The motivic Thom spectrum $M\mathbb{G}\ell$ and the algebraic cobordism $\Omega^*(-)$

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This is a report on a "work in progress" of F. Morel and M. J. Hopkins. Their work is a step toward the identification of the motivically defined theory $M\mathbb{G}\ell^{2*,*}(-)$ with the geometrically defined one $\Omega^*(-)$. Namely, they prove:

Theorem 1. *For any smooth k-variety X the natural graded homomorphism $M\mathbb{G}\ell^{2*,*}(X) \longmapsto \Omega^*(X)$ is surjective.*

The plan of the lecture was:

(1) Some basic properties of $M\mathbb{G}\ell$.
(2) The computation of $M\mathbb{G}\ell^{2*,*}(k)$.
(3) Proof of the main theorem.

From now one, the base field k is fixed and all our varieties will be k-varieties. For simplicity we shall assume k to be of characteristic zero.

1. Some basic properties of $M\mathbb{G}\ell$

In this first part, we transpose from the topological to the motivic context some classical properties of the Thom spectrum. We denote by $T = \mathbb{A}^1/\mathbb{G}m$ one of the motivic spheres. When speaking about spectra, we shall always mean T-spectra. The $\mathbb{A}^1$-homotopy category of spectra is a triangulated category denoted by $\mathbf{SH}(k)$ (cf. Morel [1]).

Let us recall that as in algebraic topology, the motivic Thom spectrum is defined by the collection: $(\mathbb{S}^0, Th(\gamma_1), \dots, Th(\gamma_n), \dots)$ together with the usual assembly maps. Here γ_n is the tautological vector bundle on the infinite Grassmanian of n-planes. For a smooth variety X, we put $M\mathbb{G}\ell^{p,q}(X) = [X_+, M\mathbb{G}\ell \wedge T^q[p-2q]]$.

Lemma 2. *$M\mathbb{G}\ell$ is an oriented ring spectrum.*

The proof is exactly the same as the classical one. It is based on the identification of $Th(\gamma_1)$ with the pointed space $(\mathbb{P}^\infty, *)$.

As a consequence, we can define for a line bundle $\mathcal{L}$ on X a first Chern class $c_1(\mathcal{L}) \in M\mathbb{G}\ell^{2,1}(X)$ by the composition: $X \overset{[\mathcal{L}]}{\longmapsto} \mathbb{P}^\infty \longmapsto M\mathbb{G}\ell \wedge T$. Using this, one obtain a projective bundle formula by the usual method, and then the other Chern classes for vector bundles. This can be used to define the Thom classes:

Definition-Construction 3. *Let $\mathcal{V}/X$ be a vector bundle of rank r. The Thom class $t(\mathcal{V})$ of $\mathcal{V}$ lives in $M\mathbb{G}\ell^{2r,r}(Th(\mathcal{V}))$. It is defined in the following manner: Recall that one model of $Th(\mathcal{V})$ is $\mathbb{P}(\mathcal{V}+1)/\mathbb{P}(\mathcal{V})$. Thus one have a long exact sequence (which breaks into short ones):*

$$\begin{array}{ccccc} M\mathbb{G}\ell^{*,*}(Th(\mathcal{V})) & \longrightarrow & M\mathbb{G}\ell^{*,*}(\mathbb{P}(\mathcal{V}+1)) & \longrightarrow & M\mathbb{G}\ell^{*,*}(\mathbb{P}(\mathcal{V})) \\ & & \| & & \| \\ & & M\mathbb{G}\ell^{*,*}(k)[1,u,\dots,u^r] & & M\mathbb{G}\ell^{*,*}(k)[1,u,\dots,u^{r-1}] \end{array}$$

We then define $t(\mathcal{V})$ to be the element of the middle group equal to $u^r - c_1.u^{r-1} + \cdots + (-1)^r c_r$ where c_i are such that the image of $t(\mathcal{V})$ became zero in the last *group. The exactness of the sequence give us a unique antecedent of $t(\mathcal{V})$ in the first group. This is the Thom class.*

A consequence of this construction is:

lemma 4. *$M\mathbb{G}\ell$ is the universal oriented ring spectrum.*

Indeed let E be such a spectrum. The construction above still make sens for E. In particular if we take the Thom classes of γ_n we get maps: $Th(\gamma_n) \longmapsto E \wedge T^n$ yielding the unique map of spectra $M\mathbb{G}\ell \longmapsto E$. Later on, we shall apply this to $E = H\mathbb{Z}$, the motivic cohomology spectrum, to get the morphism: $M\mathbb{G}\ell \longmapsto H\mathbb{Z}$.

The next step of our study is the Thom isomorphism. Let $\mathcal{V}/X$ be a vector bundle of rank r. Define (as in topology) the reduced diagonal: $Th(\mathcal{V}) \longmapsto Th(\mathcal{V}) \wedge X_+$ using the pull-back square:

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{V} \times 0 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

Theorem-Definition 5. *For any oriented ring spectrum E, the following composition:*

$$E \wedge Th(\mathcal{V}) \longrightarrow E \wedge Th(\mathcal{V}) \wedge X_+ \longrightarrow E \wedge E \wedge T^r \wedge X_+ \longrightarrow E \wedge T^r \wedge X_+$$

is an isomorphism. It is called the Thom isomorphism.

Roughly speaking, the above result says that an oriented ring spectrum does not make the difference between the Thom space of a non trivial vector bundle and the Thom space of a trivial one with the same rank. A consequence of that is a natural isomorphism: $E^{*,*}(Th(\mathcal{V})) = E^{*-2r,*-r}(X)$.

