

Construction 3.3.7. Let X be a noetherian scheme and F a Nisnevich sheaf of abelian groups on $\acute{E}t_X$. We construct an exhaustive decreasing filtration $(\Phi^c(F))_{c \geqslant 0}$ of (an injective resolution of) F by setting

$$\Phi^c(F) = \operatorname*{colim}_{Z \subset X,\, \mathrm{codim}(Z) \geqslant c} i_{Z,*} i^!_Z(F)$$

where the colimit is over the filtered poset of (not necessarily irreducible) closed subsets of codimension $\geq c$ and where $i_Z : Z \to X$ is the obvious inclusion. The functor $i^!_Z$ is the derived right adjoint of the direct image functor $i_{Z,*}$. Clearly, we have $\Phi^0(F) = F$ and $\Phi^c(F) = 0$ for $c > \dim(X)$.

We claim that $\mathrm{Gr}^c_\Phi(F) \simeq \bigoplus_{x\in X^{(c)}} x_*x^!F$. (As usual, we write $x^!F$ for $(i^!_{\overline{x}}F)_x$.) Indeed, for $Y \subset X$ of codimension c, we have a cofiber sequence

$$\operatorname*{colim}_{Z\subset Y,\, \mathrm{codim}(Z)\geqslant c+1} i_{Z,*}i^!_Z F \to i_{Y,*}i^!_Y F \to \bigoplus_{y\in Y\cap X^{(c)}} y_*y^!F.$$

The spectral sequence associated to the filtered complex $(\Phi^c(F))_{c\in\mathbb{N}}$ is the coniveau spectral sequence. For more details, see for example [CTHK97, Part 1, §1].

3.4. **The Cousin complex of a 1-strongly $\mathbb{A}^1$-invariant sheaf in dimension 2.** Let F be a Nisnevich sheaf on $\acute{\mathrm{E}}\mathrm{t}_X$, for a noetherian scheme X. The Cousin complex associated to F is the line $(q = 0)$ of the coniveau spectral sequence, namely:

$$\bigoplus_{x\in X^{(0)}} \mathrm{H}^0_x(X;F) \to \bigoplus_{x\in X^{(1)}} \mathrm{H}^1_x(X;F) \to \cdots \to \bigoplus_{x\in X^{(p)}} \mathrm{H}^p_x(X;F) \to \cdots .$$

It is denoted by $\mathrm{C}^\bullet(X;F)$. In fact, It is easy to see that

$$U \mapsto \mathrm{C}^\bullet(U;F)$$

defines a complex of flabby Nisnevich sheaves on $\acute{\mathrm{E}}\mathrm{t}_X$ which we denote by $\mathrm{C}^\bullet(-;F)$. We have:

$$\mathrm{C}^p(-;F) = \bigoplus_{x\in X^{(p)}} \mathrm{H}^p(x_*x^!F).$$

In this subsection, our goal is to prove the following.

Theorem 3.4.1. *Let M be a 1-strongly $\mathbb{A}^1$-invariant sheaf on Sm_k. Let X be an essentially smooth local k-scheme of dimension two, and let $x \in X$ be the closed point of X.*

1. *The Cousin complex $\mathrm{C}^\bullet(X;M)$ is acyclic except in degree zero where its cohomology is given by $M(X)$.*
2. *There is an isomorphism*
$$\mathrm{H}^2_x(X;M) \simeq M_{-2}(x)$$
which depends on the choice of a trivialisation of the normal bundle of N_x.

Remark 3.4.2. The proof that $\mathrm{H}^2_x(X;M)$ is isomorphic to $M_{-2}(x)$ given in [Mor12] is incomplete. The problem lies in the proof of [Mor12, Corollary 4.6]: a priori an automorphism of X^h_z might act nontrivially on $\mathrm{H}^2_z(X;M)$, and the proof given in loc. cit. does not acknowledge this issue. However, it is possible to fix F. Morel's argument by first establishing an inclusion between the images of the boundary maps; see Lemma 3.4.12 below. That said, Bachmann [Bac24, §4.2] has found a simpler and more direct argument for proving that $\mathrm{H}^2_x(X;M)$ is isomorphic to $M_{-2}(x)$. Nevertheless, we decided to stay closer to F. Morel's original argument which we find very nice.

As a warmup, we start by proving the analogous statement in dimension one.

Proposition 3.4.3. *Let M be a 1-strongly $\mathbb{A}^1$-invariant sheaf on Sm_k. Let X be an essentially smooth local k-scheme of dimension one, and let $x \in X$ be the closed point of X.*

1. *The obvious map $M(X) \to \mathrm{C}^\bullet(X;M)$ is a quasi-isomorphism and $\mathrm{H}^1(X;M) = 0$.*
2. *There is an isomorphism $\mathrm{H}^1_x(X;M) \simeq M_{-1}(x)$.*

Proof. Part (2) was obtained previously, see Proposition 3.2.5. So we only need to prove part (1). Denote by η the generic point of X. Looking at the E_1-page of the coniveau spectral sequence for X and M (see Remark 3.3.3), we see that there are at most two nonzero terms, namely $\mathrm{H}^0(\eta;M) =$

$\mathrm{H}^0_\eta(X;M)$ and $\mathrm{H}^1_x(X;M)$. It follows that the coniveau spectral sequence collapses at the E_2-page yielding isomorphisms

$$\mathrm{H}^0(X;M) \simeq \mathrm{H}^0(\mathrm{C}(X;M)) \qquad \text{and} \qquad \mathrm{H}^1(X;M) \simeq \mathrm{H}^1(\mathrm{C}(X;M)).$$

Note also that $\mathrm{C}^\bullet(X;M) = [M(\eta) \to M_{-1}(x)]$. Thus to finish the proof, we only need to show that $\mathrm{H}^1(X,M) = 0$. To do so, we use Gabber's presentation lemma. It is enough to show the following.

Claim. Given an integral smooth k-variety X' and a point $x \in X'$ of codimension one, for any class $\alpha \in \mathrm{H}^1(X';M)$ there is an open neighbourhood U of x in X' such that $\alpha|_U = 0$.

Indeed, the cohomology class $\alpha \in \mathrm{H}^1(X';M)$ is necessarily in the image of the natural map $\mathrm{H}^1_Z(X';M) \to \mathrm{H}^1(X';M)$ for some closed subset $Z \subset X'$ of codimension ≥ 1. Enlarging Z if necessary, we may assume that $x \in Z$. By Theorem 2.4.14, we may replace X' with an open neighbourhood of x and assume the existence of an étale morphism

$$e : X' \to \mathbb{A}^1_Y,$$

where Y is an integral smooth k-variety, and such that Z is finite over Y and $Z \simeq e(Z) \simeq e^{-1}e(Z)$. Note that this implies that x maps to the generic point of Y. Then, we have a Nisnevich square

$$\begin{tikzcd}
X' \smallsetminus Z \arrow[r] \arrow[d] & X' \arrow[d, "e"] \\
\mathbb{A}^1_Y \smallsetminus e(Z) \arrow[r] & \mathbb{A}^1_Y
\end{tikzcd}$$

showing that $\mathrm{H}^1_{e(Z)}(\mathbb{A}^1_Y;M) \simeq \mathrm{H}^1_Z(X';M)$. In particular, $\mathrm{H}^1_Z(X';M) \to \mathrm{H}^1(X';M)$ factors through $\mathrm{H}^1(\mathbb{A}^1_Y;M)$. This shows that α is the restriction of a class in $\mathrm{H}^1(\mathbb{A}^1_Y;M)$ which by 1-strong $\mathbb{A}^1$-invariance is the restriction of a class $\beta \in \mathrm{H}^1(Y;M)$. The class β vanishes on a dense open $V \subset Y$. This shows that α vanishes on $e^{-1}(\mathbb{A}^1_V)$ which is an open neighbourhood of x. □

Remark 3.4.4. One could ask if the previous proposition is true more generally for 0-strongly $\mathbb{A}^1$-invariant sheaves. The answer is presumably "no". A potential counterexample is $\mathbb{Z}[\mathbb{G}_\mathrm{m}]$, the free Nisnevich sheaf on $\mathbb{G}_\mathrm{m}$ viewed as a pointed sheaf of sets.

Corollary 3.4.5. *Let M be a 1-strongly $\mathbb{A}^1$-invariant sheaf on Sm_k. Let X be an essentially smooth local k-scheme of dimension one, with closed point x and generic point η. Then, the differential*

$$M(\eta) \to \mathrm{H}^1_x(X;M)$$

is the Cousin complex $\mathrm{C}^\bullet(X;M)$ is surjective.

We now go back to the two-dimensional case.

Lemma 3.4.6. *Let M be a 1-strongly $\mathbb{A}^1$-invariant sheaf and let X be an essentially smooth local k-scheme of dimension two. Then the complex $\mathrm{C}^\bullet(X;M)$ computes $\mathrm{H}^*(X;M)$. In particular, if X is henselian, then the obvious map*

$$M(X) \to \mathrm{C}^\bullet(X;M)$$

is a quasi-isomorphism.

Proof. This is obtained by inspecting the coniveau spectral sequence, see Remark 3.3.3. Indeed, by Lemma 3.3.2, if $y \in Y$ is one-codimensional then $\mathrm{H}^0_y(X;M) = 0$ and, if $x \in X$ is the closed point we have $\mathrm{H}^0_x(X;M) = \mathrm{H}^1_x(X;M) = 0$. Thus, the only nonzero terms of the E_1-page of the coniveau spectral sequence are those of the Cousin complex. □

Lemma 3.4.7. *Let M be a* 1*-strongly $\mathbb{A}^1$-invariant sheaf and let X be an essentially smooth local k-scheme of dimension two. Then* $\mathrm{H}^1(X; M) = 0$ *and hence* $\mathrm{H}^1(\mathrm{C}(X; M)) = 0$.

Proof. We apply Gabber's presentation lemma as in the proof of Proposition 3.4.3. We prove the following claim which is enough to conclude.

Claim. Given an integral smooth k-variety X' and a point $x \in X'$ of codimension two, for any class $\alpha \in \mathrm{H}^1(X'; M)$ there is an open neighbourhood U of x in X' such that $\alpha|_U = 0$.

Indeed, the cohomology class $\alpha \in \mathrm{H}^1(X'; M)$ is necessarily in the image of the natural map $\mathrm{H}^1_Z(X'; M) \to \mathrm{H}^1(X'; M)$ for some closed subset $Z \subset X'$ of codimension ≥ 1. Enlarging Z if necessary, we may assume that $x \in Z$ and that the codimension of x in Z is one. By Theorem 2.4.14, we may replace X' with an open neighbourhood of x and assume the existence of an étale morphism

$$e : X' \to \mathbb{A}^1_Y,$$

where Y is an integral smooth k-variety, and such that Z is finite over Y and $Z \simeq e(Z) \simeq e^{-1}e(Z)$. Note that this implies that x maps to a one-codimensional point $y \in Y$. Then, we have a Nisnevich square

$$\begin{tikzcd}
X' \setminus Z \arrow[r] \arrow[d] & X' \arrow[d, "e"] \\
\mathbb{A}^1_Y \setminus e(Z) \arrow[r] & \mathbb{A}^1_Y
\end{tikzcd}$$

showing that $\mathrm{H}^1_{e(Z)}(\mathbb{A}^1_Y; M) \simeq \mathrm{H}^1_Z(X'; M)$. In particular, $\mathrm{H}^1_Z(X'; M) \to \mathrm{H}^1(X'; M)$ factors through $\mathrm{H}^1(\mathbb{A}^1_Y; M)$. This shows that α is the restriction of a class in $\mathrm{H}^1(\mathbb{A}^1_Y; M)$ which by 1-strong $\mathbb{A}^1$-invariance is the restriction of a class $\beta \in \mathrm{H}^1(Y; M)$. By Proposition 3.4.3, the class β vanishes on an open neighbourhood V of y in Y. This shows that α vanishes on $e^{-1}(\mathbb{A}^1_V)$ which is an open neighbourhood of x. □

Our next goal is to identify $\mathrm{H}^2_x(X; M)$. We need the following technical lemma.

Lemma 3.4.8. *Let M be a Nisnevich sheaf on* Sm_k *(not necessarily* 1*-strongly $\mathbb{A}^1$-invariant). Let $Z \hookrightarrow Y \hookrightarrow X$ be closed immersions between essentially smooth k-schemes. Assume that Y has codimension* 1 *in X and that Z has codimension* 1 *in Y. Then, Zariski locally on X, there is a commutative square:*

$$\begin{tikzcd}
\mathrm{H}^1_{Y \setminus Z}(X \setminus Z; M) \arrow[r] & \mathrm{H}^2_Z(X; M) \\
\mathrm{H}^0(Y \setminus Z; M_{-1}) \arrow[r] \arrow[u] & \mathrm{H}^1_Z(Y; M_{-1}) \arrow[u]
\end{tikzcd}$$

where the horizontal arrows are the natural boundary maps. Moreover, assuming that M is 1*-strongly $\mathbb{A}^1$-invariant, the left vertical map coincides with the isomorphism provided by Proposition 3.2.5 for a certain choice of a trivialisation of the normal bundle of Y in X.*

Proof. It is enough to treat the case where X is a smooth k-variety since our construction will be compatible with further étale localisation. The bottom horizontal map in the statement is obtained by applying $[-, \mathrm{K}(M_{-1}, 1)]_{\mathrm{nis}}$ to the boundary map in the cofibre sequence of k-spaces

$$(Y \setminus Z)_+ \to Y_+ \to \frac{Y}{Y \setminus Z} \to \Sigma(Y \setminus Z)_+.$$

Similarly, the top horizontal map in the statement is obtained by applying $[-,\mathrm{K}(M,2)]_{\mathrm{nis}}$ to the boundary map in the cofibre sequence of k-spaces

$$\frac{X\smallsetminus Z}{X\smallsetminus Y} \to \frac{X}{X\smallsetminus Y} \to \frac{X}{X\smallsetminus Z} \to \Sigma\left(\frac{X\smallsetminus Z}{X\smallsetminus Y}\right).$$

To construct the square in the statement, we may argue locally on X. Thus, we may assume that there is a cartesian square

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & X \arrow[d, "e"] \\
Y \arrow[r, "0"] & \mathbb{A}^1_Y
\end{tikzcd}$$

with e étale. Using this square, we obtain the following commutative diagram of k-spaces

$$\begin{tikzcd}
\frac{X\smallsetminus Z}{X\smallsetminus Y} \arrow[r] \arrow[d, "\sim"'] & \frac{X}{X\smallsetminus Y} \arrow[r] \arrow[d, "\sim"'] & \frac{X}{X\smallsetminus Z} \arrow[r] \arrow[d, "\sim"'] & \Sigma\left(\frac{X\smallsetminus Z}{X\smallsetminus Y}\right) \arrow[d, "\sim"'] \\
\frac{\mathbb{A}^1_Y\smallsetminus Z}{\mathbb{A}^1_Y\smallsetminus Y} \arrow[r] & \frac{\mathbb{A}^1_Y}{\mathbb{A}^1_Y\smallsetminus Y} \arrow[r] \arrow[dd] & \frac{\mathbb{A}^1_Y}{\mathbb{A}^1_Y\smallsetminus Z} \arrow[r] \arrow[dd] & \Sigma\left(\frac{\mathbb{A}^1_Y\smallsetminus Z}{\mathbb{A}^1_Y\smallsetminus Y}\right) \arrow[dd] \\
\frac{\mathbb{A}^1_{Y\smallsetminus Z}}{\mathbb{A}^1_{Y\smallsetminus Z}\smallsetminus(Y\smallsetminus Z)} \arrow[u, "\sim"] \arrow[d] & & \arrow[r, phantom, "(\star)"] & \\
\Sigma(\mathbb{G}_{\mathrm{m}} \wedge (Y\smallsetminus Z)_+) \arrow[r] & \Sigma(\mathbb{G}_{\mathrm{m}} \wedge Y_+) \arrow[r] & \Sigma\left(\mathbb{G}_{\mathrm{m}} \wedge \frac{Y}{Y\smallsetminus Z}\right) \arrow[r] & \Sigma^2(\mathbb{G}_{\mathrm{m}} \wedge (Y\smallsetminus Z)_+).
\end{tikzcd}$$

Applying $[-,\mathrm{K}(M,2)]_{\mathrm{nis}}$ to the square $(\star)$ yields a commutative square

$$\begin{tikzcd}
\left[\Sigma\left(\frac{X\smallsetminus Z}{X\smallsetminus Y}\right), \mathrm{K}(M,2)\right]_{\mathrm{nis}} \arrow[r] & \left[\frac{X}{X\smallsetminus Z}, \mathrm{K}(M,2)\right]_{\mathrm{nis}} \\
\left[\Sigma^2\mathbb{G}_{\mathrm{m}} \wedge (Y\smallsetminus Z)_+, \mathrm{K}(M,2)\right]_{\mathrm{nis}} \arrow[r] \arrow[u] & \left[\Sigma\mathbb{G}_{\mathrm{m}} \wedge \left(\frac{Y}{Y\smallsetminus Z}\right), \mathrm{K}(M,2)\right]_{\mathrm{nis}}. \arrow[u]
\end{tikzcd}$$

Using the adjunction (Σ,Ω), we can rewrite this commutative square as follows:

$$\begin{tikzcd}
\left[\frac{X\smallsetminus Z}{X\smallsetminus Y}, \mathrm{K}(M,1)\right]_{\mathrm{nis}} \arrow[r] & \left[\frac{X}{X\smallsetminus Z}, \mathrm{K}(M,2)\right]_{\mathrm{nis}} \\
{[\mathbb{G}_{\mathrm{m}} \wedge \Sigma(Y\smallsetminus Z)_+, \mathrm{K}(M,1)]_{\mathrm{nis}}} \arrow[r] \arrow[u] & \left[\mathbb{G}_{\mathrm{m}} \wedge \frac{Y}{Y\smallsetminus Z}, \mathrm{K}(M,1)\right]_{\mathrm{nis}}. \arrow[u]
\end{tikzcd}$$

The bottom horizontal arrow can be rewritten as

$$\left[\Sigma(Y\smallsetminus Z)_+, \mathrm{K}(M,1)^{\mathbb{G}_{\mathrm{m}}}\right]_{\mathrm{nis}} \to \left[\frac{Y}{Y\smallsetminus Z}, \mathrm{K}(M,1)^{\mathbb{G}_{\mathrm{m}}}\right]_{\mathrm{nis}}.$$

The obvious map $\mathrm{K}(M_{-1},1) \to \mathrm{K}(M,1)^{\mathbb{G}_{\mathrm{m}}}$, yields a commutative square

$$\begin{tikzcd}
\left[\Sigma(Y\smallsetminus Z)_+, \mathrm{K}(M,1)^{\mathbb{G}_{\mathrm{m}}}\right]_{\mathrm{nis}} \arrow[r] & \left[\frac{Y}{Y\smallsetminus Z}, \mathrm{K}(M,1)^{\mathbb{G}_{\mathrm{m}}}\right]_{\mathrm{nis}} \\
{[\Sigma(Y\smallsetminus Z)_+, \mathrm{K}(M_{-1},1)]_{\mathrm{nis}}} \arrow[r] \arrow[u] & \left[\frac{Y}{Y\smallsetminus Z}, \mathrm{K}(M_{-1},1)\right]_{\mathrm{nis}}. \arrow[u]
\end{tikzcd}$$

Using the adjunction (Σ, Ω) and combining with the last two commutative squares, we obtain the commutative square

$$\begin{tikzcd}
\left[\frac{X\smallsetminus Z}{X\smallsetminus Y}, \mathrm{K}(M,1)\right]_{\mathrm{nis}} \arrow[r] & \left[\frac{X}{X\smallsetminus Z}, \mathrm{K}(M,2)\right]_{\mathrm{nis}} \\
{[(Y\smallsetminus Z)_+, \mathrm{K}(M_{-1},0)]_{\mathrm{nis}}} \arrow[r] \arrow[u] & \left[\frac{Y}{Y\smallsetminus Z}, \mathrm{K}(M_{-1},1)\right]_{\mathrm{nis}}. \arrow[u]
\end{tikzcd}$$

This finishes the construction of the commutative square in the statement. The last assertion follows readily from the construction. □

Corollary 3.4.9. *Let X be an essentially smooth k-scheme, $y \in X^{(1)}$ and $x \in X^{(2)}$. Assume that $x \in \overline{y}$ and that $\overline{y}$ is essentially smooth in the neighbourhood of x. Let M be a 1-strongly $\mathbb{A}^1$-invariant sheaf on Sm_k. There is a commutative square:*

$$\begin{tikzcd}
\mathrm{H}^1_y(X;M) \arrow[r, "\partial^y_x"] & \mathrm{H}^2_x(X;M) \\
M_{-1}(y) \arrow[r, "\partial^y_x"] \arrow[u, "\sim"'] & \mathrm{H}^1_x(\overline{y};M_{-1}) \simeq M_{-2}(x) \arrow[u]
\end{tikzcd}$$

where the left vertical isomorphism is provided by Proposition 3.2.5 for a certain choice of a trivialisation of the normal bundle of $\overline{y}$ in a neighbourhood of x. Moreover, the image of the map $M_{-2}(x) \to \mathrm{H}^2_x(X;M)$ depends only on y.

Proof. Only the last assertion requires a proof. By Corollary 3.4.5 applied to the 1-strongly $\mathbb{A}^1$-invariant sheaf M_{-1}, we know that the map $M_{-1}(y) \to M_{-2}(x)$ is surjective. It follows that the image of the map $M_{-2}(x) \to \mathrm{H}^2_x(X;M)$ coincides with the image of the map $\partial^y_x : \mathrm{H}^1_y(X;M) \to \mathrm{H}^2_x(X;M)$, and the latter is canonical (i.e., depends only on y). □

Proposition 3.4.10. *Let M be a 1-strongly $\mathbb{A}^1$-invariant sheaf on Sm_k. Let X be an essentially smooth k-scheme and $x \in X$ a two-codimensional point of X. Consider a one-codimensional point $y \in X$ such that x is an essentially smooth point of $\overline{y}$. Then the image of the map*

$$\partial^y_x : \mathrm{H}^1_y(X;M) \to \mathrm{H}^2_x(X;M)$$

is independent of y.

We start by proving a reduction to the local henselian case.

Lemma 3.4.11. *Keep the notations and assumptions as in Proposition 3.4.10. Let $X' \to X$ be a pro-Nisnevich neighbourhood of x in X and let $y' \in X'$ be the unique point of X' mapping to y and whose closure contains x. Then, modulo the isomorphism $\mathrm{H}^2_x(X;M) \simeq \mathrm{H}^2_x(X';M)$, the maps*

$$\partial^y_x : \mathrm{H}^1_y(X;M) \to \mathrm{H}^2_x(X;M) \qquad \textit{and} \qquad \partial^{y'}_x : \mathrm{H}^1_{y'}(X';M) \to \mathrm{H}^2_x(X';M)$$

have the same image. In particular, it suffices to prove Proposition 3.4.10 when X is local henselian with closed point x.

Proof. Without loss of generality, we may assume that X is local with closed point x. It suffices to treat the case where $X' = X^h_x$ is the henselisation of X at x. We have a commutative diagram

$$\begin{tikzcd}
& M_{-1}(y) \arrow[dl, "\sim"'] \arrow[dd] \arrow[rr] & & M_{-2}(x) \arrow[dl] \arrow[dd, "\sim"] \\
\mathrm{H}^1_y(X;M) \arrow[rr] \arrow[dd] & & \mathrm{H}^2_x(X;M) \arrow[dd, "\sim"] & \\
& M_{-1}(y') \arrow[dl, "\sim"'] \arrow[rr] & & M_{-2}(x) \arrow[dl] \\
\mathrm{H}^1_{y'}(X';M) \arrow[rr] & & \mathrm{H}^2_x(X';M) &
\end{tikzcd}$$

expressing the compatibility of the commutative square in Corollary 3.4.9 with proétale morphisms. By Corollary 3.4.5, the maps $M_{-1}(y) \to M_{-2}(x)$ and $M_{-1}(y') \to M_{-2}(x)$ are surjective, which yields the desired result. ☐

Proof of Proposition 3.4.10. By Lemma 3.4.11, we may assume that X is local henselian with closed point x. Consider two points $y_1, y_2 \in X^{(1)}$ such that $\overline{y_1}$ and $\overline{y_2}$ are essentially smooth. The relation "$\partial^{y_1}_x$ and $\partial^{y_2}_x$ have the same image" being transitive, we may assume that $\overline{y_1}$ and $\overline{y_2}$ meet transversally at x. (We are using here that a general hypersurface passing through x will meet transversally at x any given pair of essentially smooth hypersurfaces. Indeed, it suffices to lift any element of the $\kappa(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ which is not colinear to the equations defining the two given hypersurfaces. This is possible even when $\kappa(x)$ is finite.) In this case, we can find a proétale morphism

$$e : X \to \mathbb{A}^2_x$$

such that $\overline{y_1}$ and $\overline{y_2}$ are the inverse images of the coordinate hyperplanes, i.e., they are defined by the equations $t_1 = 0$ and $t_2 = 0$, where t_1 and t_2 are the coordinates on $\mathbb{A}^2_x$. (Note that this is only possible because k is perfect.) Let $p : \mathbb{A}^2_x \to \mathbb{A}^1_x$ be the linear projection given by $p(a,b) = a+b$. Note that the inverse image of the zero section along the map $p \circ e : X \to \mathbb{A}^1_x$ meets transversally the hyperserfaces $\overline{y_1}$ and $\overline{y_2}$. Let $S = (\mathbb{A}^1_x)^h_0$ be the henselisation of $\mathbb{A}^1_x$ at the zero section, and denote by $s \in S$ its closed point. Since X is local henselian, the proétale morphism e factors through a proétale morphism

$$f : X \to \mathbb{A}^1_S$$

sending x to the point $(0,s)$, and $\bar{y}_1$ and $\bar{y}_2$ to two sections of $\mathbb{A}^1_S \to S$ meeting transversally at $(0,s)$. Moreover, we have $\overline{y_1} \simeq f(\overline{y_1})$ and $\overline{y_2} \simeq f(\overline{y_2})$.

Using Lemma 3.4.11 a second time, we see that it suffices to show that the maps

$$\partial^{y_1}_x : \mathrm{H}^1_{y_1}(\mathbb{A}^1_S;M) \to \mathrm{H}^2_x(\mathbb{A}^1_S;M) \qquad \text{and} \qquad \partial^{y_2}_x : \mathrm{H}^1_{y_2}(\mathbb{A}^1_S;M) \to \mathrm{H}^2_x(\mathbb{A}^1_S;M)$$

have the same image. (Here, we are identifying the points y_1, y_2 and x with their images in $\mathbb{A}^1_S$.) We will show that

$$\mathrm{Im}(\partial^{y_1}_x) = \mathrm{Im}(\partial^{\eta}_x) = \mathrm{Im}(\partial^{y_2}_x)$$

where η is the generic point of $\mathbb{A}^1_s$ and

$$\partial^{\eta}_x : \mathrm{H}^1_{\eta}(\mathbb{A}^1_S; M) \to \mathrm{H}^2_x(\mathbb{A}^1_S; M)$$

the natural boundary map. Let K be the fraction field of S. Since M is 1-strongly $\mathbb{A}^1$-invariant, the complex

$$0 \to M(K) \to M(K(t)) \to \bigoplus_{a \in (\mathbb{A}^1_K)^{(1)}} \mathrm{H}^1_a(\mathbb{A}^1_K; M) \to 0$$

is exact. (Indeed, as argued in the proof of Proposition 3.4.3, the Cousin complex $\mathrm{C}^{\bullet}(\mathbb{A}^1_K; M)$ computes the Nisnevich cohomology of $\mathbb{A}^1_K$ with value in M, and this cohomology is given by $M(K)$ in degree zero and vanishes in nonzero degrees.) In particular, given $\alpha \in \mathrm{H}^1_{y_1}(\mathbb{A}^1_S; M) \simeq \mathrm{H}^1_{y_1}(\mathbb{A}^1_K; M)$, we can find $\beta \in M(K(t))$ such that $\partial^{K(t)}_z(\beta) = 0$ for every $z \in (\mathbb{A}^1_K)^{(1)}$, $z \neq y_1$, and $\partial^{K(t)}_{y_1}(\beta) = \alpha$. Set $\alpha' = \partial^{K(t)}_{\eta}(\beta)$. Using that $\mathrm{C}^{\bullet}(\mathbb{A}^1_S; K)$ is a complex, we deduce that $\partial^{y_1}_x(\alpha) + \partial^{\eta}_x(\alpha') = 0$. In particular, we have shown the inclusion

$$\mathrm{Im}(\partial^{y_1}_x) \subset \mathrm{Im}(\partial^{\eta}_x)$$

and, by symmetry, the same holds for y_2. To conclude, we show the following. □

Lemma 3.4.12. *Let X be an essentially smooth k-variety, $x \in X^{(2)}$ and $y_1, y_2 \in X^{(1)}$. Assume that $x \in \overline{y_1} \cap \overline{y_2}$, and that $\overline{y_1}$ and $\overline{y_2}$ are essentially smooth and intersect transversally at x. Assume also that* $\mathrm{Im}(\partial^{y_1}_x) \subseteq \mathrm{Im}(\partial^{y_2}_x)$. *Then this inclusion must be an equality.*

Proof. Using Lemma 3.4.11, we reduce to the case where X is local henselian with closed point x. Since $\overline{y_1}$ and $\overline{y_2}$ are smooth and meet transversally, we can find a proétale morphism $e : X \to \mathbb{A}^2_x$ such that $\overline{y_i} = e^{-1}(H_i)$, for $i \in \{1, 2\}$, where H_1 and H_2 are the coordinate hyperplanes in $\mathbb{A}^2_x$. Clearly, X identifies with the henselisation of $\mathbb{A}^2_x$ at the origin. In particular, there is an involution τ of X such that $\tau(y_1) = y_2$ and $\tau(y_2) = y_1$. Denote by τ_* the action of τ on the Cousin complex of X. We have $\tau_*(\mathrm{Im}(\partial^{y_1}_x)) = \mathrm{Im}(\partial^{y_2}_x)$ and $\tau_*(\mathrm{Im}(\partial^{y_2}_x)) = \mathrm{Im}(\partial^{y_1}_x)$. Thus, applying τ_* to the given inclusion $\mathrm{Im}(\partial^{y_1}_x) \subset \mathrm{Im}(\partial^{y_2}_x)$ we conclude that $\mathrm{Im}(\partial^{y_2}_x) \subset \mathrm{Im}(\partial^{y_1}_x)$, which is the desired converse inclusion. This finishes the proof of the lemma and of Proposition 3.4.10. □

We also need to understand the image of ∂^y_x when $\bar{y}$ is possibly singular at x.

Proposition 3.4.13. *Let M be a* 1*-strongly $\mathbb{A}^1$-invariant sheaf on* Sm_k. *Let X be an essentially smooth k-variety, $x \in X^{(2)}$ and $y_1, y_2 \in X^{(1)}$. Assume that $x \in \overline{y_1} \cap \overline{y_2}$, and that $\overline{y_1}$ is essentially smooth at x. Then* $\mathrm{Im}(\partial^{y_2}_x) \subset \mathrm{Im}(\partial^{y_1}_x)$.

Proof. We may assume that X is a smooth k-variety. Using Gabber's presentation lemma (i.e., Theorem 2.4.14), we may assume that there is an étale morphism $f : X \to \mathbb{A}^1_S$, where S is an integral smooth k-variety, such that f restricts to a closed immersion on $\overline{y_1} \cup \overline{y_2}$ and induces isomorphisms $\overline{y_1} \cup \overline{y_2} \simeq f(\overline{y_1} \cup \overline{y_2}) \simeq f^{-1}f(\overline{y_1} \cup \overline{y_2})$. We can also assume that $\overline{y_1} \cup \overline{y_2}$ is finite over S which implies that the image $s \in S$ of x is a one-codimensional point of S.

We can identify the points y_1, y_2 and x with their images in $\mathbb{A}^1_S$ and, by Lemma 3.4.11, we may replace X with $\mathbb{A}^1_S$. Denote by η the generic point of $\mathbb{A}^1_s$. By Proposition 3.4.10, we know that $\mathrm{Im}(\partial^{y_1}_x) = \mathrm{Im}(\partial^{\eta}_x)$. So it suffices to show that $\mathrm{Im}(\partial^{y_2}_x) \subset \mathrm{Im}(\partial^{\eta}_x)$. This is done by the same method used in the proof of Proposition 3.4.10. Let K be the fraction field of S. Given $\alpha \in \mathrm{H}^1_{y_2}(\mathbb{A}^1_K; M)$, we can find $\beta \in M(K(t))$ such that $\partial_z^{K(t)}(\beta) = 0$ for every $z \in (\mathbb{A}^1_K)^{(1)}$, $z \neq y_2$, and $\partial_{y_2}^{K(t)}(\beta) = \alpha$. Using that $\mathrm{C}^\bullet(\mathbb{A}^1_S; M)$ is a complex, we deduce that $\partial_x^y(\alpha) + \partial_x^\eta \partial_\eta^{K(t)}(\beta) = 0$. □

We are finally ready to complete the proof of the main theorem of this subsection.

Proof of Theorem 3.4.1. Let M be a 1-strongly $\mathbb{A}^1$-invariant sheaf on Sm_k. Let X be an essentially smooth local k-scheme of dimension 2 with closed point $x \in X$. In view of Lemma 3.4.7, we need to

(1) show that $\mathrm{C}^\bullet(X; M)$ is acyclic in degree 2,
(2) construct an isomorphism $\mathrm{H}^2_x(X; M) \simeq M_{-2}(x)$.

To do so, we fix $y \in X^{(1)}$ such that $\overline{y}$ is essentially smooth. We also fix trivialisations of the normal bundles of $\overline{y} \subset X$ and $x \in \overline{y}$. By Corollary 3.4.9, there is a commutative square

$$\begin{tikzcd}
\mathrm{H}^1_y(X; M) \arrow[r, "\partial_x^y"] & \mathrm{H}^2_x(X; M) \\
\mathrm{M}_{-1}(y) \arrow[u, "\sim"'] \arrow[r, "\partial_x^y"] & \mathrm{H}^1_x(\overline{y}; M_{-1}) \simeq M_{-2}(x). \arrow[u]
\end{tikzcd}$$

In particular, this gives a morphism

$$M_{-2}(x) \to \mathrm{H}^2_x(X; M). \qquad (\star).$$

We are going to show that $(\star)$ is an isomorphism. This will prove (2). It also proves that the differential $\mathrm{C}^1(X; M) \to \mathrm{C}^2(X; M)$ is surjective, and thus (1).

To prove that $(\star)$ is an isomorphism, we may assume that X is local henselian. In this case, we know that $\mathrm{H}^2(X; M) = 0$. Using Lemma 3.4.6, this implies that

$$\sum_{z \in X^{(1)}} \partial_x^z : \bigoplus_{z \in X^{(1)}} \mathrm{H}^1_z(X; M) \to \mathrm{H}^2_x(X; M)$$

is surjective. But, by Proposition 3.4.13, we have:

$$\mathrm{Im}(\partial_x^z) \subset \mathrm{Im}(\partial_x^y), \qquad \forall z \in X^{(1)}.$$

It follows that the map $\mathrm{H}^1_y(X; M) \to \mathrm{H}^2_x(X; M)$ is surjective. Since this map factors through $M_{-2}(x)$, we deduce that $M_{-2}(x) \to \mathrm{H}^2_x(X; M)$ is surjective.

It remains to prove injectivity. To do so, consider the following commutative diagram

$$\begin{tikzcd}
x \arrow[r, "j"] \arrow[rr, bend right, "x"'] & Y = \overline{\{y\}} \arrow[r, "i"] & X.
\end{tikzcd}$$

Denote by M_X, M_Y, etc., the restriction of M to the small Nisnevich sites of X, Y, etc. By Theorem 3.1.11 and Proposition 3.2.5, the complex of sheaves $i^! M_X$ on $\mathrm{\acute{E}t}_Y$ has vanishing H^0 and its H^1 is equivalent to the sheaf $(M_{-1})_Y$. Thus, we have a morphism

$$(M_{-1})_Y[-1] \to i^! M_X$$

in the derived category of Nisnevich sheaves on $\mathrm{\acute{E}t}_Y$. Applying $j^!$ yields

$$j^!(M_{-1})_Y[-1] \to j^! i^! M_X = x^! M_X.$$

Passing to H^2 and taking global sections gives the morphism

$$\mathrm{H}^1_x(Y, M_{-1}) \to \mathrm{H}^2_x(X; M). \qquad\qquad (\star')$$

It is easy to see that $(\star')$ is precisely the right vertical arrow in the square in Corollary 3.4.9 which we recalled at the beginning of the proof. Thus, $(\star')$ can be identified with the morphism $(\star)$. Now, by construction, we have an exact triangle

$$(M_{-1})_Y[-1] \to i^! M_X \to \tau^{\geqslant 2} i^! M_X$$

yielding an exact triangle

$$j^!(M_{-1})_Y[-1] \to x^! M_X \to j^! \tau^{\geqslant 2} i^! M_X$$

in the derived category of Nisnevich sheaves on $\mathrm{\acute{E}t}_x$. This shows that the kernel of the map $(\star')$ is the global sections of the image of the map

$$\mathrm{H}^1(j^! \tau^{\geqslant 2} i^! M_X) \to \mathrm{H}^2(j^!(M_{-1})_Y[-1]).$$

Since the domain of this map is zero, this proves that $(\star')$ is injective as needed. □

3.5. **Cousin complexes and** H^1**.** We now use the main result of the previous subsection to get more information about 1-strongly $\mathbb{A}^1$-invariant sheaves.

Proposition 3.5.1. *Let M be a 1-strongly $\mathbb{A}^1$-invariant sheaf on Sm_k. Let X be an essentially smooth local k-scheme. Then, we have the vanishing*

$$\mathrm{H}^1(\mathrm{C}(X; M)) = 0 \qquad \textit{and} \qquad \mathrm{H}^1(\mathrm{C}(\mathbb{A}^1_X; M)) = 0.$$

Proof. For an integer $d \geqslant 0$, consider the following two properties.

(P_d) For every local essentially smooth k-scheme X of dimension $\leqslant d$, $\mathrm{H}^1(\mathrm{C}(X; M)) = 0$.
(Q_d) For every local essentially smooth k-scheme S of dimension $\leqslant d$, $\mathrm{H}^1(\mathrm{C}(\mathbb{A}^1_S; M)) = 0$.

Note that (P_d) is known for $d = 0, 1, 2$ and that (Q_d) is known for $d = 0$. (For the former vanishing, see Proposition 3.4.3 and Lemma 3.4.7. For the latter vanishing, use 1-strong $\mathbb{A}^1$-invariance and the fact that the Cousin complex computes the cohomology of any essentially smooth k-scheme of dimension ≤ 1 with values in M.) We will prove the proposition by showing the implications

$$(P_d) \Rightarrow (Q_d) \qquad \text{and} \qquad (Q_d) \Rightarrow (P_{d+1}).$$

Step 1: $(P_d) \Rightarrow (Q_d)$. Assume that S is local of dimension $\leqslant d$. Let $\eta \in S$ be the generic point and $s \in S$ the closed point. Consider the diagram of Cousin complexes:

$$\begin{tikzcd}
\mathrm{C}^\bullet(\mathbb{A}^1_\eta; M) & : & M(\mathbb{A}^1_\eta) \arrow[r] & M(\eta(t)) \arrow[r] & \bigoplus\limits_{x \in (\mathbb{A}^1_\eta)^{(1)}} M_{-1}(x) \arrow[r] & 0 \\
\mathrm{C}^\bullet(\mathbb{A}^1_S; M) \arrow[u] & : & M(\mathbb{A}^1_S) \arrow[u] \arrow[r] & M(\eta(t)) \arrow[u, equal] \arrow[r] & \bigoplus\limits_{x \in (\mathbb{A}^1_S)^{(1)}} M_{-1}(x) \arrow[u, two heads] \arrow[r, "(\star)"] & \bigoplus\limits_{x \in (\mathbb{A}^1_S)^{(2)}} M_{-2}(x) \arrow[u] \\
\mathrm{C}^\bullet(S; M) \arrow[u] & : & M(S) \arrow[u] \arrow[r] & M(\eta) \arrow[u] \arrow[r] & \bigoplus\limits_{x \in S^{(1)}} M_{-1}(x) \arrow[u] \arrow[r] & \bigoplus\limits_{x \in S^{(2)}} M_{-2}(x). \arrow[u]
\end{tikzcd}$$

(As usual, we denote by t the coordinate on $\mathbb{A}^1$ and we write $\eta(t)$ for the generic point of $\mathbb{A}^1_\eta$.) Let

$$(\alpha_x)_{x\in(\mathbb{A}^1_S)^{(1)}} \in \bigoplus_{x\in(\mathbb{A}^1_S)^{(1)}} M_{-1}(x)$$

be the representative of a cohomology class α in $\mathrm{H}^1(\mathrm{C}(\mathbb{A}^1_S; M))$. Since $\mathrm{C}^\bullet(\mathbb{A}^1_\eta; M)$ is acyclic in degree 1, we may modify the representative of α and assume that $\alpha_x = 0$ for every $x \in (\mathbb{A}^1_\eta)^{(1)}$. Said differently, the condition that $\alpha_x \neq 0$ implies that x is the generic point of $\mathbb{A}^1_s$ for some $s \in S^{(1)}$. But then, since $(\alpha_x)_x$ is in the kernel of the map ($\star$), we deduce that each α_x is in the kernel of

$$M_{-1}(x) \to \bigoplus_{z\in(\mathbb{A}^1_s)^{(1)}} M_{-2}(z).$$

Since M_{-1} is 1-strongly $\mathbb{A}^1$-invariant, we deduce that $\alpha_x \in M_{-1}(\mathbb{A}^1_s) \simeq M_{-1}(s)$. This means that $(\alpha_x)_x$ descends to a cocycle $(\beta_s)_{s\in S^{(1)}}$ in $\mathrm{C}^\bullet(S; M)$. By (P_d), the cocycle $(\beta_s)_s$ is a coboundary, and hence the same is true for $(\alpha_x)_x$. This shows that $\alpha = 0$.

Step 2: $(Q_d) \Rightarrow (P_{d+1})$. We use Gabber's presentation lemma as in the proof of Lemma 3.4.7. More precisely, we prove the following claim which is enough to conclude.

Claim. Given an integral smooth k-variety X' and a point $x \in X'$ of codimension d, for any class $\alpha \in \mathrm{H}^1(\mathrm{C}(X'; M))$ there is an open neighbourhood U of x in X' such that $\alpha|_U = 0$.

Indeed, let $(\alpha_y)_{y\in X'^{(1)}}$ be a cocycle in $\mathrm{C}^\bullet(X'; M)$ representing α. Let $Z \subset X'$ be the closure of the finite set of points $\{y \in X'^{(1)} \mid \alpha_y \neq 0\}$. Enlarging Z if necessary, we may assume that x is a point of codimension $d-1$ in Z. By Theorem 2.4.14, we may replace X' with an open neighbourhood of x and assume the existence of an étale morphism

$$e : X' \to \mathbb{A}^1_{S'},$$

where S' is an integral smooth k-variety, and such that Z is finite over S' and $Z \simeq e(Z) \simeq e^{-1}e(Z)$. Note that this implies that x maps to a point $s \in S'$ of codimension $d-1$. Now, we may identify the points of Z with their images in $\mathbb{A}^1_{S'}$ and view $(\alpha_y)_y$ as a cocycle in $\mathrm{C}^\bullet(\mathbb{A}^1_{S'}; M)$. Using (Q_d), we can find an open neighbourhood V of s in S' such that the image of $(\alpha_y)_y$ in $\mathrm{C}^\bullet(\mathbb{A}^1_V; M)$ is a coboundary. The same is true for the image of $(\alpha_y)_y$ in $\mathrm{C}^\bullet(U; M)$ where $U = e^{-1}(\mathbb{A}^1_V)$ which is an open neighbourhood of x in X'. □

Corollary 3.5.2. *Let X be an essentially smooth k-scheme and M a 1-strongly $\mathbb{A}^1$-invariant sheaf. Then, there are natural isomorphisms*

$$\mathrm{H}^1_{\mathrm{nis}}(X; M) \simeq \mathrm{H}^1_{\mathrm{zar}}(X; M) \simeq \mathrm{H}^1(\mathrm{C}^\bullet(X; M)).$$

Proof. Indeed, by Proposition 3.5.1, the truncated complex of sheaves

$$\tau^{\leqslant 1}\mathrm{C}(-; M) = \left[\mathrm{C}^0(-; M) \to \ker(\mathrm{C}^1(-; M) \to \mathrm{C}^2(-; M))\right]$$

is a Zariski local resolution of M_X (the restriction of M to the small Nisnevich site of X) and its zero-degree term is acyclic for the Zariski and Nisnevich topologies. Thus the complex of global sections of $\tau^{\leqslant 1}\mathrm{C}(-; M)$ computes the first cohomology group of X valued in M_X in the Zariski and the Nisnevich topologies. □

Corollary 3.5.3. *Let X be an essentially smooth k-scheme and M a 1-strongly $\mathbb{A}^1$-invariant sheaf. Let $Z \subset X$ be a closed subset of codimension $\geqslant 3$. Then the obvious map $\mathrm{H}^i(X; M) \to \mathrm{H}^i(X \smallsetminus Z; M)$ is an isomorphism in degrees $i \leqslant 1$.*

Proof. By Corollary 3.5.2, it is enough to show that $\mathrm{H}^1(\mathrm{C}(X;M)) \to \mathrm{H}^1(\mathrm{C}(X\smallsetminus Z;M))$ is an isomorphism. But $\mathrm{C}^\bullet(X;M) \to \mathrm{C}^\bullet(X\smallsetminus Z;M)$ is an isomorphism in degrees $\leqslant 2$. □

Corollary 3.5.4. *Let X be an essentially smooth k-scheme and let $x \in X$ be a point of codimension $\geqslant 3$. Let M be a 1-strongly $\mathbb{A}^1$-invariant sheaf. Then $\mathrm{H}^i_x(X;M) = 0$ for $0 \leqslant i \leqslant 2$.*

Proof. Since M is unramified, the desired vanishing holds for $i = 0$ and $i = 1$ by Lemma 3.3.2. Moreover, by the same lemma, $\mathrm{H}^2_x(X;M) \simeq \mathrm{H}^1(X^h_x\smallsetminus x;M)$, where X^h_x is the henselisation of X at x. Since x has codimension $\geqslant 3$ in X^h_x, we have

$$\mathrm{H}^1(X^h_x\smallsetminus x;M) \simeq \mathrm{H}^1(X^h_x;M) = 0$$

by Corollary 3.5.3. □

3.6. **Proof that 1-strong $\mathbb{A}^1$-invariance $\Rightarrow$ 2-strong $\mathbb{A}^1$-invariance.** Let M be a 1-strongly $\mathbb{A}^1$-invariant sheaf of abelian groups on Sm_k. In this subsection, we will show that M is 2-strongly $\mathbb{A}^1$-invariant. As usual, given an essentially smooth k-scheme X, we denote by M_X the restriction of M to the small Nisnevich site of X. Here is the key proposition.

Proposition 3.6.1. *Let X be an essentially smooth k-scheme and let $i : Y \hookrightarrow X$ be the inclusion of an essentially smooth divisor. Then the complex of sheaves $i^! M_X$ satisfies $\mathrm{H}^0(i^! M_X) = 0$ and $\mathrm{H}^2(i^! M_X) = 0$. Said differently, the complex of sheaves $\tau^{\leqslant 2}(i^! M_X)$ is concentrated in cohomological degree one.*

Proof. We already know that $\mathrm{H}^0(i^! M_X) \simeq 0$ and that $\mathrm{H}^1(i^! M_X)$ is Zariski locally isomorphic to $(M_{-1})_Y$. (See Theorem 3.1.11 and Proposition 3.2.5.) Assume by contradiction that $\mathrm{H}^2(i^! M_X)$ is nonzero, and let $x \in Y$ be a point of minimal codimension such that $\underline{x}^* \mathrm{H}^2(i^! M_X)$ is nonzero. (Here, we write $\underline{x} : x \to Y$ for the obvious inclusion to distinguish it from the inclusion of x into X.) Replacing X with an étale neighbourhood of a geometric point over x, we may even assume that

$$\Gamma(x; \underline{x}^* \mathrm{H}^2(i^! M_X)) \neq 0.$$

Without loss of generality, we may also assume that X is henselian local with closed point x. Consider an exact triangle

$$\mathrm{H}^1(i^! M_X)[-1] \to i^! M_X \to \mathcal{K}$$

in the derived category of Nisnevich sheaves on Ét_Y. By construction $\mathcal{K} = \tau^{\geqslant 2}(i^! M_X)$ is concentrated in cohomological degrees $\geqslant 2$. As above, we denote by $\underline{x} : x \hookrightarrow Y$ the obvious inclusion to distinguish it from the inclusion of x into X, which we denote by x. Thus $x = i \circ \underline{x}$. Apply $\underline{x}^!$ to the previous exact triangle to get an exact triangle

$$\underline{x}^! \mathrm{H}^1(i^! M_X)[-1] \to x^! M_X \to \underline{x}^! \mathcal{K} \qquad (\triangle)$$

in the derived category of Nisnevich sheaves on Ét_x. Note also that the first term of this complex is equivalent to $\underline{x}^!(M_{-1})_Y[-1]$. Since $\mathrm{H}^2(\mathcal{K})$ is supported at the closed point $x \in Y$, we have $\mathrm{H}^2(\underline{x}^! \mathcal{K}) \simeq \underline{x}^! \mathrm{H}^2(\mathcal{K}) \simeq \underline{x}^* \mathrm{H}^2(\mathcal{K})$. Thus, in order to get a contradiction, we need to show that the sheaf $\mathrm{H}^2(\underline{x}^! \mathcal{K})$ has no nonzero global sections. From the exact triangle $(\triangle)$, we get a long exact sequence of sheaves on Ét_x:

$$\cdots \to \mathrm{H}^1(\underline{x}^!(M_{-1})_Y) \to \mathrm{H}^2(x^! M_X) \to \mathrm{H}^2(\underline{x}^! \mathcal{K}) \to \mathrm{H}^2(\underline{x}^!(M_{-1})_Y) \to \mathrm{H}^3(x^! M_X) \to \cdots .$$

Taking global sections, we obtain the long exact sequence of abelian groups:

$$\cdots \to \mathrm{H}^1_x(Y;M_{-1}) \xrightarrow{(1)} \mathrm{H}^2_x(X;M) \to \Gamma(x;\mathrm{H}^2(\underline{x}^! \mathcal{K})) \to \mathrm{H}^2_x(Y;M_{-1}) \xrightarrow{(2)} \mathrm{H}^3_x(X;M) \to \cdots . \qquad (\triangle')$$

Note that the map (1) is the one appearing in the commutative square in Corollary 3.4.9. To go further, we need to treat the cases $\dim(X) = 2$, $\dim(X) = 3$ and $\dim(X) \geqslant 4$ separately.

