Abstracts

2-Motives

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The goal of this talk is to give a reasonable candidate for a category of mixed 2-motives over a field k. By "reasonable" we mean a category $\mathbf{M}_2(k)$ that shares some of the mirific properties that the conjectural category of mixed 2-motives is expected to enjoy. The plan is as follows. First, we give the definition of $\mathbf{M}_2(k)$. Then we explain the ideas behind the verification that $\mathbf{M}_2(k)$ is an abelian category.

0.1. **Definition.** *Let k be a perfect field. An object $M \in \mathbf{DM}_{\text{eff}}(k)$ is called a* mixed 2-motive*, or simply a* 2-motive*, if it satisfies the following conditions:*

(a) $H_i(M) = 0$ *for* $i \notin \{0, -1, -2\}$*;*
(b) $H_0(M)$ *is a* 0*-motivic sheaf;*
(c) $H_{-1}(M)$ *is a* 1*-motivic sheaf;*
(d) $H_{-2}(M)$ *is a* 2*-motivic sheaf which is* 1*-connected;*
(e) *if L is a non-zero* 0*-motivic sheaf, then $L[-1]$ is not a direct factor of M and $Ext^1(H_{-2}(M), L) = 0$.*

The category of mixed 2*-motives is denoted by $\mathbf{M}_2(k)$.*

Some explanations are needed. Here, $\mathbf{DM}_{\text{eff}}(k)$ is Voevodsky's category of effective motives with rational coefficients. It can be defined as a full subcategory of $\mathbf{D}(Shv^{Nis}_{tr}(Sm/k, \mathbb{Q}))$, the derived category of Nisnevich sheaves with transfers on the category of smooth k-varieties. A complex K is in $\mathbf{DM}_{\text{eff}}(k)$ if its homology sheaves $H_i(K)$ are homotopy invariant for all $i \in \mathbb{Z}$. By a non-trivial theorem of Voevodsky, this is equivalent to the condition that the obvious maps $\mathbb{H}^n_{Nis}(X, K) \to \mathbb{H}^n_{Nis}(\mathbb{A} \times X, K)$ are isomorphisms for all $X \in Sm/k$ and $n \in \mathbb{Z}$. In particular, one sees that $\mathbf{DM}_{\text{eff}}(k)$ is a triangulated subcategory. The usual t-structure on the derived category of sheaves with transfers induces a t-structure on $\mathbf{DM}_{\text{eff}}(k)$ which is known as the *homotopy t-structure*. The heart of the homotopy t-structure is equivalent to the category $\mathbf{HI}(k)$ of homotopy invariant sheaves with transfers.

In [2], the notion of a n-motivic sheaf was introduced. Given a smooth k-variety X, we denote $h_0(X)$ the largest homotopy invariant quotient of the sheaf with transfers represented by X. Explicitly, $h_0(X)$ is the cokernel of

$$i_1^* - i_0^* : \underline{\text{hom}}(\mathbb{A}^1, \mathbb{Q}_{tr}(X)) \to \mathbb{Q}_{tr}(X).$$

Then a homotopy invariant sheaf with transfers $\mathcal{F}$ is n-motivic if it admits a presentation

$$\bigoplus_\beta h_0(Y_\beta) \to \bigoplus_\alpha h_0(X_\alpha) \to \mathcal{F} \to 0$$

where X_α and Y_β are smooth varieties of dimension $\leq n$. We denote $\mathbf{HI}_{\leq n}(k)$ the full subcategory of $\mathbf{HI}(k)$ whose objets are the n-motivic sheaves. We recall the following fact form [2].

0.2. **Proposition.** *For $n \in \{0,1\}$, $\mathbf{HI}_{\leq n}(k) \subset \mathbf{HI}(k)$ is a thick abelian subcategory, i.e., stable by subobjects, quotients and extensions. Moreover, the obvious inclusion admits a left adjoint. These are denoted by:*

$$\pi_0 : \mathbf{HI}(k) \to \mathbf{HI}_{\leq 0}(k) \quad \textit{and} \quad Alb : \mathbf{HI}(k) \to \mathbf{HI}_{\leq 1}(k).$$

We say that $\mathcal{F} \in \mathbf{HI}(k)$ is 1-*connected* if $Alb(\mathcal{F}) = 0$. It is 0-*connected* if $\pi_0(\mathcal{F}) = 0$. Now, that all the terms of Definition 0.1 are explained, we can state the main theorem of [1].

