

# PERIODS AND THE MOTIVIC GALOIS GROUP

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## 1. COMPLEX PERIODS

Let  $\mathbb{D} = \{z \in \mathbb{C}; |z| \leq 1\}$  be the closed unit disc in  $\mathbb{C}$ . For a nonnegative integer  $n$ , let  $\mathcal{O}(\mathbb{D}^n)$  be the  $\mathbb{C}$ -algebra of holomorphic functions which are defined on an open neighbourhood of  $\mathbb{D}^n$  in  $\mathbb{C}^n$ . Equivalently,  $\mathcal{O}(\mathbb{D}^n)$  is the algebra of formal power series

$$f = \sum_{(i_1, \dots, i_n) \in \mathbb{N}^n} a_{i_1, \dots, i_n} z_1^{i_1} \cdots z_n^{i_n}$$

such that, for some radius  $\rho > 1$ , the sequence  $|a_{i_1, \dots, i_n}| \rho^{i_1 + \dots + i_n}$  tends to zero when  $i_1 + \dots + i_n$  goes to infinity. We set

$$\mathcal{O}(\mathbb{D}^\infty) = \operatorname{colim}_{n \in \mathbb{N}} \mathcal{O}(\mathbb{D}^n).$$

The elements of  $\mathcal{O}(\mathbb{D}^\infty)$  are formal power series in finitely many variables with coefficients in  $\mathbb{C}$  and with radii of convergence  $> 1$ .

For  $n \geq 0$ , let  $\Omega^\bullet(\mathbb{D}^n)$  be the de Rham complex of holomorphic forms on an open neighbourhood of  $\mathbb{D}^n$  in  $\mathbb{C}^n$ . This complex is concentrated in degrees  $0 \leq m \leq n$  where it is given by

$$\Omega^m(\mathbb{D}^n) = \bigoplus_{I \subset \{1, \dots, n\}, \#(I)=m} \mathcal{O}(\mathbb{D}^n) \cdot dz_I,$$

where  $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_m}$  if  $I = \{i_1 < \cdots < i_m\}$ . We denote by  $\Omega^{n-\bullet}(\mathbb{D}^n)$  the connective complex obtained by shifting  $\Omega^\bullet(\mathbb{D}^n)$  to the left  $n$  times while keeping the differential unchanged. Thus, in homological degree  $m$ , this shifted complex is given by  $\Omega^{n-m}(\mathbb{D}^n)$ . We have a sequence

$$\Omega^{0-\bullet}(\mathbb{D}^0) \xrightarrow{-\wedge dz_1} \cdots \xrightarrow{-\wedge dz_n} \Omega^{n-\bullet}(\mathbb{D}^n) \xrightarrow{-\wedge dz_{n+1}} \Omega^{n+1-\bullet}(\mathbb{D}^{n+1}) \xrightarrow{-\wedge dz_{n+2}} \cdots$$

and we set

$$\Omega^{\infty-\bullet}(\mathbb{D}^\infty) = \operatorname{colim}_{n \in \mathbb{N}} \Omega^{n-\bullet}(\mathbb{D}^n).$$

By construction, the complex  $\Omega^{\infty-\bullet}(\mathbb{D}^\infty)$  is connective, i.e., concentrated in nonnegative homological degrees. Moreover, for  $m \geq 0$ , we have

$$\Omega^{\infty-m}(\mathbb{D}^\infty) = \bigoplus_{I \subset \mathbb{N} \setminus \{0\}, \#(I)=m} \mathcal{O}(\mathbb{D}^\infty) \cdot \widehat{dz}_I,$$

where  $\widehat{dz}_I$  is the codegree- $m$  form on  $\mathbb{D}^\infty$  obtained by wedging all the  $dz_i$ 's, for  $i \notin I$ . In fact, it is easy to see that the complex  $\Omega^{\infty-\bullet}(\mathbb{D}^\infty)$  is acyclic. We are rather concerned with the subcomplex

$$\widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty) \subset \Omega^{\infty-\bullet}(\mathbb{D}^\infty)$$