Case 1: $\dim(X) = 2$. In this case, the group $\mathrm{H}^2_x(Y; M_{-1})$ is zero since $\dim(Y) = 1$. On the other hand, the morphism (1) in the sequence $(\triangle')$ is an isomorphism by Theorem 3.4.1.

Case 2: $\dim(X) \geqslant 4$. In this case, both X and Y have dimension $\geqslant 3$ and we may use Corollary 3.5.4 to get the vanishing of the local cohomology groups $\mathrm{H}^2_x(X; M)$ and $\mathrm{H}^2_x(Y; M_{-1})$.

Case 3: $\dim(X) = 3$. This is the most difficult case. Here, we have $\mathrm{H}^2_x(X; M) = 0$ by Corollary 3.5.4 but, a priori, the group $\mathrm{H}^2_x(Y; M_{-1})$ could be nonzero. Thus, to prove the requested vanishing, we need to show that the map

$$\mathrm{H}^2_x(Y; M_{-1}) \to \mathrm{H}^3_x(X; M)$$

is injective. This map can be identified with the Gysin map

$$\mathrm{H}^1(Y \smallsetminus x; M_{-1}) \to \mathrm{H}^2(X \smallsetminus x; M).$$

There is a commutative square

$$\begin{tikzcd}
\mathrm{H}^1(Y \smallsetminus x; M_{-1}) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^2(X \smallsetminus x; M) \arrow[d] \\
\mathrm{H}^1(\mathrm{C}(Y \smallsetminus x; M_{-1})) \arrow[r, "(\star)"] & \mathrm{H}^2(\mathrm{C}(X \smallsetminus x; M))
\end{tikzcd}$$

where the left vertical arrow is an isomorphism by Corollary 3.5.2. Thus, it suffices to show that $(\star)$ is injective. Let α be in the kernel of $(\star)$ represented by a cocycle $(\alpha_y)_{y \in Y^{(1)}} \in \bigoplus_{y \in Y^{(1)}} M_{-2}(y)$ in the Cousin complex $\mathrm{C}^\bullet(Y \smallsetminus x; M_{-1})$. By assumption, there is $(\beta_z)_{z \in X^{(1)}} \in \bigoplus_{z \in X^{(1)}} M_{-1}(z)$ such that:

(1) if $y \in X^{(2)} \smallsetminus Y^{(1)}$, then

$$\sum_{z \in X^{(1)},\, y \in \bar{z}} \partial^z_y(\beta_z) = 0;$$

(2) if $y \in Y^{(1)}$, then

$$\sum_{z \in X^{(1)},\, y \in \bar{z}} \partial^z_y(\beta_z) = \alpha_y.$$

In particular, we see that $(\beta_z)_{z \in X^{(1)} \smallsetminus Y^{(0)}}$ defines a cocycle in $\mathrm{C}^\bullet(X \smallsetminus Y; M)$ of degree one. If we could write it as a coboundary, then we would be able to modify $(\beta_z)_{z \in X^{(1)}}$, so that $\beta_z = 0$ except when z is the generic point of Y. This would imply that $(\alpha_y)_{y \in Y^{(1)}}$ is a coboundary in $\mathrm{C}^\bullet(Y; M_{-1})$ as desired. Thus, to finish the proof of the proposition, it suffices to establish the vanishing of the group $\mathrm{H}^1(\mathrm{C}(X \smallsetminus Y; M))$ which, by Corollary 3.5.2, is isomorphic to the group $\mathrm{H}^1(X \smallsetminus Y; M)$. We conclude using the next lemma. □

Lemma 3.6.2. *Let X be an essentially smooth, local henselian k-scheme and $Y \subset X$ an essentially smooth hypersurface. Then, we have* $\mathrm{H}^1(X \smallsetminus Y; M) = 0$.

Proof. The proof relies on a variant of Gabber's presentation lemma, namely Lemma 3.6.3 below. Let $\alpha \in \mathrm{H}^1(X \smallsetminus Y; M)$. Then, we can find a closed subset $Z \subset X$, not containing Y, such that α belongs to the image of

$$\mathrm{H}^1_{Z \smallsetminus Y}(X \smallsetminus Y; M) \to \mathrm{H}^1(X \smallsetminus Y; M).$$

By Lemma 3.6.3, applied to a well-chosen k-variety over which X is proétale, there exists a proétale morphism $f : X \to \mathbb{A}^1_S$, with S essentially smooth local henselian, such that

(1) the induced morphism $Y \to S$ is an isomorphism and $f(Y)$ is the zero section of $\mathbb{A}^1_S$;

(2) Z is finite over S and f induces isomorphisms $Z \simeq f(Z) \simeq f^{-1}f(Z)$.

Thus, we have a Nisnevich square

$$\begin{tikzcd}
X\smallsetminus(Y \cup Z) \arrow[r] \arrow[d] & X\smallsetminus Y \arrow[d] \\
\mathbb{A}^1_S\smallsetminus(0_S \cup f(Z)) \arrow[r] & \mathbb{A}^1_S\smallsetminus 0_S,
\end{tikzcd}$$

inducing a commutative square

$$\begin{tikzcd}
\mathrm{H}^1_{Z\smallsetminus Y}(X\smallsetminus Y; M) \arrow[r] & \mathrm{H}^1(X\smallsetminus Y; M) \\
\mathrm{H}^1_{f(Z\smallsetminus Y)}(\mathbb{A}^1_S\smallsetminus 0_S; M) \arrow[r] \arrow[u, "\sim"] & \mathrm{H}^1(\mathbb{A}^1_S\smallsetminus 0_S; M) \arrow[u]
\end{tikzcd}$$

where the left vertical arrow is an isomorphism. Thus, it suffices to show that the group

$$\mathrm{H}^1(\mathbb{A}^1_S\smallsetminus 0_S; M)$$

vanishes. We can identify this group with

$$[(\mathbb{A}^1 \smallsetminus 0)_+ \wedge S_+, \mathrm{K}(M,1)]_{\mathrm{nis}} = [S_+, \mathrm{K}(M,1)^{\mathbb{A}^1\smallsetminus 0}]_{\mathrm{nis}}.$$

The unit section of $\mathbb{A}^1 \smallsetminus 0$ yields a decomposition into a direct sum

$$[S_+, \mathrm{K}(M,1)^{\mathbb{A}^1\smallsetminus 0}]_{\mathrm{nis}} \simeq [S_+, \mathrm{K}(M,1)]_{\mathrm{nis}} \oplus [S_+, \mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}}]_{\mathrm{nis}}.$$

We claim that the k-space $\mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}}$ is equivalent to $\mathrm{K}(M_{-1},1)$. Assuming this claim, it is easy to conclude. Indeed, we then have that $\mathrm{H}^1(\mathbb{A}^1_S\smallsetminus 0_S; M)$ is the direct sum of $\mathrm{H}^1(S; M)$ and $\mathrm{H}^1(S; M_{-1})$, and both groups vanish since S is local henselian.

It remains to see that the natural morphism of k-spaces

$$\mathrm{K}(M_{-1},1) \to \mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}}$$

is an equivalence. Since M is 1-strongly $\mathbb{A}^1$-invariant, the pointed k-space $\mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}}$ is motivic and $\mathrm{K}(M_{-1},1)$ is its 1-connective cover, i.e., $\mathrm{K}(M_{-1},1) \simeq \tau_{\geqslant 1}\mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}}$. Thus, to conclude, it is enough to show that $\mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}}$ is 1-connective, i.e., that $\pi_0\mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}} = 0$. By Theorem 2.4.13, if U is a local essentially smooth k-scheme with generic point η, the map

$$\Gamma(U; \pi_0\mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}}) \to \Gamma(\eta; \pi_0\mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}})$$

has trivial kernel. Since this is a morphism of abelian groups, this map is even injective. Therefore, to prove the vanishing of $\pi_0\mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}}$, it is enough to show that $\Gamma(K; \pi_0\mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}}) = 0$ for every essentially smooth extension K/k. But, we have

$$\Gamma(K; \pi_0\mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}}) \simeq [\mathrm{Spec}(K)_+, \mathrm{K}(M,1)^{\mathbb{G}_\mathrm{m}}]_{\mathrm{nis}} \simeq \mathrm{H}^1(\mathbb{A}^1_K\smallsetminus 0; M).$$

The group $\mathrm{H}^1(\mathbb{A}^1_K\smallsetminus 0; M)$ is a quotient of the group $\mathrm{H}^1(\mathbb{A}^1_K; M)$ (use for example the Cousin complex) and the latter vanishes by 1-strong $\mathbb{A}^1$-invariance. Thus $\mathrm{H}^1(\mathbb{A}^1_K\smallsetminus 0; M)$ is zero as needed. □

In proving of Lemma 3.6.2, we used the following variant of Gabber's presentation lemma. (This also appears in [Bac24, Lemma 4.16].)

Lemma 3.6.3. *Let X be a smooth k-variety, $Y \subset X$ a hypersurface and $x \in Y$ a smooth point of Y. Let $Z \subset X$ be a closed subset that does not contain any connected component of Y. Then, there exists a Nisnevich neighbourhood $X' \to X$ of $x \in X$ and an étale morphism $f : X' \to \mathbb{A}^1_S$, with S a smooth k-variety, such that the following conditions are satisfied:*

(1) $Y' = X' \times_X Y$ maps isomorphically to the zero section of $\mathbb{A}^1_S$;
(2) $Z' = X' \times_X Z$ is finite over S;
(3) $Z' \simeq f(Z') \simeq f^{-1}f(Z')$.

Proof. The lemma remains true if we replace the running assumption "k perfect" with the weaker assumption "x is essentially smooth over k". For the proof it, will be convenient to work under the latter assumption. We split the proof into several steps.

Step 1. We first reduce to the case where x is a k-rational point. Denote by D the closure of x in X. Replacing X with an open neighbourhood of x, we may assume that D is a smooth. It is then standard that, by replacing X with a Nisnevich neighbourhood of x, the inclusion of D admits a smooth retraction $r : X \to D$, and we may arrange that $r|_Y : Y \to D$ is smooth too. Let K be the fraction field of D, i.e., the residue field of x, and let $X_0 = X \times_D \operatorname{Spec}(K)$, $Y_0 = Y \times_D \operatorname{Spec}(K)$ and $Z_0 = Z \times_D \operatorname{Spec}(K)$. Then, x is naturally a K-rational point of the smooth K-variety X_0 lying on the smooth hypersurface $Y_0 \subset X_0$. Assuming that the lemma is known for X_0, Y_0 and Z_0 and the K-rational point x, it can be deduced for X, Y, Z and x by a standard spreading argument.

Step 2. From now on, we assume that x is a K-rational smooth point of Y. By Gabber's presentation lemma (see [CTHK97, Theorem 3.1.1] and [HK20, Theorem 1.1 & Remark 1.3]), we can find, after replacing X with an open neighbourhood of x, an étale morphism $e : X \to \mathbb{A}^d$ which restricts to a locally closed immersion on Y and Z, and such that $Y = e^{-1}e(Y)$ and $Z = e^{-1}e(Z)$. (In fact, this does not require the full strength of Gabber's presentation lemma and can be proven more easily, see [HK20, Lemma 2.4].) It is then sufficient to prove the lemma for $\mathbb{A}^d$ and the closures of the subsets $e(Y)$ and $e(Z)$.

Step 3. From now on, we assume that $X = \mathbb{A}^d = \operatorname{Spec}(k[t_1, \dots, t_d])$. Since x is a k-rational point, we may assume that $x = o$ is the origin of $\mathbb{A}^d$. Recall that Y is a hypersurface of $\mathbb{A}^d$ passing through o and smooth at o. Denote by $F = F(t_1, \dots, t_d)$ the polynomial defining Y. Then, F has no constant term and, after a linear change of coordinates, we may assume that

$$F(t_1, \dots, t_d) = t_1 + G(t_1, \dots, t_d)$$

where $G \in (t_1, \dots, t_d)^2$ is a linear combination of monomials of degree $\geqslant 2$.

Step 4. Consider the following change of variables appearing in the proof of the Noether normalisation lemma (see [Bou85, Chapitre V, §3, n° 1, Théorème 1] or [Sta18, Lemma 051N]):

$$s_1 = t_1, \quad s_2 = t_2 - t_1^{r}, \quad s_3 = t_3 - t_1^{r^2}, \quad \dots \quad s_d = t_d - t_1^{r^{d-1}}.$$

Here, $r \geqslant 1$ is an integer which is typically large. Then we have $t_1 = s_1$ and $t_i = s_i + s_1^{r^{i-1}}$, for $2 \leq i \leq d$. In particular, $F - s_1 \in (s_1, \dots, s_d)^2$. This implies that the composition of

$$Y \to \operatorname{Spec}(k[t_1, \dots, t_d]) \simeq \operatorname{Spec}(k[s_1, \dots, s_d]) \to \operatorname{Spec}(k[s_2, \dots, s_d])$$

is étale at the k-point o. On the other hand, by [Sta18, Lemma 00OX] and its proof, for r large enough, the composition of

$$Z \to \operatorname{Spec}(k[t_1, \dots, t_d]) \simeq \operatorname{Spec}(k[s_1, \dots, s_d]) \to \operatorname{Spec}(k[s_2, \dots, s_d])$$

is finite. Thus, in conclusion, after a certain nonlinear change of variables, we may assume that the projection $\mathbb{A}^d = \mathbb{A} \times \mathbb{A}^{d-1} \to \mathbb{A}^{d-1}$ induces a morphism $Y \to \mathbb{A}^{d-1}$ that is étale at o and a finite morphism $Z \to \mathbb{A}^{d-1}$.

Step 5. It is now easy to conclude. Denote by o' the origin of $\mathbb{A}^{d-1}$. Given a Nisnevich neighbourhood $S \to \mathbb{A}^{d-1}$ of o', set $\widetilde{X} = X \times_{\mathbb{A}^{d-1}} S$, $\widetilde{Y} = Y \times_{\mathbb{A}^{d-1}} S$ and $\widetilde{Z} = Z \times_{\mathbb{A}^{d-1}} S$. Since $Y \to \mathbb{A}^{d-1}$ is étale at o, we can choose S so that $\widetilde{Y}$ splits off a copy of S passing through o, say $\widetilde{Y} \simeq Y_1 \sqcup S$. Similarly, for S small enough, $\widetilde{Z}$ is a disjoint union $\widetilde{Z} = Z_1 \sqcup Z_0$ such that $o \notin Z_1$ and, if Z_0 is nonempty, then it is connected and $(Z_0 \times_S o')_{\mathrm{red}} = o$. We can also ensure that $Z_1 \cap S = \emptyset$ and $Z_0 \cap Y_1 = \emptyset$ by further shrinking S. Then, it is easy to see that $X' = \widetilde{X} \smallsetminus (Y_1 \cup Z_1)$ is a Nisnevich neighbourhood of o with all the desired properties. □

Now that the key proposition is proven, it is easy to conclude.

Proof that M is 2-strongly $\mathbb{A}^1$-invariant. Recall that our goal is to show that M is 2-strongly $\mathbb{A}^1$-invariant, i.e., that for every essentially smooth k-scheme X, the map

$$\mathrm{H}^2(X; M) \to \mathrm{H}^2(\mathbb{A}^1_X; M)$$

is an isomorphism. We argue by induction on the dimension of X. If $\dim(X) = 0$, there is nothing to show as both groups are zero. Assume that $\dim(X) = d > 0$, and that the result is known in dimension $\leqslant d-1$. We may assume that X is local henselian. In this case, we need to show that $\mathrm{H}^2(\mathbb{A}^1_X; M) = 0$. The induction hypothesis implies that

$$\mathrm{H}^2(X \smallsetminus Y; M) \simeq \mathrm{H}^2(\mathbb{A}^1_{X \smallsetminus Y}; M)$$

for any essentially smooth hypersurface $Y \subset X$. Denote by $i : Y \hookrightarrow X$ and $i' : \mathbb{A}^1_Y \to \mathbb{A}^1_X$ the obvious inclusions. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathbb{A}^1_{X \smallsetminus Y}; M) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^2(\mathbb{A}^1_Y; i'^{!} M_{\mathbb{A}^1_X}) \arrow[r] \arrow[d, "(\star)"', "\sim"] & \mathrm{H}^2(\mathbb{A}^1_X; M) \arrow[r] \arrow[d] & \mathrm{H}^2(\mathbb{A}^1_{X \smallsetminus Y}; M) \arrow[d, "\sim"] \\
\mathrm{H}^1(X \smallsetminus Y; M) \arrow[r] & \mathrm{H}^2(Y; i^{!} M_X) \arrow[r] & \mathrm{H}^2(X; M) \arrow[r] & \mathrm{H}^2(X \smallsetminus Y; M)
\end{tikzcd}$$

induced by the zero section of $\mathbb{A}^1$. By Proposition 3.6.1, there are natural equivalences

$$\tau^{\geqslant 2}(i^{!} M_X) \simeq (M_{-1})_Y[-1] \qquad \text{and} \qquad \tau^{\geqslant 2}(i'^{!} M_{\mathbb{A}^1_X}) \simeq (M_{-1})_{\mathbb{A}^1_Y}[-1].$$

It follows that the map $(\star)$ can be identified with

$$\mathrm{H}^1(\mathbb{A}^1_Y, M_{-1}) \to \mathrm{H}^1(Y; M_{-1})$$

showing that it is an isomorphism as indicated in the above diagram. (Recall that M_{-1} is also 1-strongly $\mathbb{A}^1$-invariant.) This implies that the morphism

$$\mathrm{H}^2(\mathbb{A}^1_X; M) \to \mathrm{H}^2(X; M)$$

is injective. Since X is local henselian, this proves that $\mathrm{H}^2(\mathbb{A}^1_X; M)$ is zero. □

3.7. **Proof that n-strong $\mathbb{A}^1$-invariance $\Rightarrow$ $n+1$-strong $\mathbb{A}^1$-invariance.** We finish the proof of the main theorem of this section, namely Theorem 3.0.1, by showing the implication in the title of the subsection. The proof of this implication for $n \geqslant 2$ is very similar to the case $n = 1$ established in Subsection 3.6, and it is in fact slightly simpler. Throughout this subsection, we fix once and for all an n-strongly $\mathbb{A}^1$-invariant sheaf M with $n \geqslant 2$. We start by recalling the following.

Recollection 3.7.1.

(1) Let X be an essentially smooth k-scheme and $x \in X^{(d)}$ a point of codimension $d \leqslant n+1$. Then, we have the vanishing of local cohomology

$$\mathrm{H}^i_x(X; M) = 0 \qquad \text{if} \qquad i \neq d.$$

Moreover, assuming that $d \leqslant n$, there is an isomorphism

$$\mathrm{H}^n_x(X; M) \simeq M_{-n}(x)$$

that depends on a trivialisation of the normal bundle $\mathrm{N}_x \simeq \mathbb{A}^d_x$. (See Propositions 3.1.13 and 3.2.5.)

(2) Let X be an essentially smooth k-scheme of dimension $\leqslant n+1$. Then all the nonzero terms of the coniveau spectral sequence for X and M lie on the line $(q = 0)$. In particular, the Cousin complex $\mathrm{C}^\bullet(X; M)$ computes $\mathrm{H}^*(X; M)$. This follows readily from the point (1).

(3) Let X be an essentially smooth k-variety and $Y \subset X$ an essentially smooth closed subscheme of codimension $d \leqslant n$. Denote by $i : Y \hookrightarrow X$ the inclusion. Then, a trivialisation $\mathrm{N}_Y \simeq \mathbb{A}^d_Y$ of the normal bundle of Y in X induces an equivalence

$$\tau^{\leqslant d}(i^! M_X) \simeq (M_{-d})_Y[-d]$$

in the derived category of Nisnevich sheaves on $\acute{\mathrm{E}}\mathrm{t}_Y$. This follows by combining Theorem 3.1.11 and Proposition 3.2.5.

Proposition 3.7.2. *Let X be an essentially smooth k-variety and $Y \subset X$ an essentially smooth closed subscheme of codimension $d \leqslant n$. Fix a trivialisation $\mathrm{N}_Y \simeq \mathbb{A}^d_Y$ of the normal bundle of Y in X. Then, there is a morphism of Cousin complexes*

$$\mathrm{C}^\bullet(Y; M_{-d})[-d] \to \mathrm{C}^\bullet(X; M). \qquad (\star)$$

If $y \in Y$ is a point of codimension c in Y such that $c + d \leqslant n$, the morphism $(\star)$ is given at y by the identity of $M_{-d-c}(y)$ modulo well-chosen isomorphisms provided by Proposition 3.2.5:

$$\mathrm{H}^c_y(Y; M_{-d}) \simeq M_{-c-d}(y) \qquad \textit{and} \qquad \mathrm{H}^{c+d}_y(X; M) \simeq M_{-c-d}(y).$$

(More precisely, the trivialisation of the normal bundle of y in X should be adapted to the normal bundle of y in Y and should be compatible with the restriction of the chosen trivialisation $\mathrm{N}_Y \simeq \mathbb{A}^d_Y$ at the point y.)

Proof. Denote by $i : Y \hookrightarrow X$ the inclusion. There is an equivalence $\tau^{\leqslant d}(i^! M_X) \simeq (M_{-d})_Y[-d]$ in the derived category of Nisnevich sheaves on $\acute{\mathrm{E}}\mathrm{t}_Y$ as explained in Recollection 3.7.1. In particular, we have a morphism

$$i_*(M_{-d})_Y[-d] \to M_X$$

in the derived category of Nisnevich sheaves on $\acute{\mathrm{E}}\mathrm{t}_X$. The coniveau spectral sequence for X can be extended to any complex of Nisnevich sheaves on $\acute{\mathrm{E}}\mathrm{t}_X$ and it is then functorial on the derived category of Nisnevich sheaves on $\acute{\mathrm{E}}\mathrm{t}_X$. In particular, we obtain a morphism of Cousin complexes

$$\mathrm{C}^\bullet(X; i_*(M_{-d})_Y[-d]) \to \mathrm{C}^\bullet(X; M).$$

Clearly, the domain of this morphism can be identified with $\mathrm{C}^\bullet(Y; M_{-d})[-d]$. The last assertion is left to the reader. □

Remark 3.7.3. Let X be a local essentially smooth k-scheme of dimension $n+1$ and let $x \in X$ be its closed point. Let $Y \subset X$ be an essentially smooth closed subscheme of codimension $1 \leqslant d \leqslant n$. Equivalently, we have $1 \leqslant \dim(Y) \leqslant n$ and this implies that the last term of the Cousin complex $\mathrm{C}^\bullet(Y; M_{-d})$ is isomorphic to $M_{-n-1}(x)$. Applying Proposition 3.7.2, we obtain a morphism

$$M_{-n-1}(x) \simeq \mathrm{H}^{n-d+1}_x(Y; M_{-d}) \to \mathrm{H}^{n+1}_x(X; M).$$

At this stage, we do not know to what extent this morphism depend on the choice of Y. However, note that for $Y_1 \subset Y_2$ two essentially smooth hypersurfaces of codimensions between 1 and n, the associated morphisms are the "same". More precisely, they are the same if the trivialisations of the normal bundles of $Y_1 \subset X$ and $Y_2 \subset X$ are chosen compatibly.

Lemma 3.7.4. *Keep the notations and assumptions as in Remark 3.7.3. The maps*

$$M_{-n-1}(x) \to \mathrm{H}^{n+1}_x(X; M)$$

are always injective.

Proof. We use an argument already used in the proof of Theorem 3.4.1; see the end of Subsection 3.4. As explained in Remark 3.7.3, it suffices to treat the case where Y is one-dimensional, i.e., $d = n$. (Indeed, any essentially smooth closed subscheme of X of codimension $\leqslant n$ contains one of codimension n.) Consider the following commutative diagram

$$\begin{tikzcd} x \arrow[r, "j"] \arrow[rr, bend right, "x"'] & Y \arrow[r, "i"] & X. \end{tikzcd}$$

There is an exact triangle

$$(M_{-n})_Y[-n] \to i^! M_X \to \tau^{\geqslant n+1}(i^! M_X)$$

in the derived category of Nisnevich sheaves on $\acute{\mathrm{E}}\mathrm{t}_Y$. Apply $j^!$ to get the exact triangle

$$j^!(M_{-n})_Y[-n] \to x^! M_X \to j^! \tau^{\geqslant n+1}(i^! M_X)$$

in the derived category of Nisnevich sheaves on $\acute{\mathrm{E}}\mathrm{t}_x$. The map in the statement can be identified with the map obtained from

$$\mathrm{H}^1(j^!(M_{-n})_Y) = \mathrm{H}^{n+1}(j^!(M_{-n})_Y[-n]) \to \mathrm{H}^{n+1}(x^! M_X)$$

by taking global sections. We are thus reduced to showing that the above map is injective and, for this, it suffices to show that $\mathrm{H}^n(j^! \tau^{\geqslant d+1}(i^! M_X))$ is zero. This is clear because the functor $j^!$ is left t-exact for the natural t-structures on the derived categories of sheaves. □

Proposition 3.7.5. *Keep the notations and assumptions as in Remark 3.7.3. The image of the map*

$$M_{-n-1}(x) \to \mathrm{H}^{n+1}_x(X; M)$$

is independent of the choice of the essentially smooth closed subscheme $Y \subset X$.

Proof. Recall that we are assuming that $n \geqslant 2$ since the case $n = 1$ was dealt with before. (Indeed, this follows from Proposition 3.4.10 since the image of the map $\partial^y_x : \mathrm{H}^1_y(X; M) \to \mathrm{H}^2_x(X; M)$ coincides with the image of the map $M_{-2}(x) \to \mathrm{H}^2_x(X; M)$ associated to the essentially smooth subscheme $\overline{y} \subset X$. This uses Corollary 3.4.9 and the surjectivity of the map $\partial^y_x : M_{-1}(y) \to$

$\mathrm{H}^1_x(\bar{y}; M_{-1})$.) Denote by I_Y the image of the map $M_{-n-1}(x) \to \mathrm{H}^{n+1}_x(X; M)$ constructed using Y. As explained in Remark 3.7.3, we know that $I_{Y_1} = I_{Y_2}$ if $Y_1 \subset Y_2$. Thus, it is enough to treat the case of hypersurfaces. But if Y_1 and Y_2 are two essentially smooth hypersurfaces in X, we may find a third one Y_3 intersecting both Y_1 and Y_2 transversally. Thus, it suffices to show that $I_{Y_1} = I_{Y_2}$ when Y_1 and Y_2 are two essentially smooth hypersurfaces intersecting transversally. But since $n \geqslant 2$, $Y_1 \cap Y_2$ has codimension $\leqslant n$, and we may use the equalities $I_{Y_1} = I_{Y_1 \cap Y_2} = I_{Y_2}$ to conclude. □

Notation 3.7.6. Keep the notations and assumptions as in Remark 3.7.3. We denote by $I \subset \mathrm{H}^{n+1}_x(X; M)$ the common image of the morphisms $M_{-n-1}(x) \to \mathrm{H}^{n+1}_x(X; M)$ associated to the essentially smooth subschemes $Y \subset X$ of codimension between 1 and n.

Proposition 3.7.7. *Let X be a local essentially smooth k-scheme of dimension $n + 1$ with closed point $x \in X$. For a point $y \in X^{(n)}$ of codimension n, consider the map*

$$\partial^y_x : \mathrm{H}^n_y(X; M) \to \mathrm{H}^{n+1}_x(X; M)$$

appearing in the last differential of the Cousin complex $\mathrm{C}^\bullet(X; M)$. Then, we have $\mathrm{Im}(\partial^y_x) \subset I$ and this inclusion is an equality if $\bar{y}$ is essentially smooth.

Proof. We start by showing the last assertion concerning the case where $Y = \bar{y}$ is smooth. The morphism of Cousin complexes $\mathrm{C}^\bullet(Y; M_{-n})[-n] \to \mathrm{C}^\bullet(X; M)$ from Proposition 3.7.2, yields a commutative square

$$\begin{tikzcd}
\mathrm{H}^n_y(X; M) \arrow[r, "\partial^y_x"] & \mathrm{H}^{n+1}_x(X; M) \\
M_{-n}(y) \arrow[r, "\partial^y_x"] \arrow[u, "\sim"'] & \mathrm{H}^1_x(Y; M_{-n}) \simeq M_{-n-1}(x). \arrow[u]
\end{tikzcd}$$

(When $n = 1$, this is Corollary 3.4.9.) By Proposition 3.7.5, the image of the right vertical arrow is precisely the subgroup I. The last assertion follows then from the surjectivity of the map $\partial^y_x : M_{-n}(y) \to \mathrm{H}^1_x(Y; M_{-n})$, which is provided by Corollary 3.4.5.

We now return to the case of a general n-codimensional point $y \in X^{(n)}$ and show the inclusion $\mathrm{Im}(\partial^y_x) \subset I$. We use a similar argument as in the proof of Proposition 3.4.13. Note that we may replace X with its henselisation at x, since this does not affect the local cohomology group $\mathrm{H}^{n+1}_x(X; M)$. Applying Gabber's presentation lemma (i.e., Theorem 2.4.14) to a suitable smooth k-variety over which X is proétale, we can find a proétale morphism

$$f : X \to \mathbb{A}^1_S,$$

where S is a local henselian essentially smooth k-scheme, such that $\bar{y}$ is finite over S and f induces isomorphisms $\bar{y} \simeq f(\bar{y}) \simeq f^{-1}f(\bar{y})$. Let $s \in S$ be the closed point of S and let $t \in S$ be the image of y in S. Denote by η_s and η_t the generic points of $\mathbb{A}_s$ and $\mathbb{A}^t$. Identify the points x and y with their images in $\mathbb{A}^1_S$. We then need to show that the image of

$$\partial^y_x : \mathrm{H}^n_y(\mathbb{A}^1_S; M) \to \mathrm{H}^{n+1}_x(\mathbb{A}^1_S; M)$$

is contained in I. By the discussion at the beginning of the proof, we know that I is the image of

$$\partial^{\eta_s}_x : \mathrm{H}^n_{\eta_s}(\mathbb{A}^1_S; M) \to \mathrm{H}^{n+1}_x(\mathbb{A}^1_S; M).$$

It is thus sufficient to show the inclusion $\mathrm{Im}(\partial^y_x) \subset \mathrm{Im}(\partial^{\eta_s}_x)$. Let $\alpha \in \mathrm{H}^n_y(\mathbb{A}^1_S; M) \simeq M_{-n}(y)$. Since the Cousin complex $\mathrm{C}^\bullet(\mathbb{A}^1_t; M_{-n+1})$ is acyclic in degree zero, we can find $\beta \in \mathrm{H}^{n-1}_{\eta_t}(\mathbb{A}^1_S; M) \simeq M_{-n+1}(\eta_t)$

such that $\partial_z^{\eta_t}(\beta) = 0$ for $z \in (\mathbb{A}_t^1)^{(1)} \smallsetminus \{y\}$ and $\partial_y^{\eta_t}(\beta) = \alpha$. Since $\partial \circ \partial(\beta) = 0$ in $\mathrm{C}^\bullet(\mathbb{A}_S^1; M)$, we deduce that $\partial_x^{\eta_s} \circ \partial_{\eta_s}^{\eta_t}(\beta) + \partial_x^y(\alpha) = 0$. This shows that $\partial_x^y(\alpha) \in \mathrm{Im}(\partial_x^{\eta_s})$ as desired. □

Corollary 3.7.8. *Let X be an essentially smooth local k-scheme of dimension $n+1$ with closed point $x \in X$. Then $\mathrm{C}^\bullet(X, M)$ is acyclic except in degree 0. Moreover, there are isomorphisms*

$$M_{-n-1}(x) \xrightarrow{\sim} \mathrm{H}_x^{n+1}(X; M)$$

given by the maps in Remark 3.7.3.

Proof. We first establish the isomorphism in the statement. We use the same argument as in the proof of Theorem 3.4.1; see the end of Subsection 3.4. First, notice that we may replace X with its henselisation at x. (Indeed, this does not affect the map $M_{-n-1}(x) \to \mathrm{H}_x^{n+1}(X; M)$.) As noted in Recollection 3.7.1, the Cousin complex $\mathrm{C}^\bullet(X; M)$ computes the cohomology of X with values in M. But, for henselian X, the said cohomology vanishes in positive degrees which implies that the last differential of $\mathrm{C}^\bullet(X; M)$, namely

$$\sum_{y \in X^{(n)}} \partial_x^y : \bigoplus_{y \in X^{(n)}} \mathrm{H}_y^n(X; M) \to \mathrm{H}_x^{n+1}(X; M),$$

is surjective. By Proposition 3.7.7, the image of this differential is I, which is also the image of the map $M_{-n-1}(x) \to \mathrm{H}_x^{n+1}(X; M)$. (See notation 3.7.6.) This shows that the maps in Remark 3.7.3 are surjective. But we also know that they are injective by Lemma 3.7.4.

We now go back to a general local essentially smooth k-scheme of dimension $n+1$ (i.e., not necessarily henselian). We still need to show that $\mathrm{C}^\bullet(X; M)$ is acyclic except in degree zero. The vanishing of $\mathrm{H}^{n+1}(\mathrm{C}(X; M))$ follows from the isomorphism $M_{-n-1}(x) \simeq \mathrm{H}_x^{n+1}(X; M)$ yielding the surjectivity of the last differential of the Cousin complex. Thus, it remains to show the vanishing of $\mathrm{H}^i(\mathrm{C}(X; M))$ for $1 \leqslant i \leqslant n$. Equivalently, we need to show the vanishing of $\mathrm{H}^i(X; M)$ in the same range. This follows from Theorem 2.4.13 applied to the pointed motivic space $\mathrm{K}(M, n)$. □

We continue with the following generalisation of Proposition 3.5.1.

Proposition 3.7.9. *Let X be an essentially smooth local k-scheme. Then, we have the vanishing*

$$\mathrm{H}^i(\mathrm{C}(X; M)) = 0 \qquad \text{and} \qquad \mathrm{H}^i(\mathrm{C}(\mathbb{A}_X^1; M)) = 0$$

for $1 \leqslant i \leqslant n$.

Proof. The proof is copy-pasted from the case $n = 1$ but we include it for the reader's convenience. For an integer $d \geqslant 0$, consider the following two properties.

- (P_d) For every local essentially smooth k-scheme X of dimension $\leqslant d$, we have $\mathrm{H}^i(\mathrm{C}(X; M)) = 0$, for $1 \leqslant i \leqslant n$.
- (Q_d) For every local essentially smooth k-scheme S of dimension $\leqslant d$, we have $\mathrm{H}^i(\mathrm{C}(\mathbb{A}_S^1; M)) = 0$, for $1 \leq i \leq n$.

Both assertions are true for $d = 0$. We will prove the proposition by showing the implications

$$(P_d) \Rightarrow (Q_d) \qquad \text{and} \qquad (Q_d) \Rightarrow (P_{d+1}).$$

Step 1: $(P_d) \Rightarrow (Q_d)$. Assume that S is local of dimension $\leqslant d$. Consider the diagram of Cousin complexes:

$$\begin{tikzcd}
\mathrm{C}^\bullet(\mathbb{A}^1_X; M) \arrow[r, phantom, ":"] & \cdots \arrow[r] & \bigoplus_{x \in (\mathbb{A}^1_X)^{(i)}} \mathrm{H}^i_x(\mathbb{A}^1_X; M) \arrow[r] & \cdots \\
\mathrm{C}^\bullet(X; M) \arrow[u] \arrow[r, phantom, ":"] & \cdots \arrow[r] & \bigoplus_{x \in X^{(i)}} \mathrm{H}^i_x(X; M) \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

Fix $\alpha \in \mathrm{H}^i(\mathrm{C}(\mathbb{A}^1_X; M))$, with $1 \leqslant i \leqslant n$. Choose a representative $(\alpha_x)_{x \in (\mathbb{A}^1_X)^{(i)}}$ in $\mathrm{C}^i(\mathbb{A}^1_X; M)$. Since $i \leq n$, we may view α_x as an element of the group $M_{-i}(x) \simeq \mathrm{H}^i_x(\mathbb{A}^1_X; M)$. There are two types of points $x \in (\mathbb{A}^1_X)^{(i)}$, namely:

- Type I: those which are finite over their image in X.
- Type II: those which are of transcendence degree one over their image in X.

This gives the partition

$$(\mathbb{A}^1_X)^{(i)} = \left(\coprod_{s \in X^{(i-1)}} (\mathbb{A}^1_s)^{(1)} \right) \coprod \left\{ \eta_s \mid s \in X^{(i)} \right\}$$

where η_s is the generic point of $\mathbb{A}^1_s$. Using that $\mathrm{C}^\bullet(\mathbb{A}^1_s; M_{-i+1})$ is acyclic in degree 1, we can find $\beta_s \in \mathrm{H}^{i-1}_{\eta_s}(\mathbb{A}^1_X; M) \simeq M_{-i+1}(\eta_s)$ such that $\partial^{\eta_s}_x(\beta_s) = \alpha_x$ for every $x \in (\mathbb{A}^1_s)^{(1)}$. Thus we may modify $(\alpha_x)_{x \in (\mathbb{A}^1_X)^{(i)}}$ by a coboundary and assume that the representative is supported at points of type II. Then, since $\partial((\alpha_x)_x) = 0$, we see that each α_{η_s} lies in the kernel of

$$M_{-i}(\eta_s) \to \bigoplus_{x \in (\mathbb{A}^1_s)^{(1)}} M_{-i-1}(x).$$

(Note that this is true even when $i = n$ by Corollary 3.7.8.) Thus, α_{η_s} belongs to $M_{-i}(\mathbb{A}^1_s) \simeq M_{-i}(s)$. This shows that $(\alpha_x)_x$ is the image of a cocycle in the complex $\mathrm{C}^\bullet(X; M)$, and we may use (P_d) to conclude that $(\alpha_x)_x$ is a coboundary.

Step 2: $(Q_d) \Rightarrow (P_{d+1})$. We use Gabber's presentation lemma. More precisely, we prove the following claim which is enough to conclude.

Claim. Given an integral smooth k-variety X' and a point $x \in X'$ of codimension d, for any class $\alpha \in \mathrm{H}^i(\mathrm{C}(X'; M))$, with $1 \leqslant i \leqslant n$, there is an open neighbourhood U of x in X' such that $\alpha|_U = 0$.

Indeed, let $(\alpha_y)_{y \in X'^{(i)}}$ be a cocycle in $\mathrm{C}^\bullet(X'; M)$ representing α. Let $Z \subset X'$ be a 1-codimensional closed subset containing the finite set of points $\{y \in X'^{(i)} \mid \alpha_y \neq 0\}$. Enlarging Z if necessary, we may assume that x is a point of codimension $d-1$ in Z. By Theorem 2.4.14, we may replace X' with an open neighbourhood of x and assume the existence of an étale morphism

$$e : X' \to \mathbb{A}^1_{S'},$$

where S' is an integral smooth k-variety, and such that Z is finite over S' and $Z \simeq e(Z) \simeq e^{-1}e(Z)$. Note that this implies that x maps to a point $s \in S'$ of codimension $d-1$. Now, we may identify the points of Z with their images in $\mathbb{A}^1_{S'}$ and view $(\alpha_y)_y$ as a cocycle in $\mathrm{C}^\bullet(\mathbb{A}^1_{S'}; M)$. Using (Q_d), we can find an open neighbourhood V of s in S' such that the image of $(\alpha_y)_y$ in $\mathrm{C}^\bullet(\mathbb{A}^1_V; M)$ is a coboundary. The same is true for the image of $(\alpha_y)_y$ in $\mathrm{C}^\bullet(U; M)$ where $U = e^{-1}(\mathbb{A}^1_V)$ which is an open neighbourhood of x in X'. □

We now derive three corollaries of Proposition 3.7.9, generalising Corollaries 3.5.2, 3.5.3 and 3.5.4.

Corollary 3.7.10. *Let X be an essentially smooth k-scheme. Then, there are natural isomorphisms*

$$\mathrm{H}^i_{\mathrm{nis}}(X;M) \simeq \mathrm{H}^i_{\mathrm{zar}}(X;M) \simeq \mathrm{H}^i(\mathrm{C}^\bullet(X;M))$$

for $0 \leqslant i \leqslant n$.

Proof. Indeed, by Proposition 3.7.9, the truncated complex of sheaves

$$\tau^{\leqslant n}\mathrm{C}(-;M) = \left[\mathrm{C}^0(-;M) \to \cdots \to \mathrm{C}^{n-1}(-;M) \to \ker(\mathrm{C}^n(-;M) \to \mathrm{C}^{n+1}(-;M))\right]$$

is a Zariski local resolution of M_X (the restriction of M to the small Nisnevich site of X) and all its terms are acyclic for the Zariski and Nisnevich topologies except possibly in degree n. Thus the complex of global sections of $\tau^{\leqslant n}\mathrm{C}(-;M)$ computes the cohomology groups of X valued in M_X up to degree n in the Zariski and the Nisnevich topologies. □

Corollary 3.7.11. *Let X be an essentially smooth k-scheme. Let $Z \subset X$ be a closed subset of codimension $\geqslant n+2$. Then the map $\mathrm{H}^i(X;M) \to \mathrm{H}^i(X \smallsetminus Z;M)$ is an isomorphism for $0 \leqslant i \leqslant n$.*

Proof. By Corollary 3.7.10, it is enough to show that $\mathrm{H}^i(\mathrm{C}(X;M)) \to \mathrm{H}^i(\mathrm{C}(X\smallsetminus Z;M))$ is an isomorphism for $i \leqslant n$. But $\mathrm{C}^\bullet(X;M) \to \mathrm{C}^\bullet(X\smallsetminus Z;M)$ is an isomorphism in degrees $\leqslant n+1$. □

Corollary 3.7.12. *Let X be an essentially smooth k-scheme and let $x \in X$ be a point of codimension $\geqslant n+2$. Then $\mathrm{H}^i_x(X;M) = 0$ for $0 \leqslant i \leqslant n+1$.*

Proof. Since M is unramified, the desired vanishing holds for $i = 0$ and $i = 1$ by Lemma 3.3.2. By the same lemma, for $2 \leqslant i \leqslant n+1$, we have $\mathrm{H}^i_x(X;M) \simeq \mathrm{H}^{i-1}(X^h_x \smallsetminus x;M)$, where X^h_x is the henselisation of X at x. Since x has codimension $\geqslant n+2$ in X^h_x, we have

$$\mathrm{H}^{i-1}(X^h_x \smallsetminus x;M) \simeq \mathrm{H}^{i-1}(X^h_x;M) = 0 \qquad \text{for} \qquad i \leqslant n+1$$

by Corollary 3.7.11. □

Now we arrive at the key proposition, namely the generalisation of Proposition 3.6.1. The proof is again copy-pasted from the case $n = 1$ but we include it for the reader's convenience.

Proposition 3.7.13. *Let X be an essentially smooth k-scheme and let $i : Y \hookrightarrow X$ be the inclusion of an essentially smooth divisor. Then the complex of sheaves $i^!M_X$ satisfies $\mathrm{H}^0(i^!M_X) = 0$ and $\mathrm{H}^r(i^!M_X) = 0$, for $2 \leqslant r \leqslant n+1$. Said differently, the complex of sheaves $\tau^{\leqslant n+1}(i^!M_X)$ is concentrated in cohomological degree one.*

Proof. We will argue by induction on n, assuming that the result is known for m-strongly $\mathbb{A}^1$-invariant sheaves when $1 \leqslant m \leqslant n-1$. (The case $m = 1$ is treated in Proposition 3.6.1.) Thus, we may assume that the required vanishing is known except in degree $n+1$. We argue by contradiction assuming that the sheaf $\mathrm{H}^{n+1}(i^!M_X)$ is nonzero. Let $x \in Y$ be a point of minimal codimension such that $\underline{x}^*\mathrm{H}^{n+1}(i^!M_X)$ is nonzero. (Here, we write $\underline{x} : x \to Y$ for the obvious inclusion to distinguish it from the inclusion of x into X.) Replacing X with an étale neighbourhood of a geometric point over x, we may even assume that

$$\Gamma(x; \underline{x}^*\mathrm{H}^{n+1}(i^!M_X)) \neq 0.$$

Without loss of generality, we may also assume that X is henselian local with closed point x. Necessarily, we have $\dim(X) \geqslant n+1$. Consider an exact triangle

$$\mathrm{H}^1(i^!M_X)[-1] \to i^!M_X \to \mathcal{K}$$

in the derived category of Nisnevich sheaves on $\acute{\mathrm{E}}\mathrm{t}_Y$. By construction $\mathcal{K} = \tau^{\geqslant n+1}(i^!M_X)$ is concentrated in cohomological degrees $\geqslant n+1$. As above, we denote by $\underline{x} : x \hookrightarrow Y$ the obvious inclusion to distinguish it from the inclusion of x into X, which we denote by x. Thus $x = i \circ \underline{x}$. Apply $\underline{x}^!$ to the previous exact triangle to get an exact triangle

$$\underline{x}^!\mathrm{H}^1(i^!M_X)[-1] \to x^!M_X \to \underline{x}^!\mathcal{K} \qquad (\triangle)$$

in the derived category of Nisnevich sheaves on $\acute{\mathrm{E}}\mathrm{t}_x$. Note also that the first term of this complex is equivalent to $\underline{x}^!(M_{-1})_Y[-1]$. Since $\mathrm{H}^{n+1}(\mathcal{K})$ is supported at the closed point $x \in Y$, we have $\mathrm{H}^{n+1}(\underline{x}^!\mathcal{K}) \simeq \underline{x}^!\mathrm{H}^{n+1}(\mathcal{K}) \simeq \underline{x}^*\mathrm{H}^{n+1}(\mathcal{K})$. Thus, in order to get a contradiction, we need to show that the sheaf $\mathrm{H}^{n+1}(\underline{x}^!\mathcal{K})$ has no nonzero global sections. From the exact triangle $(\triangle)$, we get a long exact sequence of sheaves on $\acute{\mathrm{E}}\mathrm{t}_x$:

$$\cdots \to \mathrm{H}^n(\underline{x}^!(M_{-1})_Y) \to \mathrm{H}^{n+1}(x^!M_X) \to \mathrm{H}^{n+1}(\underline{x}^!\mathcal{K}) \to \mathrm{H}^{n+1}(\underline{x}^!(M_{-1})_Y) \to \mathrm{H}^{n+2}(x^!M_X) \to \cdots .$$

Taking global sections, we obtain the long exact sequence of abelian groups:

$$\cdots \to \mathrm{H}^n_x(Y; M_{-1}) \xrightarrow{(1)} \mathrm{H}^{n+1}_x(X; M) \to \Gamma(x; \mathrm{H}^{n+1}(\underline{x}^!\mathcal{K})) \to \mathrm{H}^{n+1}_x(Y; M_{-1}) \xrightarrow{(2)} \mathrm{H}^{n+2}_x(X; M) \to \cdots . \quad (\triangle')$$

To go further, we treat the cases $\dim(X) = n+1$, $\dim(X) = n+2$ and $\dim(X) \geqslant n+3$ separately.

Case 1: $\dim(X) = n+1$. In this case, the group $\mathrm{H}^{n+1}_x(Y; M_{-1})$ is zero since $\dim(Y) = n$. On the other hand, the morphism (1) in the sequence $(\triangle')$ identifies with $M_{-n-1}(x) \to \mathrm{H}^{n+1}_x(X; M)$ which is an isomorphism by Corollary 3.7.8.

Case 2: $\dim(X) \geqslant n+3$. In this case, both X and Y have dimension $\geqslant n+2$ and we may use Corollary 3.7.12 to get the vanishing of the local cohomology groups $\mathrm{H}^{n+1}_x(X; M)$ and $\mathrm{H}^{n+1}_x(Y; M_{-1})$.