We end this section by constructing transfers map for $M\mathbb{G}\ell^{2*,*}(-)$. It is sufficient to consider the case of a closed immersion and the projection of a projective space over X. The second case follow easily from the projective bundle formula. For a closed immersion we need to use the Thom isomorphism. Indeed, let $i : Y \subset X$ be a closed immersion. We denote by ν_i, ν_X and ν_Y the normal bundles of i, X and Y. Note that ν_X and ν_Y are not vector bundles in the usual sens but only virtual one (that is of negative rank). As in topology, we can form the composition in $\mathbf{SH}(k)$: $Th(\nu_X) \longrightarrow Th(i^*\nu_X \oplus \nu_i) = Th(\nu_Y)$ When applying $E^{*,*}$ we get a map in the opposite direction: $E^{*,*}(Th(\nu_Y)) \longmapsto E^{*,*}(Th(\nu_X))$. Now using the Thom isomorphism, we have the identifications

$$E^{*,*}Th(\nu_Y) \simeq E^{*+2d_Y,*+d_Y}(Y) \qquad \text{and} \qquad E^{*,*}Th(\nu_X) \simeq E^{*+2d_X,*+d_X}(X)$$

Where d_X and d_Y are the dimension of X and Y. Then denoting $c = d_X - d_Y$ the codimension of Y in X, we obtain the wanted transfer map: $E^{*,*}(Y) \longmapsto E^{*+2c,*+c}(X)$. As a consequence, $E^{2*,*}(-)$ is an oriented Borel-Moore cohomology theory. In particular using the universality of $\Omega^*(-)$ we get the natural homomorphism in theorem 1.

2. THE COMPUTATION OF $M\mathbb{G}\ell^{2*,*}(k)$

The main step of the proof of theorem 1 is the following proposition:

Proposition 6. *The canonical homomorphism given by the formal group law:* $\mathbb{L}_* \longmapsto M\mathbb{G}\ell^{-2*,-*}(k)$ *is an isomorphism.*

The injectivity of the above homomorphism is easy: one use for example a complex realization. There is also a purely algebraic proof based on a Quillen trick... The main difficulty is to show the surjectivity. For this one need a difficult lemma:

Lemma 7. *The canonical morphism of spectra:* $M\mathbb{G}\ell \longmapsto H\mathbb{Z}$ *induce an isomorphism*[1]*:*

$$M\mathbb{G}\ell/(x_1, \dots, x_n, \dots) \xrightarrow{\sim} H\mathbb{Z}$$

Where x_i *are generator of the Lazard ring.*

Assuming lemma 7, the proof of proposition 6 goes by induction on $*$. The point is that for $N > 0$, one have $[T^N, H\mathbb{Z}] = 0$ by Voevodsky cancellation theorem. Using a stability argument, the lemma 7 implies that $[T^N, M\mathbb{G}\ell/(x_1, \dots, x_N)] = 0$. Then if we apply $[T^N, -]$ to the distinguished triangle:

$$M\mathbb{G}\ell/(x_1, \dots, x_{N-1}) \wedge T^N \xrightarrow{x_N} M\mathbb{G}\ell/(x_1, \dots, x_{N-1}) \to M\mathbb{G}\ell/(x_1, \dots, x_N) \to$$

we get a surjection: $x_N : \mathbb{Z} \longmapsto M\mathbb{G}\ell^{-2N,-N}/(x_1, \dots, x_{N-1})(k)$. Using the induction hypothesis, one deduce that: $\mathbb{L}_N \longmapsto M\mathbb{G}\ell^{-2N,-N}(k)$ is indeed a surjection.

3. PROOF OF THE MAIN THEOREM

A consequence of proposition 6 is that $M\mathbb{G}\ell^{2*,*}(-)$ is generically constant. Moreover, we have a weak form of the localization property, namely: *given a smooth pair* $(Y \subset X)$ *with* Y *of codimension* c*, one have an exact sequence:*

$$M\mathbb{G}\ell^{2*-2c,*-c}(Y) \longrightarrow M\mathbb{G}\ell^{2*,*}(X) \longrightarrow M\mathbb{G}\ell^{2*,*}(X - Y)$$

These properties suffices to derive a generalized degree formula (see [2], [3]) for $M\mathbb{G}\ell^{2*,*}(-)$. In particular, this implies that $M\mathbb{G}\ell^{2*,*}(X)$ is generated as a $M\mathbb{G}\ell^{-2*,-*}(k) = \mathbb{L}_*$-module by cobordism cycles: $[Z \longmapsto X]$ with Z a desingularization of a closed subset of X. This clearly implies Theorem 1.

REFERENCES

[1] F. Morel, An introduction to $\mathbb{A}^1$-homotopy theory, ICTP Trieste July 2002. http://www.ictp.trieste.it/~pub_off/lectures/vol15.html

[2] M. Levine et F. Morel, *Cobordisme algébrique I*, Note aux C.R. Acad. Sci. Paris, **332** Série I, p. 723-728, 2001.

[3] M. Levine and F. Morel, Algebraic Cobordism I. revised version. http://www.math.neu.edu/~levine/publ/Publ.html

[4] M. Levine, Algebraic Cobordism II. http://www.math.neu.edu/~levine/publ/Publ.html

[1] The quotient ring spectrum $M\mathbb{G}\ell/(x_1, \dots, x_n, \dots)$ is not so easy to construct. Some serious technical difficulties arise if one try to do this naively. One way to overcome these difficulties is to work in a category of $M\mathbb{G}\ell$-modules.