given in homological degree  $m$  by those codegree- $m$  forms on  $\mathbb{D}^\infty$  that vanish when restricted to the hypersurfaces defined by the equations  $z_i(1 - z_i) = 0$  for  $i \geq 1$ . More explicitly, a codegree- $m$  form

$$\sum_{I \subset \mathbb{N} \setminus \{0\}, \#(I)=m} f_I \cdot \widehat{dz}_I$$

belongs to  $\widetilde{\Omega}^{\infty-m}(\mathbb{D}^\infty)$  if and only if, for every  $I \subset \mathbb{N} \setminus \{0\}$  of cardinality  $m$ , we have  $f_I|_{z_i=0} = f_I|_{z_i=1} = 0$  for all  $i \in I$ .

**Lemma 1.** *There is a quasi-isomorphism*

$$(1.1) \quad \widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty) \xrightarrow{q \cdot i} \mathbb{C}[0], \quad f \cdot \widehat{dz}_\emptyset \mapsto \int_{[0,1]^\infty} f.$$

**Definition.** Let  $\mathcal{O}_{\text{alg}}(\mathbb{D}^\infty) \subset \mathcal{O}(\mathbb{D}^\infty)$  be the sub- $\mathbb{Q}$ -algebra whose elements are those holomorphic functions which are algebraic over the field of rational functions  $\mathbb{Q}(z_1, \dots, z_n, \dots)$ . Similarly, let

$$\Omega_{\text{alg}}^{\infty-\bullet}(\mathbb{D}^\infty) \subset \Omega^{\infty-\bullet}(\mathbb{D}^\infty)$$

be the subcomplex given in homological degree  $m$  by

$$\Omega_{\text{alg}}^{\infty-m}(\mathbb{D}^\infty) = \bigoplus_{I \subset \mathbb{N} \setminus \{0\}, \#(I)=m} \mathcal{O}_{\text{alg}}(\mathbb{D}^\infty) \cdot \widehat{dz}_I.$$

Finally, let

$$\widetilde{\Omega}_{\text{alg}}^{\infty-\bullet}(\mathbb{D}^\infty) \subset \widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty)$$

be the intersection of  $\Omega_{\text{alg}}^{\infty-\bullet}(\mathbb{D}^\infty)$  and  $\widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty)$  in  $\Omega^{\infty-\bullet}(\mathbb{D}^\infty)$ .

The following conjecture was made in [1]. (See [1, Remarque 2.109 & Conjecture A].)

**Conjecture 2.** The complex  $\Omega_{\text{alg}}^{\infty-\bullet}(\mathbb{D}^\infty)$  is acyclic except in degree zero and the map

$$(1.2) \quad H_0(\Omega_{\text{alg}}^{\infty-\bullet}(\mathbb{D}^\infty)) \rightarrow \mathbb{C}, \quad f \cdot \widehat{dz}_\emptyset \mapsto \int_{[0,1]^\infty} f,$$

deduced from (1.1), is injective.

*Remark 3.* The image of the map (1.2) is precisely the ring of effective periods. The second assertion in Conjecture 2 is essentially a reformulation of the Kontsevich–Zagier conjecture on periods [6]. (This is not completely trivial and relies on [5].) The first assertion is closely related to the existence of a motivic  $t$ -structure.

## 2. ADIC PERIODS

Let  $K$  be a field endowed with a nontrivial discrete valuation. Denote by  $k$  the residue field of  $K$  and by  $\widehat{K}$  its completion. Assume also that  $K$  has characteristic zero. The two main examples for us are  $K = \mathbb{Q}$ , with the  $p$ -adic valuation, and  $K = k(\pi)$ , with the  $\pi$ -adic valuation. Let  $\mathbb{B}$  be the Tate ball over  $\widehat{K}$  considered as a rigid analytic

variety. Let  $\mathcal{O}(\mathbb{B}^n)$  be the  $\widehat{K}$ -algebra of power series in  $n$  variables which are convergent in an open neighbourhood of  $\mathbb{B}^n$ . Then, as in the previous section, we may form the  $\widehat{K}$ -algebra  $\mathcal{O}(\mathbb{B}^\infty)$  and the complexes  $\Omega^{\infty-\bullet}(\mathbb{B}^\infty)$  and  $\widetilde{\Omega}^{\infty-\bullet}(\mathbb{B}^\infty)$ . The former complex is acyclic while the latter is quasi-isomorphic to  $\widehat{K}[0]$ . Indeed, we have the following analog of Lemma 1.