Case 3: $\dim(X) = n+2$. This is the most difficult case. Here, we have $\mathrm{H}^{n+1}_x(X; M) = 0$ by Corollary 3.7.12 but, a priori, the group $\mathrm{H}^{n+1}_x(Y; M_{-1})$ could be nonzero. Thus, to prove the requested vanishing, we need to show that the map

$$\mathrm{H}^{n+1}_x(Y; M_{-1}) \to \mathrm{H}^{n+2}_x(X; M)$$

is injective. This map can be identified with the Gysin map

$$\mathrm{H}^n(Y \smallsetminus x; M_{-1}) \to \mathrm{H}^{n+1}(X \smallsetminus x; M).$$

There is a commutative square

$$\begin{tikzcd}
\mathrm{H}^n(Y \smallsetminus x; M_{-1}) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^{n+1}(X \smallsetminus x; M) \arrow[d] \\
\mathrm{H}^n(\mathrm{C}(Y \smallsetminus x; M_{-1})) \arrow[r, "(\star)"] & \mathrm{H}^{n+1}(\mathrm{C}(X \smallsetminus x; M))
\end{tikzcd}$$

where the left vertical arrow is an isomorphism by Corollary 3.7.10. Thus, it suffices to show that $(\star)$ is injective. Let α be in the kernel of $(\star)$ represented by a cocycle $(\alpha_y)_{y \in Y^{(n)}} \in \bigoplus_{y \in Y^{(n)}} M_{-n-1}(y)$ in the Cousin complex $\mathrm{C}^\bullet(Y \smallsetminus x; M_{-1})$. By assumption, there is $(\beta_z)_{z \in X^{(n)}} \in \bigoplus_{z \in X^{(n)}} M_{-n}(z)$ such that:

(1) if $y \in X^{(n+1)} \smallsetminus Y^{(n)}$, then

$$\sum_{z \in X^{(n)},\, y \in \bar{z}} \partial^z_y(\beta_z) = 0;$$

(2) if $y \in Y^{(n)}$, then

$$\sum_{z \in X^{(n)},\, y \in \overline{z}} \partial^z_y(\beta_z) = \alpha_y.$$

In particular, we see that $(\beta_z)_{z \in X^{(n)} \smallsetminus Y^{(n-1)}}$ defines a cocycle in $\mathrm{C}^\bullet(X \smallsetminus Y; M)$ of degree n. If we could write it as a coboundary, then we would be able to modify $(\beta_z)_{z \in X^{(n)}}$, so that $\beta_z = 0$ except when $z \in Y$. This would imply that $(\alpha_y)_{y \in Y^{(n)}}$ is a coboundary in $\mathrm{C}^\bullet(Y; M_{-1})$ as desired. Thus, to finish the proof of the proposition, it suffices to establish the vanishing of the group $\mathrm{H}^n(\mathrm{C}(X \smallsetminus Y; M))$ which, by Corollary 3.7.10, is isomorphic to the group $\mathrm{H}^n(X \smallsetminus Y; M)$. We conclude using the next lemma. □

Lemma 3.7.14. *Let X be an essentially smooth, local henselian k-scheme and $Y \subset X$ an essentially smooth hypersurface. Then, we have* $\mathrm{H}^m(X \smallsetminus Y; M) = 0$ *for* $1 \leqslant m \leqslant n$.

Proof. The proof relies on Lemma 3.6.3 and it is copy-pasted from the case $n = 1$. Let $\alpha \in \mathrm{H}^m(X \smallsetminus Y; M)$, with $1 \leqslant m \leqslant n$. Then, we can find a closed subset $Z \subset X$, not containing Y, such that α belongs to the image of

$$\mathrm{H}^m_{Z \smallsetminus Y}(X \smallsetminus Y; M) \to \mathrm{H}^m(X \smallsetminus Y; M).$$

By Lemma 3.6.3, applied to a well-chosen k-variety over which X is proétale, there exists a proétale morphism $f : X \to \mathbb{A}^1_S$, with S essentially smooth local henselian, such that

(1) the induced morphism $Y \to S$ is an isomorphism and $f(Y)$ is the zero section of $\mathbb{A}^1_S$;
(2) Z is finite over S and f induces isomorphisms $Z \simeq f(Z) \simeq f^{-1}f(Z)$.

Thus, we have a Nisnevich square

$$\begin{tikzcd}
X \smallsetminus (Y \cup Z) \arrow[r] \arrow[d] & X \smallsetminus Y \arrow[d] \\
\mathbb{A}^1_S \smallsetminus (0_S \cup f(Z)) \arrow[r] & \mathbb{A}^1_S \smallsetminus 0_S,
\end{tikzcd}$$

inducing a commutative square

$$\begin{tikzcd}
\mathrm{H}^m_{Z \smallsetminus Y}(X \smallsetminus Y; M) \arrow[r] & \mathrm{H}^m(X \smallsetminus Y; M) \\
\mathrm{H}^m_{f(Z \smallsetminus Y)}(\mathbb{A}^1_S \smallsetminus 0_S; M) \arrow[r] \arrow[u, "\sim"] & \mathrm{H}^m(\mathbb{A}^1_S \smallsetminus 0_S; M) \arrow[u]
\end{tikzcd}$$

where the left vertical arrow is an isomorphism. Thus, it suffices to show that the group

$$\mathrm{H}^m(\mathbb{A}^1_S \smallsetminus 0_S; M)$$

vanishes. We can identify this group with

$$[(\mathbb{A}^1 \smallsetminus 0)_+ \wedge S_+, \mathrm{K}(M, m)]_{\mathrm{nis}} = [S_+, \mathrm{K}(M, m)^{\mathbb{A}^1 \smallsetminus 0}]_{\mathrm{nis}}.$$

The unit section of $\mathbb{A}^1 \smallsetminus 0$ yields a decomposition into a direct sum

$$[S_+, \mathrm{K}(M, m)^{\mathbb{A}^1 \smallsetminus 0}]_{\mathrm{nis}} \simeq [S_+, \mathrm{K}(M, m)]_{\mathrm{nis}} \oplus [S_+, \mathrm{K}(M, i)^{\mathbb{G}_{\mathrm{m}}}]_{\mathrm{nis}}.$$

We claim that the k-space $\mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}}$ is equivalent to $\mathrm{K}(M_{-1}, m)$. Assuming this claim, it is easy to conclude. Indeed, we then have that $\mathrm{H}^m(\mathbb{A}^1_S \smallsetminus 0_S; M)$ is the direct sum of $\mathrm{H}^m(S; M)$ and $\mathrm{H}^m(S; M_{-1})$, and both groups vanish since S is local henselian.

It remains to see that the natural morphism of k-spaces

$$\mathrm{K}(M_{-1}, m) \to \mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}}$$

is an equivalence. Since M is n-strongly $\mathbb{A}^1$-invariant and $m \leqslant n$, the pointed k-space $\mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}}$ is motivic and $\mathrm{K}(M_{-1}, m)$ is its m-connective cover, i.e., $\mathrm{K}(M_{-1}, m) \simeq \tau_{\geqslant m}\mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}}$. Thus, to conclude, it is enough to show that $\mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}}$ is m-connective, i.e., that $\pi_i\mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}} = 0$ for $0 \leqslant i \leqslant m-1$. By Theorem 2.4.13, if U is a local essentially smooth k-scheme with generic point η, the map

$$\Gamma(U; \pi_i\mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}}) \to \Gamma(\eta; \pi_i\mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}})$$

has trivial kernel. Since this is a morphism of abelian groups, this map is even injective. Therefore, to prove the vanishing of $\pi_i\mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}}$, it is enough to show that $\Gamma(K; \pi_i\mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}}) = 0$ for every essentially smooth extension K/k. But, we have

$$\Gamma(K; \pi_i\mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}}) \simeq [\Sigma^i\mathrm{Spec}(K)_+, \mathrm{K}(M, m)^{\mathbb{G}_{\mathrm{m}}}]_{\mathrm{nis}} \simeq \mathrm{H}^{m-i}(\mathbb{A}^1_K \smallsetminus 0; M).$$

For $0 \leq i \leq m-2$, the desired vanishing is clear and, for $i = m-1$, the vanishing was explained in the proof of Lemma 3.6.2. □

We can now conclude the proof exactly as in the case $n = 1$.

Proof that M is $n+1$-strongly $\mathbb{A}^1$-invariant. We need to show that, for every essentially smooth k-scheme X, the map

$$\mathrm{H}^{n+1}(X; M) \to \mathrm{H}^{n+1}(\mathbb{A}^1_X; M)$$

is an isomorphism. We argue by induction on the dimension of X. If $\dim(X) = 0$, there is nothing to show as both groups are zero. Assume that $\dim(X) = d > 0$, and that the result is known in dimension $\leqslant d-1$. We may assume that X is local henselian. In this case, we need to show that $\mathrm{H}^{n+1}(\mathbb{A}^1_X; M) = 0$. The induction hypothesis implies that

$$\mathrm{H}^{n+1}(X \smallsetminus Y; M) \simeq \mathrm{H}^{n+1}(\mathbb{A}^1_{X \smallsetminus Y}; M)$$

for any essentially smooth hypersurface $Y \subset X$. Denote by $i : Y \hookrightarrow X$ and $i' : \mathbb{A}^1_Y \to \mathbb{A}^1_X$ the obvious inclusions. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^n(\mathbb{A}^1_{X \smallsetminus Y}; M) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^{n+1}(\mathbb{A}^1_Y; i'^!M_{\mathbb{A}^1_X}) \arrow[r] \arrow[d, "\sim", "(\star)"'] & \mathrm{H}^{n+1}(\mathbb{A}^1_X; M) \arrow[r] \arrow[d] & \mathrm{H}^{n+1}(\mathbb{A}^1_{X \smallsetminus Y}; M) \arrow[d, "\sim"] \\
\mathrm{H}^n(X \smallsetminus Y; M) \arrow[r] & \mathrm{H}^{n+1}(Y; i^!M_X) \arrow[r] & \mathrm{H}^{n+1}(X; M) \arrow[r] & \mathrm{H}^{n+1}(X \smallsetminus Y; M)
\end{tikzcd}$$

induced by the zero section of $\mathbb{A}^1$. By Proposition 3.7.13, there are natural equivalences

$$\tau^{\geqslant n+1}(i^!M_X) \simeq (M_{-1})_Y[-1] \qquad \text{and} \qquad \tau^{\geqslant n+1}(i'^!M_{\mathbb{A}^1_X}) \simeq (M_{-1})_{\mathbb{A}^1_Y}[-1].$$

It follows that the map $(\star)$ can be identified with

$$\mathrm{H}^n(\mathbb{A}^1_Y, M_{-1}) \to \mathrm{H}^n(Y; M_{-1})$$

showing that it is an isomorphism as indicated in the above diagram. (Recall that M_{-1} is also n-strongly $\mathbb{A}^1$-invariant.) This implies that the morphism

$$\mathrm{H}^{n+1}(\mathbb{A}^1_X; M) \to \mathrm{H}^{n+1}(X; M)$$

is injective. Since X is local henselian, this proves that $\mathrm{H}^{n+1}(\mathbb{A}^1_X; M)$ is zero. □

3.8. **Concluding remarks.** We end this section with a list of the results obtained so far, and some complements on the $\pi_1^{\mathbb{A}^1}$ of pointed k-spaces. First, since we have learned that the notions of n-strong $\mathbb{A}^1$-invariance, for $n \geqslant 1$, are all equivalent for sheaves of abelian groups, we will only use the following terminology from now on.

Definition 3.8.1.

(1) If F is a sheaf of abelian groups on Sm_k, we say that F is strictly $\mathbb{A}^1$-invariant if

$$\mathrm{H}^n(X; F) \to \mathrm{H}^n(\mathbb{A}^1_X; F)$$

is an isomorphism for every $X \in \mathrm{Sm}_k$ and every integer $n \geqslant 0$.

(2) If G is a sheaf of groups (not necessarily abelian) on Sm_k, we say that G is strongly $\mathbb{A}^1$-invariant if

$$\mathrm{H}^n(X; G) \to \mathrm{H}^n(\mathbb{A}^1_X; G)$$

is an isomorphism for every $X \in \mathrm{Sm}_k$ and every $n \in \{0, 1\}$.

In this section, we proved that a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups is automatically strictly $\mathbb{A}^1$-invariant. We also essentially proved the following "purity" theorem.

Theorem 3.8.2 (Morel)**.** *Let M be a strictly $\mathbb{A}^1$-invariant sheaf on* Sm_k*. Then, given a closed immersion $i : Y \hookrightarrow X$ between essentially smooth k-schemes and a trivialisation $\mathrm{N}_Y \simeq \mathbb{A}^d_Y$ of its normal bundle, we have an equivalence*

$$(M_{-d})_Y[-d] \xrightarrow{\sim} i^! M_X$$

in the derived category of sheaves on $\mathrm{\acute{E}t}_Y$*.*

Proof. Indeed, by Theorem 3.1.11 and Proposition 3.2.5, we have $\tau^{\leqslant d}(i^! M_X) \simeq (M_{-d})_Y[-d]$. So, it remains to see that $\tau^{\geqslant d+1}(i^! M_X)$ vanishes. The problem is local on X. Thus, we may assume that i can be written as a composition of one-codimensional closed immersions between essentially smooth k-schemes. By a simple induction, we are thus reduced to the case where Y is a hypersurface in X. This case was treated in Proposition 3.7.13. □

Theorem 3.8.3 (Morel)**.** *Let M be a strictly $\mathbb{A}^1$-invariant sheaf on* Sm_k*, and let X be an essentially smooth k-scheme.*

(1) The E_1*-page of the coniveau spectral sequence for X and M vanishes except at the line ($q = 0$).*

(2) The Cousin complex $\mathrm{C}^\bullet(X; M)$ *computes the cohomology of X. In fact we have*

$$\mathrm{H}^n_{\mathrm{zar}}(X; M) \simeq \mathrm{H}^n_{\mathrm{nis}}(X; M) = \mathrm{H}^n(\mathrm{C}(X; M)) \qquad \forall n \in \mathbb{Z}.$$

(3) The complex of Nisnevich sheaves $U \in \mathrm{\acute{E}t}_X \mapsto \mathrm{C}^\bullet(U; M)$ is a Zariski local resolution of M_X and its terms are all acyclic for the Zariski and Nisnevich topologies.

Proof. The first assertion follows readily from Theorem 3.8.2. The last two assertions follow from Proposition 3.7.9 and Corollary 3.7.10. □

Theorem 3.8.4 (Morel)**.** *Let $\mathcal{X}$ be a pointed k-space. Then, for $n \geqslant 2$, the sheaves $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ are strictly $\mathbb{A}^1$-invariant.*

Proof. As a consequence of the unstable $\mathbb{A}^1$-connectivity theorem, we have seen that $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is $(n-1)$-strongly $\mathbb{A}^1$-invariant. (See Corollary 2.3.10.) Thus, for $n \geqslant 2$, $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant and hence strictly $\mathbb{A}^1$-invariant by Theorem 3.0.1. □

This leaves the case $n = 1$ which needs a special treatment. In fact, we also have the following.

Theorem 3.8.5 (Morel)**.** *Let $\mathcal{X}$ be a pointed k-space. Then the sheaf $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant. In particular, if $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is abelian, then it is also strictly $\mathbb{A}^1$-invariant.*

Remark 3.8.6. We will present a proof of the above theorem that relies on Theorems 3.8.3 and 3.8.4. A more elementary approach can be found in [Bac24, §2]. In particular, Bachmann's argument shows that Theorem 3.8.5 is still valid over imperfect fields.

The remainder of this subsection is devoted to proving Theorem 3.8.5. We fix a pointed k-space $\mathcal{X}$ and set $G = \pi_1^{\mathbb{A}^1}(\mathcal{X})$. The $\pi_1^{\mathbb{A}^1}$ does not change if we replace $\mathcal{X}$ by its motivic localisation and its 1-connective cover. Thus we may assume that $\mathcal{X}$ is motivic and $\mathbb{A}^1$-connected. As usual, we denote by $\mathrm{B}G$ the classifying k-space of G. (This is an object of the ∞-topos $\mathrm{Spc}(k)$.)

Lemma 3.8.7. *To prove Theorem 3.8.5, it suffices to show that the obvious map of k-spaces $\mathcal{X} \to \mathrm{B}G$ induces surjections*

$$\pi_0(\mathcal{X}(U)) \twoheadrightarrow \pi_0(\mathrm{B}G(U)) = \mathrm{H}^1(U;G)$$

for every $U \in \mathrm{Sm}_k$. In fact, it suffices to do so when $U = \mathbb{A}^1_X$, with X a local henselian essentially smooth k-scheme.

Proof. We want to show that G is strongly $\mathbb{A}^1$-invariant. We already know that G is $\mathbb{A}^1$-invariant by Corollary 2.3.10. So we are left to showing that the map $\mathrm{H}^1(U;G) \to \mathrm{H}^1(\mathbb{A}^1_U;G)$ is a bijection for every $U \in \mathrm{Sm}_k$, and it is enough to do so when U is a local henselian essentially smooth k-scheme (in which case we need to show that $\mathrm{H}^1(\mathbb{A}^1_U;G)$ is a singleton). Assuming the condition in the statement, we have a commutative square

$$\begin{tikzcd}
\pi_0(\mathcal{X}(U)) \arrow[r, two heads] \arrow[d, "\sim"] & \pi_0(\mathrm{B}G(U)) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{H}^1(U;G) \arrow[d] \\
\pi_0(\mathcal{X}(\mathbb{A}^1_U)) \arrow[r, two heads] & \pi_0(\mathrm{B}G(\mathbb{A}^1_U)) \arrow[r, phantom, "\simeq"] & \mathrm{H}^1(\mathbb{A}^1_U;G)
\end{tikzcd}$$

with surjective horizontal arrows; the left-hand side vertical arrow is a bijection since $\mathcal{X}$ is motivic. This shows that the map $\mathrm{H}^1(U;G) \to \mathrm{H}^1(\mathbb{A}^1_U;G)$ is surjective. Since U is local henselian, this is a surjection from a singleton, and hence $\mathrm{H}^1(\mathbb{A}^1_U;G)$ is a singleton as needed. □

We now fix a henselian local essentially smooth k-scheme X and set $Y = \mathbb{A}^1_X$. We also fix a G-torsor T_α on Y classified by an element $\alpha \in \mathrm{H}^1(Y;G)$. Explicitly, T_α as a sheaf of sets on $\mathrm{\acute{E}t}_Y$ endowed with a locally simply transitive action of G_Y. We need to show that α is the base point of the pointed set $\mathrm{H}^1(Y;G)$. Equivalently, we need to show that T_α is trivial, i.e., admits a global section over Y. We turn our problem into a lifting problem in the ∞-topos $\mathrm{Spc}(k)$:

$$\begin{tikzcd}
 & \mathcal{X} \arrow[d] \\
Y \arrow[r, "\alpha"] \arrow[ur, dotted] & \mathrm{B}G.
\end{tikzcd}$$

If we can solve this lifting problem, the result follows because $[Y_+, \mathcal{X}]_{\mathrm{nis}}$ is a singleton. (Indeed, since $\mathcal{X}$ is motivic, this set is in bijection with $[X_+, \mathcal{X}]_{\mathrm{nis}}$. But X is local henselian and $\mathcal{X}$ is 1-connective.) We will use the Postnikov tower to solve this lifting problem. For $n \geqslant 2$, we set

$$M_n = \pi_n^{\mathbb{A}^1}(\mathcal{X}).$$

This a sheaf of abelian groups on Sm_k with an action of G. We need some recollections on Postnikov towers.

Recollection 3.8.8. Let (C, τ) be a site. Given a sheaf of groups H acting on a sheaf of abelian groups A, we have the so-called twisted Eilenberg–Mac Lane spaces $\mathrm{K}^H(A, n)$, for $n \geqslant 2$. These are sheaves of Kan complexes on C defined as the quotient of $\mathrm{K}(A, n)$ by the action of H. (Explicitly, $\mathrm{K}^H(A, n)$ is the sheafification of $U \in C \mapsto \mathrm{E}(H(U)) \times^{H(U)} \mathrm{K}(A(U), n)$.) For every $n \geqslant 2$, we have a fibre sequence

$$\mathrm{K}(A, n) \to \mathrm{K}^H(A, n) \to \mathrm{B}H.$$

In particular, we have

$$\pi_i \mathrm{K}^H(A, n) = \begin{cases} H & \text{if } i = 1, \\ A & \text{if } i = n, \\ * & \text{else.} \end{cases}$$

Moreover, the natural action of $\pi_1 \mathrm{K}^H(A, n)$ on $\pi_n \mathrm{K}^H(A, n)$ recovers the given action of H on A. Note that $\mathrm{K}^H(A, n) \to \mathrm{B}H$ admits a natural section which participates in a fibre sequence

$$\mathrm{K}^H(A, n-1) \to \mathrm{B}H \to \mathrm{K}^H(A, n).$$

Recollection 3.8.9. Let (C, τ) be a site. Let $\mathcal{Z}$ be a sheaf of pointed Kan complexes on C, and assume that $\mathcal{Z}$ is 1-connective. Set $H = \pi_1(\mathcal{Z})$ and consider the Postnikov tower of $\mathcal{Z}$:

$$\cdots \to \tau_{\leqslant n}\mathcal{Z} \to \cdots \to \tau_{\leqslant 2}\mathcal{Z} \to \tau_{\leqslant 1}\mathcal{Z} = \mathrm{B}H.$$

Then, for every $n \geq 2$, we have commutative diagram with a cartesian square

$$\begin{tikzcd}
\tau_{\leqslant n}\mathcal{Z} \arrow[r] \arrow[d] & \mathrm{B}H \arrow[d] \\
\tau_{\leqslant n-1}\mathcal{Z} \arrow[r, "k_n"] \arrow[dr] & \mathrm{K}^H(\pi_n(\mathcal{Z}), n+1) \arrow[d] \\
 & \mathrm{B}H.
\end{tikzcd}$$

Said differently, the fibration $\tau_{\leqslant n}\mathcal{Z} \to \tau_{\leqslant n-1}\mathcal{Z}$ with fibre $\mathrm{K}(\pi_n(\mathcal{Z}), n)$ is classified by the map

$$k_n : \tau_{\leqslant n-1}\mathcal{Z} \to \mathrm{K}^H(\pi_n(\mathcal{Z}), n+1)$$

called the n-th k-invariant. (See for example [GJ99, Chapter VI, §5].)

We now go back to our lifting problem. We want to lift the map $\alpha : Y \to \mathrm{B}G$ to $\tilde{\alpha} : Y \to \mathcal{X}$. By induction it suffices to find lifts as follows (with $n \geqslant 2$)

$$\begin{tikzcd}
 & \tau_{\leqslant n}\mathcal{X} \arrow[d] \\
Y \arrow[r, "\alpha_{n-1}"] \arrow[ur, dotted] & \tau_{\leqslant n-1}\mathcal{X}.
\end{tikzcd}$$

By Recollection 3.8.9, the existence of such a lift is equivalent to the existence of a commutative square

$$\begin{tikzcd}
Y \arrow[r, "\alpha"] \arrow[d, "\alpha_{n-1}"] & \mathrm{B}G \arrow[d] \\
\tau_{\leqslant n-1}\mathcal{X} \arrow[r, "k_n"] & \mathrm{K}^G(M_n, n+1).
\end{tikzcd}$$

Equivalently, we need to show that

$$Y \xrightarrow{k_n \circ \alpha_{n-1}} \mathrm{K}^G(M_n, n+1) \times_{\mathrm{B}G,\alpha} Y$$

is homotopic to the obvious morphism. (Here we are using the projection $\mathrm{K}^G(M_n, n+1) \to \mathrm{B}G$.) To do so we translate the problem as follows.

(1) We have a sheaf of abelian groups M_n^α on $\mathrm{Ét}_Y$ obtained by twisting $(M_n)_Y$ by the torsor T_α. Explicitly, it is given by:

$$M_n^\alpha = (M_n)_Y \otimes_{\mathbb{Z}[G_Y]} \mathbb{Z}[T_\alpha].$$

Note that M_n^α is locally isomorphic to $(M_n)_Y$.

(2) There is an equivalence of sheaves of Kan complexes on $\mathrm{Ét}_Y$:

$$\mathrm{K}^G(M_n, n+1) \times_{\mathrm{B}G,\alpha} Y \xrightarrow{\sim} \mathrm{K}(M_n^\alpha, n+1).$$

(The left-hand side is viewed as a sheaf on $\mathrm{Ét}_Y$ by sending an étale scheme $Y' \to Y$ to the fibre of the map $\mathrm{K}^G(M_n, n+1)(Y') \to \mathrm{B}G(Y')$ at the point $\alpha|_{Y'}$.)

(3) The map $k_n \circ \alpha_{n-1}$ is classified by a cohomology class in $\mathrm{H}^{n+1}(Y; M_n^\alpha)$. The triviality of $k_n \circ \alpha_{n-1}$ translates into the vanishing of this class.

Thus, to conclude, it is enough to show the following.

Lemma 3.8.10. *Let M be a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups on Sm_k endowed with an action of the sheaf of groups $G = \pi_1^{\mathbb{A}^1}(\mathcal{X})$. Let $Y = \mathbb{A}^1_X$ with X a local henselian essentially smooth k-scheme. Fix $\alpha \in \mathrm{H}^1(Y; G)$, and let M^α be the α-twisted sheaf associated to M_Y. Then, we have $\mathrm{H}^i(Y; M^\alpha) = 0$ for $i \geqslant 1$.*

Proof. We start by noticing that, for every point $x \in X$, the class α is trivial on $Y_x = \mathbb{A}^1_x$. Indeed, the problem of lifting $\alpha_x = \alpha|_{Y_x} : Y_x \to \mathrm{B}G$ to a map $Y_x \to \mathcal{X}$ can be solved since the groups $\mathrm{H}^{n+1}(Y_x; M_n^{\alpha_x})$ automatically vanish for $n \geqslant 2$ by cohomological dimension. But then, any map $Y_x = \mathbb{A}^1_x \to \mathcal{X}$ factors through a map $x \to \mathcal{X}$ (since $\mathcal{X}$ is motivic) and hence is nullhomotopic (since $\mathcal{X}$ is 1-connective).

We now go back to the statement we need to prove. Since M^α is locally isomorphic to M_Y, the Cousin complex $\mathrm{C}^\bullet(-; M^\alpha)$ is a resolution of M^α by acyclic sheaves. In degree i, it is given by

$$\mathrm{C}^i(-; M^\alpha) = \bigoplus_{y \in Y^{(i)}} \mathrm{H}^i_y(Y; M^\alpha),$$

and $\mathrm{H}^i_y(Y; M^\alpha)$ is non-canonically isomorphic to $\mathrm{H}^i_y(Y; M)$ which is non-canonically isomorphic to $M_{-i}(y)$. Let $\beta \in \mathrm{H}^i(Y; M^\alpha) \simeq \mathrm{H}^i(\mathrm{C}(Y; M^\alpha))$ be represented by a cocycle $(\beta_y)_{y \in Y^{(i)}}$ in $\mathrm{C}^\bullet(-; M^\alpha)$. Since $Y = \mathbb{A}^1_X$, we have a partition

$$Y^{(i)} = \left(\coprod_{x \in X^{(i-1)}} (\mathbb{A}^1_x)^{(1)} \right) \coprod \left\{ \eta_x \mid x \in X^{(i)} \right\}$$

into points of types I and II as in Step 1 of the proof of Proposition 3.7.9. Using the fact that $\alpha \in \mathrm{H}^1(Y; G)$ vanishes on $\mathbb{A}^1_x$, for $x \in X$, we see that the map

$$\mathrm{H}^{i-1}_{\eta_x}(\mathbb{A}^1_X; M^\alpha) \to \bigoplus_{z \in (\mathbb{A}^1_x)^{(1)}} \mathrm{H}^i_z(\mathbb{A}^1_X; M^\alpha)$$

is surjective for every $x \in X^{(i-1)}$. Thus, we may change the cocycle $(\beta_y)_y$ and assume it is supported on type II points. Using that $\partial((\beta_y)_y) = 0$, it follows that

$$\beta_{\eta_x} \in \mathrm{H}^i_{\mathbb{A}^1_x}(\mathbb{A}^1_X; M^\alpha) \simeq M_{-i}(x)$$

for every $x \in X^{(i)}$. Thus, $(\beta_y)_y$ defines a cocycle in the subcomplex $D^\bullet \subset \mathrm{C}^\bullet(\mathbb{A}^1_X; M^\alpha)$ given in degree i by:

$$D^i = \bigoplus_{x \in X^{(i)}} \mathrm{H}^i_{\mathbb{A}^1_x}(\mathbb{A}^1_X; M^\alpha).$$

There is an isomorphism of complexes $D^\bullet \simeq \mathrm{C}^\bullet(X; M^{\alpha_0})$ induced by the zero section $X \to \mathbb{A}^1_X$. The class α_0 is the restriction of α along the zero section and, since X is henselian local, this class is trivial which gives rise to an isomorphism $M^{\alpha_0} \simeq M_X$ of sheaves on $\acute{\mathrm{E}}\mathrm{t}_X$. This shows that $D^\bullet$ is acyclic in degree $\geqslant 1$. Therefore, $(\beta_y)_y$ is a coboundary in $D^\bullet$ and hence also in $\mathrm{C}^\bullet(\mathbb{A}^1_X; M^\alpha)$. □

We end the section with the following corollary.

Corollary 3.8.11 (Morel). *Let $\mathcal{X}$ be a pointed connected k-space. Then the following conditions are equivalent:*

(1) $\mathcal{X}$ is motivic;
(2) $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant for all $n \geqslant 1$.

Proof. The implication (1) ⇒ (2) follows from Corollary 2.3.10 and Theorem 3.8.5. For the converse, we use the Postnikov tower of $\mathcal{X}$. It is enough to show that $\tau_{\leqslant n}\mathcal{X}$ is motivic for every $n \geqslant 1$. We argue by induction on n. When $n = 1$, this is clear. (Indeed, a sheaf of groups H is strongly $\mathbb{A}^1$-invariant if and only if the k-space $\mathrm{B}H$ is motivic.) Using Recollection 3.8.9, we reduce to showing that, given a sheaf of groups G acting on a sheaf of groups M, the k-space $\mathrm{K}^G(M, n)$ is motivic if G and M are strongly $\mathbb{A}^1$-invariant. It is enough to show that the map

$$\mathrm{K}^G(M, n)(X) \to \mathrm{K}^G(M, n)(\mathbb{A}^1_X)$$

is an equivalence for every local henselian essentially smooth k-scheme X. Clearly, $\mathrm{K}^G(M, n)(X)$ is the n-th twisted Eilenberg-Mac Lane space associated to the abelian group $M(X)$ endowed with the action of $G(X)$, and we need to prove the same for $\mathrm{K}^G(M, n)(\mathbb{A}^1_X)$. But we have a fibre sequence

$$\mathrm{K}(M, n)(\mathbb{A}^1_X) \to \mathrm{K}^G(M, n)(\mathbb{A}^1_X) \to \mathrm{B}G(\mathbb{A}^1_X),$$

so it is enough to show the equivalences $\mathrm{K}(M, n)(\mathbb{A}^1_X) \simeq \mathrm{K}(M(X), n)$ and $\mathrm{B}G(\mathbb{A}^1_X) \simeq \mathrm{B}(G(X))$. Both equivalences follow from the fact that G and M are strongly $\mathbb{A}^1$-invariant. (In the case of M, we actually need that M is strictly $\mathbb{A}^1$-invariant which requires Theorem 3.0.1.) □

4. STRICTLY $\mathbb{A}^1$-INVARIANT SHEAVES OF MILNOR–WITT K-THEORY

In this section, we explicitly construct some of the most fundamental examples of strictly $\mathbb{A}^1$-invariant sheaves, namely the sheaves of Milnor–Witt K-theory. These sheaves play a central role because they arise as the first homotopy sheaves of motivic spheres; however, we defer the proof of this fact to the next section.

We begin with the basic definition of the Milnor–Witt K-groups of fields, given by generators and relations, and then construct their standard operations (residue and norm maps) and prove their key properties (homotopy invariance, reciprocity, Gersten resolution, etc.). This culminates in the construction of the sheaves of unramified Milnor–Witt K-theory, which we show to be strictly $\mathbb{A}^1$-invariant. For the sake of simplicity, some of the main results of this section are discussed under the extra assumption that the characteristic is different from 2. This restriction is not essential and can be removed, but the case of characteristic 2 requires special arguments (see [Mor12, §5.1] and [Fel23]). Besides F. Morel's original work [Mor12], we refer the reader to [Fel23], [Car23], [Dég25] and [BCD+25] for other accounts of the theory.

4.1. **Milnor–Witt K-theory of fields.** The following definition was found by Hopkins and Morel.

Definition 4.1.1. Let F be a field. The Milnor–Witt K-theory ring of F, denoted by $\mathrm{K}_*^{\mathrm{MW}}(F)$, is the $\mathbb{Z}$-graded ring, with generators $[u]$ in degree 1, for each $u \in F^\times$, and a generator η in degree -1, satisfying the following relations.

1. (Steinberg) For $u, v \in F^\times$ with $u + v = 1$, we have $[u][v] = 0$ in $\mathrm{K}_2^{\mathrm{MW}}(F)$.
2. For every $u, v \in F^\times$, we have $[uv] = [u] + [v] + \eta[u][v]$.
3. For every $u \in F^\times$, we have $\eta[\mu] = [\mu]\eta$.
4. $\eta^2[-1] + 2\eta = 0$. (We also set $h = \eta[-1] + 2$, and write this relation as $\eta h = 0$.)

Remark 4.1.2. Let $\mathrm{K}_*^{\mathrm{M}}(F) := \mathrm{K}_*^{\mathrm{MW}}(F)/(\eta)$ be the graded ring obtained by modding out by the two-sided ideal generated by η. Then $\mathrm{K}_*^{\mathrm{M}}(F)$ is quotient of the tensor algebra on $F^\times$ by the Steinberg relation. This is precisely the Milnor K-theory ring of F. As usual, for $u \in F^\times$, we denote by $\{u\} \in \mathrm{K}_1^{\mathrm{M}}(F)$ the image of $[u] \in \mathrm{K}_1^{\mathrm{MW}}(F)$.

Definition 4.1.3. Let F be a field. For $n \in \mathbb{Z}$, let $\widetilde{\mathrm{K}}_n^{\mathrm{MW}}(F)$ be the abelian group generated by symbols of the form $[\eta^m, u_1, \dots, u_r]$, for $u_1, \dots, u_r \in F^\times$ and $m, r \geqslant 0$ integers with $r - m = n$, subject to the following relations.

(1_n) (Steinberg) $[\eta^m, u_1, \dots, u_r] = 0$ if $u_i + u_{i+1} = 1$ for some $1 \leq i \leq r - 1$.

(2_n) For $a, b \in F^\times$ and any $1 \leqslant i \leqslant r$, we have:

$$\begin{aligned}[\eta^m, u_1, \dots u_{i-1}, ab, u_{i+1}, \dots, u_r] &= [\eta^m, u_1, \dots u_{i-1}, a, u_{i+1}, \dots, u_r] \\ &\quad + [\eta^m, u_1, \dots u_{i-1}, b, u_{i+1}, \dots, u_r] \\ &\quad + [\eta^{m+1}, u_1, \dots u_{i-1}, a, b, u_{i+1}, \dots, u_r].\end{aligned}$$

(4_n) For $1 \leqslant i \leqslant r + 2$, we have:

$$[\eta^{m+2}, u_1, \dots, u_{i-1}, -1, u_{i+1}, \dots, u_{r+2}] + 2[\eta^{m+1}, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{r+2}] = 0.$$

For later use, we record the following fact.

Lemma 4.1.4. *Let F be a field. For every $n \in \mathbb{Z}$, there is an isomorphism*

$$\widetilde{\mathrm{K}}_n^{\mathrm{MW}}(F) \xrightarrow{\sim} \mathrm{K}_n^{\mathrm{MW}}(F)$$

sending a generator $[\eta^m, u_1, \dots, u_r]$ to $\eta^m[u_1] \cdots [u_r]$.

Proof. It is easy to see that the assignment

$$([\eta^m, u_1, \dots, u_r], [\eta^{m'}, v_1, \dots, v_{r'}]) \mapsto [\eta^{m+m'}, u_1, \dots, u_r, v_1, \dots, v_{r'}]$$

turn $\widetilde{\mathrm{K}}_*^{\mathrm{MW}}(F) = \bigoplus_{n \in \mathbb{Z}} \widetilde{\mathrm{K}}_n^{\mathrm{MW}}(F)$ into a $\mathbb{Z}$-graded ring generated by the symbols $\eta := [\eta^1]$ and $[u] := [\eta^0, u]$, for $u \in F^\times$. Moreover, the relations (1) – (4) in Definition 4.1.1 are satisfied by these generators. This yields a $\mathbb{Z}$-graded ring homomorphism

$$\mathrm{K}_*^{\mathrm{MW}}(F) \to \widetilde{\mathrm{K}}_*^{\mathrm{MW}}(F)$$

which is surjective. On the other hand, we also have surjective maps

$$\widetilde{\mathrm{K}}_n^{\mathrm{MW}}(F) \to \mathrm{K}_n^{\mathrm{MW}}(F)$$

sending a symbol $[\eta^m, u_1, \dots, u_r]$ to $\eta^m[u_1]\cdots[u_r]$. The two maps defined above are clearly inverse of each other. □

From now on in this subsection, the letter F will always denote a field.

Notation 4.1.5. For $a \in F^\times$, we set

$$\langle a\rangle = 1 + \eta[a] \quad \in \quad \mathrm{K}_0^{\mathrm{MW}}(F).$$

Observe that $h = 1 + \langle -1\rangle$.

Remark 4.1.6. Using the previous notation, we can rewrite relation (2) in Definition 4.1.1 as follows:

$$[uv] = [u] + \langle u\rangle[v] = [u]\langle v\rangle + [v].$$

The second equality uses that η is central.

Lemma 4.1.7. *In* $\mathrm{K}_1^{\mathrm{MW}}(F)$ *we have* $[1] = 0$.

Proof. Writing $1 = (-1)\cdot(-1)$, we obtain:

$$\begin{aligned} [1] &= [-1] + [-1] + \eta[-1][-1] \\ &= [-1](2 + \eta[-1]) = [-1]h. \end{aligned}$$

Thus, $\eta[1] = 0$ by relation (4). Next, we write $1 = 1\cdot 1$, to obtain

$$[1] = [1] + [1] + \eta[1][1] = 2\cdot[1].$$

This implies that $[1] = 0$ as desired. □

Corollary 4.1.8. *In* $\mathrm{K}_0^{\mathrm{MW}}(F)$*, we have* $\langle 1\rangle = 1$.

Proposition 4.1.9.

(1) The subring $\mathrm{K}_0^{\mathrm{MW}}(F)$ *is generated by* $\langle a\rangle$ *for* $a \in F^\times$*. Moreover, it is central in* $\mathrm{K}_*^{\mathrm{MW}}(F)$*.*

(2) For $a, b \in F^\times$*, we have* $\langle ab\rangle = \langle a\rangle\langle b\rangle$*. Thus* $\langle a\rangle$ *is invertible in* $\mathrm{K}_0^{\mathrm{MW}}(F)$ *with inverse* $\langle a^{-1}\rangle$*.*

Proof. Clearly, $\mathrm{K}_0^{\mathrm{MW}}(F)$ is generated as a ring by the symbols $\eta[a] = \langle a\rangle - 1$, for $a \in F^\times$, and hence also by the elements $\langle a\rangle$, for $a \in F^\times$. To show that $\mathrm{K}_0^{\mathrm{MW}}(F)$ is central, it is enough to show that each symbol $\eta[a]$ is central. For this, we need to check that

$$(\eta[a])\cdot[b] = [b]\cdot(\eta[a])$$

for all $b \in F^\times$. Said differently, we need to see that

$$\eta[a][b] = \eta[b][a]$$

for all $a, b \in F^\times$. But the left-hand side is $[ab]-[a]-[b]$, while the right-hand side is $[ba]-[b]-[a]$. This proves the claim. We now check the formula

$$\langle ab\rangle = \langle a\rangle\langle b\rangle$$

for $a, b \in F^\times$. This is essentially obvious. Indeed, we have

$$\begin{aligned} \langle ab\rangle = 1 + \eta[ab] &= 1 + \eta([a] + [b] + \eta[a][b]) \\ &= 1 + \eta[a] + \eta[b] + \eta^2[a][b] \\ &= (1 + \eta[a])(1 + \eta[b]) = \langle a\rangle\langle b\rangle. \end{aligned}$$

This finishes the proof of the proposition. □

Corollary 4.1.10. *For $a, b \in F^\times$, we have*

$$\begin{aligned} \left[\frac{a}{b}\right] &= [a] - \left\langle\frac{a}{b}\right\rangle[b] \\ &= \langle b^{-1}\rangle([a]-[b]). \end{aligned}$$

In particular, we have $[a^{-1}] = -\langle a^{-1}\rangle[a]$.

Proof. We write $a = \left(\frac{a}{b}\right)\cdot b$ to get

$$\begin{aligned} [a] &= \left[\frac{a}{b}\right] + \left\langle\frac{a}{b}\right\rangle[b] \\ &= \langle b\rangle\left[\frac{a}{b}\right] + [b]. \end{aligned}$$

This gives the result using that $\langle b\rangle$ is invertible with inverse $\langle b^{-1}\rangle$. □

Lemma 4.1.11.

1. *For $n \geqslant 1$, the group $\mathrm{K}_n^{\mathrm{MW}}(F)$ is generated by the products $[u_1]\cdots[u_n]$ with $u_1, \ldots, u_n \in F^\times$.*
2. *For $n \leqslant 0$, the group $\mathrm{K}_n^{\mathrm{MW}}(F)$ is generated by the elements $\eta^{-n}\langle u\rangle$, with $u \in F^\times$.*

Proof. First assume that $n \geqslant 1$. Clearly, $\mathrm{K}_n^{\mathrm{MW}}(F)$ is generated by elements of the form

$$\eta^m[u_1]\cdots[u_{n+m}].$$

We need to show that this element belongs to the subgroup of $\mathrm{K}_n^{\mathrm{MW}}(F)$ generated by the products in Part (1). We do this by induction on m. If $m = 0$, there is nothing to prove. If $m \geqslant 1$, then $n + m \geqslant 2$, and we can write:

$$\eta^m[u_1]\cdots[u_{n+m}] = \eta^{m-1}[u_1]\cdots[u_{n+m-2}]([u_{n+m-1}u_{n+m}] - [u_{n+m-1}] - [u_{n+m}]).$$

We then conclude by induction.

Assume now that $n \leqslant 0$. By the same reasoning as above, an element of the form

$$\eta^{m-n}[u_1]\cdots[u_m],$$

with $m \geqslant 2$, can be expressed as a linear combination of elements of the form $\eta^{m-n-1}[v_1]\cdots[v_{m-1}]$. So, in conclusion, we see that $\mathrm{K}_n^{\mathrm{MW}}(F)$, with $n \leqslant 0$, is generated as a group by η^{-n} and $\eta^{-n+1}[a]$, for $a \in F^\times$. Clearly, this set of generators can be replaced with $\eta^{-n}\langle a\rangle$, for $a \in F^\times$. □

Corollary 4.1.12. *For $n \leqslant 0$, the map*

$$\eta : \mathrm{K}_n^{\mathrm{MW}}(F) \to \mathrm{K}_{n-1}^{\mathrm{MW}}(F)$$

is surjective.

Notation 4.1.13. We set

$$(-1)_\epsilon := -\langle -1\rangle \in \mathrm{K}_0^{\mathrm{MW}}(F).$$

Observe that $(-1)_\epsilon^2 = 1$.

Lemma 4.1.14. *We have $(-1)_\epsilon \cdot \eta = \eta$.*

Proof. Indeed, we have

$$\begin{aligned} (-1)_\epsilon \cdot \eta &= -(1 + \eta[-1])\eta \\ &= -(2 + \eta[-1])\eta + \eta = \eta \end{aligned}$$

as desired. □

Proposition 4.1.15.

(1) For $a \in F^\times$, we have

$$[a][-a] = 0 \qquad \textit{and} \qquad [a][a] = [a][-1] = [-1][a].$$

(2) For $a, b \in F^\times$, we have

$$[a][b] = (-1)_\epsilon \cdot [b][a].$$

Corollary 4.1.16. *The $\mathrm{K}_0^{\mathrm{MW}}(F)$-algebra $\mathrm{K}_*^{\mathrm{MW}}(F)$ is ϵ-graded commutative in the sense that*

$$\alpha \cdot \beta = (-1)_\epsilon^{\deg(\alpha)\deg(\beta)} \beta \cdot \alpha$$

for all homogenous elements α and β in $\mathrm{K}_*^{\mathrm{MW}}(F)$.

Proof of Proposition 4.1.15, Part (1). This relies on the Steinberg relation (which we have not used so far). If $a = 1$, there is nothing to prove. So we assume that $a \neq 1$, so that $1 - a$ and $1 - a^{-1}$ are in $F^\times$. By the Steinberg relation we have:

$$\left[\frac{1}{a}\right]\left[1 - \frac{1}{a}\right] = 0.$$

Recall that

$$\left[\frac{1}{a}\right] = -\langle a^{-1}\rangle[a] \qquad \text{and} \qquad \left[1 - \frac{1}{a}\right] = \left[\frac{1-a}{-a}\right] = [1-a] - \left\langle \frac{1-a}{-a} \right\rangle [-a].$$

Multiplying these and using that $[a][1-a] = 0$, we obtain that:

$$\langle a^{-1}\rangle \left\langle \frac{1-a}{-a} \right\rangle [a][-a] = 0.$$

Since $\langle u\rangle$ is invertible for every $u \in F^\times$, this gives $[a][-a] = 0$ as needed.

The second relation follows easily. Indeed,

$$\begin{aligned} [a][a] &= [a][(-1)(-a)] \\ &= [a]([-1] + [-a] + \eta[-a][-1]) \\ &= [a][-1] \end{aligned}$$

Similar reasoning gives $[a][a] = [-1][a]$. In particular, this shows that $[-1] \in \mathrm{K}_1^{\mathrm{MW}}(F)$ is a central element (in the non-graded sense!). This finishes the proof of (1). □

We record the following consequence of Part (1) of Proposition 4.1.15.

Corollary 4.1.17. *For every $a \in F^\times$, we have $\langle a^2\rangle = 1$.*

Proof. We need to show that $\eta[a^2] = 0$. Using Part (1) of Proposition 4.1.15, we have

$$\begin{aligned} \eta[a^2] &= \eta(2[a] + \eta[a][a]) \\ &= \eta(2[a] + \eta[-1][a]) \\ &= \eta(2 + \eta[-1])[a] = 0 \end{aligned}$$

as desired. □

Proof of Proposition 4.1.15, Part (2). Using Part (1) of Proposition 4.1.15, we have

$$\begin{aligned} 0 &= [ab][-ab] \\ &= ([a] + \langle a\rangle[b])([-a] + \langle -a\rangle[b]) \\ &= \langle a\rangle[b][-a] + \langle -a\rangle[a][b] + \langle -a^2\rangle[b][b] \\ &= \langle a\rangle([b][-a] + \langle -1\rangle[a][b]) + \langle -1\rangle[b][-1]. \end{aligned}$$

Now using that $[-a] = [a] + \langle a\rangle[-1]$, we obtain:

$$\begin{aligned} 0 &= \langle a\rangle([b][a] + \langle -1\rangle[a][b]) + \langle a^2\rangle[b][-1] + \langle -1\rangle[b][-1] \\ &= \langle a\rangle([b][a] + \langle -1\rangle[a][b]) + [b][-1] + \langle -1\rangle[b][-1]. \end{aligned}$$

So to conclude, it is enough to show that

$$[b][-1] + \langle -1\rangle[b][-1] = 0.$$

To do so, recall that $[b][b] = [b][-1]$. We can compute $[b][b]$ as follows:

$$\begin{aligned} 0 = [b][-b] &= [b]([-1] + \langle -1\rangle[b]) \\ &= [b][-1] + \langle -1\rangle[b][b]. \end{aligned}$$

This shows that $[b][b] = -\langle -1\rangle[b][-1]$ and finishes the proof. □

Corollary 4.1.18. *For $a \in F^\times$, we have*

$$\langle a\rangle + \langle -a\rangle = \langle 1\rangle + \langle -1\rangle = h.$$

Proof.

$$\begin{aligned} \langle a\rangle + \langle -a\rangle &= 1 + \eta[a] + 1 + \eta[-a] \\ &= 2 + \eta([a] + [-a]). \end{aligned}$$

Now, observe that

$$[-a^2] = [-a] + [a] + \eta[a][-a] = [a] + [-a].$$

Thus we have

$$\begin{aligned} \langle a\rangle + \langle -a\rangle &= 2 + \eta[-a^2] \\ &= 1 + \langle -a^2\rangle = 1 + \langle -1\rangle \end{aligned}$$

as desired. □

We now recall some facts about the Grothendieck–Witt ring $\mathrm{GW}(F)$ of the field F. This is the group completion of the commutative monoid of isomorphism classes of symmetric inner product spaces (i.e., finite-dimensional vector spaces endowed with a non-degenerate symmetric bilinear form). For $u \in F^\times$, we denote by $\langle u\rangle \in \mathrm{GW}(F)$ the class of the bilinear form $(x, y) \mapsto uxy$ on F. These generate $\mathrm{GW}(F)$ as a group. We will assume the following presentation of $\mathrm{GW}(F)$.

Proposition 4.1.19. *The group $\mathrm{GW}(F)$ is generated by the elements $\langle u\rangle$, for $u \in F^\times$, subject to the following relations:*

(1) $\langle uv^2\rangle = \langle u\rangle$,
(2) $\langle u\rangle + \langle -u\rangle = 1 + \langle -1\rangle$,
(3) $\langle u\rangle + \langle v\rangle = \langle u + v\rangle + \langle (u + v)uv\rangle$ if $u + v \neq 0$.

Remark 4.1.20. A proof of Proposition 4.1.19 can be found in [Lam05, Theorem 4.3] (see also [Car23, Theorem 2.8]). Here, we only explain the origin of relation (3). In the symmetric inner product space

$$\left(F^2, \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right),$$

we may consider the vector

$$\alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Assuming that $(\alpha,\alpha) = u+v$ is nonzero, its orthogonal is spanned by

$$\beta = \begin{pmatrix} v \\ -u \end{pmatrix}.$$

We have $(\beta,\beta) = (u+v)uv$. This means that

$$\left(F^2, \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right) \simeq \left(F^2, \begin{pmatrix} u+v & 0 \\ 0 & (u+v)uv \end{pmatrix}\right),$$

yielding the third relation in the above proposition.

Corollary 4.1.21. *There is a unique morphism of rings*

$$\mathrm{GW}(F) \to \mathrm{K}_0^{\mathrm{MW}}(F)$$

sending $\langle u\rangle$ to $\langle u\rangle$ for $u \in F^\times$.

Proof. Uniqueness is clear. For existence, we need to show that relation (3) is satisfied in $\mathrm{K}_0^{\mathrm{MW}}(F)$. Let $u, v \in F^\times$ such that $u+v \neq 0$. We must verify that

$$\langle u\rangle + \langle v\rangle = \langle u+v\rangle + \langle (u+v)uv\rangle \qquad \text{in} \qquad \mathrm{K}_0^{\mathrm{MW}}(F).$$

Set $a = \dfrac{u}{u+v}$ so that $1-a = \dfrac{v}{u+v}$. Dividing by $\langle u+v\rangle \in \mathrm{K}_0^{\mathrm{MW}}(F)^\times$, we may as well prove that

$$\langle a\rangle + \langle 1-a\rangle = 1 + \langle a(1-a)\rangle.$$

But we have

$$\begin{aligned} 1 + \langle a(1-a)\rangle &= 2 + \eta[a(1-a)] \\ &= 2 + \eta([a] + [1-a] + \underbrace{\eta[a][1-a]}_{=0}) \\ &= (1+\eta[a]) + (1+\eta[1-a]) \\ &= \langle a\rangle + \langle 1-a\rangle \end{aligned}$$

as needed. □

Construction 4.1.22. Recall that $h = 1 + \langle -1\rangle$. In $\mathrm{GW}(F)$, this is the class of the hyperbolic plane. The subgroup $\mathbb{Z}h \subset \mathrm{GW}(F)$ generated by h is also an ideal (by the second relation in Proposition 4.1.19). The Witt ring of the field F is the quotient

$$\mathrm{W}(F) := \mathrm{GW}(F)/\mathbb{Z}h.$$

The rank map $\mathrm{rk} : \mathrm{GW}(F) \to \mathbb{Z}$ induces a morphism of rings $\mathrm{W}(F) \to \mathbb{Z}/2$ and the square

$$\begin{tikzcd} \mathrm{GW}(F) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathrm{W}(F) \arrow[r] & \mathbb{Z}/2 \end{tikzcd}$$

is cartesian. (Indeed, this square is a pushout in the category of abelian groups.) We set

$$\begin{aligned} \mathrm{I}(F) &= \ker(\mathrm{W}(F) \to \mathbb{Z}/2) \\ &\simeq \ker(\mathrm{GW}(F) \to \mathbb{Z}). \end{aligned}$$

This is called the fundamental ideal. Denote by $\phi_0 : \mathrm{GW}(F) \to \mathrm{K}_0^{\mathrm{MW}}(F)$ the map given by Corollary 4.1.21. This map is surjective by Lemma 4.1.11. For $n > 0$, the composition

$$\mathrm{GW}(F) \to \mathrm{K}_0^{\mathrm{MW}}(F) \xrightarrow{\eta^n} \mathrm{K}_{-n}^{\mathrm{MW}}(F)$$

takes h to zero (by the relation $\eta h = 0$). This induces a map

$$\phi_n : \mathrm{W}(F) \to \mathrm{K}^{\mathrm{MW}}_{-n}(F)$$

participating in a commutative square (for $n > 0$):

$$\begin{tikzcd}
\mathrm{GW}(F) \arrow[r, two heads, "\phi_0"] \arrow[d, two heads] & \mathrm{K}^{\mathrm{MW}}_0(F) \arrow[d, two heads, "\eta^n"] \\
\mathrm{W}(F) \arrow[r, two heads, "\phi_n"] & \mathrm{K}^{\mathrm{MW}}_{-n}(F).
\end{tikzcd}$$

Proposition 4.1.23. *For every $n \geqslant 0$, the map ϕ_n is an isomorphism.*

Proof. Set $\mathrm{i}^n(F) = \mathrm{I}^n(F)/\mathrm{I}^{n+1}(F)$. Note that this is a $\mathbb{Z}/2$-vector space. There is a natural morphism

$$s_n : \mathrm{K}^{\mathrm{M}}_n(F)/2 \to \mathrm{i}^n(F)$$

called the Milnor morphism. It is characterised by the property that $s_* : \mathrm{K}^{\mathrm{M}}_*(F)/2 \to \mathrm{i}^*(F)$ is a graded ring homomorphism given in degree one by $s_1(u) = \langle u \rangle - 1 \in \mathrm{W}(F)$ for every $u \in F^\times$. In fact, the morphism s_* is known to be an isomorphism by the Milnor conjecture on quadratic forms, now a theorem of Orlov–Vishik–Voevodsky [OVV07, Theorem 4.1] (see also [Mor05a, Theorem 1.1]). However, we do not need to use the Milnor conjecture on quadratic forms here. We set

$$\mathrm{J}^n(F) = \mathrm{I}^n(F) \times_{\mathrm{i}^n(F)} \mathrm{K}^{\mathrm{M}}_n(F),$$

with the convention that $\mathrm{I}^n(T) = \mathrm{W}(T)$ for $n \leqslant 0$. Note that this convention yields $\mathrm{i}^n(F) = 0$ for $n < 0$ and $\mathrm{i}^0(F) = \mathbb{Z}/2$. Thus, we have $\mathrm{J}^n(F) = \mathrm{W}(F)$ for $n < 0$ and $\mathrm{J}^0(F) = \mathrm{GW}(F)$.

In the $\mathbb{Z}$-graded ring $\mathrm{J}^*(F)$ we have the following elements:

- $\eta \in \mathrm{J}^{-1}(F)$ corresponding to $1 \in \mathrm{W}(F)$ modulo the isomorphism $\mathrm{J}^{-1}(F) \simeq \mathrm{W}(F)$.
- $[u] = (\langle u \rangle - 1, \{u\}) \in \mathrm{J}^1(F)$, for $u \in F^\times$.

It is easy to see that these elements satisfy the four relations defining $\mathrm{K}^{\mathrm{MW}}_*(F)$. This yields a morphism of graded rings

$$\mathrm{K}^{\mathrm{MW}}_*(F) \to \mathrm{J}^*(F).$$

In degree $* \leqslant 0$, this yields retractions to the maps ϕ_* defined previously. Since the latter maps are surjective, this proves the proposition. □

Remark 4.1.24. In fact, using the Milnor conjecture [OVV07, Theorem 4.1], F. Morel proves that the map $\mathrm{K}^{\mathrm{MW}}_*(F) \to \mathrm{J}^*(F)$ is an isomorphism in all degrees. See [Mor04, Théorème 5.3].

We end this subsection with the following useful result.

Notation 4.1.25. For $n \in \mathbb{Z}$, we define the element $n_\epsilon \in \mathrm{GW}(F)$ as follows. If $n \geqslant 0$, we set

$$\begin{aligned} n_\epsilon &= \overbrace{1 + \langle -1 \rangle + 1 + \cdots}^{n \text{ terms}} \\ &= \sum_{i=1}^{n} \langle (-1)^{i-1} \rangle. \end{aligned}$$

In particular, we have $0_\epsilon = 0$ and $1_\epsilon = 1$. For $n \leqslant 0$, we set

$$\begin{array}{rcl} n_\epsilon & = & \overbrace{-\langle -1\rangle - 1 - \langle -1\rangle - \cdots}^{-n \text{ terms}} \\ & = & -\sum_{i=1}^{-n} \langle (-1)^i \rangle. \end{array}$$

In particular, we have $(-1)_\epsilon = -\langle -1\rangle$, which is compatible with Notation 4.1.13.

Lemma 4.1.26. *For $a \in F^\times$ and $n \in \mathbb{Z}$, we have $[a^n] = n_\epsilon [a]$.*

Proof. We first assume that $n \geqslant 0$ and argue by induction on n. The result is clear for $n = 0$. For $n \geqslant 1$, we have

$$\begin{array}{rcl} [a^n] & = & [a^{n-1} \cdot a] \\ & = & [a^{n-1}] + [a] + \eta[a^{n-1}][a] \\ & = & (n-1)_\epsilon[a] + [a] + \eta(n-1)_\epsilon[a][a] \\ & = & (n-1)_\epsilon[a] + [a] + \eta[-1](n-1)_\epsilon[a] \\ & = & ((n-1)_\epsilon + 1 + \eta[-1](n-1)_\epsilon)[a]. \end{array}$$

Also, using the induction hypothesis a second time, we have $(n-1)_\epsilon[-1] = [(-1)^{n-1}]$ so that

$$1 + \eta[-1](n-1)_\epsilon = \langle (-1)^{n-1} \rangle.$$

Thus, the last line in the above chain of equalities can be rewritten as $n_\epsilon[a]$.

Next, assume that $n < 0$. Then, we have

$$\begin{array}{rcl} [a^n] \; = \; [(a^{-1})^{-n}] & = & (-n)_\epsilon[a^{-1}] \\ & = & -(-n_\epsilon)\langle a^{-1}\rangle[a] \\ & = & -(-n)_\epsilon\langle a\rangle[a]. \end{array}$$

But

$$\begin{array}{rcl} \langle a\rangle[a] & = & [a] + \eta[a][a] \\ & = & [a] + \eta[-1][a] \\ & = & \langle -1\rangle[a], \end{array}$$

so we get

$$-(-n)_\epsilon\langle a\rangle[a] = -(-n)_\epsilon\langle -1\rangle[a] = n_\epsilon[a]$$

as needed. □

4.2. **Residue homomorphisms.** In this subsection we construct and study residue homomorphisms in the context of Milnor–Witt K-theory. Throughout this subsection, F will be a field endowed with a nontrivial discrete valuation v. We denote by $\mathcal{O}_v \subset F$ the ring of integers of the valuation v and by k_v its residue field.

Remark 4.2.1. In Milnor K-theory, we have by [Mil70, Lemma 2.1] a residue homomorphism

$$\partial_v : \mathrm{K}^{\mathrm{M}}_*(F) \to \mathrm{K}^{\mathrm{M}}_{*-1}(k_v)$$

characterised by the formulae

$$\partial_v(\{\pi\}\{u_1\}\cdots\{u_n\}) = \{\bar{u}_1\}\cdots\{\bar{u}_n\} \qquad \text{and} \qquad \partial_v(\{u_1\}\cdots\{u_n\}) = 0$$

for any uniformizer $\pi \in \mathcal{O}_v$ and invertible elements $u_1, \ldots, u_n \in \mathcal{O}_v^\times$. (Here and below, we denote by $\overline{a} \in k_v$ the residue class of an element $a \in \mathcal{O}_v$.) We seek a similar homomorphism for Milnor–Witt K-theory. It turns out that in this case, the residue homomorphism depends (in a controlled way) on the choice of a uniformizer. More precisely, for each uniformizer $\pi \in O_v$, there is a residue homomorphism ∂_v^π in Milnor–Witt K-theory.

Theorem 4.2.2. *Let $\pi \in O_v$ be a uniformizer. There exists a unique morphism of graded groups*

$$\partial_v^\pi : \mathrm{K}_*^{\mathrm{MW}}(F) \to \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v)$$

satisfying the following properties:

(1) ∂_v^π commutes with multiplication by η;
(2) $\partial_v^\pi([\pi][u_1]\cdots[u_n]) = [\bar{u}_1]\cdots[\bar{u}_n]$ for all $u_1, \cdots, u_n \in O_v^\times$;
(3) $\partial_v^\pi([u_1]\cdots[u_n]) = 0$ for all $u_1, \ldots, u_n \in O_v^\times$.

Proof of unicity. Recall that $\mathrm{K}_n^{\mathrm{MW}}(F)$ is generated by

$$\eta^m[u_1]\cdots[u_r] \qquad (\star)$$

with $m \geqslant 0$, $r \geqslant 0$, $u_1, \ldots, u_r \in F^\times$ and $n = r - m$. Write $u_i = a_i\pi^{e_i}$ with $a_i \in O_v^\times$ and $e_i \in \mathbb{Z}$. Then, we have

$$\begin{aligned} [u_i] &= [a_i\pi^{e_i}] \\ &= [a_i] + [\pi^{e_i}] + \eta[a_i][\pi^{e_i}] \\ &= [a_i] + (e_i)_\epsilon[\pi] + \eta([a_i](e_i)_\epsilon[\pi]) \\ &= [a_i] + (e_i)_\epsilon[\pi] + (e_i)_\epsilon\eta[a_i][\pi]. \end{aligned}$$

Noting that $(e_i)_\epsilon$ is a linear combination of 1 and $\eta[-1]$, and using that $[\pi][\pi] = [\pi][-1]$, we deduce that the element $(\star)$ lies in the subgroup spanned by elements of the form

$$\eta^{m'}[\pi^{0/1}][a_1]\cdots[a_{r'}]$$

with $a_1, \ldots, a_{r'} \in O_v^\times$. Using the properties (1) – (3) of the statement, it follows immediately that ∂_v^π is unique if it exists. □

Our next task is to establish the existence of ∂_v^π. The proof is an adaptation to Milnor–Witt K-theory of the proof of [Mil70, Lemma 2.1] (see also [BT73, Chapter I, §4]).

Notation 4.2.3. Let ξ be a formal variable in degree 1 which we adjoin to $\mathrm{K}_*^{\mathrm{MW}}(k_v)$ with the relation $\xi^2 = \xi[-1]$. We denote by

$$\mathrm{K}_*^{\mathrm{MW}}(k_v)[\xi]$$

the resulting $(-1)_\epsilon$-graded commutative ring. In particular we have

$$\langle a\rangle\xi = \xi\langle a\rangle \qquad \text{and} \qquad [a]\xi = (-1)_\epsilon\xi[a]$$

for every $a \in k_v^\times$. As a graded group, we have $\mathrm{K}_*^{\mathrm{MW}}(k_v)[\xi] = \mathrm{K}_*^{\mathrm{MW}}(k_v) \oplus \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v)\cdot\xi$.

Lemma 4.2.4. *For $u \in F^\times$, denote by $[u]'$ the degree-one element in $\mathrm{K}_*^{\mathrm{MW}}(k_v)[\xi]$ given by*

$$[u]' := [\bar{a}] + n_\epsilon\langle\bar{a}\rangle\cdot\xi,$$

where $u = a\pi^n$ with $a \in O_v^\times$ and $n \in \mathbb{Z}$. (Note that $[a\pi^n] = [a] + \langle a\rangle[\pi^n] = [a] + n_\epsilon\langle a\rangle[\pi]$.) Then the elements $[u]'$, for $u \in F^\times$, satisfy the relations (1)–(4) defining Milnor–Witt K-theory.

Proof. Since $[-1]' = [-1]$, relation (4) is obviously satisfied. Similarly, relation (3) asserting that η is central is clearly satisfied. So we only need to check the relations (1) and (2).

The Steinberg relation. Write $u = a\pi^n$, with $a \in O_v^\times$ and $n \in \mathbb{Z}$, and assume that $u \neq 1$. We distinguish three cases.

Case 1: $n > 0$. In this case $1 - u \in \mathcal{O}_v^\times$ and $\overline{1-u} = 1$ in k_v. Thus $[1-u]' = 0$, and $[u]'[1-u]' = 0$ as needed.

Case 2: $n < 0$. In this case

$$1 - u = \underbrace{(\pi^{-n} - a)}_{\in \mathcal{O}_v^\times}\pi^n.$$

Thus

$$\begin{aligned} [1-u]' &= [\overline{\pi^{-n} - a}] + n_\epsilon\langle\overline{\pi^{-n} - a}\rangle \cdot \xi \\ &= [-\bar{a}] + n_\epsilon\langle -\bar{a}\rangle \cdot \xi. \end{aligned}$$

This gives

$$\begin{aligned} [u]'[1-u]' &= ([\bar{a}] + n_\epsilon\langle\bar{a}\rangle \cdot \xi)([-\bar{a}] + n_\epsilon\langle -\bar{a}\rangle \cdot \xi) \\ &= [\bar{a}][-\bar{a}] + n_\epsilon\langle -\bar{a}\rangle[\bar{a}]\xi + n_\epsilon\langle\bar{a}\rangle \overbrace{\xi[-\bar{a}]}^{=-\langle -1\rangle[-\bar{a}]\xi} + \overbrace{\langle\bar{a}\rangle\langle -\bar{a}\rangle}^{=\langle -1\rangle} n_\epsilon^2 \overbrace{\xi^2}^{=[-1]\xi}. \end{aligned}$$

Also, notice that $n_\epsilon^2[-1] = [(-1)^{n^2}] = [(-1)^n] = n_\epsilon[-1]$. Thus, we may continue the chain of equalities as follows:

$$\begin{aligned} &= n_\epsilon\langle -\bar{a}\rangle[\bar{a}]\xi - n_\epsilon\langle -\bar{a}\rangle[-\bar{a}]\xi + n_\epsilon\langle -1\rangle[-1]\xi \\ &= n_\epsilon(\langle -\bar{a}\rangle[\bar{a}] - \langle -\bar{a}\rangle[-\bar{a}] + \langle -1\rangle[-1])\xi. \end{aligned}$$

Thus, it suffices to show that $\langle -\bar{a}\rangle[\bar{a}] - \langle -\bar{a}\rangle[-\bar{a}] + \langle -1\rangle[-1] = 0$. Using that $[-\bar{a}] = [\bar{a}] + \langle\bar{a}\rangle[-1]$, we have

$$\begin{aligned} \langle -\bar{a}\rangle[\bar{a}] - \langle -\bar{a}\rangle[-\bar{a}] + \langle -1\rangle[-1] &= -\langle -\bar{a}\rangle\langle\bar{a}\rangle[-1] + \langle -1\rangle[-1] \\ &= -\langle -1\rangle[-1] + \langle -1\rangle[-1] \;=\; 0 \end{aligned}$$

as needed.

Case 3: $n = 0$. We write $1 - u = b\pi^m$ with $b \in \mathcal{O}_v^\times$ and $m \geqslant 0$. If $m > 0$, then $\bar{u} = 1$ and $[u]' = 0$, so there is nothing to prove. If $m = 0$, then $u, 1-u \in \mathcal{O}_v^\times$ and $[u]' = [\bar{u}]$ and $[1-u]' = [1-\bar{u}]$. In this case, the Steinberg relation in $\mathrm{K}_*^{\mathrm{MW}}(k_v)$ can be used to conclude.

The relation (2). Assume that $u = a\pi^m$ and $v = b\pi^n$ with $a, b \in \mathcal{O}_v^\times$ and $m, n \in \mathbb{Z}$. We compute

$$\begin{aligned} & [u]' + [v]' + \eta[u]'[v]' \\ = \;& [\bar{a}] + m_\epsilon\langle\bar{a}\rangle\xi + [\bar{b}] + n_\epsilon\langle\bar{b}\rangle\xi + \eta([\bar{a}] + m_\epsilon\langle\bar{a}\rangle\xi)([\bar{b}] + n_\epsilon\langle\bar{b}\rangle\xi) \\ = \;& [\bar{a}] + [\bar{b}] + m_\epsilon\langle\bar{a}\rangle\xi + n_\epsilon\langle\bar{b}\rangle\xi + \eta([\bar{a}][\bar{b}] + n_\epsilon\langle\bar{b}\rangle[\bar{a}]\xi + m_\epsilon\langle\bar{a}\rangle\xi[\bar{b}] + m_\epsilon n_\epsilon\langle\bar{a}\bar{b}\rangle\xi^2) \\ = \;& [\bar{a}] + [\bar{b}] + \eta[\bar{a}][\bar{b}] + m_\epsilon\langle\bar{a}\rangle\xi + n_\epsilon\langle\bar{b}\rangle\xi + n_\epsilon\langle\bar{b}\rangle(\langle\bar{a}\rangle - 1)\xi \\ & -m_\epsilon\langle -\bar{a}\rangle(\langle\bar{b}\rangle - 1)\xi + m_\epsilon n_\epsilon\langle\bar{a}\bar{b}\rangle(\langle -1\rangle - 1)\xi \\ = \;& [\bar{a}] + [\bar{b}] + \eta[\bar{a}][\bar{b}] + (m_\epsilon\langle\bar{a}\rangle + n_\epsilon\langle\bar{a}\bar{b}\rangle - m_\epsilon\langle -\bar{a}\bar{b}\rangle + m_\epsilon\langle -\bar{a}\rangle \\ & +m_\epsilon n_\epsilon\langle -\bar{a}\bar{b}\rangle - m_\epsilon n_\epsilon\langle\bar{a}\bar{b}\rangle)\xi. \end{aligned}$$

We need to compare this with

$$[uv]' = [\bar{a}\bar{b}] + \langle\bar{a}\bar{b}\rangle(m+n)_\epsilon\xi.$$

Thus, we need to show that

$$m_\epsilon\langle\bar{a}\rangle + n_\epsilon\langle\bar{a}\bar{b}\rangle - m_\epsilon\langle -\bar{a}\bar{b}\rangle + m_\epsilon\langle -\bar{a}\rangle + m_\epsilon n_\epsilon\langle -\bar{a}\bar{b}\rangle - m_\epsilon n_\epsilon\langle\bar{a}\bar{b}\rangle - (m+n)_\epsilon\langle\bar{a}\bar{b}\rangle = 0.$$

To do so, we note that

$$\langle\bar{a}\rangle + \langle -\bar{a}\rangle = \langle 1\rangle + \langle -1\rangle = \langle\bar{a}\bar{b}\rangle + \langle -\bar{a}\bar{b}\rangle.$$

So, the task becomes to show that

$$m_\epsilon\langle\bar{a}\bar{b}\rangle + n_\epsilon\langle\bar{a}\bar{b}\rangle - m_\epsilon\langle-\bar{a}\bar{b}\rangle + m_\epsilon\langle-\bar{a}\bar{b}\rangle + m_\epsilon n_\epsilon\langle-\bar{a}\bar{b}\rangle - m_\epsilon n_\epsilon\langle\bar{a}\bar{b}\rangle - (m+n)_\epsilon\langle\bar{a}\bar{b}\rangle = 0$$

This is equivalent to showing that

$$(m_\epsilon + n_\epsilon + m_\epsilon n_\epsilon\langle-1\rangle - m_\epsilon n_\epsilon - (m+n)_\epsilon)\langle\bar{a}\bar{b}\rangle = 0$$

which is also equivalent to showing that

$$(m+n)_\epsilon = m_\epsilon + n_\epsilon + (m_\epsilon n_\epsilon)\langle-1\rangle - m_\epsilon n_\epsilon.$$

This is an easy exercise that can be done by distinguishing four cases according to the parities of m and n. □

Corollary 4.2.5. *There exists a unique graded ring homomorphism*

$$\Theta_v^\pi : \mathrm{K}_*^{\mathrm{MW}}(F) \to \mathrm{K}_*^{\mathrm{MW}}(k_v)[\xi]$$

sending η to η, $[\pi]$ to ξ and $[a]$, for $a \in \mathcal{O}_v^\times$, to $[\bar{a}]$.

Proof. If Θ_v^π exists, then it must send $[a\pi^n]$, for $a \in \mathcal{O}_v^\times$ and $n \in \mathbb{Z}$, to $[\bar{a}] + n_\epsilon\langle\bar{a}\rangle\xi$. Moreover, by the previous lemma, this assignment factors through $\mathrm{K}_*^{\mathrm{MW}}(F)$. □

Definition 4.2.6. Let $\pi \in \mathcal{O}_v$ be a uniformizer. For $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(F)$, write

$$\Theta_v^\pi(\alpha) = \mathrm{s}_v^\pi(\alpha) + \xi \cdot \partial_v^\pi(\alpha).$$

This defines two graded homomorphisms

$$\mathrm{s}_v^\pi : \mathrm{K}_*^{\mathrm{MW}}(F) \to \mathrm{K}_*^{\mathrm{MW}}(k_v) \qquad \text{and} \qquad \partial_v^\pi : \mathrm{K}_*^{\mathrm{MW}}(F) \to \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v).$$

In fact s_v^π is a ring homomorphism.

Proposition 4.2.7. *For any $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(F)$ we have:*

(1) $\partial_v^\pi([-\pi]\cdot\alpha) = \langle-1\rangle\cdot \mathrm{s}_v^\pi(\alpha)$;
(2) $\partial_v^\pi([u]\cdot\alpha) = (-1)_\epsilon[\bar{u}]\cdot\partial_v^\pi(\alpha)$, for $u \in \mathcal{O}_v^\times$;
(3) $\partial_v^\pi(\langle u\rangle\cdot\alpha) = \langle\bar{u}\rangle\cdot\partial_v^\pi(\alpha)$, for $u \in \mathcal{O}_v^\times$.

Proof. Since Θ_v^π is a ring homomorphism, we have

$$\begin{aligned}\Theta_v^\pi([u]\cdot\alpha) &= \Theta_v^\pi([u])\cdot\Theta_v^\pi(\alpha)\\ &= [\bar{u}]\cdot(\mathrm{s}_v^\pi(\alpha) + \xi\cdot\partial_v^\pi(\alpha))\\ &= [\bar{u}]\cdot \mathrm{s}_v^\pi(\alpha) + (-1)_\epsilon\xi[\bar{u}]\partial_v^\pi(\alpha).\end{aligned}$$

This proves (2). Note also that the same argument gives:

(2′) $\partial_v^\pi(\alpha\cdot[u]) = \partial_v^\pi(\alpha)\cdot[\bar{u}]$.

It follows that $\partial_v^\pi(\alpha\cdot\langle u\rangle) = \partial_v^\pi(\alpha)\cdot\langle\bar{u}\rangle$. Since $\langle u\rangle$ and $\langle\bar{u}\rangle$ are central, we obtain (3).

It remains to prove (1). By construction, we have $\Theta_v^\pi([\pi]) = \xi$. Thus

$$\begin{aligned}\Theta_v^\pi([\pi]\cdot\alpha) &= \Theta_v^\pi([\pi])\cdot\Theta_v^\pi(\alpha)\\ &= \xi\cdot(\mathrm{s}_v^\pi(\alpha) + \xi\cdot\partial_v^\pi(\alpha))\\ &= \xi \mathrm{s}_v^\pi(\alpha) + \xi^2\cdot\partial_v^\pi(\alpha)\\ &= \xi(\mathrm{s}_v^\pi(\alpha) + [-1]\cdot\partial_v^\pi(\alpha))\end{aligned}$$

Using that $[-1]$ is central, we can write

$$[-1]\cdot\partial_v^\pi(\alpha) = \partial_v^\pi([-1]\cdot\alpha).$$

This gives:

$$\partial_v^\pi([\pi]\cdot\alpha) = \mathrm{s}_v^\pi(\alpha) + \partial_v^\pi([-1]\cdot\alpha).$$

Thus, $\partial_v^\pi(([\pi]-[-1])\cdot\alpha) = \mathrm{s}_v^\pi(\alpha)$. Now $[\pi] = [(-1)\cdot(-\pi)] = [-1] + \langle -1\rangle[-\pi]$. So, we finally get

$$\partial_v^\pi(\langle -1\rangle[-\pi]\alpha) = \mathrm{s}_v^\pi(\alpha)$$

as desired. □

Corollary 4.2.8.

(1) For $u_1,\dots,u_n \in O_v^\times$, we have

$$\partial_v^\pi([u_1]\cdots[u_n]) = 0 \qquad \text{and} \qquad \partial_v^\pi([\pi][u_1]\cdots[u_n]) = [\bar u_1]\cdots[\bar u_n].$$

We also have $\mathrm{s}_v^\pi([u_1]\cdots[u_n]) = [\bar u_1]\cdots[\bar u_n]$.

(2) For any $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(F)$, we have $\mathrm{s}_v^\pi([\pi]\cdot\alpha) = 0$ and $\mathrm{s}_v^\pi([u]\cdot\alpha) = [\bar u]\cdot \mathrm{s}_v^\pi(\alpha)$, for $u \in O_v^\times$.

(3) For any $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(F)$ and any $u \in O_v^\times$, we have $\partial_v^\pi(\alpha\cdot[u]) = \partial_v^\pi(\alpha)\cdot[\bar u]$.

Proof. This follows readily from Proposition 4.2.7. □

We now describe the dependence of ∂_v^π on the choice of the uniformizer.

Lemma 4.2.9. *Let $F \subset E$ be a field extension and let w be a valuation of E extending v. Assume that the ramification index at w is $e \geqslant 1$ and let $\rho \in O_w$ be a uniformizer. Write $\pi = u\rho^e$ with $u \in O_w^\times$. Then we have a commutative square of ring homomorphisms*

$$\begin{tikzcd} \mathrm{K}_*^{\mathrm{MW}}(F) \arrow[r, "\Theta_v^\pi"] \arrow[d] & \mathrm{K}_*^{\mathrm{MW}}(k_v)[\xi] \arrow[d, "(\star)"] \\ \mathrm{K}_*^{\mathrm{MW}}(E) \arrow[r, "\Theta_w^\rho"] & \mathrm{K}_*^{\mathrm{MW}}(k_w)[\xi'] \end{tikzcd}$$

where $(\star)$ restricts to the obvious morphism $\mathrm{K}_*^{\mathrm{MW}}(k_v) \to \mathrm{K}_*^{\mathrm{MW}}(k_w)$ and sends ξ to $[\bar u] + e_\epsilon\langle \bar u\rangle\xi'$.

Proof. Since $\mathrm{K}_*^{\mathrm{MW}}(k_v)[\xi]$ is the free $(-1)_\epsilon$-graded commutative $\mathrm{K}_*^{\mathrm{MW}}(k_v)$-algebra on one generator ξ satisfying $\xi^2 = \xi[-1]$, the existence of $(\star)$ would follow if we can prove that

$$([\bar u] + e_\epsilon\langle\bar u\rangle\xi')^2 = ([\bar u] + e_\epsilon\langle\bar u\rangle\xi')[-1].$$

We compute:

$$\begin{aligned} ([\bar u] + e_\epsilon\langle\bar u\rangle\xi')^2 &= [\bar u][\bar u] + e_\epsilon\langle\bar u\rangle[\bar u]\xi' + e_\epsilon\langle\bar u\rangle\xi'[\bar u] + e_\epsilon^2\langle\bar u\rangle^2\xi'^2 \\ &= [\bar u][-1] + e_\epsilon\langle\bar u\rangle[\bar u]\xi' - e_\epsilon\langle -\bar u\rangle[\bar u]\xi' + e_\epsilon\xi'[-1]. \end{aligned}$$

Thus, to conclude, we need to show that

$$\langle\bar u\rangle[\bar u] - \langle -\bar u\rangle[\bar u] + [-1] = \langle\bar u\rangle[-1].$$

But

$$\begin{aligned} \langle\bar u\rangle[\bar u] - \langle -\bar u\rangle[\bar u] &= (\eta[\bar u] - \eta[-\bar u])[\bar u] \\ &= \eta[\bar u][\bar u] \ = \ \eta[\bar u][-1] \ = \ (\langle\bar u\rangle - 1)[-1]. \end{aligned}$$

This proves our claim.

Now that we know that $(\star)$ exists, we can easily check that the square in the statement is commutative. Indeed, it is enough to check that the two possible maps from $\mathrm{K}_*^{\mathrm{MW}}(F)$ to $\mathrm{K}_*^{\mathrm{MW}}(k_w)[\xi']$ agree on $[u]$, for $u \in O_v^\times$, and on $[\pi]$. But, this is clear by construction. □

The next result is the Milnor–Witt K-theory version of [BT73, Chapter I, Proposition 4.8].

Corollary 4.2.10. *Keep the notations and assumptions as in lemma 4.2.9. Then for every* $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(F)$*, we have the identity*

$$\partial_w^{\rho}(\alpha|_E) = e_\epsilon\langle\bar{u}\rangle\partial_v^{\pi}(\alpha)|_{k_w}.$$

Proof. By the previous lemma, we have

$$\begin{aligned}\Theta_w^{\rho}(\alpha|_E) &= \mathrm{s}_v^{\pi}(\alpha)|_{k_w} + ([\bar{u}] + e_\epsilon\langle\bar{u}\rangle\xi')\partial_v^{\pi}(\alpha)|_{k_w} \\ &= (\mathrm{s}_v^{\pi}(\alpha)|_{k_w} + [\bar{u}]\partial_v^{\pi}(\alpha)|_{k_w}) + \xi'(e_\epsilon\langle\bar{u}\rangle\partial_v^{\pi}(\alpha)|_{k_w}).\end{aligned}$$

This proves the corollary. □

Remark 4.2.11. A particular case of Corollary 4.2.10 yields the formula

$$\partial_v^{u\pi} = \langle\bar{u}\rangle\partial_v^{\pi} \qquad \text{for} \qquad u \in \mathcal{O}_v^{\times},$$

describing the dependence of the residue homomorphism on the choice of the uniformizer. Thus, for $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(F)$, the condition that $\partial_v^{\pi}(\alpha) = 0$ is independent of the choice of π. If satisfied, we say that α is unramified at v.

Definition 4.2.12. Let X be an irreducible noetherian regular scheme with field of rational functions F. We denote by $\mathrm{K}_*^{\mathrm{MW}}(X)$ the graded subgroup of $\mathrm{K}_*^{\mathrm{MW}}(F)$ consisting of those elements which are unramified at every valuation of F corresponding to a one-codimensional point of X. This is actually a graded subring since the residue homomorphisms are derivations in the following sense.

Lemma 4.2.13. *For homogenous elements* $\alpha, \beta \in \mathrm{K}_*^{\mathrm{MW}}(F)$*, we have*

$$\partial_v^{\pi}(\alpha\beta) = \partial_v^{\pi}(\alpha)\mathrm{s}_v^{\pi}(\beta) + (-1)_\epsilon^{\deg(\alpha)}\mathrm{s}_v^{\pi}(\alpha)\partial_v^{\pi}(\beta) + \partial_v^{\pi}(\alpha)\partial_v^{\pi}(\beta)[-1].$$

In particular, if α *and* β *are unramified at* v*, so is* $\alpha\beta$*.*

Proof. This follows readily from the fact that Θ_v^{π} is a ring homomorphism. □

Remark 4.2.14. If X is not necessarily irreducible, we let $\mathrm{K}_*^{\mathrm{MW}}(X)$ be the direct product of the rings of unramified Milnor–Witt K-theory of the components of X. We will see in Subsection 4.7 that the assignment $X \mapsto \mathrm{K}_*^{\mathrm{MW}}(X)$ extends to a Nisnevich sheaf $\mathrm{K}_*^{\mathrm{MW}}$ on Sm_k, which is strictly $\mathbb{A}^1$-invariant. At this point, we can only see that the assignment $X \mapsto \mathrm{K}_*^{\mathrm{MW}}(X)$ is contravariantly functorial for smooth morphisms (or, more generally, for flat morphisms). Also, it is easy to see that $X \mapsto \mathrm{K}_*^{\mathrm{MW}}(X)$ defines a Nisnevich sheaf on $\mathrm{\acute{E}t}_X$ for every regular noetherian scheme X.

Theorem 4.2.15. *The subring*

$$\mathrm{K}_*^{\mathrm{MW}}(\mathcal{O}_v) \subset \mathrm{K}_*^{\mathrm{MW}}(F)$$

is generated by η *and the symbols* $[u]$*, for* $u \in \mathcal{O}_v^{\times}$*. Consequently, as a subgroup,* $\mathrm{K}_n^{\mathrm{MW}}(\mathcal{O}_v)$ *is generated by the symbols* $[u_1]\cdots[u_n]$*, for* $u_1, \cdots, u_n \in \mathcal{O}_v^{\times}$ *if* $n \geqslant 1$*, and the symbols* $\eta^{-n}\langle u\rangle$*, with* $u \in \mathcal{O}_v^{\times}$*, if* $n \leqslant 0$*.*

Proof. The second assertion follows from the first one using the same argument as in the proof of Lemma 4.1.11. Thus, we only treat the first assertion following the proof of [Mor12, Theorem 3.22].

Let $A_* \subset \mathrm{K}_*^{\mathrm{MW}}(F)$ be the subring generated by η and the symbols $[u]$, for $u \in \mathcal{O}_v^{\times}$. Let $Q_* = \mathrm{K}_*^{\mathrm{MW}}(F)/A_*$. The residue homomorphism ∂_v^{π} is zero on A_* and thus induces a surjective map

$$Q_* \twoheadrightarrow \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v). \qquad (1)$$

We will construct a surjection

$$\mathrm{K}_{*-1}^{\mathrm{MW}}(k_v) \twoheadrightarrow Q_*, \qquad (2)$$

and show that (1) $\circ$ (2) is the identity of $\mathrm{K}^{\mathrm{MW}}_{*-1}(k_v)$. This would show that (1) is an isomorphism, proving that $A_* = \mathrm{K}^{\mathrm{MW}}_*(O_v)$ as desired.

To construct (2), consider the ring $\mathcal{E}_*$ of graded endomorphisms of Q_*. We denote by $\mu_\eta \in \mathcal{E}_{-1}$ the endomorphism

$$Q_* \xrightarrow{\eta\cdot -} Q_{*-1}$$

and, for $u \in O_v^\times$, we denote by $\mu_u \in \mathcal{E}_1$ the endomorphism

$$\mu_u : Q_* \xrightarrow{[u]\cdot -} Q_{*+1}.$$

We claim that μ_u depends only on $\bar{u}$. To prove this, we need to check that

$$([u] - [u(1+\rho)]) \cdot \alpha \in A_*$$

for every $\alpha \in \mathrm{K}^{\mathrm{MW}}_*(F)$ and $\rho \in \mathfrak{m}_v$. We may assume that

$$\alpha = \eta^m [\pi][a_1] \cdots [a_r]$$

with $a_1, \dots, a_r \in O_v^\times$. (See the proof of the unicity part in Theorem 4.2.2.) So we are left to showing that

$$([u] - [u(1+\rho)]) \cdot [\pi] \in A_2.$$

Expanding the first factor of the product, it is enough to show that $[1+\rho][\pi] \in A_2$. Write $\rho = -a\pi^n$ with $a \in O_v^\times$ and $n \geq 1$. If $n = 1$, the Steinberg relation gives:

$$0 = [1 - a\pi][a\pi] = [1 - a\pi]([a] + [\pi]\langle a\rangle)$$

showing that $[1 - a\pi][\pi] = -\langle a\rangle[1 - a\pi][a] \in A_2$. If $n \geqslant 2$, we argue as follows:

$$\begin{aligned} 1 - a\pi^n &= 1 - \pi + (1 - a\pi^{n-1})\pi \\ &= (1-\pi)\left(1 + \frac{1 - a\pi^{n-1}}{1-\pi}\pi\right) \\ &= (1-\pi)(1 - b\pi) \quad \text{with} \quad b \in O_v^\times. \end{aligned}$$

Thus

$$\begin{aligned} [1 - a\pi^n][\pi] &= [1-\pi][\pi] + [1 - b\pi][\pi] + \eta[1-\pi][\pi][1 - b\pi] \\ &= [1 - b\pi][\pi] \end{aligned}$$

which belongs to A_* as explained above. This proves our claim.

By the previous discussion, given $u \in O_v^\times$, we may write $\mu_{\bar{u}}$ instead of μ_u. We next notice that μ_η and the $\mu_{\bar{u}}$, for $\bar{u} \in k_v^\times$, satisfy the four relations defining the Milnor–Witt K-theory of k_v. Thus, we have a morphism of graded rings

$$\mathrm{K}^{\mathrm{MW}}_*(k_v) \to \mathcal{E}_*.$$

Composing with the evaluation at the class of $[\pi]$, we obtain a morphism

$$\mathrm{K}^{\mathrm{MW}}_*(k_v) \to Q_{*+1}, \quad [\bar{u}_1] \cdots [\bar{u}_n] \mapsto [u_1] \cdots [u_n][\pi] \bmod A_*.$$

It is clear that the composition with ∂_v^π is, up to a power of $(-1)_\epsilon$, the identity. Thus, to finish the proof, it remains to see that the map we just constructed is surjective. But we know that $\mathrm{K}^{\mathrm{MW}}_*(F)$ is generated by A_* and symbols of the form

$$\eta^m [\pi][u_1] \cdots [u_n],$$

for $u_1, \cdots, u_n \in O_v^\times$. This finishes the proof of the theorem. □

Remark 4.2.16. The restriction of s_v^π to $\mathrm{K}_*^{\mathrm{MW}}(O_v)$ is independent of the choice of π. This can be seen using the previous theorem, but one can also prove it more directly using the relation between s_v^π and s_v^ρ. (See Remark 4.2.11.) In any case, we simply denote by

$$\mathrm{s}_v : \mathrm{K}_*^{\mathrm{MW}}(O_v) \to \mathrm{K}_*^{\mathrm{MW}}(k_v)$$

the restriction of s_v^π. This map takes the symbol $[u]$, for $O_v^\times$, to $[\bar{u}]$.

4.3. $\mathbb{A}^1$**-Invariance, transfers and reciprocity.** As in the previous two subsections, we denote by F a field. We let T be the coordinate on $\mathbb{A}_F^1$ and, for $x \in (\mathbb{A}_F^1)^{(1)}$, we denote by $P_x \in F[T]$ the monic polynomial generating the ideal of definition of x. Then P_x is a uniformizer of the valuation of $F(T)$ corresponding to x. In particular we have the residue homomorphism

$$\partial_x^{P_x} : \mathrm{K}_*^{\mathrm{MW}}(F(T)) \to \mathrm{K}_{*-1}^{\mathrm{MW}}(\kappa(x)).$$

The next result is the Milnor–Witt K-theory version of a theorem of Milnor [Mil70, Theorem 2.3] (see also [BT73, Chapter I, Theorem 5.1]).

Theorem 4.3.1. *There is a split short exact sequence of graded abelian groups*

$$0 \to \mathrm{K}_*^{\mathrm{MW}}(F) \to \mathrm{K}_*^{\mathrm{MW}}(F(T)) \xrightarrow{(\partial_x^{P_x})_x} \bigoplus_{x\in(\mathbb{A}_F^1)^{(1)}} \mathrm{K}_{*-1}^{\mathrm{MW}}(\kappa(x)) \to 0.$$

We will adapt Milnor's proof of [Mil70, Theorem 2.3] to Milnor–Witt K-theory. We only need to show the exactness of the sequence. Indeed, the splitting of the sequence is clear since the morphism $\mathrm{K}_*^{\mathrm{MW}}(F) \to \mathrm{K}_*^{\mathrm{MW}}(F(T))$ admits a retraction, namely

$$\partial_0^T([T]\cdot -) : \mathrm{K}_*^{\mathrm{MW}}(F(T)) \to \mathrm{K}_*^{\mathrm{MW}}(F),$$

where $0 \in (\mathbb{A}_F^1)^{(1)}$ is the origin. For the proof, we introduce a filtration by subrings:

$$L_*^{(0)} = \mathrm{K}_*^{\mathrm{MW}}(F) \subset L_*^{(1)} \subset \cdots \subset L_*^{(d)} \subset \cdots \subset \mathrm{K}_*^{\mathrm{MW}}(F(T))$$

such that $L_*^{(d)}$ is the graded subring generated by η and the symbols $[P] \in \mathrm{K}_1^{\mathrm{MW}}(F(T))$ with $P \in F[T]\smallsetminus\{0\}$ of degree $\leqslant d$. Clearly $\mathrm{K}_*^{\mathrm{MW}}(F(T)) = \bigcup_d L_*^{(d)}$. Using the relation $[ab] = [a]+[b]+\eta[a][b]$, we see that a symbol $[Q_1]\cdots[Q_r]$ belongs to $L_*^{(d)}$ if the Q_i's are fractions with numerator and denominator a product of polynomials of degree $\leqslant d$.

Proposition 4.3.2.

1. *For $n \geqslant 1$, $L_n^{(d)}$ is generated by $L_n^{(d-1)}$ and symbols*

$$\eta^m[P_1]\cdots[P_{n+m}],$$

   with P_1 of degree d and P_i, for $2 \leqslant i \leqslant n+m$, of degree $\leq d-1$.
2. *Let $P \in F[T]$ be a monic polynomial of degree $d > 0$. Let $G_1, \ldots, G_r$ be polynomials of degree $\leqslant d-1$. Finally, let G be the remainder of the Euclidean division of $\prod_{i=1}^r G_i$ by P. Then in $L_2^{(d)}/L_2^{(d-1)}$, one has:*

$$[P][G_1\cdots G_r] = [P][G].$$

Proof. For the first assertion, we need to show that a symbol $[P_1][P_2]$, with $\deg(P_1) = \deg(P_2) = d$, can be expressed, modulo $L_2^{(d-1)}$, as a linear combination of symbols as in Part (1). Write

$$P_2 = -aP_1 + G,$$

with $a \in F^\times$ and G of degree $\leqslant d-1$. If $G = 0$, then

$$\begin{aligned}[P_1][P_2] &= [P_1][-aP_1] \\ &= [P_1]([-a] + \langle -a\rangle[P_1]) \\ &= [P_1][-a] + \langle -a\rangle[-1][P_1]\end{aligned}$$

as desired. Thus, we may assume that $G \neq 0$. Dividing by G we have:

$$1 = \frac{P_2}{G} + \frac{aP_1}{G}.$$

By the Steinberg relation, we deduce that

$$\left[\frac{aP_1}{G}\right]\left[\frac{P_2}{G}\right] = 0.$$

On the other hand, we have

$$\left[\frac{aP_1}{G}\right] = \left[\frac{a}{G}\right] + \left\langle\frac{a}{G}\right\rangle[P_1] \qquad \text{and} \qquad \left[\frac{P_2}{G}\right] = \left[\frac{1}{G}\right] + \left\langle\frac{1}{G}\right\rangle[P_2].$$

It follows that

$$\left[\frac{a}{G}\right]\left[\frac{1}{G}\right] + \left\langle\frac{1}{G}\right\rangle\left[\frac{a}{G}\right][P_2] + \left\langle\frac{a}{G}\right\rangle\left[\frac{1}{G}\right][P_1] + \langle a\rangle[P_1][P_2] = 0.$$

This gives that

$$[P_1][P_2] = -(\langle a^{-1}\rangle[aG^{-1}][G^{-1}] + \langle a^{-1}G^{-1}\rangle[aG^{-1}][P_2] + \langle G^{-1}\rangle[G^{-1}][P_1]),$$

and the right-hand side can be easily written modulo $L_2^{(d-1)}$ as a linear combination of symbols as in Part (1).

Now we prove the second assertion. If $r = 1$, there is nothing to prove. Assume that $r = 2$, and form the Euclidean division $G_1G_2 = QP + G$ of G_1G_2 by P. Note that $\deg(Q) \leqslant d-2$ and $\deg(G) \leqslant d-1$. Dividing by G_1G_2, we have

$$\frac{QP}{G_1G_2} + \frac{G}{G_1G_2} = 1.$$

By the Steinberg relation, we deduce that

$$\left[\frac{QP}{G_1G_2}\right]\left[\frac{G}{G_1G_2}\right] = 0.$$

Using that

$$\left[\frac{QP}{G_1G_2}\right] = \left\langle\frac{Q}{G_1G_2}\right\rangle[P] + \left[\frac{Q}{G_1G_2}\right],$$

we obtain

$$\left\langle\frac{Q}{G_1G_2}\right\rangle[P]\left[\frac{G}{G_1G_2}\right] + \left[\frac{Q}{G_1G_2}\right]\left[\frac{G}{G_1G_2}\right] = 0.$$

Notice that

$$\left\langle\frac{Q}{G_1G_2}\right\rangle^{-1}\left[\frac{Q}{G_1G_2}\right]\left[\frac{G}{G_1G_2}\right] \in L_{d-1}.$$

Thus, in $L_*^{(d)}/L_*^{(d-1)}$, we have the relation

$$[P]\left[\frac{G}{G_1G_2}\right] \equiv 0.$$

Combining this with Lemma 4.3.3 below, we get that $[P][G] - [P][G_1G_2]$ belongs to $L_*^{(d-1)}$ as needed.

To finish the proof, we assume that $r \geqslant 3$ and argue by induction on r. We have

$$[P][G_1 \cdots G_r] = [P][G_1 \cdots G_{r-1}] + [P][G_r] + \eta[P][G_1 \cdots G_{r-1}][G_r].$$

Let G' be the remainder of the division of $G_1 \cdots G_{r-1}$ by P. By the induction hypothesis, the right-hand side is equal in $L_*^{(d)}/L_*^{(d-1)}$ to

$$[P][G'] + [P][G_r] + \eta[P][G'][G_r] = [P][G'G_r].$$

We can now use the case $r = 2$ to conclude. □

Lemma 4.3.3. *For $a, b \in F^\times$, we have*

$$\left[\frac{a}{b}\right] = \langle -a\rangle([a] - [b]).$$

Proof. Indeed, we have

$$\begin{aligned}\left[\frac{a}{b}\right] &= [a] + \langle a\rangle[b^{-1}] \\ &= [a] - \langle a\rangle\langle -1\rangle[b] \\ &= \langle a\rangle(\langle a^{-1}\rangle[a] - \langle -1\rangle[b]).\end{aligned}$$

Now we have $\langle a\rangle[a] = \langle -1\rangle[a]$, and hence $\langle a^{-1}\rangle[a] = \langle -1\rangle[a]$. This yields the desired formula. □

Proof of Theorem 4.3.1. Let $d \geqslant 1$ be an integer and let $P \in F[T]$ be a monic irreducible polynomial of degree d. We denote by $\overline{K}_*^P$ the graded subgroup of $L_*^{(d)}/L_*^{(d-1)}$ generated by the classes of elements of the form

$$\eta^m[P][G_1] \cdots [G_n]$$

with $G_1, \ldots, G_n$ of degree $\leqslant d-1$. For any nonzero polynomial G of degree $\leqslant d-1$, multiplication by $[G]$ yields a degree-one map

$$\epsilon[G] : \overline{K}_*^P \to \overline{K}_*^P, \qquad \eta^m[P][G_1] \cdots [G_n] \mapsto \eta^m[P][G][G_1] \cdots [G_n].$$

(That this is well-defined is clear since multiplication by $[G]$ maps $L_*^{(d-1)}$ to itself.) Let $\mathcal{E}_*^P$ be the graded endomorphism ring of $\overline{K}_*^P$. We claim that multiplication by η and the $\epsilon[G]$, for $G \in F[T]_{\leqslant d-1} \setminus \{0\} \simeq (F[T]/P)^\times$, satisfy the four relations defining Milnor–Witt K-theory of the field $F[T]/P$. Conditions (3) and (4) are obvious. The Steinberg relation is also clear. It remains to prove relation (3), i.e., that

$$\epsilon[G_1G_2] = \epsilon[G_1] + \epsilon[G_2] + \eta \cdot \epsilon[G_1] \cdot \epsilon[G_2].$$

Note that, by construction, $\epsilon[G_1G_2] = \epsilon[G]$, where G is the remainder of the division of G_1G_2 by P. (Indeed, this is how the multiplication in the field $F[T]/P$ is transported via the isomorphism $F[T]_{\leqslant d-1} \simeq F[T]/P$.) By Proposition 4.3.2, we have in $\overline{K}_*^P$:

$$\begin{aligned}\epsilon[G](\eta^m[P][H_1] \cdots [H_n]) &= \eta^m[P][G][H_1] \cdots [H_n] \\ &= \eta^m[P][G_1G_2][H_1] \cdots [H_n] \\ &= \eta^m[P]([G_1] + [G_2] + \eta[G_1][G_2])[H_1] \cdots [H_n] \\ &= (\epsilon[G_1] + \epsilon[G_2] + \eta\epsilon[G_1][G_2])(\eta^m[P][H_1] \cdots [H_n]).\end{aligned}$$

This proves our claim. We thus have a graded ring homomorphism

$$\mathrm{K}_*^{\mathrm{MW}}(F[T]/P) \to \mathcal{E}_*^P.$$

Evaluating at the symbol $[P]$, we get a graded homomorphism

$$\mathrm{K}^{\mathrm{MW}}_{*-1}(F[T]/P) \to \overline{K}^P_*$$

sending a symbol $\eta^m[G_1]\cdots[G_n]$, with $\deg(G_i) \leqslant d-1$, to the class of $\eta^m[P][G_1]\cdots[G_n]$. In particular, we see that this map is surjective. On the other hand, the residue homomorphism $\partial^P_{x_P}$, associated to the P-adic valuation x_P and the uniformizer P, provides a map

$$\overline{K}^P_* \to \mathrm{K}^{\mathrm{MW}}_{*-1}(F[T]/P)$$

which is also surjective and is a retraction of the previous map. Thus, these two maps are isomorphisms, inverse of each other.