**Lemma 4.** *There is a quasi-isomorphism*

$$(2.1) \quad \widetilde{\Omega}^{\infty-\bullet}(\mathbb{B}^\infty) \xrightarrow{\text{q.-i.}} \widehat{K}[0], \quad f \cdot \widehat{dz}_\emptyset \mapsto \int_{[0,1]^\infty} f.$$

Note that the map (2.1) is given by “formal integration” of power series, i.e., by taking formal primitives and evaluating at 0 and 1.

**Definition.** Let  $\mathcal{O}_{K\text{-alg}}(\mathbb{B}^\infty) \subset \mathcal{O}(\mathbb{B}^\infty)$  be the sub- $K$ -algebra whose elements are those power series which are algebraic over the field of rational functions  $K(z_1, \dots, z_n, \dots)$ . As in Definition 1, this yields sub-complexes  $\Omega_{K\text{-alg}}^{\infty-\bullet}(\mathbb{B}^\infty)$  and  $\widetilde{\Omega}_{K\text{-alg}}^{\infty-\bullet}(\mathbb{B}^\infty)$  of  $\Omega^{\infty-\bullet}(\mathbb{B}^\infty)$ .

By analogy with Conjecture 2, we make the following conjecture.

**Conjecture 5.** Assume that  $K = \mathbb{Q}$ , with the  $p$ -adic valuation, or  $K = k(\pi)$ , with the  $\pi$ -adic valuation. Then, the complex  $\Omega_{K\text{-alg}}^{\infty-\bullet}(\mathbb{B}^\infty)$  is acyclic except in degree zero and the map

$$(2.2) \quad H_0(\Omega_{K\text{-alg}}^{\infty-\bullet}(\mathbb{B}^\infty)) \rightarrow \widehat{K}, \quad f \cdot \widehat{dz}_\emptyset \mapsto \int_{[0,1]^\infty} f,$$

deduced from (2.1), is injective.

*Remark 6.* In the  $p$ -adic case, Conjecture 5 is wide-open. In the  $\pi$ -adic case, it is essentially a theorem by [3, 4]. More precisely, it is known to hold after stabilisation; see [4, Théorème 1.7].

### 3. THE MOTIVIC GALOIS GROUP

For every field  $k$  endowed with a complex embedding  $k \hookrightarrow \mathbb{C}$ , there is an associated motivic Galois group  $\mathcal{G}_{\text{mot}}(k)$ . It is a derived affine group scheme, i.e., the spectrum of a connective derived Hopf algebra  $\mathcal{H}_{\text{mot}}(k)$ . Roughly speaking,  $\mathcal{G}_{\text{mot}}(k)$  is the group of autoequivalences of the Betti realisation

$$B^* : \text{DM}(k; \mathbb{Q}) \rightarrow \text{D}(\mathbb{Q}), \quad M(X) \mapsto C_*(X(\mathbb{C})).$$

It turns out that the complex  $\widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty)$  is, up to inverting the formal period corresponding to  $2\pi i$ , the Betti realisation of the (big) motive representing de Rham cohomology in  $\text{DM}(k; \mathbb{Q})$ . This shows that  $\widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty)$  is essentially the algebra of a torsor over  $\mathcal{G}_{\text{mot}}(k)$ . This perspective is crucial in proving Conjecture 5 in the  $\pi$ -adic case (after

stabilisation). In fact, the said conjecture becomes a translation of the property that the derived affine group scheme

$$\mathcal{G}_{\text{rel}}(K/k) := \ker\{\mathcal{G}_{\text{mot}}(K) \rightarrow \mathcal{G}_{\text{mot}}(k)\}$$

is classical (i.e., its Hopf algebra is concentrated in degree zero) and is a quotient of the pro-algebraic completion of the fundamental group of  $\mathbb{C} \setminus k$ , considered as a pro-Zariski open. These results are proven in [2, Théorèmes 2.55 & 2.57].

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