In fact, more is true. By Proposition 4.3.2, the map

$$\bigoplus_{P\,\text{irred.},\ \deg(P)=d} \mathrm{K}^{\mathrm{MW}}_{*-1}(F[T]/P) \to L^{(d)}_*/L^{(d-1)}_*$$

is surjective and admits a surjective retraction given by

$$L^{(d)}_*/L^{(d-1)}_* \xrightarrow{(\partial^P_{x_P})_P} \bigoplus_{P\,\text{irred.},\ \deg(P)=d} \mathrm{K}^{\mathrm{MW}}_{*-1}(F[T]/P).$$

Thus, both maps are isomorphisms, inverse of each other. Since $\mathrm{K}^{\mathrm{MW}}_*(F(T))/L^{(0)}_*$ is a successive extensions of $L^{(d)}_*/L^{(d-1)}_*$, this proves that

$$\mathrm{K}^{\mathrm{MW}}_*(F(T))/L^{(0)}_* \xrightarrow{(\partial^{P_x}_x)_x} \bigoplus_{x\in(\mathbb{A}^1_F)^{(1)}} \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(x))$$

is an isomorphism, and finishes the proof of the theorem. □

The previous theorem can be used to define transfers (aka., norm homomorphisms) on Milnor–Witt K-theory. For Milnor K-theory, this is done in [BT73, Chapter I, §5] and [Kat80, Section 1.7]. We follow closely F. Morel's general treatment in [Mor12, §4.2].

Construction 4.3.4. Let E/F be a finite extension generated by an element $u \in E$. We identify u with the closed point of $\mathbb{A}^1_F$ corresponding to the surjection $F[T] \twoheadrightarrow E, T \mapsto u$. Consider the valuation v_∞ on $F(T)$ with ring of integers $O_{v_\infty} = F[T^{-1}]_{(T^{-1})}$, residue field $k_{v_\infty} \simeq F$ and uniformizer $-T^{-1}$. The residue homomorphism

$$\partial^{-T^{-1}}_\infty : \mathrm{K}^{\mathrm{MW}}_*(F(T)) \to \mathrm{K}^{\mathrm{MW}}_{*-1}(T)$$

is zero on $\mathrm{K}^{\mathrm{MW}}_*(F)$. By Theorem 4.3.1, there is an induced map

$$\bigoplus_{x\in(\mathbb{A}^1_F)^{(1)}} \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(x)) \xrightarrow{"-\partial^{-T^{-1}}_\infty"} \mathrm{K}^{\mathrm{MW}}_{*-1}(F).$$

Restricting to the direct summand corresponding to the point u, we deduce a graded homomorphism

$$\mathrm{N}^u_{E/F} : \mathrm{K}^{\mathrm{MW}}_*(E) \to \mathrm{K}^{\mathrm{MW}}_*(F)$$

called the transfers (aka., the norm homomorphism).

We will study transfers in details. We first note the following simple but very useful fact.

Theorem 4.3.5 (Reciprocity for $\mathbb{P}^1$)**.** *For $x \in \mathbb{P}^1_F \setminus \infty$ a closed point, denote by P_x the monic polynomial generating the ideal of x in $F[T]$. Also, set $P_\infty := -T^{-1}$. Then, for every $\alpha \in \mathrm{K}^{\mathrm{MW}}_*(F(T))$, we have the identity*

$$\sum_{x \in (\mathbb{P}^1_F)^{(1)}} \mathrm{N}^x_{\kappa(x)/F} \circ \partial^{P_x}_x(\alpha) = 0,$$

where, by convention, $\mathrm{N}^\infty_{\kappa(\infty)/F}$ is the obvious isomorphism $\mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(\infty)) \simeq \mathrm{K}^{\mathrm{MW}}_{*-1}(F)$.

Proof. By Theorem 4.3.1, we may assume that there exists $x_0 \in (\mathbb{A}^1_F)^{(1)}$ such that $\partial^{P_x}_x(\alpha) = 0$ for all $x \in (\mathbb{A}^1_F)^{(1)}$ with $x \neq x_0$. Set $\beta = \partial^{P_{x_0}}_{x_0}(\alpha)$. Then the class of α modulo $\mathrm{K}^{\mathrm{MW}}_*(F)$ is the image of β by the composite map

$$\mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(x_0)) \hookrightarrow \bigoplus_{x \in (\mathbb{A}^1_F)^{(1)}} \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(x)) \simeq \mathrm{K}^{\mathrm{MW}}_*(F(T))/\mathrm{K}^{\mathrm{MW}}_*(F).$$

Thus, by construction, we have

$$\mathrm{N}^{x_0}_{\kappa(x_0)/F}(\beta) = -\partial^{-T^{-1}}_\infty(\alpha).$$

This can be rewritten as

$$\mathrm{N}^{x_0}_{\kappa(x_0)/F} \circ \partial^{P_{x_0}}_{x_0}(\alpha) + \partial^{-T^{-1}}_\infty(\alpha) = 0$$

as needed. □

Lemma 4.3.6. *Let E/F be a finite extension generated by an element $u \in E$. Then $\mathrm{N}^u_{E/F}$ is a morphism of right $\mathrm{K}^{\mathrm{MW}}_*(F)$-modules. More explicitly, for $\alpha \in \mathrm{K}^{\mathrm{MW}}_*(E)$ and $\beta \in \mathrm{K}^{\mathrm{MW}}_*(F)$, we have*

$$\mathrm{N}^u_{E/F}(\alpha \cdot \beta|_E) = \mathrm{N}^u_{E/F}(\alpha) \cdot \beta.$$

Proof. Indeed, by Corollary 4.2.8, the short exact sequence

$$0 \to \mathrm{K}^{\mathrm{MW}}_*(F) \to \mathrm{K}^{\mathrm{MW}}_*(F(T)) \to \bigoplus_{x \in (\mathbb{A}^1_F)^{(1)}} \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(x)) \to 0$$

is a sequence of right $\mathrm{K}^{\mathrm{MW}}_*(F)$-modules. Similarly, the homomorphism $-\partial^{-T^{-1}}_\infty$ is a morphism of right $\mathrm{K}^{\mathrm{MW}}_*(F)$-modules. □

Corollary 4.3.7. *The composition of*

$$\mathrm{K}^{\mathrm{MW}}_*(F) \to \mathrm{K}^{\mathrm{MW}}_*(E) \xrightarrow{\mathrm{N}^u_{E/F}} \mathrm{K}^{\mathrm{MW}}_*(F)$$

is given by multiplication by d_ϵ where $d = [E : F]$ is the degree of the extension E/F.

Proof. By Lemma 4.3.6, it is enough to show that $\mathrm{N}^u_{E/F}(1) = d_\epsilon$. To compute $\mathrm{N}^u_{E/F}(1)$, we use the reciprocity law (i.e., Theorem 4.3.5) applied to the symbol $[P_u] \in \mathrm{K}^{\mathrm{MW}}_1(F(T))$. For $x \in (\mathbb{A}^1_F)^{(1)}$, with $x \neq u$, we have $P_u \in \mathcal{O}^\times_x$ and thus

$$\partial^{P_x}_x(P_u) = 0.$$

On the other hand $\partial^{P_u}_u(P_u) = 1$. Thus we get from Theorem 4.3.5 the identity:

$$\mathrm{N}^u_{E/F}(1) + \partial^{-T^{-1}}_\infty(P_u) = 0.$$

In order to compute $\partial_\infty^{-T^{-1}}(P_u)$, we write $P_u = T^d \cdot Q(T^{-1})$ where Q is the reverse polynomial of P_u. Since P_u is monic, the constant term of Q is 1. Said differently, $Q(T^{-1}) \in \mathcal{O}_\infty^\times$ and its residue in $\kappa(\infty) = F$ is 1. It follows that

$$\begin{aligned}\partial_\infty^{-T^{-1}}[P_u] &= \langle -1\rangle \partial_\infty^{T^{-1}}[(T^{-1})^{-d}\cdot Q(T^{-1})]\\ &= \langle -1\rangle \partial_\infty^{T^{-1}}((-d)_\epsilon[T^{-1}]\langle Q(T^{-1})\rangle + [Q(T^{-1})])\\ &= \langle -1\rangle(-d)_\epsilon\langle Q(0)\rangle + [Q(0)]\\ &= -d_\epsilon.\end{aligned}$$

This proves that $\mathrm{N}^u_{E/F}(1) = d_\epsilon$ as needed. □

Corollary 4.3.8. *Let E_1/F and E_2/F be two finite extensions of coprime degrees. Then the map*

$$\mathrm{K}_*^{\mathrm{MW}}(F) \to \mathrm{K}_*^{\mathrm{MW}}(E_1) \times \mathrm{K}_*^{\mathrm{MW}}(E_2)$$

has trivial kernel.

Proof. It is enough to show that, for $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(F)$, the relations $n_\epsilon \cdot \alpha = 0$ and $m_\epsilon \cdot \alpha = 0$, with $\gcd(n,m) = 1$, imply that $\alpha = 0$. To do so, it is enough to show that the subgroup $A \subset \mathrm{K}_0^{\mathrm{MW}}(F)$ spanned by n_ϵ, $n_\epsilon\langle -1\rangle$, m_ϵ and $m_\epsilon\langle -1\rangle$ contains 1. Note that

$$n_\epsilon + n_\epsilon\langle -1\rangle = n(1+\langle -1\rangle) \qquad \text{and} \qquad m_\epsilon + m_\epsilon\langle -1\rangle = m(1+\langle -1\rangle).$$

Since $\gcd(m,n) = 1$, we deduce that $2_\epsilon = 1 + \langle -1\rangle$ belongs to A. But, assuming that n is odd, the element 1 clearly belongs to the subgroup generated by 2_ϵ and n_ϵ. □

We next describe the compatibility between the norm and the restriction homomorphisms.

Proposition 4.3.9. *Let E/F be a finite extension and let F'/F be any extension. Write*

$$E \otimes_F F' = E_1'' \times \cdots \times E_r''$$

where $E_1'', \ldots, E_r''$ are local artinian with residue fields $E_1', \ldots, E_r'$ and lengths $e_1, \ldots, e_r$. Assume that E/F is generated by an element $u \in E$, and let $u_1, \ldots, u_r$ be the images of u in $E_1', \ldots, E_r'$. Then, for $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(E)$, we have:

$$\mathrm{N}^u_{E/F}(\alpha)|_{F'} = \sum_{i=1}^r (e_i)_\epsilon \cdot \mathrm{N}^{u_i}_{E_i'/F'}\left(\left\langle \frac{P_u}{P_{u_i}^{e_i}}(u_i)\right\rangle \cdot \alpha|_{E_i'}\right).$$

Proof. Note that $P_u = P_{u_1}^{e_1} \cdots P_{u_r}^{e_r}$. By Corollary 4.2.10, there is a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_*^{\mathrm{MW}}(F) \arrow[r] \arrow[d] & \mathrm{K}_*^{\mathrm{MW}}(F(T)) \arrow[r, "(\partial_x^{P_x})_x"] \arrow[d] & \displaystyle\bigoplus_{x \in (\mathbb{A}^1_F)^{(1)}} \mathrm{K}_{*-1}^{\mathrm{MW}}(\kappa(x)) \arrow[r] \arrow[d, "(\star)"] & 0 \\
0 \arrow[r] & \mathrm{K}_*^{\mathrm{MW}}(F') \arrow[r] & \mathrm{K}_*^{\mathrm{MW}}(F'(T)) \arrow[r, "(\partial_{x'}^{P_{x'}})_{x'}"] & \displaystyle\bigoplus_{x' \in (\mathbb{A}^1_{F'})^{(1)}} \mathrm{K}_{*-1}^{\mathrm{MW}}(\kappa(x')) \arrow[r] & 0.
\end{tikzcd}$$

The map $(\star)$ is given by a matrix $(\mu_{x',x})_{x',x}$, such that $\mu_{x',x} = 0$ unless x' is mapped to x in which case we have

$$\mu_{x',x} = e_\epsilon \left\langle \frac{P_x}{P_{x'}^e}(x')\right\rangle \qquad \in \qquad \mathrm{GW}(\kappa(x'))$$

where e is the length of the local ring $\mathcal{O}_{x'} \otimes_{\mathcal{O}_x} \kappa(x)$. It follows that $-\partial_\infty^{-T^{-1}}$ induces a commutative square

$$\begin{tikzcd}
\bigoplus_{x\in(\mathbb{A}^1_F)^{(1)}} \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(x)) \arrow[r, "(\mathrm{N}^{x}_{\kappa(x)/F})_x"] \arrow[d, "(\star)"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(F) \arrow[d] \\
\bigoplus_{x'\in(\mathbb{A}^1_{F'})^{(1)}} \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(x')) \arrow[r, "(\mathrm{N}^{x'}_{\kappa(x')/F'})_{x'}"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(F')
\end{tikzcd}$$

where the right-hand vertical arrow is the obvious restriction map. This proves the formula in the statement. □

As an application of Proposition 4.3.9, we record the following formula which we use in the proof of Theorem 4.5.13.

Corollary 4.3.10. *Let $q \geqslant 1$ be an odd integer which is coprime to the characteristic of F, and assume that F contains a primitive q-th root of unity. Then, in $\mathrm{K}^{\mathrm{MW}}_0(F)$, we have the identity $q_\epsilon = q\langle q\rangle$.*

Proof. Fix a primitive q-th root of unity ζ. Let ρ and π be indeterminates such that $\rho^q = \pi$ and consider the finite Galois extension $F(\rho)/F(\pi)$ with Galois group generated by the automorphism σ sending ρ to $\zeta\rho$. We apply Proposition 4.3.9 with "E/F" and "F'/F" both equal to $F(\rho)/F(\pi)$. Note that we have a pushout square of commutative rings

$$\begin{tikzcd}
F(\rho) \arrow[r, "(\sigma^{\circ i})_i"] & \prod_{i=0}^{q-1} F(\rho) \\
F(\pi) \arrow[r] \arrow[u] & F(\rho) \arrow[u, "\Delta"']
\end{tikzcd}$$

where Δ is the diagonal embedding. Thus, for $\beta \in \mathrm{K}^{\mathrm{MW}}_*(F(\rho))$, we have

$$\mathrm{N}^{\rho}_{F(\rho)/F(\pi)}(\beta)|_{F(\rho)} = \sum_{i=0}^{q-1} \mathrm{N}^{\zeta^i\rho}_{F(\rho)/F(\rho)}\left(\left\langle \frac{T^q-\pi}{T-\zeta^i\rho}(\zeta^i\rho)\right\rangle \cdot \sigma^i(\beta)\right).$$

By Corollary 4.3.7, we have $\mathrm{N}^{\zeta^i\rho}_{E/E} = \mathrm{id}$. Moreover, we have

$$\begin{aligned}
\frac{T^q-\pi}{T-\zeta^i\rho}(\zeta^i\rho) &= \left(\prod_{0\leqslant j\leqslant q-1,\ j\neq i} (\zeta^i-\zeta^j)\right)\cdot \rho^{q-1} \\
&= \zeta^{-i}\left(\prod_{1\leqslant j\leqslant q-1} (1-\zeta^j)\right)\cdot \rho^{q-1} \\
&= (-1)^{q-1}\zeta^{-i}q\rho^{q-1}.
\end{aligned}$$

Since q is odd, ζ is a square. Since $q-1$ is even, we have $\langle(-1)^{q-1}\zeta^{-i}q\rho^{q-1}\rangle = \langle q\rangle$. Thus, we have shown the identity

$$\mathrm{N}^{\rho}_{F(\rho)/F(\pi)}(\beta)|_{F(\rho)} = \sum_{i=0}^{q-1}\langle q\rangle\sigma^i(\beta).$$

The result follows by taking $\beta = 1$ and using Corollary 4.3.7. $\square$

Remark 4.3.11. The method of proof of Corollary 4.3.10 yields also interesting information even when F is not assumed to contain a primitive root of unity. For simplicity, we explain this when q is an odd prime different from the characteristic of F. We are thus assuming that F does not contain a primitive q-th root of unity, i.e., that $F(\zeta)/F$ is an extension of degree $q-1$. Note that we have a pushout square

$$\begin{tikzcd}
F(\rho) \arrow[r, "{(\mathrm{id},\rho\mapsto\zeta\rho)}"] & F(\rho)\times F(\zeta,\rho) \\
F(\pi) \arrow[u] \arrow[r] & F(\rho). \arrow[u, "{(\mathrm{id},\rho\mapsto\rho)}"']
\end{tikzcd}$$

Applying Proposition 4.3.9, we get

$$q_\epsilon = \mathrm{N}^{\rho}_{F(\rho)/F(\pi)}(1)|_{F(\rho)} = \left\langle \frac{T^q-\pi}{T-\rho}(\rho)\right\rangle + \mathrm{N}^{\zeta\rho}_{F(\zeta,\rho)/F(\rho)}\left\langle \frac{T^q-\pi}{P_{\zeta\rho}}(\zeta\rho)\right\rangle.$$

Note that $T^q - \pi = (T-\rho)P_{\zeta\rho}$. Thus, we obtain

$$q_\epsilon = \langle q\rangle + \mathrm{N}^{\zeta\rho}_{F(\zeta,\rho)/F(\rho)}(\langle(1-\zeta)\rho\rangle) = \langle q\rangle + \langle\rho\rangle\mathrm{N}^{\zeta\rho}_{F(\zeta,\rho)/F(\rho)}(\langle 1-\zeta\rangle).$$

Now, by Proposition 4.4.2 below, we have $\mathrm{N}^{\zeta\rho}_{F(\zeta,\rho)/F(\rho)} = \langle\rho\rangle^{d-2}\mathrm{N}^{\zeta}_{F(\zeta,\rho)/F(\rho)}$. Thus, at the end we obtain the formula

$$q_\epsilon = \langle q\rangle + \mathrm{N}^{\zeta}_{F(\zeta)/F}(\langle 1-\zeta\rangle).$$

(Here, as in the proof of Corollary 4.3.10, we are using that $\mathrm{K}^{\mathrm{MW}}_*(F) \to \mathrm{K}^{\mathrm{MW}}_*(F(T))$ is injective.)

Following [Mor12, Definition 4.26], we introduce a normalised version of the norm homomorphism. This will allow for a nicer reformulation of the previous proposition.

Definition 4.3.12. Let E/F be a finite extension generated by an element $u \in E$.

(1) Assume that E/F is separable, so that $P'_u \neq 0$. For $\alpha \in \mathrm{K}^{\mathrm{MW}}_*(E)$, we set

$$\widetilde{\mathrm{N}}^u_{E/F}(\alpha) = \mathrm{N}^u_{E/F}\left(\alpha\cdot\langle P'_u(u)\rangle\right).$$

(2) Assume that E/F is non-separable and that F has characteristic $p \neq 2$. Then, we can write uniquely $P_u = R_u(T^{p^m})$ with $R'_u \neq 0$. For $\alpha \in \mathrm{K}^{\mathrm{MW}}_*(E)$, we set

$$\widetilde{\mathrm{N}}^u_{E/F}(\alpha) = \mathrm{N}^u_{E/F}\left(\alpha\cdot\langle R'_u(u^{p^m})\rangle\right).$$

The map $\widetilde{\mathrm{N}}^u_{E/F}$ will be called the normalised transfers or the normalised norm homomorphism.

Corollary 4.3.13. *Keep the notations and assumptions as in Proposition 4.3.9. Assume furthermore that E/F is separable or that the characteristic of F is $\neq 2$. Then, for $\alpha \in \mathrm{K}^{\mathrm{MW}}_*(E)$, we have*

$$\widetilde{\mathrm{N}}^u_{E/F}(\alpha)|_{F'} = \sum_{i=1}^r (e_i)_\epsilon \cdot \widetilde{\mathrm{N}}^{u_i}_{E'_i/F'}(\alpha|_{E'_i}).$$

Proof. We first prove the corollary assuming that E/F is separable. In this case, $e_i = 1$ for $1 \leqslant i \leqslant r$ and, by Proposition 4.3.9, we have

$$\widetilde{\mathrm{N}}^u_{E/F}(\alpha)|_{F'} = \sum_{i=1}^r \mathrm{N}^{u_i}_{E'_i/F'}(\alpha|_{E'_i}\langle Q_i(u_i)\rangle\langle P'_u(u_i)\rangle)$$

where $P_u = Q_i P_{u_i}$. Thus, it suffices to show that $\langle Q_i(u_i)\rangle\langle P'_u(u_i)\rangle = \langle P'_{u_i}(u_i)\rangle$ or, equivalently, that

$$\langle P'_u(u_i)\rangle = \langle Q_i(u_i)\rangle\langle P'_{u_i}(u_i)\rangle.$$

But $P'_u = Q_i P'_{u_i} + Q'_i P_{u_i}$, so $P'_u(u_i) = Q_i(u_i)P'_{u_i}(u_i)$ as needed.

Now, assume that E/F is non-separable and let $p > 2$ be the characteristic of F. Write $P_u = R_u(T^{p^m})$ with $R'_u \neq 0$. Similarly, for $1 \leqslant i \leqslant r$, write $P_{u_i} = R_{u_i}(T^{p^{m_i}})$ with $R'_{u_i} \neq 0$. Since $P_u = P_{u_1}^{e_1} \cdots P_{u_r}^{e_r}$, we see that $e_i = p^{m-m_i}$ and that $R_u = \widetilde{R}_{u_1} \cdots \widetilde{R}_{u_r}$ where $\widetilde{R}_{u_i}$ is a polynomial characterised by the property that $\widetilde{R}_{u_i}(T^{p^{m-m_i}}) = R_{u_i}(T)^{p^{m-m_i}}$. By Proposition 4.3.9, we have

$$\widetilde{\mathrm{N}}^u_{E/F}(\alpha)|_{F'} = \sum_{i=1}^{r} (e_i)_\epsilon \cdot \mathrm{N}^{u_i}_{E'_i/F'}(\alpha|_{E'_i}\langle Q_i(u_i)\rangle\langle R'_u(u_i^{p^m})\rangle)$$

where $P_u = Q_i P_{u_i}^{e_i}$. As before, we need to show that

$$\langle R'_u(u_i^{p^m})\rangle = \langle Q_i(u_i)\rangle\langle R'_{u_i}(u_i^{p^{m_i}})\rangle.$$

Write $R_u = \widetilde{Q}_i\widetilde{R}_{u_i}$ so that $R'_u = \widetilde{Q}_i\widetilde{R}'_{u_i} + \widetilde{Q}'_i\widetilde{R}_{u_i}$. Note that

$$\widetilde{R}_{u_i}(u_i^{p^m}) = \widetilde{R}_{u_i}((u_i^{p^{m_i}})^{p^{m-m_i}}) = R_{u_i}(u_i^{p^{m_i}})^{p^{m-m_i}} = P_{u_i}(u_i)^{p^{m-m_i}} = 0.$$

It follows that

$$R'_u(u_i^{p^m}) = \widetilde{Q}_i(u_i^{p^m})\widetilde{R}'_{u_i}(u_i^{p^m}).$$

But we also have $\widetilde{R}'_{u_i}(T^{p^{m-m_i}}) = R'_{u_i}(T)^{p^{m-m_i}}$ which gives as before

$$\widetilde{R}'_{u_i}(u_i^{p^m}) = \widetilde{R}'_{u_i}((u_i^{p^{m_i}})^{p^{m-m_i}}) = R'_{u_i}(u_i^{p^{m_i}})^{p^{m-m_i}}.$$

As p is odd, we deduce that

$$\langle R'_u(u_i^{p^m})\rangle = \langle \widetilde{Q}_i(u_i^{p^m})\rangle\langle R'_{u_i}(u_i^{p^{m_i}})\rangle.$$

Thus, it remains to see that $\langle Q_i(u_i)\rangle = \langle \widetilde{Q}_i(u_i^{p^m})\rangle$. For this, it suffices to show that for $j \neq i$, we have

$$\langle P_{u_j}(u_i)^{p^{m-m_j}}\rangle = \langle \widetilde{R}_{u_j}(u_i^{p^m})\rangle.$$

But, as before, we have

$$\widetilde{R}_{u_j}(u_i^{p^m}) = \widetilde{R}_{u_j}((u_i^{p^{m_j}})^{p^{m-m_j}}) = R_{u_j}(u_i^{p^{m_j}})^{p^{m-m_j}} = P_{u_j}(u_i)^{p^{m-m_j}}.$$

This finishes the proof of the corollary. □

Remark 4.3.14. Definition 4.3.12 for non-separable extensions makes perfect sense when the characteristic of F is 2. However, in the proof of Corollary 4.3.13, one needs the characteristic of F to be odd. This suggests that a different definition for the normalised transfers is needed in characteristic 2. Such a definition can be found in [Mor12, §5.1] and [Fel23]. We have decided not to include the characteristic 2 case in these notes.

4.4. **Computing norms in Milnor–Witt K-theory.**

In this subsection, we present some techniques for computing the norm homomorphism in Milnor–Witt K-theory. As usual, F will denote a field throughout this subsection.

Lemma 4.4.1. *Let E/F be a finite extension generated by an element $u \in E$. Denote by $\mathrm{Nm}_{E/F} : E^\times \to F^\times$ the norm homomorphism. Then, we have*

$$\mathrm{N}^u_{E/F}([-u]) = [\mathrm{Nm}_{E/F}(-u)] = [P_u(0)].$$

Proof. Consider the symbol $[T][P_u]$ in $\mathrm{K}_2^{\mathrm{MW}}(F(T))$. We want to apply the reciprocity formula to this symbol, see Theorem 4.3.5. For this, we note the following:

- if $x \in (\mathbb{A}^1_F)^{(1)} \smallsetminus \{0, u\}$, then T and P_u belong to $\mathcal{O}_x^\times$ and we have $\partial_x^{P_x}([T][P_u]) = 0$;
- $\partial_0^T([T][P_u]) = [P_u(0)]$;
- $\partial_u^{P_u}([T][P_u]) = -\langle -1\rangle [u]$.

On the other hand, $\partial_\infty^{-T^{-1}}([T][P_u])$ can be computed as follows. Let $d = [E : F]$ be the degree of P_u and write $P_u = T^d Q_u(T^{-1})$ where Q_u is the reverse polynomial of P_u. Then, we have:

$$\begin{aligned}
[P_u] &= [T^d Q_u(T^{-1})] \\
&= [T^d]\langle Q_u(T^{-1})\rangle + [Q_u(T^{-1})] \\
&= (-d)_\epsilon [T^{-1}]\langle Q_u(T^{-1})\rangle + [Q_u(T^{-1})].
\end{aligned}$$

This gives

$$\begin{aligned}
[T][P_u] &= (-1)_\epsilon [T^{-1}][P_u] \\
&= d_\epsilon [T^{-1}][-1]\langle Q_u(T^{-1})\rangle + (-1)_\epsilon [T^{-1}][Q_u(T^{-1})].
\end{aligned}$$

It follows that $\partial_\infty^{-T^{-1}}([T][P_u]) = \langle -1\rangle d_\epsilon [-1] = -d_\epsilon [-1]$. Putting the previous observations together, we obtain:

$$[P_u(0)] - \langle -1\rangle \mathrm{N}^u_{E/F}([u]) - d_\epsilon [-1] = 0.$$

Now, by Corollary 4.3.7, we have $\mathrm{N}^u_{E/F}([-1]) = d_\epsilon [-1]$. Thus we may rewrite the previous relation as:

$$\begin{aligned}
0 &= [P_u(0)] - \langle -1\rangle \mathrm{N}^u_{E/F}([u]) - \mathrm{N}^u_{E/F}([-1]) \\
&= [P_u(0)] - \mathrm{N}^u_{E/F}([u]\langle -1\rangle + [-1]) \\
&= [P_u(0)] - \mathrm{N}^u_{E/F}([-u]).
\end{aligned}$$

This finishes the proof of the lemma. □

Proposition 4.4.2. *Let E/F be a finite extension of degree d generated by an element $u \in E$. Then, for every $a \in F^\times$ and $b \in F$, we have*

$$\mathrm{N}^{au+b}_{E/F} = \langle a\rangle^{d-1} \mathrm{N}^u_{E/F}.$$

Proof. Consider the ring homomorphism

$$\tau^* : F[T] \xrightarrow{\sim} F[T], \quad T \mapsto aT + b$$

and let $\tau : \mathbb{A}^1_F \xrightarrow{\sim} \mathbb{A}^1_F$ be the induced automorphism of $\mathbb{A}^1_F$. Note that $\tau(u) = au + b$ and that we have a commutative triangle

$$\begin{tikzcd}
F[T] \arrow[r, "T\mapsto u"] & E. \\
F[T] \arrow[u, "\tau^*"] \arrow[ur, "T\mapsto au+b"'] &
\end{tikzcd}$$

Also, for $x \in (\mathbb{A}^1_F)^{(1)}$, we have $\tau(x) = ax + b$. Moreover,

$$\tau^* P_{ax+b} = P_{ax+b}(aT + b) = a^{\deg(x)} P_x(T).$$

Thus, we have commutative squares

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[r, "\partial_x^{P_x}"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(x)) \\
\mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[u, "\sim", "\tau^*"'] \arrow[r, "\partial_{ax+b}^{P_{ax+b}}"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(ax+b)), \arrow[u, "\sim", "\langle a\rangle^{\deg(x)}"']
\end{tikzcd}$$

yielding a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}^{\mathrm{MW}}_*(F) \arrow[r] \arrow[d, equal] & \mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[r, "(\partial_x^{P_x})_x"] & \bigoplus\limits_{x\in(\mathbb{A}^1_F)^{(1)}} \mathrm{K}^{\mathrm{MW}}_*(\kappa(x)) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{K}^{\mathrm{MW}}_*(F) \arrow[r] & \mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[u, "\sim", "\tau^*"'] \arrow[r, "(\partial_{ax+b}^{P_{ax+b}})_x"] & \bigoplus\limits_{x\in(\mathbb{A}^1_F)^{(1)}} \mathrm{K}^{\mathrm{MW}}_*(\kappa(ax+b)) \arrow[u, "\sim", "(\langle a\rangle^{\deg(x)})_x"'] \arrow[r] & 0.
\end{tikzcd}$$

On the other hand, τ extends to an isomorphism of $\mathbb{P}^1_F$ sending ∞ to ∞ and we have a commutative square

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[r, "-\partial_\infty^{-(aT+b)^{-1}}"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(F) \\
\mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[u, "\sim", "\tau^*"'] \arrow[r, "-\partial_\infty^{-T^{-1}}"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(F). \arrow[u, equal]
\end{tikzcd}$$

Since

$$(aT+b)^{-1} = T^{-1}\cdot\left(\frac{T}{aT+b}\right) = T^{-1}\left(\frac{1}{a+bT^{-1}}\right),$$

we see that $\partial_\infty^{-(aT+b)^{-1}} = \langle a\rangle\cdot\partial_\infty^{-T^{-1}}$ using Corollary 4.2.10. Thus, we also have the following commutative square

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[r, "-\partial_\infty^{-T^{-1}}"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(F) \\
\mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[u, "\sim", "\tau^*"'] \arrow[r, "-\partial_\infty^{-T^{-1}}"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(F). \arrow[u, "\sim", "\langle a\rangle"']
\end{tikzcd}$$

Applying Construction 4.3.4, we obtain a commutative square

$$\begin{tikzcd}
\bigoplus\limits_{x\in(\mathbb{A}^1_F)^{(1)}} \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(x)) \arrow[r, "(\mathrm{N}^x_{\kappa(x)/F})_x"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(F) \\
\bigoplus\limits_{x\in(\mathbb{A}^1_F)^{(1)}} \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(ax+b)) \arrow[u, "(\langle a\rangle^{\deg(x)})_x"'] \arrow[r, "(\mathrm{N}^{ax+b}_{\kappa(ax+b)/F})_x"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(F). \arrow[u, "\langle a\rangle"']
\end{tikzcd}$$

This gives the desired formula. □

Corollary 4.4.3. *Let E/F be a finite extension of degree d generated by an element $u \in E$. Then, for every $a \in F^\times$ and $b \in F$, we have*

$$\mathrm{N}^u_{E/F}([au+b]) = \langle -a\rangle^{d-1} \cdot [\mathrm{Nm}_{E/F}(au+b)].$$

Proof. Combining Lemma 4.4.1 and Proposition 4.4.2, we obtain

$$\begin{aligned} \mathrm{N}^u_{E/F}([au+b]) &= \langle -a\rangle^{d-1}\mathrm{N}^{-au-b}_{E/F}([au+b]) \\ &= \langle -a\rangle^{d-1}[\mathrm{Nm}_{E/F}(au+b)] \end{aligned}$$

as desired. □

The previous corollary can be used to compute the norm homomorphism in Milnor–Witt K-theory for any degree-two extension.

Proposition 4.4.4. *Let E/F be a finite extension of degree two, and let $u \in E\setminus F$. Then $\mathrm{K}^{\mathrm{MW}}_*(E)$ is generated by symbols of the form*

- $\eta^m[a_1]\cdots[a_n]$, *with* $a_1,\dots,a_n \in F^\times$,
- $\eta^m[au+b][a_1]\cdots[a_n]$, *with* $a, a_1, \cdots, a_n \in F^\times$ *and* $b \in F$.

Moreover, we have:

(1) $\mathrm{N}^u_{E/F}(\eta^m[a_1]\cdots[a_n]) = 2_\epsilon\eta^m[a_1]\cdots[a_n]$;

(2) $\mathrm{N}^u_{E/F}(\eta^m[au+b][a_1]\cdots[a_n]) = \langle -a\rangle\eta^m[\mathrm{Nm}_{E/F}(au+b)][a_1]\cdots[a_n]$.

Proof. Only the first assertion needs a proof. It follows from the Steinberg relation by a standard argument □

Proposition 4.4.4 explicitly shows the dependence of $\mathrm{N}^u_{E/F}$ on u in the case of quadratic extensions: we need the generator u to write a general element as $au+b$, with $a, b \in F$, so that we can twist the usual norm by $\langle a\rangle$.

Proposition 4.4.5. *Assume that the characteristic of F is different from 2 and let E/F be a quadratic extension. Then $\widetilde{\mathrm{N}}^u_{E/F}$ is independent of the choice of a generator u. In fact, for $a_1 \in E^\times\setminus F^\times$ and $a_2,\dots,a_n \in F^\times$, we have:*

$$\widetilde{\mathrm{N}}_{E/F}([a_1]\cdots[a_n]) = \widetilde{\mathrm{N}}_{E/F}([a_1])\cdot[a_2]\cdots[a_n]$$

where $\widetilde{\mathrm{N}}_{E/F}([a_1]) = [\mathrm{Nm}_{E/F}(a_1)]$ *if* $\mathrm{tr}_{E/F}(a_1) = 0$ *and is equal to*

$$(1+\langle -1\rangle - \langle -\mathrm{tr}_{E/F}(a_1)\rangle)[\mathrm{Nm}_{E/F}(a_1)] + (\langle -\mathrm{tr}_{E/F}(a_1)\rangle - \langle -2\rangle)[\mathrm{Nm}_{E/F}(a_1) - \tfrac{1}{4}\mathrm{tr}_{E/F}(a_1)^2]$$

if $\mathrm{tr}_{E/F}(a_1) \neq 0$.

Proof. Recall that $\widetilde{\mathrm{N}}^u_{E/F} = \mathrm{N}^u_{E/F}(\langle P'_u(u)\rangle\cdot -)$. Now, for $b \in F$, we have $P_{u+b}(T) = P_u(T-b)$ so that $P'_{u+b}(T) = P'_u(T-b)$ and $P'_{u+b}(u+b) = P'_u(u)$. Since $\mathrm{N}^{u+b}_{E/F} = \mathrm{N}^u_{E/F}$ (by Proposition 4.4.2), we see that $\widetilde{\mathrm{N}}^u_{E/F} = \widetilde{\mathrm{N}}^{u+b}_{E/F}$. Thus, we may assume that $u^2 \in F$ and we need to show that

$$\widetilde{\mathrm{N}}^{au}_{E/F} = \widetilde{\mathrm{N}}^u_{E/F}$$

for $a \in F^\times$. But $P_{au} = a^2P_u(a^{-1}T)$ so that $P'_{au} = aP'_u(a^{-1}T)$, and hence $P'_{au}(au) = aP'_u(u) = 2au$. Thus, we have

$$\begin{aligned} \widetilde{\mathrm{N}}^{au}_{E/F} &= \mathrm{N}^{au}_{E/F}(\langle 2au\rangle\cdot -) \\ &= \langle a\rangle\mathrm{N}^u_{E/F}(\langle 2au\rangle\cdot -) \\ &= \mathrm{N}^u_{E/F}(\langle 2u\rangle\cdot -) \\ &= \widetilde{\mathrm{N}}^u_{E/F}. \end{aligned}$$

For the second part of the statement, we need to compute $\widetilde{\mathrm{N}}_{E/F}$. Fix a generator $u \in E$ such that $u^2 \in F$. For $a \in F^\times$ and $b \in F$, we have:

$$\begin{aligned}
\widetilde{\mathrm{N}}_{E/F}([au+b]) &= \widetilde{\mathrm{N}}^{v}_{E/F}([v+b]) \qquad \text{with} \quad v = au \\
&= \mathrm{N}^{v}_{E/F}(\langle 2v\rangle \cdot [v+b]) \\
&= \mathrm{N}^{v}_{E/F}(\langle v\rangle [v+b])\langle 2\rangle \\
&= \mathrm{N}^{v}_{E/F}([v+b] + \eta[v][v+b])\langle 2\rangle \\
&= \langle -2\rangle[\mathrm{Nm}_{E/F}(v+b)] + \langle 2\rangle\eta\mathrm{N}^{v}_{E/F}([v][v+b]).
\end{aligned}$$

Note that $b = \mathrm{tr}_{E/F}(au+b)/2$ and, if $b = 0$, we obtain $\widetilde{\mathrm{N}}_{E/F}([v]) = [\mathrm{Nm}_{E/F}(v)]$ as needed in the statement. In the remainder of the proof, we assume that $b \neq 0$. By Lemma 4.4.6 below, we have the identity:

$$[v][v+b] = [v][b] + [-b][b+v].$$

This allows us to continue the computation as follows:

$$\begin{aligned}
&= \langle -2\rangle[\mathrm{Nm}_{E/F}(v+b)] + \langle 2\rangle\eta\langle -1\rangle[\mathrm{Nm}_{E/F}(v)][b] \\
&\quad +\langle 2\rangle\eta(-1)_{\epsilon}\langle -1\rangle[\mathrm{Nm}_{E/F}(v+b)][-b] \\
&= \langle -2\rangle[\mathrm{Nm}_{E/F}(v+b)] + \langle -2\rangle(\langle b\rangle - 1)[\mathrm{Nm}_{E/F}(v)] \\
&\quad -\langle 2\rangle(\langle -b\rangle - 1)[\mathrm{Nm}_{E/F}(v+b)] \\
&= (1 + \langle -1\rangle - \langle -2b\rangle)[\mathrm{Nm}_{E/F}(v+b)] + (\langle -2b\rangle - \langle -2\rangle)[\mathrm{Nm}_{E/F}(v)].
\end{aligned}$$

Now, we note that $b = \mathrm{tr}_{E/F}(v+b)/2$ and that

$$\mathrm{Nm}_{E/F}(v) = -v^2 = \mathrm{Nm}_{E/F}(v+b) - b^2 = \mathrm{Nm}_{E/F}(v+b) - \tfrac{1}{4}\mathrm{tr}_{E/F}(v+b)^2.$$

This finishes the proof. □

The following simple identity was used in the proof of Proposition 4.4.5.

Lemma 4.4.6. *For $a, b \in F^\times$, with $a + b \neq 0$, we have*

$$[a][a+b] = [a][b] + [-b][a+b]$$

Proof. Indeed, we compute as follows:

$$\begin{aligned}
[a][a+b] &= \left(\left[-\frac{a}{b}\right]\langle -b\rangle + [-b]\right)\left(\left[\frac{a}{b}+1\right]\langle b\rangle + [b]\right) \\
&= \left[-\frac{a}{b}\right][b]\langle -b\rangle + [-b]\left[\frac{a}{b}+1\right]\langle b\rangle \\
&= ([a]\langle -b^{-1}\rangle + [-b^{-1}])[b] + [-b]([a+b]\langle b^{-1}\rangle + [b^{-1}]) \\
&= [a][b] + [-b][a+b].
\end{aligned}$$

This proves the lemma. □

Corollary 4.4.7. *Let E/F be a quadratic extension. Let v be a nontrivial discrete valuation on F which is unramified in E. Assume that the characteristic of k_v is different from 2. Let π be a uniformizer of v and hence of every valuation w over v on E. (There are at most two such* *valuations!) Then the following square commutes*

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(E) \arrow[r, "(\partial^\pi_w)_w"] \arrow[d, "\widetilde{\mathrm{N}}_{E/F}"] & \bigoplus_{w|v} \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(w)) \arrow[d, "\sum_{w|v} \widetilde{\mathrm{N}}_{k_w/k_v}"] \\
\mathrm{K}^{\mathrm{MW}}_*(F) \arrow[r, "\partial^\pi_v"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(k_v).
\end{tikzcd}$$

Proof. When v splits in E, we may replace F with F^h_v (the fraction field of the henselisation of $\mathcal{O}_v$) and use the compatibilities of restrictions with residues and norms, provided by Corollaries 4.2.10 and 4.3.13, to reduce to the case where $E = F$. So it is enough to treat the case where v is inert, i.e., we may assume that there is a unique valuation w above v. In this case k_w/k_v is also a quadratic extension. Fix $u \in \mathcal{O}^\times_w \smallsetminus \mathcal{O}^\times_v$. Then u is a generator of E/F and $\bar{u}$ is a generator of k_w/k_v. We need to check that

$$\widetilde{\mathrm{N}}_{k_w/k_v} \circ \partial^\pi_w = \partial^\pi_v \circ \widetilde{\mathrm{N}}_{E/F}.$$

Equivalently, we need to check that

$$\mathrm{N}^{\bar{u}}_{k_w/k_v}(\langle P'_{\bar{u}}(\bar{u})\rangle \cdot \partial^\pi_w(-)) = \partial^\pi_v \circ \mathrm{N}^u_{E/F}(\langle P'_u(u)\rangle \cdot -).$$

Since $P'_u(u) \in \mathcal{O}^\times_w$, we have

$$\langle P'_{\bar{u}}(\bar{u})\rangle \cdot \partial^\pi_w(-) = \partial^\pi_w(\langle P'_u(u)\rangle \cdot -).$$

Thus, we are reduced to showing that

$$\mathrm{N}^{\bar{u}}_{k_w/k_v} \circ \partial^\pi_w = \partial^\pi_v \circ \mathrm{N}^u_{E/F}.$$

Now, it is easy to see that $\mathrm{K}^{\mathrm{MW}}_*(E)$ is generated by the following symbols (for $a_1, \cdots, a_n \in \mathcal{O}^\times_v$):

(1) $\eta^m[a_1]\cdots[a_n]$,
(2) $\eta^m[\pi][a_1]\cdots[a_n]$,
(3) $\eta^m[u][a_1]\cdots[a_n]$,
(4) $\eta^m[\pi][u][a_1]\cdots[a_n]$.

The desired equality is clear on the generators of the types (1) and (2). For generators of type (3), we need to show that

$$\partial^\pi_v \mathrm{N}^u_{E/F}(\eta^m[u][a_1]\cdots[a_n]) = 0.$$

This follows from Corollary 4.4.3 using that $\mathrm{Nm}_{E/F}(u) \in \mathcal{O}^\times_v$. For generators of type (4), we compute as follows:

$$\begin{aligned}
\partial^\pi_v \circ \mathrm{N}^u_{E/F}(\eta^m[\pi][u][a_1]\cdots[a_n]) &= \partial^\pi_v(-\eta^m[\mathrm{Nm}_{E/F}(u)][\pi][a_1]\cdots[a_n]) \\
&= \langle -1\rangle \eta^m[\mathrm{Nm}_{E/F}(\bar{u})][\bar{a}_1]\cdots[\bar{a}_n] \\
&= \mathrm{N}^{\bar{u}}_{k_w/k_v}(\eta^m[\bar{u}][\bar{a}_1]\cdots[\bar{a}_n]).
\end{aligned}$$

This finishes the proof. □

The next two results are particular cases of [Mor12, Theorem 4.27].

Proposition 4.4.8. *Assume that the characteristic of F is different from 2 and let E/F be a finite extension admitting a filtration*

$$F = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

such that E_i/E_{i-1} is quadratic for every $1 \leqslant i \leqslant r$. Then $\widetilde{\mathrm{N}}^u_{E/F}$ is independent of the choice of a generator u of E. Moreover, we have:

$$\widetilde{\mathrm{N}}_{E/F} = \widetilde{\mathrm{N}}_{E_1/E_0} \circ \cdots \circ \widetilde{\mathrm{N}}_{E_r/E_{r-1}}.$$

Proof. We argue by induction on $[E : F] = 2^r$. If E/F is a quadratic extension, the result has been already proven in Proposition 4.4.5. If $[E : F] \geqslant 4$, we fix a quadratic subextension H/F of E/F and a generator $u \in E$ of E/F (and hence also of E/H). Let $P_u \in F[T]$ and $Q_u \in H[T]$ be the minimal polynomials of u. Consider the finite étale morphism $\mathbb{A}^1_H \to \mathbb{A}^1_F$. By Corollary 4.4.7, we have a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}^{\mathrm{MW}}_*(H) \arrow[r] \arrow[d, "\widetilde{\mathrm{N}}_{H/F}"] & \mathrm{K}^{\mathrm{MW}}_*(H(T)) \arrow[r, "(\partial_y^{P_x})_{x,y|x}"] \arrow[d, "\widetilde{\mathrm{N}}_{H(T)/F(T)}"] & \bigoplus_{x \in (\mathbb{A}^1_F)^{(1)}} \bigoplus_{y|x} \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(y)) \arrow[r] \arrow[d, "(\sum_{y|x} \widetilde{\mathrm{N}}_{\kappa(y)/\kappa(x)})_x"] & 0 \\
0 \arrow[r] & \mathrm{K}^{\mathrm{MW}}_*(F) \arrow[r] & \mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[r, "(\partial_x^{P_x})_x"] & \bigoplus_{x \in (\mathbb{A}^1_F)^{(1)}} \mathrm{K}^{\mathrm{MW}}_{*-1}(\kappa(x)) \arrow[r] & 0.
\end{tikzcd}$$

Using Corollary 4.2.10 and contemplating Construction 4.3.4, we deduce the following commutative diagram

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_{*-1}(E) \arrow[r, "\langle \frac{P_u}{Q_u}(u) \rangle"] \arrow[d, equal] & \mathrm{K}^{\mathrm{MW}}_{*-1}(E) \arrow[r, "\mathrm{N}^u_{E/H}"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(H) \arrow[d, "\widetilde{\mathrm{N}}_{H/F}"] \\
\mathrm{K}^{\mathrm{MW}}_{*-1}(E) \arrow[rr, "\mathrm{N}^u_{E/F}"] & & \mathrm{K}^{\mathrm{MW}}_{*-1}(F).
\end{tikzcd}$$

Arguing as in the proof of Corollary 4.3.13 (for separable extensions), we can transform the above commutative diagram into the following commutative triangle:

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(E) \arrow[r, "\widetilde{\mathrm{N}}^u_{E/H}"] \arrow[dr, "\widetilde{\mathrm{N}}^u_{E/F}"'] & \mathrm{K}^{\mathrm{MW}}_*(H) \arrow[d, "\widetilde{\mathrm{N}}_{H/F}"] \\
& \mathrm{K}^{\mathrm{MW}}_*(F).
\end{tikzcd}$$

By induction, $\widetilde{\mathrm{N}}^u_{E/H}$ is independent of the choice of u. This proves that $\widetilde{\mathrm{N}}^u_{E/F}$ is independent of u, and gives the formula $\widetilde{\mathrm{N}}_{E/F} = \widetilde{\mathrm{N}}_{H/F} \circ \widetilde{\mathrm{N}}_{E/H}$. □

We are now ready to prove the main result of this subsection. (The analogous result for Milnor K-theory is due to Kato [Kat80, §1.7, Proposition 5].)

Theorem 4.4.9. *Assume that the characteristic of F is different from 2 and let E/F be a finite extension admitting a filtration*

$$F = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

such that E_i/E_{i-1} is generated by one element $u_i \in E_i$ for each $1 \leqslant i \leqslant r$. Then the composite morphism

$$\widetilde{\mathrm{N}}^{u_r}_{E_r/E_{r-1}} \circ \cdots \circ \widetilde{\mathrm{N}}^{u_1}_{E_1/E_0} : \mathrm{K}^{\mathrm{MW}}_*(E) \to \mathrm{K}^{\mathrm{MW}}_*(F)$$

depends only on the extension E/F, and we denote it by $\widetilde{\mathrm{N}}_{E/F}$. Moreover, given another finite extension D/E, we have:

$$\widetilde{\mathrm{N}}_{D/F} = \widetilde{\mathrm{N}}_{E/F} \circ \widetilde{\mathrm{N}}_{D/E}.$$

Proof. Let $\widetilde{N}'$ and $\widetilde{N}''$ be two norm homomorphisms $\mathrm{K}_*^{\mathrm{MW}}(E) \to \mathrm{K}_*^{\mathrm{MW}}(F)$ obtained by filtering E/F and choosing generators as in the statement. Our goal is to show that $\widetilde{N}' = \widetilde{N}''$. Fix $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(E)$ and assume by contradiction that $\widetilde{N}'(\alpha) \neq \widetilde{N}''(\alpha)$. Let $\beta = \widetilde{N}'(\alpha) - \widetilde{N}''(\alpha)$ be the difference. If G/F is a normal extension containing E/F (so that $(G \times_F E)_{\mathrm{red}}$ is a finite product of copies of G), Corollary 4.3.13 implies that $\widetilde{N}'(\alpha)|_G = \widetilde{N}''(\alpha)|_G$. In particular, by Corollary 4.3.13, we have $n_\epsilon \cdot \beta = 0$ where $n = [G : F]$. Now let $m \geqslant 1$ be the smallest positive integer such that $m_\epsilon \cdot \beta = 0$. We want to show that $m = 1$. Changing α if necessary, we may assume that α is chosen so that:

- $\beta = \widetilde{N}'(\alpha) - \widetilde{N}''(\alpha) \neq 0$;
- $m_\epsilon \cdot \beta = 0$, with $m \geqslant 1$ the smallest possible (among all the possible counterexamples to the statement).

If m is even, then $m_\epsilon = (\frac{m}{2}) \cdot 2_\epsilon$. Thus, replacing α with $(\frac{m}{2}) \cdot \alpha$, we see that $m = 2$. If m is odd, using that $p_\epsilon q_\epsilon = (pq)_\epsilon$ for p and q odd, we see that $m = p$ is an odd prime. In conclusion, we may assume that $p_\epsilon \cdot \beta = 0$, with p a prime number. Using Corollary 4.3.13, we may replace F with a pro-p'-extension, and assume that every finite extension of F has degree a power of p.

Case 1: $p = 2$. This was done in Proposition 4.4.8.

Case 2: $p > 2$. In this case, every element of F is a square. It follows that $\langle a \rangle = 1$, for every $a \in F^\times$. This implies the following:

- $\mathrm{K}_0^{\mathrm{MW}}(F) = \mathbb{Z}$;
- $\mathrm{K}_{-n}^{\mathrm{MW}} = \mathbb{Z}/2$, for $n \geqslant 1$;
- $\mathrm{K}_n^{\mathrm{MW}}(F) = \mathrm{K}_n^{\mathrm{M}}(F)$ for $n \geqslant 1$.

Thus, we can use the analogous statement for Milnor K-theory from [Kat80, §1.7, Proposition 5] to conclude. For the reader's convenience, we include a sketch of the argument in the case where E/F is separable. We first treat the case where $[E : F] = p$. Then, given $u \in E$ a generator, every element $v \in E^\times$ is a product of an element of $F^\times$ and elements of the form $u - a$, for $a \in F$. (Indeed, every polynomial in $F[T]$ of degree $\leqslant p - 1$ is a product of linear factors.) By a standard argument, it follows that every $\alpha \in \mathrm{K}_n^{\mathrm{M}}(E)$ is a linear combination of symbols of the form $\{v\}\{b_2\} \cdots \{b_n\}$, with $v \in E^\times$ and $b_2, \cdots, b_n \in F^\times$, and using Corollary 4.4.3, we have

$$\mathrm{N}_{E/F}^u(\{v\}\{b_2\} \cdots \{b_n\}) = \{\mathrm{Nm}_{E/F}(v)\}\{b_2\} \cdots \{b_n\}$$

which is independent of u as needed. This shows that $\mathrm{N}_{E/F}^u$ is independent of u.

If $[E : F] > p$, one argues by induction as in the proof of Proposition 4.4.8 using the following version of Corollary 4.4.7. □

Lemma 4.4.10. *Let E/F be a Galois extension of degree a prime number p. Let v be a nontrivial discrete valuation on F which is unramified in E. Assume that there is a unique valuation w on E extending v. Then the following square commutes*

$$\begin{tikzcd}
\mathrm{K}_*^{\mathrm{M}}(E) \arrow[r, "\partial_w"] \arrow[d, "\mathrm{N}"] & \mathrm{K}_{*-1}^{\mathrm{M}}(k_w) \arrow[d, "\mathrm{N}"] \\
\mathrm{K}_*^{\mathrm{M}}(F) \arrow[r, "\partial_v"] & \mathrm{K}_{*-1}^{\mathrm{M}}(k_v).
\end{tikzcd}$$

Proof. We may assume that O_v is henselian. Denote by $\pi \in O_v$ a uniformizer. Using the same reasoning as above, we may assume that every finite extension of k_v has degree a power of p. In this case, we see that $E^\times$ is generates by π, $O_v^\times$, and elements of the form $u - a$, where $u \in O_w^\times$ is a fixed generator of E and $a \in F$. Thus it is enough to check commutativity of the square at symbols of the form

$$[\pi]^{0/1}[u-a]^{0/1}[b_1]\cdots[b_r]$$

for $a \in F$ and $b_1, \ldots, b_r \in F^\times$. This is an easy computation left to the reader. $\square$

4.5. **Twisted Milnor–Witt K-theory and canonical residue homomorphisms.** As usual, we denote by F a field. Starting from Remark 4.5.6, we fix a perfect ground field k and assume that F is essentially smooth over k.

Construction 4.5.1. Let L be a line over F (i.e., a one-dimensional F-vector space). The Milnor–Witt K-theory of F twisted by L is given by

$$\mathrm{K}_*^{\mathrm{MW}}(F, L) = \mathrm{K}_*^{\mathrm{MW}}(F) \otimes_{\mathbb{Z}[F^\times]} \mathbb{Z}[L^\times]$$

where $L^\times = L \smallsetminus \{0\}$, and where $F^\times$ acts on $L^\times$ by multiplication and on $\mathrm{K}_*^{\mathrm{MW}}(F)$ by $(u, \alpha) \mapsto \langle u\rangle \cdot \alpha$ for $u \in F^\times$ and $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(F)$. A trivialisation of L induces an isomorphism $\mathrm{K}_*^{\mathrm{MW}}(F, L) \simeq \mathrm{K}_*^{\mathrm{MW}}(F)$, and a different choice of trivialisation multiplies this isomorphism by $\langle u\rangle$ for some $u \in F^\times$.

Notation 4.5.2. An element of $\mathrm{K}_*^{\mathrm{MW}}(F, L)$ can be written as $\alpha \otimes \langle e\rangle$, or just $\alpha\langle e\rangle$, with $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(F)$ and $e \in L \smallsetminus \{0\}$. We then have the identity $\alpha \otimes \langle ue\rangle = \alpha\langle u\rangle \otimes \langle e\rangle$ for $u \in F^\times$.

Remark 4.5.3. Given two lines L and L' over F, there is a canonical isomorphism

$$\mathrm{K}_*^{\mathrm{MW}}(F, L^{\otimes 2} \otimes L') \simeq \mathrm{K}_*^{\mathrm{MW}}(F, L').$$

This follows immediately from the fact that $\langle a^2\rangle = \langle 1\rangle$ for every $a \in F^\times$. In particular, we also have a canonical isomorphism $\mathrm{K}_*^{\mathrm{MW}}(F, L) \simeq \mathrm{K}_*^{\mathrm{MW}}(F, L^{-1})$.

Remark 4.5.4. Let E/F be any extension and let L be a line over F. Then, there is an obvious restriction homomorphism

$$\mathrm{K}_*^{\mathrm{MW}}(F, L) \to \mathrm{K}_*^{\mathrm{MW}}(E, L \otimes_F E)$$

sending $\alpha \otimes \langle e\rangle$ to $\alpha|_E \otimes \langle e\rangle$. If E/F is finite, we also have an obvious norm homomorphism

$$\widetilde{\mathrm{N}}_{E/F} : \mathrm{K}_*^{\mathrm{MW}}(E, L \otimes_F E) \to \mathrm{K}_*^{\mathrm{MW}}(F, L)$$

given by $\widetilde{\mathrm{N}}_{E/F}(\beta \otimes \langle e\rangle) = \widetilde{\mathrm{N}}_{E/F}(\beta) \otimes \langle e\rangle$, for $\beta \in \mathrm{K}_*^{\mathrm{MW}}(E)$ and $e \in L^\times$. (This relies on the fact that the normalised transfers $\widetilde{\mathrm{N}}_{E/F} : \mathrm{K}_*^{\mathrm{MW}}(E) \to \mathrm{K}_*^{\mathrm{MW}}(F)$ is $\mathrm{GW}(F)$-linear.)

We use Construction 4.5.1 to define a canonical residue homomorphism.

Lemma 4.5.5. *Let v be a nontrivial discrete valuation on F. Let $\mathfrak{n}_v = \mathfrak{m}_v/\mathfrak{m}_v^2$ be the normal line over k_v. Then, there is a morphism*

$$\partial_v : \mathrm{K}_*^{\mathrm{MW}}(F) \to \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v, \mathfrak{n}_v)$$

such that, for every uniformizer $\pi \in \mathfrak{m}_v$, the following triangle

$$\begin{tikzcd}
\mathrm{K}_*^{\mathrm{MW}}(F) \arrow[r, "\partial_v^\pi"] \arrow[dr, "\partial_v"'] & \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v) \arrow[d, "\langle\bar{\pi}\rangle"] \\
& \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v, \mathfrak{n}_v)
\end{tikzcd}$$

commutes. Said differently, for $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(F)$, we have $\partial_v(\alpha) = \partial_v^\pi(\alpha) \otimes \langle\bar{\pi}\rangle$.

Proof. We need to show that the expression

$$\partial_v^\pi(\alpha) \otimes \langle \bar{\pi} \rangle \in \mathrm{K}^{\mathrm{MW}}_{*-1}(k_v, \mathfrak{n}_v)$$

is independent of the choice of π. But if $\pi' = u\pi$ is another uniformizer (with $u \in O_v^\times$), then

$$\begin{aligned} \partial_v^{\pi'}(\alpha) \otimes \langle \bar{\pi}' \rangle &= \langle \bar{u} \rangle \partial_v^\pi(\alpha) \otimes \langle \bar{u} \rangle \langle \bar{\pi} \rangle \\ &= \langle \bar{u}^2 \rangle \partial_v^\pi(\alpha) \otimes \langle \bar{\pi} \rangle \\ &= \partial_v^\pi(\alpha) \otimes \langle \bar{\pi} \rangle. \end{aligned}$$

This finishes the proof. □

Remark 4.5.6. More generally, for every invertible O_v-module L, there is a canonical residue homomorphism

$$\partial_v : \mathrm{K}^{\mathrm{MW}}_*(F, L_F) \to \mathrm{K}^{\mathrm{MW}}_{*-1}(k_v, \mathfrak{n}_v \otimes L_v).$$

Here and below, we write L_F and L_v for $L \otimes_{O_v} F$ and $L \otimes_{O_v} k_v$. There is a natural choice of such an L when O_v is essentially smooth over the ground field k. Indeed, in this case we can take the so-called canonical bundle.

Recollection 4.5.7. Let X be an essentially smooth k-scheme. The canonical bundle of X, denoted by ω_X, is the top exterior power of the locally free O_X-module Ω_X of Kähler differentials on X. (If $X = \mathrm{Spec}(A)$, we also write ω_A instead of $\omega_{\mathrm{Spec}(A)}$.) Recall that for $Y \subset X$ a locally closed essentially smooth subscheme of codimension one, we have an isomorphism

$$\omega_Y \otimes_{O_Y} \mathcal{N}_Y \xrightarrow{\sim} \omega_X \otimes_{O_X} O_Y$$

given by $v \otimes \overline{a} \mapsto v \wedge \mathrm{d}a$, for v a top-degree differential form on Y and a a regular function that vanishes at Y. (Here, we denote by $\mathcal{N}_Y$ the normal sheaf of Y.) In particular, for $x \in X^{(1)}$, we have an isomorphism

$$\omega_x \otimes \mathfrak{n}_x \simeq (\omega_X) \otimes_{O_X} \kappa(x).$$

This immediately gives the following.

Lemma 4.5.8. *Let v be a nontrivial discrete valuation on F such that O_v is essentially smooth over k. Let L be an invertible O_v-module. Then, there is a canonical residue homomorphism*

$$\partial_v : \mathrm{K}^{\mathrm{MW}}_*(F, \omega_F \otimes L_F) \to \mathrm{K}^{\mathrm{MW}}_{*-1}(k_v, \omega_{k_v} \otimes L_v).$$

The following construction appears in [Sch97, §2.2.4].

Construction 4.5.9. Let E/F be a separable extension between fields that are essentially smooth over k. Then, we have a canonical isomorphism $\Omega_F \otimes_F E \simeq \Omega_E$ yielding an isomorphism of determinant lines $\omega_F \otimes_F E \simeq \omega_E$. Thus, by Remark 4.5.4, there is a canonical norm homomorphism

$$\widetilde{\mathrm{N}}_{E/F} : \mathrm{K}^{\mathrm{MW}}_*(E, \omega_E) \to \mathrm{K}^{\mathrm{MW}}_*(F, \omega_F)$$

for every separable extension E/F.

Construction 4.5.10. Assume that k has positive characteristic $p > 0$. Let E/F be a totally inseparable extension between fields that are essentially smooth over k. Assume also that $E^p \subset F$. Then, the inclusions $E^p \subset F \subset E$ and $F^p \subset E^p \subset F$ give rise to two short exact sequences

$$0 \to \Omega_{F/E^p} \otimes_F E \to \Omega_{E/E^p} \to \Omega_{E/F} \to 0 \qquad \text{and}$$

$$0 \to \Omega_{E^p/F^p} \otimes_{E^p} F \to \Omega_{F/F^p} \to \Omega_{F/E^p} \to 0,$$

which we can combine into a four-term exact sequence as follows

$$0 \to \Omega_{E^p/F^p} \otimes_{E^p} E \to \Omega_{F/F^p} \otimes_F E \to \Omega_{E/E^p} \to \Omega_{E/F} \to 0.$$

Since k is perfect, we have canonical isomorphisms $\Omega_F \simeq \Omega_{F/F^p}$ and $\Omega_E \simeq \Omega_{E/E^p}$. Set $F' = F \otimes_{k,\phi} k$ and $E' = E \otimes_{k,\phi} k$ where $\phi : k \to k$ is the Frobenius isomorphism. Then, $\Omega_{E^p/F^p} \otimes_{E^p} E$ is isomorphic to the base change of $\Omega_{E'/F'} \simeq \Omega_{E/F} \otimes_{k,\phi} k$ along the relative Frobenius morphism $E' \to E$. Thus, if $\nu_{E/F}$ denotes the determinant line of $\Omega_{E/F}$, then $\nu_{E/F}^{\otimes p}$ is canonically isomorphic to the determinant line of $\Omega_{E^p/F^p} \otimes_{E^p} E$. Thus, passing to the determinant lines in the above four-term exact sequence, we obtain a canonical isomorphism

$$\omega_E \simeq \omega_F \otimes_F \nu_{E/F}^{\otimes 1-p}.$$

In particular, if p is odd, we have canonical isomorphisms

$$\mathrm{K}_*^{\mathrm{MW}}(E, \omega_F \otimes_F E) \simeq \mathrm{K}_*^{\mathrm{MW}}(E, \omega_F \otimes_F \nu_{E/F}^{\otimes 1-p}) \simeq \mathrm{K}_*^{\mathrm{MW}}(E, \omega_E).$$

By Remark 4.5.4, this enables us to define a canonical norm homomorphism

$$\widetilde{\mathrm{N}}_{E/F} : \mathrm{K}_*^{\mathrm{MW}}(E, \omega_E) \to \mathrm{K}_*^{\mathrm{MW}}(F, \omega_F)$$

for inseparable extensions E/F such that $E^p \subset F$.

Lemma 4.5.11. *Assume that k has positive odd characteristic $p > 2$ and let E/F be a finite extension between fields that are essentially smooth over k. Consider a filtration*

$$F = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

such that E_i/E_{i-1} is separable or totally inseparable with $(E_i)^p \subset E_{i-1}$. Then the composite morphism

$$\widetilde{\mathrm{N}}_{E_r/E_{r-1}} \circ \cdots \circ \widetilde{\mathrm{N}}_{E_1/E_0} : \mathrm{K}_*^{\mathrm{MW}}(E, \omega_E) \to \mathrm{K}_*^{\mathrm{MW}}(F, \omega_F)$$

depends only on the extension E/F, and we denote it by $\widetilde{\mathrm{N}}_{E/F}$.

Proof. We first consider the case where E/F is totally inseparable. By an easy argument, we reduce to showing that

$$\widetilde{\mathrm{N}}_{E/F} = \widetilde{\mathrm{N}}_{E/H} \circ \widetilde{\mathrm{N}}_{H/F} \qquad (\star)$$

for extensions $F \subset H \subset E$ such that $E^p \subset F$. In this case we have three isomorphisms

$$\omega_H \simeq \omega_F \otimes_F \nu_{H/F}^{\otimes 1-p}, \quad \omega_E \simeq \omega_F \otimes_F \nu_{E/F}^{\otimes 1-p} \quad \text{and} \quad \omega_E \simeq \omega_H \otimes_H \nu_{E/H}^{\otimes 1-p}$$

provided by Construction 4.5.10. We also have a short exact sequence

$$0 \to \Omega_{H/F} \otimes_H E \to \Omega_{E/F} \to \Omega_{E/H} \to 0$$

yielding an isomorphism of determinant lines $\nu_{E/F} \simeq \nu_{H/F} \otimes_H \nu_{E/H}$. Moreover, the following square

$$\begin{tikzcd}
\omega_E \arrow[r, "\sim"] \arrow[d, "\sim"] & \omega_H \otimes_H \nu_{E/H}^{\otimes 1-p} \arrow[d, "\sim"] \\
\omega_F \otimes_F \nu_{E/F}^{\otimes 1-p} \arrow[r, "\sim"] & \omega_F \otimes_F \otimes \nu_{H/F}^{\otimes 1-p} \otimes_H \nu_{E/H}^{\otimes 1-p}
\end{tikzcd}$$

is commutative. This readily implies the identity $(\star)$.

We now return to the general case. We know the result if E/F is separable or totally inseparable. Thus, to conclude, it remains to see that if $E = H \otimes_F G$ with H/F totally inseparable and G/F separable, then we have the equality

$$\widetilde{\mathrm{N}}_{H/F} \circ \widetilde{\mathrm{N}}_{E/H} = \widetilde{\mathrm{N}}_{G/F} \circ \widetilde{\mathrm{N}}_{E/G}.$$

By an easy induction, we reduce to the case where $H^p \subset F$. Then the desired result follows from the fact that the isomorphism $\omega_E \simeq \omega_G \otimes_G \nu_{E/G}^{\otimes 1-p}$ coincides with the base change of the isomorphism $\omega_H \simeq \omega_F \otimes_F \nu_{H/F}^{\otimes 1-p}$ along $H \to E$ modulo the obvious identifications. □

Lemma 4.5.12. *Assume that the characteristic of k is different from* 2*. Let E/F be a finite extension between fields that are essentially smooth over k. Let F'/F be a separable algebraic extension and write*

$$E \otimes_F F' = E'_1 \times \cdots \times E'_r$$

as a product of finite extensions of F'. Then, there is a commutative square

$$\begin{tikzcd}
\mathrm{K}_*^{\mathrm{MW}}(E;\omega_E) \arrow[r] \arrow[d, "\widetilde{\mathrm{N}}_{E/F}"] & \bigoplus_{i=1}^r \mathrm{K}_*^{\mathrm{MW}}(E'_i,\omega_{E'_i}) \arrow[d, "(\widetilde{\mathrm{N}}_{E'_i/F'})_i"] \\
\mathrm{K}_*^{\mathrm{MW}}(F,\omega_F) \arrow[r] & \mathrm{K}_*^{\mathrm{MW}}(F',\omega_{F'})
\end{tikzcd}$$

where the horizontal arrows are the obvious restriction maps modulo the natural isomorphisms $\omega_{F'} \simeq \omega_F \otimes_F F'$ and $\omega_{E'_i} \simeq \omega_E \otimes_E E'_i$.

Proof. We can treat separately the case where E/F is separable and the case where E/F is totally inseparable with $E^p \subset F$ (where p is the characteristic of k). In the first case, the result follows from Corollary 4.3.13. In the second case, $E' = E \otimes_F F'$ is already a field and the result follows also from Corollary 4.3.13 by noticing that the isomorphism $\omega_{E'} \simeq \omega_{F'} \otimes_{F'} \nu_{E'/F'}^{\otimes 1-p}$ is the base change of the isomorphism $\omega_E \simeq \omega_F \otimes_F \nu_{E/F}^{\otimes 1-p}$ along $E \to E'$. □

We can now state the first main result of this subsection.

Theorem 4.5.13. *Assume that the characteristic of k is different from* 2*. Let v be a nontrivial discrete valuation on F such that O_v is essentially smooth over k. Let E/F be a finite extension, and let $w_1, \dots, w_r$ be the discrete valuations on E extending v. Then there is a commutative square*

$$\begin{tikzcd}
\mathrm{K}_*^{\mathrm{MW}}(E,\omega_E) \arrow[r, "(\partial_{w_i})_i"] \arrow[d, "\widetilde{\mathrm{N}}_{E/F}"] & \bigoplus_{i=1}^r \mathrm{K}_{*-1}^{\mathrm{MW}}(k_{w_i},\omega_{k_{w_i}}) \arrow[d, "(\widetilde{\mathrm{N}}_{k_{w_i}/k_v})_i"] \\
\mathrm{K}_*^{\mathrm{MW}}(F,\omega_F) \arrow[r, "\partial_v"] & \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v,\omega_{k_v}).
\end{tikzcd}$$

Proof. Using Lemma 4.5.12, we may replace F with its henselisation $F_v^h = \mathrm{Frac}(O_v^h)$ at v and assume that O_v is local henselian. In this case, there is only one valuation w on E extending v, and we need to show the commutativity of the square

$$\begin{tikzcd}
\mathrm{K}_*^{\mathrm{MW}}(E,\omega_E) \arrow[r, "\partial_w"] \arrow[d, "\widetilde{\mathrm{N}}_{E/F}"] & \mathrm{K}_{*-1}^{\mathrm{MW}}(k_w,\omega_{k_w}) \arrow[d, "\widetilde{\mathrm{N}}_{k_w/k_v}"] \\
\mathrm{K}_*^{\mathrm{MW}}(F,\omega_F) \arrow[r, "\partial_v"] & \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v,\omega_{k_v}).
\end{tikzcd} \qquad (\star)$$

The proof is rather long and we split it into several steps.

Step 1. Assume that k has positive characteristic $p > 2$. Then it suffices to show that ($\star$) commutes up to p_ϵ-torsion. More precisely, it suffices to show that for every $\beta \in \mathrm{K}^{\mathrm{MW}}_*(E, \omega_E)$ there is an integer $m \geqslant 0$ such that $(p^m)_\epsilon \cdot (\widetilde{\mathrm{N}}_{k_w/k_v} \circ \partial_w(\beta) - \partial_v \circ \widetilde{\mathrm{N}}_{E/F}(\beta)) = 0$.

Indeed, assuming that this is satisfied, then it suffices to show that

$$\gamma = (\widetilde{\mathrm{N}}_{k_w/k_v} \circ \partial_w(\beta) - \partial_v \circ \widetilde{\mathrm{N}}_{E/F}(\beta))$$

vanishes over some finite extension of k_v of degree a power of 2. Since $p > 2$, any such extension is separable and is the residue extension of an unramified separable extension of F. Thus, using Lemma 4.5.12, we may replace F with an unramified pro-2-extension and assume that every element of k_v is a square. As explained in the proof of Theorem 4.4.9, the map $\mathrm{K}^{\mathrm{MW}}_*(k_v) \to \mathrm{K}^{\mathrm{M}}_*(k_v)$ is then an isomorphism in nonnegative degrees, and we are reduced to the analogous problem for Milnor K-theory. This is well known (see [Kat80]), but for the reader's convenience, we include a sketch of the proof.

In Milnor K-theory, there are no nontrivial orientations, i.e., the analogue of the square ($\star$) is

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{M}}_*(E) \arrow[r, "\partial_w"] \arrow[d, "\mathrm{N}_{E/F}"] & \mathrm{K}^{\mathrm{M}}_{*-1}(k_w) \arrow[d, "\mathrm{N}_{k_w/k_v}"] \\
\mathrm{K}^{\mathrm{M}}_*(F) \arrow[r, "\partial_v"] & \mathrm{K}^{\mathrm{M}}_{*-1}(k_v).
\end{tikzcd}$$

Given $\beta \in \mathrm{K}^{\mathrm{M}}_*(E)$, we know by assumption that $\gamma = \mathrm{N}_{k_w/k_v} \circ \partial_w(\beta) - \partial_v \circ \mathrm{N}_{E/F}(\beta)$ is p-torsion. We need to show that γ is zero, and it is enough to do so after restricting to a finite extension of k_v of degree prime to p (which is then automatically separable). Thus, using Lemma 4.5.12, we may replace F with an algebraic unramified pro-p'-extension and assume that any finite extension of k_v has degree a power of p. This implies that the Galois group of F is a direct product of a pro-p-group and $\widehat{\mathbb{Z}}^{(p)}(1) = \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$. In particular, any finite extension of F is a composition of extensions of prime degrees. Thus, we may assume that $E = F[\sqrt[q]{\pi}]$, where $\pi \in O_v$ is a uniformizer and $q \neq p$ a prime number, or that $[E : F] = p$. In the first case, we conclude by applying Corollaries 4.2.10 and 4.3.13 with base change along $F \to F[\sqrt[q]{\pi}]$. (This yields that $q \cdot \gamma = 0$.)

In the second case, we argue as follows. Using Corollaries 4.2.10 and 4.3.13, we may replace F with $F[\sqrt[n]{\pi}]$ for any integer n prime to p. But, any finite extension of the field $F_\infty = F[\sqrt[n]{\pi} \mid (n, p) = 1]$ has degree a power of p. In particular, any polynomial of degree $\leqslant p - 1$ with coefficients in F_∞ is a product of linear factors. Let $u \in E^\times \smallsetminus F^\times$ be a generator of the extension E/F. If π is not a uniformizer of O_w, then assume u to be a uniformizer of O_w. If π is a uniformizer, then take $u \in O_w^\times$. Then, any symbol in $\mathrm{K}^{\mathrm{M}}_*(E_\infty)$, with $E_\infty = E \otimes_F F_\infty$, can be written as a linear combinations of symbols of the form $\{u\}^{0/1}\{a_1\} \cdots \{a_r\}$ where

- $a_1, \ldots, a_r \in F_\infty$;
- all the elements $u, a_1, \ldots, a_r$, except possible one, are units with respect to the unique valuation of E_∞ extending v.

Thus, starting with an element $\beta \in \mathrm{K}^{M}_*(E)$, we may replace F with $F[\sqrt[n]{\pi}]$, and reduce to the case where $\beta = \{u\}^{0/1}\{a_1\} \cdots \{a_r\}$, with $a_1, \ldots, a_r \in F$. This is then treated by a direct computation.

Step 2. Assume that k has positive characteristic $p > 2$. Then it is enough to show that ($\star$) commutes up to p_ϵ-torsion in the following two cases:

- E/F is totally inseparable of degree p;
- E/F is separable and unfiercely ramified, i.e., k_w/k_v is also separable.

Indeed, fix an essentially smooth retraction $\mathrm{Spec}(O_v) \to \mathrm{Spec}(k_v)$ corresponding to a section $k_v \to O_v$. (This is possible since k is perfect). If $(k_v)^{\mathrm{perf}}$ denotes the perfect closure of k_v, then any finite extension of the field $F \otimes_{k_v} (k_v)^{\mathrm{perf}}$ is residually separable. It follows that we can find a finite totally unseparable extension H/F such that, if $G = (E \otimes_F H)_{\mathrm{red}}$ is the composite field, then the extension G/H is residually separable.

Now, since $[G : E] = p^m$ is a power of p, the image of $\widetilde{\mathrm{N}}_{G/E} : \mathrm{K}_*^{\mathrm{MW}}(G, \omega_G) \to \mathrm{K}_*^{\mathrm{MW}}(E, \omega_E)$ contains $(p^m)_\epsilon \cdot \mathrm{K}_*^{\mathrm{MW}}(E, \omega_E)$. Thus, by Step 1, it suffices to show that the equality

$$\widetilde{\mathrm{N}}_{k_w/k_v} \circ \partial_w \circ \widetilde{\mathrm{N}}_{G/E} = \partial_v \circ \widetilde{\mathrm{N}}_{E/F} \circ \widetilde{\mathrm{N}}_{G/E}$$

up to p_ϵ-torsion. Thus, it is enough to treat the case of the extensions G/H, H/F and G/E. The last two are totally inseparable. The first one is residually separable. Thus, it is a composition of a totally inseparable extension and a separable unfiercely ramified one.

Step 3. In this step and the next one, we treat the case where E/F is totally inseparable of degree p. The ideal $\mathfrak{m}_v O_w$ is either $\mathfrak{m}_w$ or $(\mathfrak{m}_w)^p$. (This follows from [Bou85, Chapitre VI, §8, n° 5, Théorème 2] using that O_v is excellent.) In the first case, we can find $b \in O_w$ such that $\overline{b} \in k_w \smallsetminus k_v$. This implies that the element $a = b^p$ of O_v has residue which is not a p-th power in k_v. In the second case, if $\rho \in O_w$ is a uniformizer of O_w, then $\pi = \rho^p$ is a uniformizer of O_v. Thus, there are two possibilities for E/F:

- $E = F[T]/(T^p - a)$ with $a \in O_v^\times$ such that $\overline{a} \in k_v \smallsetminus (k_v)^p$;
- $E = F[T]/(T^p - \pi)$ with $\pi \in O_v$ a uniformizer.

In this step, we treat the case $E = F[T]/(T^p - a)$. Note that $\mathrm{d}a \in \Omega_{O_v}$ is not divisible by π since it maps to a nonzero differential form in Ω_{k_v}. It follows that the O_v-module $\Omega_{O_v}/\mathrm{d}a$ is torsion-free, and hence free. Therefore,

$$\Omega_{O_v[T]/(T^p - a)} \simeq (O_v[T]/(T^p - a)) \otimes_{O_v} (\Omega_{O_v}/\mathrm{d}a) \oplus (O_v[T]/(T^p - a)) \cdot \mathrm{d}T.$$

is locally free of the right dimension. This shows that $O_v[T]/(T^p - a)$ is essentially smooth, and hence isomorphic to O_w. In particular, we may rewrite the above isomorphism as follows

$$\Omega_{O_w} \simeq O_w \otimes_{O_v} (\Omega_{O_v}/\mathrm{d}a) \oplus O_w \cdot \mathrm{d}T.$$

Taking determinant lines, we obtain the isomorphism

$$\omega_{O_w} \simeq \omega_{O_v} \otimes_{O_v} O_w.$$

In fact, modulo the isomorphism $\Omega_{O_w/O_v} \simeq O_w \cdot \mathrm{d}T$ yielding the trivialisation $\nu_{O_w/O_v} \simeq O_w$, the above isomorphism coincides with the isomorphism

$$\omega_{O_w} \simeq \omega_{O_v} \otimes_{O_v} \nu_{O_w/O_v}^{\otimes 1-p}$$

one gets by repeating Construction 4.5.10. Moreover, we have a commutative diagram of isomorphisms

$$\begin{tikzcd}
k_w \otimes_{O_w} \omega_{O_w} \arrow[d, "\sim"] & \omega_{k_w} \otimes \mathfrak{n}_w \arrow[l, "\sim"'] & \omega_{k_w} \arrow[l, "\mathrm{id}\otimes\overline{\pi}"', "\sim"] \arrow[d, "\sim"] \\
k_w \otimes_{O_v} \omega_{O_v} & k_w \otimes_{k_v} \omega_{k_v} \otimes \mathfrak{n}_v \arrow[l, "\sim"'] & k_w \otimes_{k_v} \omega_{k_v} \arrow[l, "\mathrm{id}\otimes\overline{\pi}"', "\sim"]
\end{tikzcd}$$

and, as before, the right-hand vertical isomorphism coincides with the one from Construction 4.5.10 modulo the trivialisation of $\nu_{k_w/k_v}^{\otimes 1-p}$ induced by $\Omega_{k_w/k_v} \simeq k_w \cdot \mathrm{d}T$. This shows that a trivialisation of ω_{O_v} yields a trivialisation of ω_{O_w}, ω_{k_w} and ω_{k_v} modulo which the square ($\star$) can be then identified with the square

$$\begin{tikzcd}
\mathrm{K}_*^{\mathrm{MW}}(E) \arrow[r, "\partial_w^\pi"] \arrow[d, "\widetilde{\mathrm{N}}_{E/F}"] & \mathrm{K}_{*-1}^{\mathrm{MW}}(k_w) \arrow[d, "\widetilde{\mathrm{N}}_{k_w/k_v}"] \\
\mathrm{K}_*^{\mathrm{MW}}(F) \arrow[r, "\partial_v^\pi"] & \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v).
\end{tikzcd}$$

The commutation of this square up to p_ϵ-torsion follows by applying Corollaries 4.2.10 and 4.3.13 to the base change along $F \to E$.

Step 4. Here, we treat the case where $E = F[T]/(T^p - \pi)$ with $\pi \in O_v$ a uniformizer. We denote by $\rho = \sqrt[p]{\pi}$ the uniformizer of O_w. As in Step 3, we have an isomorphism

$$\Omega_{O_w} \simeq O_w \otimes_{O_v} (\Omega_{O_v}/\mathrm{d}\pi) \oplus O_w \cdot \mathrm{d}\rho$$

yielding an isomorphism $\omega_{O_w} \simeq \omega_{O_v} \otimes_{O_v} O_w$. The latter coincides over E with the isomorphism $\omega_E \simeq \omega_F \otimes_F \nu_{E/F}^{\otimes 1-p}$ of Construction 4.5.10 modulo the trivialisation $\nu_{E/F} = \Omega_{E/F} = E \cdot \mathrm{d}\rho$. Thus, using the isomorphism $\omega_{O_w} \simeq \omega_{O_v} \otimes_{O_v} O_w$, any trivialisation of ω_F yields a trivialisation of ω_E modulo which the canonical norm $\widetilde{\mathrm{N}}_{E/F} : \mathrm{K}_*^{\mathrm{MW}}(E; \omega_E) \to \mathrm{K}_*^{\mathrm{MW}}(F, \omega_F)$ coincides with the normalised norm $\widetilde{\mathrm{N}}_{E/F} : \mathrm{K}_*^{\mathrm{MW}}(E) \to \mathrm{K}_*^{\mathrm{MW}}(F)$. On the other hand, $\mathrm{d}\pi$ is the image of $\overline{\pi} \in \mathfrak{n}_v$ while $\mathrm{d}\rho$ is the image of $\overline{\rho} \in \mathfrak{n}_w$. Thus, over $k_v \simeq k_w$, we have a commutative diagram of isomorphisms

$$\begin{tikzcd}
k_w \otimes_{O_w} \omega_{O_w} \arrow[d, "\sim"] & \omega_{k_w} \otimes \mathfrak{n}_w \arrow[l, "\sim"'] & \omega_{k_w} \arrow[l, "\mathrm{id}\otimes\bar{\rho}"', "\sim"] \arrow[d, equal] \\
k_v \otimes_{O_v} \omega_{O_v} & \omega_{k_v} \otimes \mathfrak{n}_v \arrow[l, "\sim"'] & \omega_{k_v}. \arrow[l, "\mathrm{id}\otimes\bar{\pi}"', "\sim"]
\end{tikzcd}$$

Thus, we are reduced to showing the commutativity of square

$$\begin{tikzcd}
\mathrm{K}_*^{\mathrm{MW}}(E) \arrow[r, "\partial_w^\rho"] \arrow[d, "\widetilde{\mathrm{N}}_{E/F}"] & \mathrm{K}_{*-1}^{\mathrm{MW}}(k_w) \arrow[d, equal] \\
\mathrm{K}_*^{\mathrm{MW}}(F) \arrow[r, "\partial_v^\pi"] & \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v)
\end{tikzcd}$$

up to p_ϵ-torsion. This again follows by applying Corollaries 4.2.10 and 4.3.13 to the base change along the extension $F \to E$.

Step 5. When k has positive characteristic $p > 2$, it remains to consider the case where E/F and k_w/k_v are both separable. On the other hand, this is automatically the case when k has characteristic zero. Thus, in the remainder of the proof, we may assume that both E/F and k_w/k_v are separable. We want to reduce to the case where $k_v = k_w$, i.e., to the totally ramified case. Since O_v is henselian, the extension E/F is a composition of an unramified separable extension and a totally ramified one.

In this step, we treat the case where E/F is unramified. Let $\pi \in O_v$ be a uniformizer. Then π is also a uniformizer of O_w and we are reduced to showing the commutativity of the square

$$\begin{tikzcd}
\mathrm{K}_*^{\mathrm{MW}}(E) \arrow[r, "\partial_w^\pi"] \arrow[d, "\widetilde{\mathrm{N}}_{E/F}"] & \mathrm{K}_{*-1}^{\mathrm{MW}}(k_w) \arrow[d, "\widetilde{\mathrm{N}}_{k_w/k_v}"] \\
\mathrm{K}_*^{\mathrm{MW}}(F) \arrow[r, "\partial_v^\pi"] & \mathrm{K}_{*-1}^{\mathrm{MW}}(k_v).
\end{tikzcd}$$

Using Corollaries 4.2.10 and 4.3.13, we know that the square commutes up to d_ϵ-torsion, where $d = [E : F]$. We argue by contradiction, and fix $\beta \in \mathrm{K}_*^{\mathrm{MW}}(E)$ such that, if

$$\gamma = \widetilde{\mathrm{N}}_{k_w/k_v} \circ \partial_w^\pi(\beta) - \partial_v^\pi \circ \widetilde{\mathrm{N}}_{E/F}(\beta),$$

then $\gamma \neq 0$ but $q_\epsilon \cdot \gamma = 0$ with $q \geqslant 1$ an integer. We also assume that β is chosen so that q is the smallest possible. Then, q is a prime number. We may then assume that every finite separable extension of k_v has degree a power of q. If q is odd, then every element of k_v is a square and the morphism $\mathrm{K}_*^{\mathrm{MW}}(k_v) \to \mathrm{K}^{\mathrm{M}}(k_v)$ is an isomorphism in nonnegative degrees. We conclude as in Step 1. If $q = 2$, then we may assume that E/F is a quadratic unramified extension. This case has been treated before in Corollary 4.4.7.

Step 6. It remains to treat the case where E/F is totally ramified, i.e., where $k_v \simeq k_w$. In this step, we start discussing the case where E/F is tamely ramified, i.e., $E = F[T]/(T^q - \pi) = F[\sqrt[q]{\pi}]$ where $\pi \in O_v$ is a uniformizer and q is a prime number different from the characteristic of k. We have a pushout square of O_w-modules

$$\begin{tikzcd}
O_w \cdot \mathrm{d}\pi \arrow[r] \arrow[d] & O_w \cdot \mathrm{d}\rho = O_w \cdot \dfrac{\mathrm{d}\pi}{q \cdot \rho^{q-1}} \arrow[d] \\
O_w \otimes_{O_v} \Omega_{O_v} \arrow[r] & \Omega_{O_w},
\end{tikzcd}$$

yielding an isomorphism of determinant lines

$$\omega_{O_v} \otimes_{O_v} (O_w \cdot \mathrm{d}\pi)^{-1} \simeq \omega_{O_w} \otimes_{O_w} (O_w \cdot \mathrm{d}\rho)^{-1}.$$

By construction, we have a commutative diagram of isomorphisms

$$\begin{tikzcd}
\omega_{O_v} \otimes_{O_v} (k_w \cdot \mathrm{d}\pi)^{-1} \arrow[r, "\sim"] \arrow[d, "\sim"] & \omega_{O_v} \otimes_{O_v} (k_w \cdot \overline{\pi})^{-1} \arrow[r, "\sim"] & \omega_{k_v} \arrow[d, equal] \\
\omega_{O_w} \otimes_{O_w} (k_w \cdot \mathrm{d}\rho)^{-1} \arrow[r, "\sim"] & \omega_{O_w} \otimes_{O_w} (k_w \cdot \overline{\rho})^{-1} \arrow[r, "\sim"] & \omega_{k_w}.
\end{tikzcd}$$

Choose a generator g of ω_{O_v}. The above isomorphism takes $g \otimes (\mathrm{d}\pi)^{-1}$ to $g' \otimes (\mathrm{d}\rho)^{-1}$ where g' is a generator of ω_{O_w}. It follows from the previous commutative diagram that the pair of generators $g \in \omega_{O_v}$ and $\overline{\pi} \in \mathfrak{n}_v$ induces the same trivialisation of ω_{k_v} as the pair of generators $g' \in \omega_{O_v}$ and $\overline{\rho} \in \mathfrak{n}_v$. Finally, note that over E, we have two generators g and g' of $\omega_E \simeq E \otimes_F \omega_F$, and they are related by the formula

$$g = q \cdot \rho^{q-1} g'.$$

Thus, to conclude, we need to show that the square

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(E) \arrow[r, "\partial^\rho_w"] \arrow[d, "\widetilde{\mathrm{N}}_{E/F}(\langle q\cdot\rho^{q-1}\rangle\cdot -)"'] & \mathrm{K}^{\mathrm{MW}}_{*-1}(k_w) \arrow[d, equal] \\
\mathrm{K}^{\mathrm{MW}}_*(F) \arrow[r, "\partial^\pi_v"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(k_v)
\end{tikzcd}$$

commutes. Since $q\cdot\rho^{q-1}$ is the value at ρ of the derivative of $T^q-\pi$, we see that we are reduced to showing that the following square commutes:

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(E) \arrow[r, "\partial^\rho_w"] \arrow[d, "\mathrm{N}^\rho_{E/F}"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(k_w) \arrow[d, equal] \\
\mathrm{K}^{\mathrm{MW}}_*(F) \arrow[r, "\partial^\pi_v"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(k_v).
\end{tikzcd}$$

Step 7. We continue the discussion from Step 6. Let $\beta\in\mathrm{K}^{\mathrm{MW}}_*(E)$ and consider

$$\gamma=\partial^\rho_w(\beta)-\partial^\pi_v\circ\mathrm{N}^\rho_{E/F}(\beta).$$

In this step, we show that $q_\epsilon\cdot\gamma=0$. For simplicity, we assume that F has a primitive q-th root of unity. (The case where F has no primitive q-th root of unity is not really harder but requires knowing the commutativity of ($\star$) when E/F is unramified, a case that was solved in Step 5. It also uses Remark 4.3.11 instead of Corollary 4.3.10.) Applying Corollary 4.2.10 and Proposition 4.3.9 to the base change along $F\to E$, we have

$$q_\epsilon\cdot\partial^\pi_v\circ\mathrm{N}^\rho_{E/F}(\beta)=\partial^\rho_w(\mathrm{N}^\rho_{E/F}(\beta)|_E)=\sum_{i=0}^{q-1}\partial^\rho_w\left(\left\langle\frac{T^q-\pi}{T-\zeta_q^i\rho}(\zeta_q^i\rho)\right\rangle\cdot\sigma^i(\beta)\right)$$

where ζ_q is a q-th root of unity and $\sigma: E\to E$ is the automorphism sending ρ to $\zeta_q\rho$. Moreover, as shown in the proof of Corollary 4.3.10, we have

$$\frac{T^q-\pi}{T-\zeta_q^i\rho}(\zeta_q^i\rho)=(-1)^{q-1}\zeta_q^{-i}q\rho^{q-1}.$$

Using Corollary 4.2.10, this gives:

$$\begin{aligned}
q_\epsilon\cdot\partial^\pi_v\circ\mathrm{N}^\rho_{E/F}(\beta) &= \sum_{i=0}^{q-1}\partial^\rho_w\left(\left\langle(-1)^{q-1}\zeta_q^{-i}q\rho^{q-1}\right\rangle\cdot\sigma^i(\beta)\right)\\
&= \sum_{i=0}^{q-1}\partial^{\zeta_q^i\rho}_w\left(\left\langle(-1)^{q-1}q\rho^{q-1}\right\rangle\cdot\sigma^i(\beta)\right)\\
&= \sum_{i=0}^{q-1}\partial^{\zeta_q^i\rho}_w\circ\sigma^i\left(\left\langle(-1)^{q-1}q\zeta_q^i\rho^{q-1}\right\rangle\cdot\beta\right).
\end{aligned}$$

Since σ^i acts trivially on the residue field k_w and sends ρ to $\zeta_q^i\rho$, we have $\partial^\rho_w=\partial^{\zeta_q^i\rho}_w\circ\sigma^i$. In conclusion, we obtain the formula:

$$q_\epsilon\cdot\partial^\pi_v\circ\mathrm{N}^\rho_{E/F}(\beta)=\left(\sum_{i=0}^{q-1}\langle\zeta_q^i\rangle\right)\langle(-1)^{q-1}q\rangle\partial^\rho_w(\langle\rho^{q-1}\rangle\cdot\beta).$$

We claim that the right-hand side is equal to $q_\epsilon \cdot \partial^\rho_w(\beta)$. If q is odd, then $q-1$ is even and $\langle \rho^{q-1}\rangle = 1$. Also, ζ_q is a square so that the right hand side is just $q\langle q\rangle\partial^\rho_w(\beta)$. By Corollary 4.3.10, $q\langle q\rangle = q_\epsilon$ as needed. Next, assume that $q = 2$. Then, the right-hand side is $2_\epsilon \cdot \partial^\rho_w(\langle\rho\rangle\beta)$ and we want to verify the identity

$$2_\epsilon \cdot \partial^\rho_w(\beta) = 2_\epsilon \cdot \partial^\rho_w(\langle\rho\rangle\beta).$$

To do so, we may assume that $\beta = [\rho]^{0/1}[b_1]\cdots[b_n]$ with $b_1,\dots,b_n \in O_w^\times$. By Corollary 4.2.8, it even suffices to consider the two cases $\beta = 1$ and $\beta = [\rho]$. If $\beta = 1$, the left-hand side is zero and the same is true for right-hand side by the following computation

$$2_\epsilon \cdot \partial^\rho_w(\langle\rho\rangle) = 2_\epsilon \cdot \partial^\rho_w(1+\eta[\rho]) = 2_\epsilon \cdot \eta = 0.$$

If $\beta = [\rho]$ the left-hand side is 2_ϵ and the same is true for the right-hand side by the following computation

$$2_\epsilon \cdot \partial^\rho_w(\langle\rho\rangle[\rho]) = 2_\epsilon \cdot \partial^\rho_w(\langle -1\rangle[\rho]) = 2_\epsilon\langle -1\rangle = 2_\epsilon.$$

This completes the proof that $q_\epsilon \cdot \gamma = 0$.

Step 8. In this step, we complete the proof of the commutativity of $(\star)$ in the case where $E = F[T]/(T^q - \pi)$, which we started in Step 6. We need to show that the square

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(E) \arrow[r, "\partial^\rho_w"] \arrow[d, "\mathrm{N}^\rho_{E/F}"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(k_w) \arrow[d, equal] \\
\mathrm{K}^{\mathrm{MW}}_*(F) \arrow[r, "\partial^\pi_v"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(k_v)
\end{tikzcd}$$

commutes, and have shown that it commutes up to q_ϵ-torsion. Thus, it suffices to show that γ (as on Step 7) vanishes over some separable extension of k_v of degree prime to q. Thus, we may replace F by a separable unramified pro-q'-extension and assume that every finite extension of k_v has degree a power of q.

If q is odd, every element of k_v is a square and the morphism $\mathrm{K}^{\mathrm{MW}}_*(k_v) \to \mathrm{K}^{\mathrm{M}}_*(k_v)$ is an isomorphism in nonnegative degrees. Thus, we may conclude as in Step 1. So, it remains to treat the case $q = 2$. We argue as in the proof of Corollary 4.4.7. Using Proposition 4.4.4, it is easy to see that $\mathrm{K}^{\mathrm{MW}}_*(E)$ is generated by the following symbols (for $a, b, a_1, \cdots, a_n \in O_v^\times$, $m \in \mathbb{N}$ and $e \geq 1$ odd):

(1) $\eta^m[\rho]^{0/1}[a_1]\cdots[a_n]$,
(2) $\eta^m[a+b\rho^e][a_1]\cdots[a_n]$.

Then the result can be checked by direct computation. Indeed, we have

$$\begin{aligned}
\partial^\pi_v \mathrm{N}^\rho_{E/F}([\rho]) &= \partial^\pi_v(\langle -1\rangle[\mathrm{Nm}_{E/F}(\rho)]) \\
&= \partial^\pi_v(\langle -1\rangle[-\pi]) = 1 = \partial^\rho_w([\rho])
\end{aligned}$$

and

$$\begin{aligned}
\partial^\pi_v \mathrm{N}^\rho_{E/F}([a+b\rho^e]) &= \partial^\pi_v(\langle -1\rangle[\mathrm{Nm}(a+b\rho^e)]) \\
&= \partial^\pi_v(\langle -1\rangle[a^2 - b^2\pi^e]) = 0 = \partial^\rho_w([a+b\rho^e])
\end{aligned}$$

as needed.

Step 9. To complete the proof, it remains to consider the case of a widely totally ramified separable extension E/F. Thus, k has positive characteristic $p > 2$ and we may assume that $[E : F] = p$. We also have $k_v \simeq k_w$, and the ramification index of E/F is p. Fix a uniformizer $\rho \in O_w$. Then P_ρ is an Eisenstein polynomial of degree p with constant term $P_\rho(0)$ a uniformizer of O_v. We set $\pi = -P_\rho(0)$. Recall that, by Step 1, it suffices to show that the square $(\star)$ commutes up to p_ϵ-torsion.

Since k is perfect, we may find a section $k_v \to O_v$ making O_v essentially smooth over k_v. Then, O_w is also essentially smooth over k_v as it the local ring of a regular curve at a k_v-rational point. This yields short exact sequences

$$0 \to \Omega_{k_v} \otimes_{k_v} O_v \to \Omega_{O_v} \to \Omega_{O_v/k_v} \to 0 \qquad \text{and}$$

$$0 \to \Omega_{k_v} \otimes_{k_v} O_w \to \Omega_{O_w} \to \Omega_{O_w/k_v} \to 0.$$

Moreover, we have isomorphisms

$$\Omega_{O_v/k_v} \simeq O_v \cdot \mathrm{d}\pi \qquad \text{and} \qquad \Omega_{O_w/k_v} \simeq O_w \cdot \mathrm{d}\rho.$$

Passing to the determinant lines, we obtain isomorphisms

$$\omega_{O_v} \simeq \omega_{k_v} \otimes_{k_v} O_v \cdot \mathrm{d}\pi \qquad \text{and} \qquad \omega_{O_w} \simeq \omega_{k_v} \otimes_{k_v} O_w \cdot \mathrm{d}\rho.$$

Thus, fixing a generator g of ω_{k_v}, we deduce generators $g\mathrm{d}\pi$ and $g\mathrm{d}\rho$ of ω_{O_v} and ω_{O_w}. On the other hand, since E/F is separable, there is a canonical isomorphism $E \otimes_F \omega_F \simeq \omega_E$. Thus, we may view $g\mathrm{d}\pi$ and $g\mathrm{d}\rho$ as generators of ω_E which differs by

$$\frac{\mathrm{d}\rho}{\mathrm{d}\pi} \in E^\times$$

(The above fraction makes sense since $\mathrm{d}\pi$ and $\mathrm{d}\rho$ are generators of the lines Ω_{O_v/k_v} and Ω_{O_w/k_v} which can be canonically identified after base change to E.) Write $P_\rho = T^p + a_{p-1}T^{p-1} + \cdots + a_0$, with $a_0 = -\pi$. In Ω_{O_w/k_w}, we then have the relation

$$P'_\rho(\rho)\mathrm{d}\rho + \left(\frac{\mathrm{d}a_{p-1}}{\mathrm{d}\pi}\rho^{p-1} + \cdots + \frac{\mathrm{d}a_1}{\mathrm{d}\pi}\rho - 1\right)\mathrm{d}\pi = 0.$$

Thus, there is a unit $u \in O_w^\times$ with $\bar{u} = 1$, such that $\mathrm{d}\pi = uP'_\rho(\rho)\mathrm{d}\rho$. Therefore, using the trivialisations g, $g\mathrm{d}\pi$ and $g\mathrm{d}\rho$, we can identify the square $(\star)$ with the following one:

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(E) \arrow[r, "\partial^\rho_w"] \arrow[d, "\widetilde{\mathrm{N}}_{E/F}(\langle uP'_\rho(\rho)\rangle \cdot -)"'] & \mathrm{K}^{\mathrm{MW}}_{*-1}(k_w) \arrow[d, equal] \\
\mathrm{K}^{\mathrm{MW}}_*(F) \arrow[r, "\partial^\pi_v"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(k_v).
\end{tikzcd}$$

Since O_v is henselian and p is odd, u is a square in $O_v^\times$. Thus, in conclusion, we need to show that the square

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(E) \arrow[r, "\partial^\rho_w"] \arrow[d, "\mathrm{N}^\rho_{E/F}"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(k_w) \arrow[d, equal] \\
\mathrm{K}^{\mathrm{MW}}_*(F) \arrow[r, "\partial^\pi_v"] & \mathrm{K}^{\mathrm{MW}}_{*-1}(k_v)
\end{tikzcd}$$

commutes up to p_ϵ-torsion.

Step 10. Let $\beta \in \mathrm{K}_*^{\mathrm{MW}}(E)$ and consider

$$\gamma = \partial_w^\rho(\beta) - \partial_v^\pi \circ \mathrm{N}_{E/F}^\rho(\beta).$$

In this step, we show that $p_\epsilon \cdot \gamma = 0$. For simplicity, we assume that E/F is Galois and denote by σ a generator of the Galois group. (The argument below can be adapted to the non-Galois case, but we leave this to the reader.) We have a pushout square of commutative rings

$$\begin{tikzcd}
E \arrow[r, "(\sigma^i)_i"] & \prod_{i=0}^{p-1} E \\
F \arrow[u] \arrow[r] & E. \arrow[u, "\Delta"']
\end{tikzcd}$$

Applying Corollary 4.2.10 and Proposition 4.3.9, we have

$$\begin{aligned}
p_\epsilon \cdot \partial_v^\pi \circ \mathrm{N}_{E/F}^\rho(\beta) \;=\; \partial_w^\rho(\mathrm{N}_{E/F}^\rho(\beta)|_E) \;&=\; \sum_{i=0}^{p-1} \partial_w^\rho\left(\left\langle \frac{P_\rho}{T-\sigma^i(\rho)}(\sigma^i(\rho))\right\rangle \cdot \sigma^i(\beta)\right) \\
&=\; \sum_{i=0}^{p-1} \partial_w^\rho \sigma^i \left(\left\langle \frac{P_\rho}{T-\rho}(\rho)\right\rangle \cdot \beta\right) \\
&=\; \sum_{i=0}^{p-1} \partial_w^{\sigma^{-i}(\rho)} \left(\left\langle \frac{P_\rho}{T-\rho}(\rho)\right\rangle \cdot \beta\right).
\end{aligned}$$

Recall that $\sigma^{-i}(\rho)/\rho$ belongs to $O_w^\times$ and has residue 1 in $k_w^\times$. (See [Ser04, Chapitre IV, §1 & §2].) Thus, by Corollary 4.2.10, we have $\partial_w^{\sigma^{-i}(\rho)} = \partial_w^\rho$, for every $0 \leqslant i \leqslant p-1$. Also, we have

$$\frac{P_\rho}{T-\rho}(\rho) = \prod_{i=1}^{p-1}(\rho - \sigma^i(\rho))$$

and, for $1 \leq i \leq p-1$, we can write

$$\frac{\rho - \sigma^i(\rho)}{\rho - \sigma(\rho)} = \sum_{j=1}^{i} \frac{\sigma^{j-1}(\rho - \sigma(\rho))}{\rho - \sigma(\rho)}.$$

As mentioned before, each of the fractions

$$\frac{\sigma^{j-1}(\rho - \sigma(\rho))}{\rho - \sigma(\rho)}$$

belongs to $O_w^\times$ and has residue $1 \in k^\times$. This shows that, for $1 \leq i \leq p-1$, the fraction

$$\frac{\rho - \sigma^i(\rho)}{\rho - \sigma(\rho)}$$

belongs to $O_w^\times$ and has residue $i \in k_w^\times$. This shows that

$$\frac{P_\rho}{T-\rho}(\rho) = \mu \cdot (\rho - \sigma(\rho))^{p-1}$$

where $\mu \in O_w^\times$ with residue $(p-1)! = (-1)^{\frac{p-1}{2}} \in k_w^\times$. In conclusion, we obtain the formula

$$p_\epsilon \cdot \partial_v^\pi \circ \mathrm{N}_{E/F}^\rho(\beta) = p\langle(-1)^{\frac{p-1}{2}}\rangle \partial_w^\rho(\beta).$$

Thus, to conclude, we need to check the identity $p_\epsilon = p\langle(-1)^{\frac{p-1}{2}}\rangle$ in $\mathrm{K}_0^{\mathrm{MW}}(\mathbb{F}_p)$. If p is congruent to 1 modulo 4, then -1 is a square in $\mathbb{F}_p$ and both sides of the desired identity are equal to p. If p is congruent to 3 modulo 4, then we need that $p_\epsilon = p\langle -1\rangle$. Equivalently, we need to show that

$$\left(\frac{p+1}{2}\right)(1 - \langle 1\rangle) = 0$$

and it is enough to show that $2(1 - \langle 1\rangle)$. This follows from [Lam05, Corollary 3.6] asserting that $\mathrm{W}(\mathbb{F}_p) \simeq \mathbb{Z}/4$. (Indeed, $1 - \langle -1\rangle$ belongs to augmentation ideal of $\mathrm{GW}(\mathbb{F}_p)$ which is thus isomorphic to $\mathbb{Z}/2$.) This finishes the proof of Theorem 4.5.13. □

4.6. **The Gersten complex for Milnor–Witt K-theory.** We fix a perfect ground field k of characteristic different from 2. In this subsection, we introduce the Gersten complex for Milnor–Witt K-theory. We start by defining the underlying "complex". (Here, a "complex" is a graded object with a degree-one endomorphism which does not necessarily square to zero.)

Construction 4.6.1. Let X be an essentially smooth k-scheme. The Gersten "complex" of X with values in Milnor–Witt K-theory is the "complex"

$$\mathrm{C}^\bullet(X, \omega_X; \mathrm{K}_*^{\mathrm{MW}})$$

defined as follows. It is zero in negative degrees and in degree $d \geq 0$ it is given by:

$$\bigoplus_{x \in X^{(d)}} \mathrm{K}_{*-d}^{\mathrm{MW}}(\kappa(x), \omega_{\kappa(x)}).$$

The differential is the sum of the maps

$$\partial_x^y : \mathrm{K}_*^{\mathrm{MW}}(\kappa(y), \omega_{\kappa(y)}) \to \mathrm{K}_{*-1}^{\mathrm{MW}}(\kappa(x), \omega_{\kappa(x)}),$$

for $x, y \in X$ of consecutive codimension and $x \in \overline{y}$. These maps are defined as follows. Let Y be the normalisation of $\overline{y}$, and let $x_1, \ldots, x_r \in Y$ be the codimension-one points of Y that are above x. Then, for $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(\kappa(y))$, we set:

$$\partial_x^y(\alpha) = \sum_{i=1}^{r} \widetilde{\mathrm{N}}_{\kappa(x_i)/\kappa(x)} \circ \partial_{x_i}^y(\alpha).$$

Said differently, there is a commutative triangle as follows:

$$\begin{tikzcd}
\mathrm{K}_*^{\mathrm{MW}}(\kappa(y), \omega_{\kappa(y)}) \arrow[r, "(\partial_{x_i}^y)_i"] \arrow[dr, "\partial_x^y"'] & \bigoplus_{i=1}^{r} \mathrm{K}_{*-1}^{\mathrm{MW}}(\kappa(x_i), \omega_{\kappa(x_i)}) \arrow[d, "(\widetilde{\mathrm{N}}_{\kappa(x_i)/\kappa(x)})_i"] \\
& \mathrm{K}_{*-1}^{\mathrm{MW}}(\kappa(x), \omega_{\kappa(x)}).
\end{tikzcd}$$

It is easy to see that $X \mapsto \mathrm{C}^\bullet(X, \omega_X; \mathrm{K}_*^{\mathrm{MW}})$ is contravariantly functorial for étale maps. This defines a sheaf $\mathrm{C}^\bullet(-, \omega_-; \mathrm{K}_*^{\mathrm{MW}})$ on the small Nisnevich site of any essentially smooth k-scheme.

Remark 4.6.2. Given a line bundle L on X, we can define more generally the Gersten "complex"

$$\mathrm{C}^\bullet(X, \omega_X \otimes L; \mathrm{K}_*^{\mathrm{MW}})$$

by replacing each summand $\mathrm{K}_{*-d}^{\mathrm{MW}}(\kappa(x), \omega_{\kappa(x)})$ with $\mathrm{K}_{*-d}^{\mathrm{MW}}(\kappa(x), \omega_{\kappa(x)} \otimes L_x)$. In particular, taking $L = \omega_X^{-1}$, we obtain the untwisted Gersten "complex" $\mathrm{C}^\bullet(X; \mathrm{K}_*^{\mathrm{MW}})$. For $x \in X$, we set

$$\omega_{x/X} := \omega_{\kappa(x)} \otimes x^\star(\omega_X^{-1}),$$

which is canonically isomorphic to the normal bundle of the inclusion $x \hookrightarrow X$. Thus, we have

$$\mathrm{C}^d(X; \mathrm{K}_*^{\mathrm{MW}}) = \bigoplus_{x \in X^{(d)}} \mathrm{K}_{*-d}^{\mathrm{MW}}(\kappa(x), \omega_{x/X}).$$

In general, all these Gersten "complexes" are contravariant for étale morphisms and they are isomorphic to each other locally on X.

Theorem 4.6.3. *The Gersten "complex"* $\mathrm{C}^*(X; \mathrm{K}_*^{\mathrm{MW}})$ *is a complex.*

Remark 4.6.4. The property that a "complex" of sheaves is a complex can be checked locally. Thus, Theorem 4.6.3 implies that the twisted Gersten "complexes" $\mathrm{C}^*(X, L; \mathrm{K}_*^{\mathrm{MW}})$ are also complexes.

Theorem 4.6.3 is a particular case of [Mor12, Theorem 5.31]. For Milnor K-theory, the analogous result was proven by Kato [Kat86], and we will adapt his proof to the case of Milnor–Witt K-theory. We first need to verify the reciprocity formula for the canonical norms and residues.

Theorem 4.6.5 (Reciprocity for $\mathbb{P}^1$)**.** *Let F be a field essentially smooth over k. Then, for all $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(F(T), \omega_{F(T)})$, we have the identity*

$$\sum_{x \in (\mathbb{P}^1_F)^{(1)}} \widetilde{\mathrm{N}}_{\kappa(x)/F} \circ \partial_x(\alpha) = 0$$

in $\mathrm{K}_{*-1}^{\mathrm{MW}}(F, \omega_F)$. The same formula holds if we twist further by a line defined over F.

Proof. By Theorem 4.3.1, the group $\mathrm{K}_*^{\mathrm{MW}}(F(T), \omega_{F(T)})$ is spanned by elements having at most one nonzero residue at closed points of $\mathbb{A}^1_F$. Thus, it is enough to show the formula assuming that $\partial_x(\alpha) = 0$ for $x \in (\mathbb{A}^1_F)^{(1)} \setminus \{z\}$, where z is a fixed closed point of $\mathbb{A}^1_F$.

Step 1. We want to reduce to the case where z is an F-rational point. Let $E = \kappa(z)$ and denote by $z' \in (\mathbb{A}^1_E)^{(1)}$ the obvious E-rational point of $\mathbb{A}^1_E$ associated to z. Then, by Theorem 4.5.13, we have a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_*^{\mathrm{MW}}(E, \omega_E) \arrow[r] \arrow[d, "\widetilde{\mathrm{N}}_{E/F}"] & \mathrm{K}_*^{\mathrm{MW}}(E(T), \omega_{E(T)}) \arrow[r, "(\partial_{x'})_{x'}"] \arrow[d, "\widetilde{\mathrm{N}}_{E(T)/F(T)}"] & \bigoplus_{x' \in (\mathbb{A}^1_E)^{(1)}} \mathrm{K}_*^{\mathrm{MW}}(\kappa(x'), \omega_{x'}) \arrow[r] \arrow[d, "(\widetilde{\mathrm{N}}_{\kappa(x')/\kappa(x)})_{x,x'}"] & 0 \\
0 \arrow[r] & \mathrm{K}_*^{\mathrm{MW}}(F, \omega_F) \arrow[r] & \mathrm{K}_*^{\mathrm{MW}}(F(T), \omega_{F(T)}) \arrow[r, "(\partial_x)_x"] & \bigoplus_{x \in (\mathbb{A}^1_F)^{(1)}} \mathrm{K}_*^{\mathrm{MW}}(\kappa(x), \omega_x) \arrow[r] & 0.
\end{tikzcd}$$

Thus, we can find $\alpha' \in \mathrm{K}_*^{\mathrm{MW}}(E(T))$ such that:

- $\partial_{z'}(\alpha') = \partial_z(\alpha)$;
- $\partial_{x'}(\alpha') = 0$ for $x' \in (\mathbb{A}^1_E)^{(1)} \setminus \{z'\}$.

By an easy diagram chase, this implies that $\alpha - \widetilde{\mathrm{N}}_{E(T)/F(T)}(\alpha')$ belongs to the image of

$$\mathrm{K}_*^{\mathrm{MW}}(F, \omega_F) \to \mathrm{K}_*^{\mathrm{MW}}(F(T), \omega_{F(T)}).$$

In particular, it has zero residue at all closed points of $\mathbb{P}^1_F$. It is thus sufficient to prove the reciprocity formula for $\widetilde{\mathrm{N}}_{E(T)/F(T)}(\alpha')$. But Theorem 4.5.13, gives also a commutative square

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(E(T), \omega_{E(T)}) \arrow[r, "\partial_\infty"] \arrow[d, "\widetilde{\mathrm{N}}_{E(T)/F(T)}"] & \mathrm{K}^{\mathrm{MW}}_*(E, \omega_\infty) \arrow[d, "\widetilde{\mathrm{N}}_{E/F}"] \\
\mathrm{K}^{\mathrm{MW}}_*(F(T), \omega_{F(T)}) \arrow[r, "\partial_\infty"] & \mathrm{K}^{\mathrm{MW}}_*(F, \omega_\infty).
\end{tikzcd}$$

This shows that the reciprocity formula for α' implies the reciprocity formula for $\widetilde{\mathrm{N}}_{E(T)/F(T)}(\alpha')$.

Step 2. By Step 1, we may assume that z is a rational point. By Theorem 4.3.5, it is enough to show that for every F-rational point $x \in \mathbb{P}^1(F)$ we have a commutative square

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(F(T), \omega_{F(T)}) \arrow[r, "\partial_x"] \arrow[d, "\sim"] & \mathrm{K}^{\mathrm{MW}}_*(F, \omega_F) \arrow[d, "\sim"] \\
\mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[r, "\partial_x^{P_x}"] & \mathrm{K}^{\mathrm{MW}}_*(F),
\end{tikzcd}$$

where the vertical isomorphisms are deduced from a generator $g \in \omega_F$ and the associated generator

$$g \cdot \mathrm{d}T \in \omega_{F(T)} \simeq \omega_F \otimes_F F(T) \cdot \mathrm{d}T.$$

If $x \in \mathbb{A}^1(F)$, then $P_x = T - x$ and the isomorphism

$$\omega_F \otimes \mathfrak{n}_x \to \omega_{F[T]} \otimes_{F[T], x} F$$

sends $g \otimes \overline{T - x}$ to $g \cdot \mathrm{d}T$. Thus, the generators $g \cdot \mathrm{d}T \in \omega_{F[T]}$ and $\overline{T - x} \in \mathfrak{n}_x$ induce the generator $g \in \omega_F$. This proves the commutation of the above square when $x \neq \infty$.

It remains to treat the case of $\infty \in \mathbb{P}^1(F)$. We can deduce this from the case of $0 \in \mathbb{P}^1(F)$ by using the automorphism τ of $\mathbb{P}^1_F$ given by $T \mapsto T^{-1}$. Indeed, we have two commutative squares

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(F(T), \omega_{F(T)}) \arrow[r, "\partial_0"] \arrow[d, "\tau", "\sim"'] & \mathrm{K}^{\mathrm{MW}}_*(F, \omega_F) \arrow[d, equal] \\
\mathrm{K}^{\mathrm{MW}}_*(F(T), \omega_{F(T)}) \arrow[r, "\partial_\infty"] & \mathrm{K}^{\mathrm{MW}}_*(F, \omega_F)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[r, "\partial_0^T"] \arrow[d, "\tau", "\sim"'] & \mathrm{K}^{\mathrm{MW}}_*(F) \arrow[d, equal] \\
\mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[r, "\partial_\infty^{T^{-1}}"] & \mathrm{K}^{\mathrm{MW}}_*(F).
\end{tikzcd}$$

The first one is given by the functoriality of the canonical residue. The second one follows from the fact that τ takes T to T^{-1}. Using that $\partial_\infty^{-T^{-1}} = \langle -1 \rangle \partial_\infty^{T^{-1}}$, it suffices to show that the square

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{MW}}_*(F(T), \omega_{F(T)}) \arrow[r, "\sim"] \arrow[d, "\tau", "\sim"'] & \mathrm{K}^{\mathrm{MW}}_*(F(T)) \arrow[d, "\tau", "\sim"'] \\
\mathrm{K}^{\mathrm{MW}}_*(F(T), \omega_{F(T)}) \arrow[r, "\sim"] & \mathrm{K}^{\mathrm{MW}}_*(F(T))
\end{tikzcd}$$

commutes up to $\langle -1 \rangle$. This follows from the fact that τ maps the generator $g \cdot \mathrm{d}T \in \omega_{F(T)}$ to the generator $-T^2 g\mathrm{d}T \in \omega_{F(T)}$ and the identity $\langle -T^2 \rangle = \langle -1 \rangle$. □

Proof of Theorem 4.6.3. Although the line bundle ω_X was only defined when X is essentially smooth, the Gersten "complex" $\mathrm{C}^\bullet(X, \omega_X; \mathrm{K}^{\mathrm{MW}}_*)$ makes sense for any k-scheme X which is essentially of finite type (i.e., proétale over a k-variety). To show that the differential squares to zero, it is enough to treat the case where X is local of dimension 2 (but possibly singular). Let x be the closed point of X. The Gersten "complex" $\mathrm{C}^\bullet(X, \omega_X; \mathrm{K}_*^{\mathrm{MW}})$ has three nonzero terms and the last one does not change if we replace X with its henselisation at x. Thus, we may assume that X is local henselian. In particular, we can assume that there is a finite morphism

$$e : X \to \mathrm{Spec}(F\{S, T\}) = X_0,$$

where $F\{S, T\}$ denotes the henselisation of $F[S, T]$ at the origin, inducing an isomorphism on the residue fields $F \simeq \kappa(x)$. By Theorem 4.5.13, there is a morphism of "complexes"

$$e_* : \mathrm{C}^\bullet(X, \omega_X; \mathrm{K}_*^{\mathrm{MW}}) \to \mathrm{C}^\bullet(X_0, \omega_{X_0}; \mathrm{K}_*^{\mathrm{MW}})$$

given by the canonical norms $\widetilde{\mathrm{N}}_{\kappa(y)/\kappa(y_0)} : \mathrm{K}_*^{\mathrm{MW}}(y, \omega_y) \to \mathrm{K}_*^{\mathrm{MW}}(y_0, \omega_{y_0})$, for $y \in X$ and $y_0 = e(y)$ its image in X_0. Since e_* induces an isomorphism in degree 2, it is enough to show that the differential of $\mathrm{C}^\bullet(X_0, \omega_{X_0}; \mathrm{K}_*^{\mathrm{MW}})$ squares to zero. Thus, we may assume that $X = \mathrm{Spec}(F\{S, T\})$.

For the remainder of the proof, we set $R = F\{S\}$, $A = F\{S, T\} = R\{T\}$, $K = \mathrm{Frac}(R)$ and $L = \mathrm{Frac}(A)$. Fix a uniformizer $\pi \in R$ (e.g., $\pi = S$), and let $\mathfrak{m}_R \subset R$ be the maximal ideal.

Lemma 4.6.6. *The group $L^\times$ is generated by $A^\times$, π, T, and elements of the form:*

$$1 + a_1 T^{-1} + \cdots + a_n T^{-n} \qquad \textit{where} \qquad a_1, \ldots, a_n \in \mathfrak{m}_R.$$

Proof. Indeed, every irreducible divisor of $X = \mathrm{Spec}(A)$ is either the special fibre of the morphism $X \to \mathrm{Spec}(R)$ or is finite over $\mathrm{Spec}(R)$. In the former case its ideal is generated by π. In the latter case it is the inverse image of an irreducible closed subscheme of $\mathbb{A}^1_R = \mathrm{Spec}(R[T])$ which is finite over $\mathrm{Spec}(R)$ and whose closed point is the origin of $\mathbb{A}^1_F$. But, such a closed subscheme is the zero set of a polynomial of the form $T^n + a_1 T^{n-1} + \cdots + a_n$ with $a_i \in \mathfrak{m}_R$ for $1 \leqslant i \leqslant n$. □

Denote by $x \in X$ the closed point of X and $\eta \in X$ its generic point. Let $y \in X$ be a one-codimensional point of X. Then, for $a \in A^\times$, we have

$$\partial^\eta_y(\alpha \cdot [a]) = \partial^\eta_y(\alpha) \cdot [a(y)] \qquad \text{and} \qquad \partial^y_x(\beta \cdot [a(y)]) = \partial^y_x(\beta) \cdot [a(x)]$$

for every $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(L, \omega_L)$ and $\beta \in \mathrm{K}_{*-1}^{\mathrm{MW}}(\kappa(y), \omega_{\kappa(y)})$. Therefore, it is enough to check the vanishing $\partial \circ \partial = 0$ on products of symbols of elements taken from the following list:

$$\pi, \quad T \quad \text{and} \quad 1 + a_1 T^{-1} + \cdots + a_n T^{-n} \quad \text{with} \quad a_1, \ldots, a_n \in \mathfrak{m}_R. \qquad (\star)$$

(Here we are identifying $\mathrm{K}_*^{\mathrm{MW}}(L, \omega_L)$ with $\mathrm{K}_*^{\mathrm{MW}}(L)$ using the generator $\nu = \mathrm{d}T \wedge \mathrm{d}S$ of the invertible A-module ω_A.) The case of $[\pi][T]\langle \nu \rangle$ can be handled by a direct computation which we leave to the reader. Instead, we consider the case of a symbol

$$\alpha = [f][g_1] \cdots [g_r]\langle \nu \rangle,$$

where $f = 1 + a_1 T^{-1} + \cdots + a_n T^{-n}$, with $a_1, \ldots, a_n \in \mathfrak{m}_R$, and the g_i's are chosen from the list $(\star)$. Note that we may view α as a symbol in $\mathrm{K}_*^{\mathrm{MW}}(K(T), \omega_{K(T)})$.

Let $\xi \in X^{(1)}$ be the generic point of the special fibre of the morphism $X \to \mathrm{Spec}(R)$. As explained in the proof of Lemma 4.6.6, any one-codimensional point $y \in X^{(1)} \smallsetminus \{\xi\}$ maps isomorphically to a point in $\mathbb{A}^1_K$, which we also call y. In fact, we may identify $X^{(1)} \smallsetminus \{\xi\}$ with the subset of $(\mathbb{A}^1_K)^{(1)}$ consisting of those points whose closures in $\mathbb{A}^1_R$ are finite over $\mathrm{Spec}(R)$ and contain the origin of $\mathbb{A}^1_F$ (i.e., the image of the closed point of X). Denote by $\eta_0 = \mathrm{Spec}(K(T))$ the generic point of $\mathbb{A}^1_K$ (i.e., the image of $\eta = \mathrm{Spec}(L)$, the generic point of X). Observe that for $y \in (\mathbb{A}^1_K)^{(1)} \smallsetminus (X^{(1)} \smallsetminus \{\xi\})$ we have $\partial^{\eta_0}_y(\alpha) = 0$. Indeed, for such a point, the closure $\overline{y} \subset \mathbb{A}^1_R$ does not contain the origin of $\mathbb{A}^1_F$. This implies that f and the g_i's are all in $\mathcal{O}^\times_y$.

Recall that our goal is to show that $\partial \circ \partial(\alpha) = 0$, with ∂ the differential of the Gersten "complex" $\mathrm{C}^\bullet(X, \omega_X; \mathrm{K}_*^{\mathrm{MW}})$. We have

$$\partial \circ \partial(\alpha) = \partial_x^{\xi} \circ \partial_{\xi}^{\eta}(\alpha) + \sum_{y \in X^{(1)} \smallsetminus \{\xi\}} \partial_x^y \circ \partial_y^{\eta}(\alpha).$$

Note that $\partial_{\xi}^{\eta}(\alpha) = 0$ because $f \in \mathcal{O}_{X,\xi}^{\times}$ and has residue class 1 in $\kappa(\xi)$. On the other hand, for $y \in X^{(1)} \smallsetminus \{\xi\}$, its closure $\bar{y}$ in X is finite over $\operatorname{Spec}(R)$ and we may consider the commutative diagram

$$\begin{tikzcd}
y \arrow[r] \arrow[d] & \widetilde{\overline{y}} \arrow[d] & \tilde{x} \arrow[l] \arrow[r] \arrow[d] & x \arrow[dl, equal] \\
\operatorname{Spec}(K) \arrow[r] & \operatorname{Spec}(R) & s, \arrow[l] &
\end{tikzcd}$$

where s is the closed point of $\operatorname{Spec}(R)$, $\widetilde{\overline{y}}$ is the normalisation of $\bar{y}$, which is the spectrum of a discrete valuation ring, and $\tilde{x}$ is the closed point of $\widetilde{\overline{y}}$. Note that the above diagram has cartesian squares up to nil-immersions. By Theorem 4.5.13, we have

$$\partial_x^y := \widetilde{\mathrm{N}}_{\kappa(\tilde{x})/\kappa(x)} \circ \partial_{\tilde{x}}^y = \partial_v \circ \widetilde{\mathrm{N}}_{\kappa(y)/K},$$

where ∂_v is the canonical residue associated to the valuation v with valuation ring $R \subset K$. This gives the equality

$$\begin{aligned}
\partial \circ \partial(\alpha) &= \sum_{y \in X^{(1)} \smallsetminus \{\xi\}} \partial_v \circ \widetilde{\mathrm{N}}_{\kappa(y)/K} \circ \partial_y^{\eta}(\alpha) \\
&= \partial_v \left(\sum_{y \in (\mathbb{A}_K^1)^{(1)}} \widetilde{\mathrm{N}}_{\kappa(y)/K} \circ \partial_y^{\eta_0}(\alpha) \right).
\end{aligned}$$

Since $\partial_{\infty}^{\eta_0}(\alpha) = 0$ (because $f \in \mathcal{O}_{\infty}^{\times}$ and has residue class $1 \in \kappa(\infty) = K$), the desired vanishing follows from the reciprocity formula for $\mathbb{P}^1$ as stated in Theorem 4.6.5. □

4.7. **Milnor–Witt K-theory as a strictly $\mathbb{A}^1$-invariant sheaf.** In this subsection, we use what we have learned so far about Milnor–Witt K-theory to promote the assignment $X \to \mathrm{K}_*^{\mathrm{MW}}(X)$ to a strictly $\mathbb{A}^1$-invariant sheaf on Sm_k. As before, we fix a perfect ground field k. In principle, we need to assume that the characteristic of k is different from 2 since we only discussed some of the basic properties of Milnor–Witt K-theory in this case. However, as said before, there is a parallel story in characteristic 2 that was developed by F. Morel in [Mor12, §5.1].

For an essentially smooth k-scheme X, we have $\mathrm{K}_*^{\mathrm{MW}}(X) = \mathrm{H}^0(\mathrm{C}(X; \mathrm{K}_*^{\mathrm{MW}}))$. (Compare with Remark 4.2.14.) Also, note that the assignment $X \mapsto \mathrm{C}^\bullet(X; \mathrm{K}_*^{\mathrm{MW}})$ is functorial for essentially smooth morphisms. We first prove the $\mathbb{A}^1$-invariance of the cohomology of the Gersten complex.

Proposition 4.7.1. *Let X be an essentially smooth local k-scheme. Then the complexes* $\mathrm{C}^\bullet(X; \mathrm{K}_*^{\mathrm{MW}})$ *and* $\mathrm{C}^\bullet(\mathbb{A}_X^1; \mathrm{K}_*^{\mathrm{MW}})$ *are acyclic except in degree* 0. *Moreover, the map*

$$\mathrm{H}^0(\mathrm{C}(X; \mathrm{K}_*^{\mathrm{MW}})) \to \mathrm{H}^0(\mathrm{C}(\mathbb{A}_X^1; \mathrm{K}_*^{\mathrm{MW}}))$$

is an isomorphism.

Proof. We prove this by induction on $d = \dim(X)$ using the same argument as in Propositions 3.5.1 and 3.7.9. When X is the spectrum of a field, this is Theorem 4.3.1. Now assume that $d \geqslant 1$.

We first prove that $\mathrm{C}^\bullet(X;\mathrm{K}^{\mathrm{MW}}_*)$ is acyclic in positive degrees. For this, we use Gabber's presentation lemma. More precisely, we prove the following claim. Given an integral smooth k-variety X' and a point $x \in X'$ of codimension d, for any class $\alpha \in \mathrm{H}^i(\mathrm{C}(X';\mathrm{K}^{\mathrm{MW}}_*))$, with $i \geqslant 1$, there is an open neighbourhood U of x in X' such that $\alpha|_U = 0$. Indeed, let $(\alpha_y)_{y\in X'^{(i)}}$ be a cocycle in $\mathrm{C}^\bullet(X';\mathrm{K}^{\mathrm{MW}}_*)$ representing α. Let $Z \subset X'$ be a 1-codimensional closed subset containing the finite set of points $\{y \in X'^{(i)} \mid \alpha_y \neq 0\}$. Enlarging Z if necessary, we may assume that x is a point of codimension $d-1$ in Z. By Theorem 2.4.14, we may replace X' with an open neighbourhood of x and assume the existence of an étale morphism $e : X' \to \mathbb{A}^1_{S'}$, where S' is an integral smooth k-variety, and such that Z is finite over S' and $Z \simeq e(Z) \simeq e^{-1}e(Z)$. Note that this implies that x maps to a point $s \in S'$ of codimension $d-1$. Now, we may identify the points of Z with their images in $\mathbb{A}^1_{S'}$ and view $(\alpha_y)_y$ as a cocycle in $\mathrm{C}^\bullet(\mathbb{A}^1_{S'};\mathrm{K}^{\mathrm{MW}}_*)$. By the induction hypothesis, we can find an open neighbourhood V of s in S' such that the image of $(\alpha_y)_y$ in $\mathrm{C}^\bullet(\mathbb{A}^1_V;\mathrm{K}^{\mathrm{MW}}_*)$ is a coboundary. The same is true for the image of $(\alpha_y)_y$ in $\mathrm{C}^\bullet(U;\mathrm{K}^{\mathrm{MW}}_*)$ where $U = e^{-1}(\mathbb{A}^1_V)$ which is an open neighbourhood of x in X'.

We finish the proof by showing that $\mathrm{C}^\bullet(X;\mathrm{K}^{\mathrm{MW}}_*) \to \mathrm{C}^\bullet(\mathbb{A}^1_X;\mathrm{K}^{\mathrm{MW}}_*)$ is a quasi-isomorphism. In degree 0, this follows directly from Theorem 4.3.1. In degree $i \geqslant 1$, consider a cocycle $\alpha = (\alpha_y)_{y\in(\mathbb{A}^1_X)^{(i)}}$ in $\mathrm{C}^\bullet(\mathbb{A}^1_X;\mathrm{K}^{\mathrm{MW}}_*)$. As in the proof of Proposition 3.7.9, we partition the i-codimensional points of $\mathbb{A}^1_X$ in two types

$$(\mathbb{A}^1_X)^{(i)} = \left(\coprod_{x\in X^{(i-1)}} (\mathbb{A}^1_x)^{(1)}\right) \coprod \left\{\eta_x \mid x \in X^{(i)}\right\}$$

where η_x is the generic point of $\mathbb{A}^1_x$. Viewing $(\alpha_y)_{y\in\mathbb{A}^1_x}$ as a cocycle in $\mathrm{C}^\bullet(\mathbb{A}^1_x,\omega_{x/X};\mathrm{K}^{\mathrm{MW}}_{*-i})$ and using the acyclicity of this complex (by Theorem 4.3.1), we can modify α by a coboundary and assume it is supported on points of type II. It is then easy to see that $\alpha_{\eta_x} \in \mathrm{K}^{\mathrm{MW}}_*(\mathbb{A}^1_x,\omega_{x/X}) \subset \mathrm{K}^{\mathrm{MW}}_*(\eta_x,\omega_{x/X})$. Said differently α belongs to the image of $\mathrm{C}^\bullet(X;\mathrm{K}^{\mathrm{MW}}_*) \to \mathrm{C}^\bullet(\mathbb{A}^1_X;\mathrm{K}^{\mathrm{MW}}_*)$, and hence is a coboundary by the previous discussion. □

Corollary 4.7.2. *Let X be an essentially smooth k-scheme. Then, the complex of Nisnevich sheaves $U \mapsto \mathrm{C}^\bullet(U;\mathrm{K}^{\mathrm{MW}}_*)$ is a Zariski local resolution of $\mathrm{K}^{\mathrm{MW}}_*$ on $\acute{\mathrm{E}}\mathrm{t}_X$ and its terms are acyclic for the Zariski and Nisnevich topologies.*

Proof. This is immediate consequence of Proposition 4.7.1. □

Below, we denote $\mathrm{Sm}'_k \subset \mathrm{Sm}_k$ the wide subcategory spanned by smooth morphisms. The notion of strict $\mathbb{A}^1$-invariance (from Definition 3.8.1) extends in the obvious way to sheaves of abelian groups on Sm'_k.

Corollary 4.7.3. *Let X be an essentially smooth k-scheme. For all $i \geqslant 0$, there are isomorphisms:*

$$\mathrm{H}^i_{\mathrm{zar}}(X;\mathrm{K}^{\mathrm{MW}}_*) \simeq \mathrm{H}^i_{\mathrm{nis}}(X;\mathrm{K}^{\mathrm{MW}}_*) \simeq \mathrm{H}^i(\mathrm{C}(X;\mathrm{K}^{\mathrm{MW}}_*)).$$

In particular $\mathrm{K}^{\mathrm{MW}}_*$ is a strictly $\mathbb{A}^1$-invariant Nisnevich sheaf on Sm'_k.

Proof. The isomorphisms follow from Corollary 4.7.2. To show that $\mathrm{K}^{\mathrm{MW}}_*$ is strictly $\mathbb{A}^1$-invariant, we need to show that

$$\mathrm{H}^i(X;\mathrm{K}^{\mathrm{MW}}_*) \simeq \mathrm{H}^i(\mathbb{A}^1_X;\mathrm{K}^{\mathrm{MW}}_*),$$

and it is enough to do so when X is local henselian. We then use Proposition 4.7.1 to conclude. □

We finish this section by explaining how to extend $\mathrm{K}^{\mathrm{MW}}_*$ from Sm'_k to Sm_k.

Lemma 4.7.4. *Let X be an essentially smooth k-scheme and let $Y \subset X$ be an essentially smooth hypersurface. Assume that X and Y are irreducible, and denote by η_X and η_Y their generic points. Let $S \subset X$ be the localisation of X at η_Y. Note that $\mathcal{O}(S)$ is a discrete valuation ring with fraction field $\kappa(\eta_X)$, and we have the homomorphism*

$$\mathrm{s}_{\eta_Y} : \mathrm{K}_*^{\mathrm{MW}}(S) \to \mathrm{K}_*^{\mathrm{MW}}(\eta_Y)$$

defined in Remark 4.2.16. Then, the homomorphism s_{η_Y} maps $\mathrm{K}_*^{\mathrm{MW}}(X)$ inside $\mathrm{K}_*^{\mathrm{MW}}(Y)$.

Proof. Since $\mathrm{K}_*^{\mathrm{MW}}$ is a sheaf on Sm'_k, we may argue locally on X. In particular, we may assume that the ideal defining $Y \subset X$ is principal, and we choose a generator $\rho \in \mathcal{O}(X)$ of this ideal. Note that ρ is also a uniformizer of the discrete valuation ring $\mathcal{O}_{X,\eta_Y}$.

Fix an element $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(X)$. Given $y \in Y^{(1)}$, we need to show that $\partial_y^\pi \mathrm{s}_{\eta_Y}(\alpha) = 0$, where $\pi \in \mathcal{O}_{Y,y}$ is a uniformizer. Recall from Proposition 4.2.7 that

$$\mathrm{s}_{\eta_Y}(\alpha) = \langle -1 \rangle \partial_{\eta_Y}^\rho([-\rho] \cdot \alpha).$$

Thus we need to show that $\partial_y^\pi \circ \partial_{\eta_Y}^\rho([-\rho] \cdot \alpha) = 0$ or, equivalently, using the canonical residues homomorphisms, that

$$\partial_y^{\eta_Y} \circ \partial_{\eta_Y}^{\eta_X}([-\rho] \cdot \alpha) = 0.$$

(Here, we use $\overline{\rho}$ to trivialise the normal line bundle of $Y \subset X$, and then use any trivialisation of the normal line bundle of $y \subset Y$.) By Theorem 4.6.3, asserting that $\mathrm{C}^\bullet(X; \mathrm{K}_*^{\mathrm{MW}})$ is a complex, we have $\partial \circ \partial([-\rho] \cdot \alpha) = 0$. Since ρ is invertible on $X \smallsetminus Y$, for every $x \in X^{(1)} \smallsetminus \{\eta_Y\}$, we have $\rho \in \mathcal{O}_{X,x}^\times$ and hence $\partial_x^{\eta_X}([-\rho] \cdot \alpha) = 0$. Thus, we have

$$\partial([-\rho] \cdot \alpha) = \partial_{\eta_Y}^{\eta_X}([-\rho] \cdot \alpha).$$

This finishes the proof. □

In the situation of the previous lemma, we have a restriction morphism

$$\mathrm{s}_Y^X : \mathrm{K}_*^{\mathrm{MW}}(X) \to \mathrm{K}_*^{\mathrm{MW}}(Y).$$

We want to extend this to the case where $Y \subset X$ has arbitrary codimension.

Proposition 4.7.5. *Given an essentially smooth k-scheme X and a closed essentially smooth subscheme $Y \subset X$, there is a well-defined restriction morphism*

$$\mathrm{s}_Y^X : \mathrm{K}_*^{\mathrm{MW}}(X) \to \mathrm{K}_*^{\mathrm{MW}}(Y).$$

These morphisms satisfy the following conditions.

(1) *If $\mathrm{codim}_X(Y) = 1$, and X and Y are connected, then s_Y^X is the morphism previously constructed in Lemma 4.7.4.*

(2) *If $Z \subset Y \subset X$ is a sequence of closed essentially smooth subschemes, then $\mathrm{s}_Z^X = \mathrm{s}_Z^Y \circ \mathrm{s}_Y^X$.*

Proof. Since $\mathrm{K}_*^{\mathrm{MW}}$ is a sheaf on Sm'_k, it is enough to define s_Y^X locally on X. In particular, we can assume that there is chain of irreducible closed essentially smooth subschemes

$$X = H_0 \supset H_1 \supset \cdots \supset H_d = Y,$$

with H_i of codimension 1 in H_{i-1} for all $1 \leqslant i \leqslant d$. We set $\mathrm{s}_Y^X = \mathrm{s}_{H_d}^{H_{d-1}} \circ \cdots \circ \mathrm{s}_{H_1}^{H_0}$ and the whole point is to show that s_Y^X is independent of the choice of the H_i's for $1 \leqslant i \leqslant d-1$.

By a standard argument, one reduces to the case where X is local of dimension 2. We denote by $x \in X$ the closed point of X. Let Y_1 and Y_2 be two essentially smooth hypersurfaces intersecting transversally at x. We want to show that

$$\mathrm{s}_x^{Y_1} \circ \mathrm{s}_{Y_1}^X = \mathrm{s}_x^{Y_2} \circ \mathrm{s}_{Y_2}^X.$$

We denote by ξ_1 and ξ_2 the generic points of Y_1 and Y_2. Let π_1 and π_2 be generators of the ideals in $\mathcal{O}(X)$ defining Y_1 and Y_2, which we view also as uniformizers of the discrete valuation rings $\mathcal{O}_{X,\xi_1}$ and $\mathcal{O}_{X,\xi_2}$. We set $\bar{\pi}_1 = \pi_1|_{Y_2}$ and $\bar{\pi}_2 = \pi_2|_{Y_1}$. Then, $\bar{\pi}_1$ is a uniformizer of $\mathcal{O}_{Y_2,x}$ while $\bar{\pi}_2$ is a uniformizer of $\mathcal{O}_{Y_1,x}$. Now, for $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(X)$, we have

$$\begin{aligned}
\mathrm{s}_x^{Y_1} \circ \mathrm{s}_{Y_1}^X(\alpha) &= \mathrm{s}_x^{Y_1}(\langle -1\rangle \partial_{\xi_1}^{\pi_1}([-\pi_1] \cdot \alpha)) \\
&= \langle -1\rangle \partial_x^{\bar{\pi}_2}([-\bar{\pi}_2]\langle -1\rangle \partial_{\xi_1}^{\pi_1}([-\pi_1] \cdot \alpha)) \\
&= (-1)_\epsilon \partial_x^{\bar{\pi}_2} \partial_{\xi_1}^{\pi_1}([-\pi_2][-\pi_1] \cdot \alpha),
\end{aligned}$$

and similarly for Y_2. Thus, to conclude, we need to show that

$$\partial_x^{\bar{\pi}_1} \circ \partial_{\xi_2}^{\pi_2}([-\pi_1][-\pi_2] \cdot \alpha) = \partial_x^{\bar{\pi}_2} \circ \partial_{\xi_1}^{\pi_1}([-\pi_2][-\pi_1] \cdot \alpha).$$

Equivalently, we need to show that

$$\partial_x^{\bar{\pi}_1} \circ \partial_{\xi_2}^{\pi_2}([-\pi_1][-\pi_2] \cdot \alpha) + \langle -1\rangle \partial_x^{\bar{\pi}_2} \circ \partial_{\xi_1}^{\pi_1}([-\pi_1][-\pi_2] \cdot \alpha) = 0.$$

To do so, we again use Theorem 4.6.3 asserting that $\mathrm{C}^\bullet(X, \omega_X; \mathrm{K}_*^{\mathrm{MW}})$ is a complex. More precisely, we use that

$$\partial \circ \partial([-\pi_1][-\pi_2] \cdot \alpha \cdot \langle \mathrm{d}\pi_1 \wedge \mathrm{d}\pi_2\rangle) = 0$$

where $\mathrm{d}\pi_1 \wedge \mathrm{d}\pi_2$ is viewed as a generator of $\omega_{X/\kappa(x)} = \Omega^2_{X/\kappa(x)}$ (using an essentially smooth retraction $X \to x$ of the obvious inclusion).

Denote by $\eta \in X$ the generic point of X. Since $\alpha \in \mathrm{K}_*^{\mathrm{MW}}(X)$ is unramified over X, we have

$$\partial_y^\eta([-\pi_1][-\pi_2] \cdot \alpha \cdot \langle \mathrm{d}\pi_1 \wedge \mathrm{d}\pi_2\rangle) = 0$$

for every $y \in X^{(1)} \setminus \{\xi_1, \xi_2\}$. On the other hand, we have

$$\begin{aligned}
\partial_{\xi_1}^\eta([-\pi_1][-\pi_2] \cdot \alpha \cdot \langle \mathrm{d}\pi_1 \wedge \mathrm{d}\pi_2\rangle) &= \langle -1\rangle \cdot \partial_{\xi_1}^\eta([-\pi_1][-\pi_2] \cdot \alpha \cdot \langle \mathrm{d}\pi_2 \wedge \mathrm{d}\pi_1\rangle) \\
&= \langle -1\rangle \cdot \partial_{\xi_1}^{\pi_1}([-\pi_1] \cdot [-\pi_2] \cdot \alpha)\langle \mathrm{d}\bar{\pi}_2\rangle
\end{aligned}$$

whereas

$$\partial_{\xi_2}^\eta([-\pi_1][-\pi_2] \cdot \alpha \cdot \langle \mathrm{d}\pi_1 \wedge \mathrm{d}\pi_2\rangle) = \partial_{\eta_{Y_2}}^{\pi_2}([-\pi_1][-\pi_2] \cdot \alpha)\langle \mathrm{d}\bar{\pi}_1\rangle.$$

Thus, the relation $\partial \circ \partial([-\pi_1][-\pi_2] \cdot \alpha \cdot \langle \mathrm{d}\pi_1 \wedge \mathrm{d}\pi_2\rangle) = 0$ becomes

$$(\partial_x^{\bar{\pi}_1} \circ \partial_{\xi_2}^{\pi_2} + \langle -1\rangle \cdot \partial_x^{\bar{\pi}_2} \circ \partial_{\xi_1}^{\pi_1})([-\pi_1][-\pi_2] \cdot \alpha) = 0$$

as needed. □

Combining Proposition 4.7.5 with the easy functoriality of $X \mapsto \mathrm{K}_*^{\mathrm{MW}}(X)$ for smooth morphisms, we obtain the desired extension of $\mathrm{K}_*^{\mathrm{MW}}$ as a sheaf on Sm_k. Together with Corollary 4.7.3, this gives the following result.

Theorem 4.7.6 (F. Morel). *For every $n \in \mathbb{Z}$, there is a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups $\mathrm{K}_n^{\mathrm{MW}}$ on Sm_k whose value on an essentially smooth field extension F/k is the Milnor–Witt K-theory group $\mathrm{K}_n^{\mathrm{MW}}(F)$. Moreover, the contraction $(\mathrm{K}_n^{\mathrm{MW}})_{-1}$ of $\mathrm{K}_n^{\mathrm{MW}}$ is canonically isomorphic to $\mathrm{K}_{n-1}^{\mathrm{MW}}$.*

5. HOMOTOPY SHEAVES OF MOTIVIC SPHERES

In this section, we show that the Milnor–Witt K-theory sheaves introduced in Theorem 4.7.6 arise naturally as homotopy sheaves of motivic spheres. We work over a field k. When applying Theorem 4.7.6, one would a priori need to assume that the characteristic of k is different from 2, since this assumption was in force throughout the second half of Section 4. However, as mentioned previously, Theorem 4.7.6 remains valid in characteristic 2; its proof in this case requires additional arguments, which are given in [Mor12].

5.1. Preliminaries on suspension.

We gather here some simple well-known facts about suspensions in an ∞-category. Throughout this subsection, we let C be an ∞-category having finite limits and colimits. We denote by $*$ a final object of C and we let C_* be the ∞-category of pointed objects in C. We are mostly interested in the case where C is the ∞-category $\mathrm{Spc}(k)$ of k-spaces or the Morel–Voevodsky ∞-category $\mathcal{H}(k)$ of motivic spaces.

Recollection 5.1.1. The suspension of an object $A \in C$, denoted by ΣA, is defined by the pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma A. \end{tikzcd}$$

If A is a pointed object, then its suspension ΣA is also naturally pointed by the composite map

$$* \simeq \Sigma(*) \to \Sigma A.$$

This can be promoted into endofunctors Σ of C and C_*. For $n \geqslant 0$, we write Σ^n for n-th suspension endofunctor obtained by repeatedly applying the suspension endofunctor n times.

Recollection 5.1.2. The coproduct of two pointed objects $A, B \in C_*$ is denoted by $A \vee B$. As an object of C, it is defined by the pushout square

$$\begin{tikzcd} * \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A \vee B, \end{tikzcd}$$

and it is pointed by the composite map

$$* \simeq * \vee * \to A \vee B.$$

On the other hand, the product of the pointed objects A and B is their product $A \times B$ in C pointed by the composite map

$$* \simeq * \times * \to A \times B.$$

The smash product $A \wedge B$ is then defined by the pushout square

$$\begin{tikzcd} A \vee B \arrow[r] \arrow[d] & A \times B \arrow[d] \\ * \arrow[r] & A \wedge B \end{tikzcd}$$

where the top horizontal map is induced by the two composite maps

$$A \vee B \to A \vee * \simeq A \qquad \text{and} \qquad A \vee B \to * \vee B \simeq B.$$

Note that $A \wedge B$ is naturally a pointed object of C.

Lemma 5.1.3. *Assume that finite direct products distribute over finite colimits in C.*

(1) The object $S^0 = * \coprod *$ pointed by one of the two copies of $*$ is the unit for the smash product, i.e., we have an equivalence $S^0 \wedge A \simeq A$ for every $A \in C_*$.
(2) For $A, B \in C_*$, there is an equivalence $(\Sigma A) \wedge B \simeq \Sigma(A \wedge B)$.

Proof. Part (1) is clear. For part (2), we note that there is a pushout square

$$\begin{tikzcd}
((* \coprod_A *) \times *) \vee ((* \coprod_* *) \times B) \arrow[r] \arrow[d] & (* \coprod_A *) \times B \arrow[d] \\
* \arrow[r] & (\Sigma A) \wedge B.
\end{tikzcd}$$

Using that the functor $- \times B$ preserves finite colimits, we can rewrite this square as follows:

$$\begin{tikzcd}
(* \coprod_{A \times *} *) \vee (B \coprod_{* \times B} B) \arrow[r] \arrow[d] & B \coprod_{A \times B} B \arrow[d] \\
* \arrow[r] & (\Sigma A) \wedge B.
\end{tikzcd}$$

Modding out first by $B \coprod_{* \times B} B$, we deduce the pushout square

$$\begin{tikzcd}
(* \coprod_{A \times *} *) \arrow[r] \arrow[d] & * \coprod_{A \times B / * \times B} * \arrow[d] \\
* \arrow[r] & (\Sigma A) \wedge B.
\end{tikzcd}$$

This yields an equivalence $* \coprod_{A \wedge B} * \simeq (\Sigma A) \wedge B$ as needed. □

An important fact that we will use is:

Lemma 5.1.4. *For every pointed object $A \in C_*$, ΣA is naturally a cogroup in C_*.*

Proof. We do not need the precise definition of a (co)group in an ∞-category. Instead, we just describe the comultiplication map $\Sigma A \to \Sigma A \vee \Sigma A$. Note that $\Sigma A \vee \Sigma A$ is the colimit of the solid part in the following diagram:

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d] & * \arrow[d, dotted] \\
A \arrow[r] \arrow[d] & * \arrow[r, dotted] \arrow[d, dotted] & \Sigma A \arrow[d, dotted] \\
* \arrow[r, dotted] & \Sigma A \arrow[r, dotted] & \Sigma A \vee \Sigma A.
\end{tikzcd}$$

On the other hand, ΣA can be defined as the colimit of the solid part in the following diagram:

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, equal] & * \arrow[d, dotted] \\
A \arrow[r, equal] \arrow[d] & A \arrow[r, dotted] \arrow[d, dotted] & * \arrow[d, dotted] \\
* \arrow[r, dotted] & * \arrow[r, dotted] & \Sigma A.
\end{tikzcd}$$

There is an obvious map from the second solid diagram to the first one inducing the desired comultiplication map. □

Remark 5.1.5. Lemma 5.1.4 is dual to the well-known fact that the loop space ΩA of a pointed object $A \in \mathcal{C}_*$ is naturally a group object in $\mathcal{C}_*$.

Construction 5.1.6. Let A and B be pointed objects in $\mathcal{C}$. Then we have maps

$$A \times B \to A, \quad A \times B \to B \quad \text{and} \quad A \times B \to A \wedge B$$

in $\mathcal{C}_*$. Taking suspension, these yield morphisms of cogroups:

$$\Sigma(A \times B) \to \Sigma A, \quad \Sigma(A \times B) \to \Sigma B \quad \text{and} \quad \Sigma(A \times B) \to \Sigma(A \wedge B).$$

These are morphisms of cogroups. Using the comultiplication on $\Sigma(A \times B)$, we can assemble these into one morphism in $\mathcal{C}_*$:

$$\Sigma(A \times B) \to \Sigma A \vee \Sigma B \vee \Sigma(A \wedge B) \qquad (\star).$$

We have the following well-known fact.

Lemma 5.1.7. *The map* $(\star)$ *from Construction 5.1.6 is an equivalence.*

Proof. By construction, we have a cofiber sequence

$$A \vee B \to A \times B \to A \wedge B$$

yielding a cofiber sequence

$$\Sigma(A \vee B) \to \Sigma(A \times B) \to \Sigma(A \wedge B).$$

The latter sequence splits by the map

$$\Sigma(A \times B) \to \Sigma A \vee \Sigma B \simeq \Sigma(A \vee B)$$

induced by the comultiplication of $\Sigma(A \times B)$. We conclude using the following general result. □

Lemma 5.1.8. *Let* $U \xrightarrow{\alpha} V \xrightarrow{\beta} W$ *be a cofibre sequence in* $\mathcal{C}_*$*, with* U*,* V *and* W *cogroups, and* α *and* β *morphisms of cogroups. Assume that* α *has a retraction* $\rho : V \to U$ *and consider the map*

$$V \to U \vee W \qquad (\star)$$

deduced from the comultiplication of V*. Then* $(\star)$ *is an equivalence.*

Proof. By Yoneda, it suffices to show that the induced map

$$\pi_i \mathrm{Map}_{\mathcal{C}_*}(U \vee W, X) \to \pi_i \mathrm{Map}_{\mathcal{C}_*}(V, X)$$

is a bijection for every object $X \in \mathcal{C}_*$ and every integer $i \geqslant 0$. This map can be identified with the map

$$\pi_i \mathrm{Map}_{\mathcal{C}_*}(U, X) \times \pi_i \mathrm{Map}_{\mathcal{C}_*}(W, X) \to \pi_i \mathrm{Map}_{\mathcal{C}_*}(V, X)$$

obtained from the exact sequence of groups

$$\pi_i \mathrm{Map}_{\mathcal{C}_*}(W, X) \xrightarrow{(1)} \pi_i \mathrm{Map}_{\mathcal{C}_*}(V, X) \xrightarrow{(2)} \pi_i \mathrm{Map}_{\mathcal{C}_*}(U, X)$$

together with its splitting provided by the retraction ρ. To conclude, it remains to see that the above exact sequence is a short exact sequence. Note that the existence of the retraction ρ ensures that the map (2) is surjective. It also ensures that the map (1) is injective since its kernel is in bijection with the cokernel of $\pi_{i+1} \mathrm{Map}_{\mathcal{C}_*}(V, X) \to \pi_{i+1} \mathrm{Map}_{\mathcal{C}_*}(U, X)$. □

Lemma 5.1.9. *Let $A \in C_*$ be a pointed object and consider the pointed object A_+ obtained by adding a new base point to A. Then, there is an equivalence*

$$\Sigma(A_+) \simeq S^1 \vee \Sigma A$$

where $S^1 = \Sigma(S^0)$ is the one-dimensional sphere.

Proof. There is a cofibre sequence

$$S^0 \to A_+ \to A$$

where the first map sends the free point of S_0 to the base point of A. Note that this map has a retraction induced by $A \to *$. Thus, suspending, we get a cofibre sequence

$$S^1 \to \Sigma(A_+) \to \Sigma A$$

to which Lemma 5.1.8 applies. □

5.2. **Milnor–Witt K-theory as homotopy groups of motivic spheres.** Milnor–Witt K-theory was introduced in Section 4 by generators and relations. Our goal is to show that it arises naturally in motivic homotopy theory. The next theorem, which is the main result of this section, does not require the ground field k to be perfect. See Remark 5.2.3 below.

Theorem 5.2.1 (Morel)**.** *For $n \geqslant 1$ and $m \geqslant 2$, there is an isomorphism*

$$\pi_m^{\mathbb{A}^1}(\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n}) \simeq \mathrm{K}_n^{\mathrm{MW}}$$

of strictly $\mathbb{A}^1$-invariant sheaves.

Remark 5.2.2.

1. The condition $m \geqslant 2$ is necessary and ensures that $\pi_m^{\mathbb{A}^1}(\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n})$ is abelian. In fact, the homotopy sheaf $\pi_1^{\mathbb{A}^1}(\Sigma \mathbb{G}_{\mathrm{m}})$ is nonabelian as shown in [Mor12, §7.3].
2. Also, the condition $n \geqslant 1$ is necessary. Indeed, $\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 0} = S^m$ is the m-th simplicial sphere, and its $\pi_m^{\mathbb{A}^1}$ is the constant sheaf $\mathbb{Z}$.

Remark 5.2.3. The reader might worry that the sheaf $\mathrm{K}_n^{\mathrm{MW}}$ was only defined under the assumption that the ground field k is perfect. In fact, the groups $\mathrm{K}_n^{\mathrm{MW}}(X)$ make sense for all regular noetherian schemes X; see Definition 4.2.12. Moreover, for a smooth k-variety X, the group $\mathrm{K}_n^{\mathrm{MW}}(X)$ is precisely the value on X of the left Kan extension of the sheaf $\mathrm{K}_n^{\mathrm{MW}}$ on Sm_{k_0}, where $k_0 \subset k$ is the prime subfield of k. In other words, it is sensible to define the sheaf $\mathrm{K}_n^{\mathrm{MW}}$ on Sm_k, for any field k, as being the left Kan extension of the sheaf defined in Subsection 4.7 over the perfect field k_0. Similarly, the sheaf $\pi_m^{\mathbb{A}^1}(\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n})$ on Sm_k is the left Kan extension of the analogous sheaf on Sm_{k_0}. Thus, it is enough to prove Theorem 5.2.1 when the ground field is perfect.

We start by noting the following.

Lemma 5.2.4. *There is a unique morphism of presheaves of abelian groups*

$$\mathbb{Z}[\mathbb{G}_{\mathrm{m}}^{\wedge n}] \to \mathrm{K}_n^{\mathrm{MW}}, \quad \textit{for} \quad n \geqslant 1,$$

sending a section $(u_1, \ldots, u_n)$ of $\mathbb{G}_{\mathrm{m}}^{\wedge n}$ to $[u_1] \cdots [u_n]$. Moreover, when evaluated on essentially smooth field extensions of k, this morphism is surjective.

Proof. The last assertion is contained in Lemma 4.1.11. Thus, we only need to prove the existence of a morphism as in the statement. (Uniqueness is clear.) We may assume that k is perfect. If X is an irreducible essentially smooth k-scheme with fraction field F, then for $u_1, \dots, u_n \in \mathcal{O}(X)^\times$, the symbol $[u_1]\cdots[u_n] \in \mathrm{K}_n^{\mathrm{MW}}(F)$ is unramified at every one-codimensional point of X. (See Theorem 4.2.2.) This proves that $[u_1]\cdots[u_n]$ belongs to $\mathrm{K}_n^{\mathrm{MW}}(X)$ yielding the map

$$\mathbb{Z}[\mathbb{G}_{\mathrm{m}}^{\wedge n}](X) \to \mathrm{K}_n^{\mathrm{MW}}(X).$$

It remains to see that this map is functorial in X. By an easy argument we reduce to showing this for a one-codimensional closed immersion between irreducible essentially smooth k-schemes. Looking at the generic points we are left to showing the following. Let v be a discrete valuation on a field F such that $\mathcal{O}_v$ is essentially smooth over k. Then the following square is commutative:

$$\begin{tikzcd}
\mathcal{O}_v^\times \times \cdots \times \mathcal{O}_v^\times \arrow[r] \arrow[d] & \mathrm{K}_n^{\mathrm{MW}}(\mathcal{O}_v) \arrow[d, "s_v"] \\
k_v^\times \times \cdots \times k_v^\times \arrow[r] & \mathrm{K}_n^{\mathrm{MW}}(k_v).
\end{tikzcd}$$

This follows from Corollary 4.2.8. □

Construction 5.2.5. Let M be a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups on Sm_k and suppose we are given a morphism of pointed sheaves of sets:

$$\rho : \mathbb{G}_{\mathrm{m}}^{\wedge n} \to M.$$

Then ρ determines uniquely a morphism of k-spaces

$$\rho_m : \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n} \to \mathrm{K}(M, m), \quad \text{for every} \quad m \geqslant 0.$$

Indeed, by adjunction, we have

$$\begin{aligned}
\mathrm{Map}(\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n}, \mathrm{K}(M, m)) &\simeq \mathrm{Map}(\mathbb{G}_{\mathrm{m}}^{\wedge n}, \Omega^m \mathrm{K}(M, m)) \\
&\simeq \mathrm{Map}(\mathbb{G}_{\mathrm{m}}^{\wedge n}, \mathrm{K}(M, 0)) \\
&\simeq \mathrm{Map}(\mathbb{G}_{\mathrm{m}}^{\wedge n}, M).
\end{aligned}$$

Since $\mathrm{K}(M, m)$ is a motivic space, the map ρ_m factors uniquely as:

$$\rho'_m : \mathrm{L}_{\mathbb{A}^1} \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n} \to \mathrm{K}(M, m).$$

Note that $\mathrm{L}_{\mathbb{A}^1} \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n}$ is m-connective by the $\mathbb{A}^1$-connectivity theorem (see Theorem 2.3.3). Moreover, passing to the m-th homotopy sheaves, ρ'_m gives rise to a morphism:

$$\pi_m^{\mathbb{A}^1}(\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n}) \to M$$

whose domain is strictly $\mathbb{A}^1$-invariant when $m \geqslant 2$ (by Theorem 3.8.4 and left Kan extension from a perfect ground field).

Applying Construction 5.2.5 to the morphism of pointed sheaves $\mathbb{G}_{\mathrm{m}}^{\wedge n} \to \mathrm{K}_n^{\mathrm{MW}}$ provided by Lemma 5.2.4, we obtain the following.

Lemma 5.2.6. *For $m \geqslant 2$ and $n \geqslant 1$, there is a unique morphism of strictly $\mathbb{A}^1$-invariant sheaves*

$$\pi_m^{\mathbb{A}^1}(\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n}) \to \mathrm{K}_n^{\mathrm{MW}}$$

making the following triangle

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{G}_{\mathrm{m}}^{\wedge n}] \arrow[r] \arrow[dr] & \pi_m^{\mathbb{A}^1}(\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n}) \arrow[d] \\
& \mathrm{K}_n^{\mathrm{MW}}
\end{tikzcd}$$

commutative.

We can now state slightly more precisely the main theorem of this section.

Theorem 5.2.7 (F. Morel)**.** *For $n \geqslant 1$ and $m \geqslant 2$, the map*

$$\pi_m^{\mathbb{A}^1}(\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n}) \to \mathrm{K}_n^{\mathrm{MW}},$$

from Lemma 5.2.6, is an isomorphism.

To prove Theorem 5.2.7, it will be convenient to work in the following setting.

Setting 5.2.8. We fix a morphism of sheaves of pointed sets

$$\rho : \mathbb{G}_{\mathrm{m}}^{\wedge n} \to M$$

with $n \geqslant 1$ and M a strictly $\mathbb{A}^1$-invariant sheaf. Given an essentially smooth field extension F/k, we denote by $[u_1, \ldots, u_n] \in M(F)$ the image by ρ of an n-tuple $(u_1, \ldots, u_n) \in (F^\times)^n$.

More generally, we have elements $[\eta^r, u_1, \ldots, u_{n+r}] \in M(F)$ for every $r \geqslant 0$ and $u_1, \ldots, u_{n+r} \in F^\times$. To explain the construction of these elements, we need to introduce the Hopf map.

Construction 5.2.9. For $m \geqslant 1$, there is a map

$$\eta : \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2} \to \Sigma^m \mathbb{G}_{\mathrm{m}}$$

of pointed k-spaces, called the Hopf map, which is defined as follows. By Lemma 5.1.7, there is an equivalence

$$\Sigma^m(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}) \xrightarrow{\sim} \Sigma^m \mathbb{G}_{\mathrm{m}} \vee \Sigma^m \mathbb{G}_{\mathrm{m}} \vee \Sigma^m(\mathbb{G}_{\mathrm{m}} \wedge \mathbb{G}_{\mathrm{m}}).$$

In particular, there is a map of k-spaces $\Sigma^m(\mathbb{G}_{\mathrm{m}}^{\wedge 2}) \to \Sigma^m(\mathbb{G}_{\mathrm{m}}^{\times 2})$ and the desired map η is defined to be the composition of

$$\Sigma^m(\mathbb{G}_{\mathrm{m}}^{\wedge 2}) \to \Sigma^m(\mathbb{G}_{\mathrm{m}}^{\times 2}) \xrightarrow{\Sigma^m \mu} \Sigma^m \mathbb{G}_{\mathrm{m}},$$

where $\mu : \mathbb{G}_{\mathrm{m}}^{\times 2} \to \mathbb{G}_{\mathrm{m}}$ is the multiplication of $\mathbb{G}_{\mathrm{m}}$.

Lemma 5.2.10. *If $m \geqslant 2$, the triangle*

$$\begin{tikzcd}
\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2} \arrow[r, "\eta"] \arrow[d, "\tau"] & \Sigma^m \mathbb{G}_{\mathrm{m}} \\
\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2}, \arrow[ur, "\eta"'] &
\end{tikzcd}$$

where τ is the permutation of factors, commutes (up to homotopy).

Proof. Since the multiplication of $\mathbb{G}_{\mathrm{m}}$ is commutative, it is enough to show that the square

$$\begin{tikzcd}
\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2} \arrow[r] \arrow[d, "\tau"] & \Sigma^m \mathbb{G}_{\mathrm{m}}^{\times 2} \arrow[d, "\tau"] \\
\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2} \arrow[r] & \Sigma^m \mathbb{G}_{\mathrm{m}}^{\times 2}
\end{tikzcd}$$

commutes. This follows from the constructions and the well-known fact that the comultiplication of doubly suspended k-spaces is cocommutative up to homotopy. □

Corollary 5.2.11. *For $r \geqslant 0$, there is a map of k-spaces*

$$\eta^r : \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge r+1} \to \Sigma^m \mathbb{G}_{\mathrm{m}},$$

which is unique up to homotopy. Moreover, for $r, s \geqslant 0$, the following square

$$\begin{tikzcd}
\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge r+s+1} \arrow[r, "\mathrm{id}\wedge\eta^s"] \arrow[d, "\eta^r\wedge\mathrm{id}"] & \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge r+1} \arrow[d, "\eta^r"] \\
\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge s+1} \arrow[r, "\eta^s"] & \Sigma^m \mathbb{G}_{\mathrm{m}}
\end{tikzcd}$$

commutes up to homotopy.

We now return to Setting 5.2.8 and introduce the following notation.

Notation 5.2.12. We denote by

$$\eta^r\rho : \mathbb{G}_{\mathrm{m}}^{\wedge n+r} \to M$$

the map corresponding to the composition of

$$\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n+r} \xrightarrow{\eta^r\wedge\mathrm{id}} \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n} \xrightarrow{\rho} \mathrm{K}(M, m) \qquad (\text{for } m \geqslant 2)$$

under the bijection $\pi_0\mathrm{Map}(\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n+r}, \mathrm{K}(M, m)) \simeq \pi_0\mathrm{Map}(\mathbb{G}_{\mathrm{m}}^{\wedge n+r}, M)$. For an essentially smooth field extension F/k, we denote by

$$[\eta^r, u_1, \ldots, u_{n+r}] \in M(F)$$

the image of an $n + r$-tuple $(u_1, \ldots, u_{n+r}) \in (F^\times)^{n+r}$ by the morphism $\eta^r\rho$.

The key proposition is the following. (Compare with Definition 4.1.3.)

Proposition 5.2.13. *Let F/k be an essentially smooth field extension. The symbols*

$$[\eta^r, u_1, \ldots, u_{n+r}] \in M(F)$$

satisfy all the relations we have in Milnor–Witt K-theory, namely:

(1_n) (Steinberg) $[\eta^r, u_1, \ldots, u_{n+r}] = 0$ *if* $u_i + u_{i+1} = 1$ *for some* $1 \leq i \leq n + r - 1$.

(2_n) For $a, b \in F^\times$,

$$[\eta^r, u_1, \ldots, ab, \ldots, u_{n+r}] = [\eta^r, u_1, \ldots, a, \ldots, u_{n+r}] + [\eta^r, u_1, \ldots, b, \ldots, u_{n+r}] \\ +[\eta^{r+1}, u_1, \ldots, a, b, \ldots, u_{n+r}].$$

(4_n) For each $1 \leqslant i \leqslant n + r + 1$,

$$[\eta^{r+2}, u_1, \ldots, u_{i-1}, -1, u_i, \ldots, u_{n+r+1}] + 2[\eta^{r+1}, u_1, \ldots, u_{n+r+1}] = 0.$$

The proof of Proposition 5.2.13 will be given in the next two subsections. Here, we assume this proposition and use it to prove F. Morel's theorem.

Proof of Theorem 5.2.7. As explained before, we may assume that k is perfect. We apply Proposition 5.2.13 with

$$M = \pi_m^{\mathbb{A}^1}(\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge n}), \quad \text{for} \quad m \geqslant 2 \quad \text{and} \quad n \geqslant 1,$$

and the obvious map of pointed sheaves $\rho : \mathbb{G}_{\mathrm{m}}^{\wedge n} \to M$. Combining Proposition 5.2.13, with Lemmas 4.1.4 and 5.2.6, we see that the image of the map $\rho : (F^\times)^n \to M(F)$ spans a copy of $\mathrm{K}_n^{\mathrm{MW}}(F)$ for every essentially smooth field extension F/k. More precisely, we have a commutative diagram

$$\begin{tikzcd}
& \mathbb{Z}[(F^\times)^{\wedge n}] \arrow[dl, two heads] \arrow[d] \arrow[dr, two heads] & \\
\mathrm{K}_n^{\mathrm{MW}}(F) \arrow[r] \arrow[rr, equal, bend right] & M(F) \arrow[r] & \mathrm{K}_n^{\mathrm{MW}}(F).
\end{tikzcd}$$

Moreover, given a discrete valuation v on F such that $\mathcal{O}_v$ is essentially smooth over k, the map $M(\mathcal{O}_v) \to M(F)$ is injective and, by Theorem 4.2.15, the image of the map $\rho : (\mathcal{O}_v^\times)^n \to M(F)$ spans a copy of $\mathrm{K}_n^{\mathrm{MW}}(\mathcal{O}_v)$. Thus, we also have commutative diagrams

$$\begin{tikzcd}
& \mathbb{Z}[(\mathcal{O}_v^\times)^{\wedge n}] \arrow[dl, two heads] \arrow[d] \arrow[dr, two heads] & \\
\mathrm{K}_n^{\mathrm{MW}}(\mathcal{O}_v) \arrow[r] \arrow[rr, equal, bend right] & M(\mathcal{O}_v) \arrow[r] & \mathrm{K}_n^{\mathrm{MW}}(\mathcal{O}_v)
\end{tikzcd}$$

which are contained in the previous ones. Now, if X is an irreducible essentially smooth k-scheme, we may intersect the above diagrams when $\mathcal{O}_v$ varies among the $\mathcal{O}_{X,x}$, for $x \in X^{(1)}$. This yields the following commutative diagram:

$$\begin{tikzcd}
& \mathbb{Z}[(\mathcal{O}(X)^\times)^{\wedge n}] \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{K}_n^{\mathrm{MW}}(X) \arrow[r] \arrow[rr, equal, bend right] & M(X) \arrow[r] & \mathrm{K}_n^{\mathrm{MW}}(X).
\end{tikzcd}$$

We claim that the subgroups $\mathrm{K}_n^{\mathrm{MW}}(X) \subset M(X)$, for $X \in \mathrm{Sm}_k$, define a subsheaf of M. To prove this, we reduce as usual to treat the functoriality for a one-codimensional closed immersion $Y \to X$ between irreducible essentially smooth k-schemes. Using that M and $\mathrm{K}_n^{\mathrm{MW}}$ are unramified, we can reduce further to the case where $X = \mathrm{Spec}(\mathcal{O}_v)$ and $Y = \mathrm{Spec}(k_v)$ for an essentially smooth discrete valuation k-algebra $\mathcal{O}_v$. The desired result is clear in this case because the subgroup $\mathrm{K}_n^{\mathrm{MW}}(\mathcal{O}_v) \subset M(\mathcal{O}_v)$ is spanned by the image of $(\mathcal{O}_v^\times)^n \to M(\mathcal{O}_v)$ by Theorem 4.2.15.

It is now easy to conclude. We have constructed a commutative diagram of sheaves

$$\begin{tikzcd}
& \mathbb{Z}[(\mathbb{G}_{\mathrm{m}})^{\wedge n}] \arrow[dl, two heads] \arrow[d] \arrow[dr, two heads] & \\
\mathrm{K}_n^{\mathrm{MW}} \arrow[r] \arrow[rr, equal, bend right] & M \arrow[r] & \mathrm{K}_n^{\mathrm{MW}}.
\end{tikzcd}$$

But, as explained in Construction 5.2.5, $\pi_m^{\mathbb{A}^1}(\Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge n})$ is the initial strictly $\mathbb{A}^1$-invariant sheaf receiving a map from $\mathbb{Z}[(\mathbb{G}_{\mathrm{m}})^{\wedge n}]$. This forces $M = \pi_m^{\mathbb{A}^1}(\Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge n})$ to be isomorphic to $\mathrm{K}_n^{\mathrm{MW}}$. □

5.3. **The relations (2_n) and (4_n).** Given a morphism of pointed sheaves

$$\rho : \mathbb{G}_{\mathrm{m}}^{\wedge n} \to M,$$

with $n \geqslant 1$ and M strictly $\mathbb{A}^1$-invariant, we have constructed elements

$$[\eta^r, u_1, \ldots, u_{n+r}] \in M(F)$$

for all $u_1, \ldots, u_{n+r} \in F^\times$ with F/k an essentially smooth extension. To complete the proof of Morel's theorem, we still need to check that these symbols satisfy the relations (1_n), (2_n) and (4_n). In this subsection, we do this for the relations (2_n) and (4_n). The more interesting relation (1_n) will be treated in the next subsection.

Lemma 5.3.1. *Relation (2_n) is satisfied.*

Proof. Without loss of generality, we may assume that $k = F$. (Recall that we did not assume k perfect in defining the symbols in $M(F)$.) For $a, b \in k^\times$, we need to show that the composition of

$$\mathbb{G}_{\mathrm{m}}^{\wedge i} \wedge S^0 \wedge \mathbb{G}_{\mathrm{m}}^{\wedge n+r-i-1} \xrightarrow{[ab]} \mathbb{G}_{\mathrm{m}}^{\wedge n+r} \xrightarrow{\eta^r\rho} M$$

is homotopic to the sum of the following three compositions

$$\mathbb{G}_{\mathrm{m}}^{\wedge i} \wedge S^0 \wedge \mathbb{G}_{\mathrm{m}}^{\wedge n+r-i-1} \xrightarrow{[a],\,[b]} \mathbb{G}_{\mathrm{m}}^{\wedge n+r} \xrightarrow{\eta^r\rho} M \qquad \text{and}$$

$$\mathbb{G}_{\mathrm{m}}^{\wedge i} \wedge S^0 \wedge \mathbb{G}_{\mathrm{m}}^{\wedge n+r-i-1} = \mathbb{G}_{\mathrm{m}}^{\wedge i} \wedge S^0 \wedge S^0 \wedge \mathbb{G}_{\mathrm{m}}^{\wedge n+r-i-1} \xrightarrow{[a]\wedge[b]} \mathbb{G}_{\mathrm{m}}^{\wedge n+r+1} \xrightarrow{\eta^{n+1}\rho} M.$$

Replacing M with M_{-n-r+1}, we are left to check the statement for $n = 1$ and $r = 0$. More precisely, we now assume that we are given a morphism of pointed sheaves $\rho : \mathbb{G}_{\mathrm{m}} \to M$ and our goal is to compare the composite map

$$S^0 \xrightarrow{[ab]} \mathbb{G}_{\mathrm{m}} \xrightarrow{\rho} M$$

with the sum of the following three composite maps

$$S^0 \xrightarrow{[a],\,[b]} \mathbb{G}_{\mathrm{m}} \xrightarrow{\rho} M \qquad \text{and} \qquad S^0 = S^0 \wedge S^0 \xrightarrow{[a]\wedge[b]} \mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\eta\rho} M.$$

To do so, we may as well check that, for $m \geqslant 2$, the composite map

$$S^m \xrightarrow{[ab]} \Sigma^m\mathbb{G}_{\mathrm{m}} \xrightarrow{\rho_m} \mathrm{K}(M, m) \qquad (\star)$$

is the sum of the three composite maps

$$S^m \xrightarrow{[a],\,[b]} \Sigma^m\mathbb{G}_{\mathrm{m}} \xrightarrow{\rho_m} \mathrm{K}(M, m) \qquad \text{and} \qquad S^m \xrightarrow{[a]\wedge[b]} \Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\eta} \Sigma^m\mathbb{G}_{\mathrm{m}} \xrightarrow{\rho_m} \mathrm{K}(M, m).$$

But the composite map ($\star$) is also the composition of

$$S^m \xrightarrow{[a,b]} \Sigma^m\mathbb{G}_{\mathrm{m}}^{\times 2} \xrightarrow{\mu} \Sigma^m\mathbb{G}_{\mathrm{m}} \xrightarrow{\rho_m} \mathrm{K}(M, m).$$

On the other hand, by Lemma 5.1.7, we have an equivalence

$$\Sigma^m\mathbb{G}_{\mathrm{m}}^{\times 2} \simeq \Sigma^m\mathbb{G}_{\mathrm{m}} \vee \Sigma^m\mathbb{G}_{\mathrm{m}} \vee \Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2}.$$

Modulo this equivalence, the map $\mu : \Sigma^m\mathbb{G}_{\mathrm{m}}^{\times 2} \to \Sigma^m\mathbb{G}_{\mathrm{m}}$ is given by

$$(\mathrm{id} \vee \mathrm{id} \vee \eta) : \Sigma^m\mathbb{G}_{\mathrm{m}} \vee \Sigma^m\mathbb{G}_{\mathrm{m}} \vee \Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2} \to \Sigma^m\mathbb{G}_{\mathrm{m}}$$

whereas $[a,b] : S^m \to \Sigma^m \mathbb{G}_{\mathrm{m}}^{\times 2}$ is given by

$$S^m \to S^m \vee S^m \vee S^m \xrightarrow{[a]\vee[b]\vee([a]\wedge[b])} \Sigma^m \mathbb{G}_{\mathrm{m}} \vee \Sigma^m \mathbb{G}_{\mathrm{m}} \vee \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2}.$$

This proves the lemma since the addition law on $[S^m, \mathrm{K}(M,m)]$ is induced by the comultiplication of the cogroup S^m. □

Next we check relation (4_n). As in the proof of Lemma 5.3.1, we may assume that $k = F$ and we reduce to showing that, for $m \geqslant 2$, the sum of the maps

$$\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{2\eta} \Sigma^m \mathbb{G}_{\mathrm{m}} \qquad \text{and} \qquad \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{[-1]} \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 3} \xrightarrow{\eta^2} \Sigma^m \mathbb{G}_{\mathrm{m}}$$

is $\mathbb{A}^1$-nullhomotopic. We need a construction.

Construction 5.3.2. Let $m \geqslant 1$ and $u \in k^\times$. The morphism $u : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$, $a \mapsto ua$, induces a map $u : \Sigma^m(\mathbb{G}_{\mathrm{m}+}) \to \Sigma^m(\mathbb{G}_{\mathrm{m}+})$ that we can compose with the obvious projection to get

$$u : \Sigma^m(\mathbb{G}_{\mathrm{m}+}) \to \Sigma^m \mathbb{G}_{\mathrm{m}}.$$

By Lemma 5.1.9, we have an equivalence $\Sigma^m(\mathbb{G}_{\mathrm{m}+}) \simeq S^m \vee \Sigma \mathbb{G}_{\mathrm{m}}$. The composition of

$$\Sigma^m \mathbb{G}_{\mathrm{m}} \to S^m \vee \Sigma^m \mathbb{G}_{\mathrm{m}} \simeq \Sigma^m(\mathbb{G}_{\mathrm{m}+}) \xrightarrow{u} \Sigma^m \mathbb{G}_{\mathrm{m}}$$

defines an endomorphism of $\Sigma^m \mathbb{G}_{\mathrm{m}}$ which we denote by $\langle u \rangle : \Sigma^m \mathbb{G}_{\mathrm{m}} \to \Sigma^m \mathbb{G}_{\mathrm{m}}$.

The following result justifies the above notation.

Lemma 5.3.3. *Assume that $m \geqslant 2$. Then, the endomorphism $\langle u \rangle$ of $\Sigma^m \mathbb{G}_{\mathrm{m}}$ is equal to the sum of the identity and the composition of*

$$\Sigma^m \mathbb{G}_{\mathrm{m}} \xrightarrow{[u]} \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\eta} \Sigma^m \mathbb{G}_{\mathrm{m}}.$$

Proof. By construction, $\langle u \rangle : \Sigma^m \mathbb{G}_{\mathrm{m}} \to \Sigma^m \mathbb{G}_{\mathrm{m}}$ is the composition of

$$\Sigma^m \mathbb{G}_{\mathrm{m}} \to S^m \vee \Sigma^m \mathbb{G}_{\mathrm{m}} \simeq \Sigma^m(\mathbb{G}_{\mathrm{m}+}) \xrightarrow{\mathrm{id}\times u} \Sigma^m \mathbb{G}_{\mathrm{m}}^{\times 2} \xrightarrow{\mu} \Sigma^m \mathbb{G}_{\mathrm{m}}.$$

By Lemma 5.1.7, we have an equivalence

$$\Sigma^m \mathbb{G}_{\mathrm{m}}^{\times 2} \simeq \Sigma^m \mathbb{G}_{\mathrm{m}} \vee \Sigma^m \mathbb{G}_{\mathrm{m}} \vee \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2}.$$

Modulo this equivalence, the map $\mu : \Sigma^m \mathbb{G}_{\mathrm{m}}^{\times 2} \to \Sigma^m \mathbb{G}_{\mathrm{m}}$ is given by

$$(\mathrm{id} \vee \mathrm{id} \vee \eta) : \Sigma^m \mathbb{G}_{\mathrm{m}} \vee \Sigma^m \mathbb{G}_{\mathrm{m}} \vee \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2} \to \Sigma^m \mathbb{G}_{\mathrm{m}}$$

whereas $\mathrm{id} \times u : \Sigma^m(\mathbb{G}_{\mathrm{m}+}) \to \Sigma^m \mathbb{G}_{\mathrm{m}}^{\times 2}$ is given by

$$\Sigma^m(\mathbb{G}_{\mathrm{m}+}) \to \Sigma^m(\mathbb{G}_{\mathrm{m}+}) \vee \Sigma^m(\mathbb{G}_{\mathrm{m}+}) \vee \Sigma^m(\mathbb{G}_{\mathrm{m}+}) \xrightarrow{\mathrm{id}\vee[u]\vee(\mathrm{id}\wedge[u])} \Sigma^m \mathbb{G}_{\mathrm{m}} \vee \Sigma^m \mathbb{G}_{\mathrm{m}} \vee \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2}.$$

This shows that the composite map

$$\Sigma^m(\mathbb{G}_{\mathrm{m}+}) \xrightarrow{\mathrm{id}\times u} \Sigma^m \mathbb{G}_{\mathrm{m}}^{\times 2} \xrightarrow{\mu} \Sigma^m \mathbb{G}_{\mathrm{m}}$$

is homotopic to the sum of the following three maps

$$\Sigma^m(\mathbb{G}_{\mathrm{m}+}) \xrightarrow{\mathrm{id}} \Sigma^m \mathbb{G}_{\mathrm{m}}, \quad \Sigma^m(\mathbb{G}_{\mathrm{m}+}) \to S^m \xrightarrow{[u]} \Sigma^m \mathbb{G}_{\mathrm{m}} \quad \text{and}$$

$$\Sigma^m(\mathbb{G}_{\mathrm{m}+}) \xrightarrow{\mathrm{id}} \Sigma^m \mathbb{G}_{\mathrm{m}} \xrightarrow{\mathrm{id}\wedge[u]} \Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\eta} \Sigma^m \mathbb{G}_{\mathrm{m}}.$$

Since the composition of

$$\Sigma^m\mathbb{G}_{\mathrm{m}} \to S^m \vee \Sigma^m\mathbb{G}_{\mathrm{m}} \simeq \Sigma^m(\mathbb{G}_{\mathrm{m}+}) \to S^m$$

is nullhomotopic, we deduce that $\langle u\rangle$ is homotopic to a sum of two maps as in the statement. □

Thus, to prove relation (4_n), it suffices to show the following lemma.

Lemma 5.3.4. *For $m \geqslant 2$, the sum of the following two maps*

$$\Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\eta} \Sigma^m\mathbb{G}_{\mathrm{m}} \qquad \textit{and} \qquad \Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\langle -1\rangle\wedge\mathrm{id}} \Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\eta} \Sigma^m\mathbb{G}_{\mathrm{m}}$$

is $\mathbb{A}^1$-nullhomotopic.

For this, we need the following.

Sublemma 5.3.5. *For $m \geqslant 2$, the two endomorphisms of $\Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2}$ given by $\mathrm{id}\wedge\langle -1\rangle$ and by swapping the two factors of $\mathbb{G}_{\mathrm{m}}^{\times 2}$ sum to zero in the group*

$$[\Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2}, \Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2}]_{\mathbb{A}^1}.$$

Proof of Lemma 5.3.4 assuming Sublemma 5.3.5. By Lemma 5.2.10, the map $\eta : \Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2} \to \Sigma^m\mathbb{G}_{\mathrm{m}}$ is homotopic to the composite map

$$\Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\tau} \Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\eta} \Sigma^m\mathbb{G}_{\mathrm{m}},$$

where τ is the endomorphism given by swapping the two factors of $\mathbb{G}_{\mathrm{m}}^{\times 2}$. By Sublemma 5.3.5, this composite map is $\mathbb{A}^1$-homotopic to the opposite of the composite map

$$\Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\mathrm{id}\wedge\langle -1\rangle} \Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\eta} \Sigma^m\mathbb{G}_{\mathrm{m}}.$$

This is precisely the content of Lemma 5.3.4. □

Proof of Sublemma 5.3.5. We need to use another model of $\Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2}$. There is a cocartesian square of pointed k-spaces

$$\begin{tikzcd}
\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1 \arrow[d] \\
\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}} \arrow[r] & \mathbb{A}^2\smallsetminus 0.
\end{tikzcd}$$

In $\mathcal{H}(k)_*$, we can rewrite this square as follows

$$\begin{tikzcd}
\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[d] \\
\mathbb{G}_{\mathrm{m}} \arrow[r] & \mathbb{A}^2\smallsetminus 0.
\end{tikzcd}$$

Applying suspensions, we deduce that $\Sigma^{m-1}(\mathbb{A}^2\smallsetminus 0)$ is $\mathbb{A}^1$-equivalent to the pushout of

$$\begin{tikzcd}
\Sigma^{m-1}\mathbb{G}_{\mathrm{m}}^{\times 2} \arrow[r] \arrow[d] & \Sigma^{m-1}\mathbb{G}_{\mathrm{m}} \\
\Sigma^{m-1}\mathbb{G}_{\mathrm{m}},
\end{tikzcd}$$

which is $\Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2}$ by Lemma 5.1.7. This yields an $\mathbb{A}^1$-equivalence of pointed k-spaces:

$$\Sigma^m\mathbb{G}_{\mathrm{m}}^{\wedge 2} \xrightarrow{\sim} \Sigma^{m-1}(\mathbb{A}^2\smallsetminus 0).$$

Moreover, modulo this $\mathbb{A}^1$-equivalence, the endomorphism $\mathrm{id} \wedge \langle u \rangle$ of $\Sigma^m \mathbb{G}_{\mathrm{m}}^{\wedge 2}$ corresponds to the action of the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$$

on $\Sigma^{m-1}(\mathbb{A}^2 \smallsetminus 0)$. On the other hand, the swap endomorphism τ permutes the edges $(1, 0)$ and $(0, 1)$ in the diagram

$$\begin{tikzcd}
\Sigma^{m-1}\mathbb{G}_{\mathrm{m}}^{\times 2} \arrow[r] \arrow[d] & \Sigma^{m-1}\mathbb{G}_{\mathrm{m}} \\
\Sigma^{m-1}\mathbb{G}_{\mathrm{m}} &
\end{tikzcd}$$

and hence correspond to the opposite of the action of the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on $\Sigma^{m-1}(\mathbb{A}^2 \smallsetminus 0)$. Thus, to conclude, it suffices to show that the two matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

act in the same way on the pointed k-space $\mathbb{A}^2 \smallsetminus 0$ up to $\mathbb{A}^1$-homotopy. This follows from the fact that the group $\mathrm{SL}_2(k)$ is generated by unipotent matrices. Indeed, unipotent matrices can be $\mathbb{A}^1$-deformed in SL_2 to the identity matrix. □

5.4. **The Steinberg relation.** To complete the proof of Theorem 5.2.7, we still need to show that the Steinberg relation is satisfied by the symbols

$$[\eta^r, u_1, \ldots, u_{n+r}] \in M(F)$$

obtained using a morphism of pointed k-spaces $\rho : \mathbb{G}_{\mathrm{m}}^{\wedge n} \to M$ as in Setting 5.2.8. (See Notation 5.2.12.) As usual, it is enough to do so for $F = k$. Also, replacing M with a suitable contraction (as in the proof of Lemma 5.3.1), we reduce to the case $n = 2$ and $r = 0$. The desired result follows then from the next theorem.

Theorem 5.4.1 (Hu–Kriz, Druzhinin, Hoyois)**.** *Consider the Steinberg map of k-spaces*

$$\mathrm{st} : (\mathbb{A}^1 \smallsetminus \{0, 1\})_+ \to \mathbb{G}_{\mathrm{m}}^{\wedge 2}, \qquad a \mapsto (a, 1 - a).$$

Then $\Sigma(\mathrm{st})$ is nullhomotopic in $\mathcal{H}(k)_*$.

This was first proven by Hu–Kriz [HK01] (see also [Dru21]). Here, we follow the presentation of Hoyois [Hoy18]. Roughly speaking, the proof consists in finding a replacement of st up to $\mathbb{A}^1$-homotopy which extends to $\mathbb{A}^1$. Let

$$B = \mathrm{Bl}_{\{(0,1),(1,0)\}}\mathbb{A}^2 \smallsetminus \widetilde{0 \times \mathbb{A}^1} \cup \widetilde{\mathbb{A}^1 \times 0}$$

be the result of removing the proper transforms of the coordinate axes in the blowup of $\mathbb{A}^2$ at the points $(0, 1)$ and $(1, 0)$. We have an inclusion

$$\widetilde{\mathrm{st}} : \mathbb{A}^1 \to B$$

extending $\mathrm{st} : \mathbb{A}^1 \smallsetminus \{0, 1\} \to \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$. We will see that the $\mathbb{A}^1$-homotopy type of B is closely related to the $\mathbb{A}^1$-homotopy type of the k-space $\mathbb{G}_{\mathrm{m}} \wedge \mathbb{G}_{\mathrm{m}}$.

Set theoretically, we have

$$B = \mathbb{G}_m \times \mathbb{G}_m \sqcup E_{0,1} \sqcup E_{1,0}$$

where $E_{0,1}$ and $E_{1,0}$ are the exceptional divisors over $(0,1)$ and $(1,0)$. These exceptional divisors are isomorphic to $\mathbb{A}^1$. We set:

$$U_{0,1} = \mathbb{G}_m \times \mathbb{G}_m \sqcup E_{0,1} \qquad \text{and} \qquad U_{1,0} = \mathbb{G}_m \times \mathbb{G}_m \sqcup E_{1,0}.$$

These form an open covering of B intersecting at $\mathbb{G}_m \times \mathbb{G}_m$. To determine the homotopy type of $U_{0,1}$, we note that there is a commutative square

$$\begin{tikzcd}
\mathbb{G}_m \times \{1\} \arrow[r] \arrow[d] & \mathbb{A}^1 \times \{1\} \arrow[d] \\
\mathbb{G}_m \times \mathbb{G}_m \arrow[r] & U_{0,1}
\end{tikzcd}$$

yielding a map of pointed k-spaces

$$e_{0,1} : \mathbb{G}_m \times \mathbb{G}_m \coprod_{\mathbb{G}_m \times \{1\}} \mathbb{A}^1 \times \{1\} \to U_{0,1}.$$

(The domain and codomain are both pointed by $(1,1)$.) Here is the key lemma.

Lemma 5.4.2. *The map $\Sigma e_{0,1}$ is an $\mathbb{A}^1$-equivalence.*

Proof. The map $e_{0,1}$ can be extended into a pointed map of k-spaces

$$\widetilde{e}_{0,1} : \mathbb{G}_m \times \mathbb{A}^1 \coprod_{\mathbb{G}_m \times \{1\}} \mathbb{A}^1 \times \{1\} \to \mathrm{Bl}_{(0,1)}\mathbb{A}^2 - \widetilde{0 \times \mathbb{A}^1}.$$

The domain and the codomain of $\widetilde{e}_{0,1}$ are $\mathbb{A}^1$-contractible. Therefore, the map

$$\frac{\mathbb{G}_m \times \mathbb{A}^1 \coprod_{\mathbb{G}_m \times \{1\}} \mathbb{A}^1 \times \{1\}}{\mathbb{G}_m \times \mathbb{G}_m \coprod_{\mathbb{G}_m \times \{1\}} \mathbb{A}^1 \times \{1\}} \to \frac{\mathrm{Bl}_{(0,1)}\mathbb{A}^2 - \widetilde{0 \times \mathbb{A}^1}}{U_{0,1}} \tag{$\star$}$$

is $\mathbb{A}^1$-equivalent to $\Sigma e_{0,1}$. It remains to see that the map $(\star)$ is an equivalence. But, by Zariski excision, the domain and the codomain of $(\star)$ are both equivalent to

$$\frac{\mathbb{G}_m \times \mathbb{A}^1}{\mathbb{G}_m \times \mathbb{G}_m},$$

and modulo these equivalences, the map $(\star)$ is given by the identity. □

By symmetry, the suspension of the map

$$e_{1,0} : \mathbb{G}_m \times \mathbb{G}_m \coprod_{\{1\} \times \mathbb{G}_m} \{1\} \times \mathbb{A}^1 \to U_{1,0}$$

is also an $\mathbb{A}^1$-equivalence. Since

$$B = U_{0,1} \coprod_{\mathbb{G}_m \times \mathbb{G}_m} U_{1,0},$$

we deduce the following.

Corollary 5.4.3. *Consider the map of pointed k-spaces*

$$e : \mathbb{A}^1 \times \{1\} \coprod_{\mathbb{G}_m \times \{1\}} \mathbb{G}_m \times \mathbb{G}_m \coprod_{\{1\} \times \mathbb{G}_m} \{1\} \times \mathbb{A}^1 \to B.$$

Then Σe is an $\mathbb{A}^1$-equivalence.

Now, remark that collapsing the two copies of $\mathbb{A}^1$ gives a map

$$\mathbb{A}^1 \times \{1\} \coprod_{\mathbb{G}_m \times \{1\}} \mathbb{G}_m \times \mathbb{G}_m \coprod_{\{1\} \times \mathbb{G}_m} \{1\} \times \mathbb{A}^1 \to \mathbb{G}_m \wedge \mathbb{G}_m$$

which is an $\mathbb{A}^1$-equivalence. Consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathbb{A}^1 \coprod_{\mathbb{G}_m \times \{1\}} \mathbb{G}_m \times \mathbb{G}_m \coprod_{\{1\} \times \mathbb{G}_m} \mathbb{A}^1 \arrow[d, "\simeq_{\mathbb{A}^1}"] \arrow[dd, bend left, "e"] \\
(\mathbb{A}^1 \smallsetminus \{0,1\})_+ \arrow[ur, "\mathrm{st}'"] \arrow[r, "\mathrm{st}"] & \mathbb{G}_m \wedge \mathbb{G}_m \\
 & B.
\end{tikzcd}$$

Since Σe is an $\mathbb{A}^1$-equivalence, it is enough to show that $e \circ \mathrm{st}'$ is $\mathbb{A}^1$-nullhomotopic. But the map $e \circ \mathrm{st}'$ extends to a map

$$\widetilde{\mathrm{st}} : \mathbb{A}^1_+ \to B.$$

In fact, the image of $\widetilde{\mathrm{st}}$ is contained in a chain of three copies of $\mathbb{A}^1$, namely

$$\widetilde{\mathrm{st}}(\mathbb{A}^1) \cup E_{(0,1)} \cup \widetilde{\mathbb{A}^1 \times \{1\}}.$$

This finishes the proof of Theorem 5.4.1.

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