0.3. **Theorem.** *The category $\mathbf{M}_2(k)$ is abelian.*

In the rest of the talk, we will explain the strategy of the proof of Theorem 0.3. The proof goes by first showing that some larger category ${}^2\mathcal{H}^{\mathcal{M}}(k)$ is abelian. The latter is the full subcategory of $\mathbf{DM}_{\text{eff}}(k)$ whose objects are called $(2, \mathcal{H})$-sheaves. An object $M \in \mathbf{DM}_{\text{eff}}(k)$ is a $(2, \mathcal{H})$-sheaf if it satisfies all the properties of Definition 0.1 except the one stating that $H_{-2}(M)$ is 2-motivic. In other words, instead of (d), we only ask that $H_{-2}(M)$ is 1-connected. Then we show that ${}^2\mathcal{H}^{\mathcal{M}}(k)$ is abelian by constructing a t-structure on $\mathbf{DM}_{\text{eff}}(k)$ whose heart is exactly the category of $(2, \mathcal{H})$-sheaves. The t-structure doing this job is the 2-*motivic t-structure*.

The 2-motivic t-structure is obtained from the homotopy t-structure by applying twice an abstract construction which we now explain. Let $\mathcal{T}$ be a triangulated category endowed with a t-structure $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$. Let $\mathcal{H}$ denotes the heart of $\mathcal{T}$. Assume that we are given a thick abelian subcategory $\mathcal{A} \subset \mathcal{H}$ and a left adjoint to the inclusion $\mathrm{F} : \mathcal{H} \to \mathcal{A}$. Assume also that for every exact sequence in $\mathcal{H}$:

$$0 \to A' \to A \to A'' \to 0$$

with $A'' \in \mathcal{A}$, the morphism $\mathrm{F}(A') \to \mathrm{F}(A)$ is a monomorphism. Then we have the following fact (cf. [1]).

0.4. **Lemma.** *We define a t-structure $({}^{\text{`}}\mathcal{T}_{\geq 0}, {}^{\text{`}}\mathcal{T}_{\leq 0})$ on $\mathcal{T}$ by the following conditions.*

- *An objet $P \in \mathcal{T}$ is in ${}^{\text{`}}\mathcal{T}_{\geq 0}$ iff $P \in \mathcal{T}_{\geq -1}$ and $H_{-1}(P)$ is F-connected (i.e., it is sent to 0 by F).*
- *An object $N \in \mathcal{T}$ is in ${}^{\text{`}}\mathcal{T}_{\leq 0}$ iff $N \in \mathcal{T}_{\leq 0}$ and $H_0(N)$ is in $\mathcal{A}$.*

0.5. **Remark.** The new t-structure $({}^{\text{`}}\mathcal{T}_{\geq 0}, {}^{\text{`}}\mathcal{T}_{\leq 0})$ is called a *perverted* t-structure. An objet A is in the heart of the perverted t-structure if it satisfies the following three conditions.

(a) $H_i(A) = 0$ for $i \notin \{0, -1\}$;
(b) $H_0(A)$ is on $\mathcal{A}$;
(c) $H_{-1}(A)$ is F-connected.

The construction of the n-motivic t-structures $({}^n\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^n\mathcal{T}^{\mathcal{M}}_{\leq 0}(k))$, for $n \in \{0,1,2\}$, goes by induction on n. For $n = 0$, it is simply the 0-motivic t-structure. For $n \in \{1,2\}$, it is obtained by perverting the $(n-1)$-motivic t-structure with respect to the subcategory of $(n-1)$-motives. More precisely, we set.

0.6. **Definition.** *The* 1-motivic *t-structure* $({}^1\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^1\mathcal{T}^{\mathcal{M}}_{\leq 0}(k))$ *is obtained by perverting the homotopy t-structure using the subcategory* $\mathbf{HI}_{\leq 0}(k) \subset \mathbf{HI}(k)$. *The heart of the* 1*-motivic t-structure is denoted by* ${}^1\mathcal{H}^{\mathcal{M}}(k)$ *and its objects are called* $(1,\mathcal{H})$-sheaves*.*

These are objects $M \in \mathbf{DM}_{\text{eff}}(k)$ such that $H_i(M) = 0$ for $i \notin \{0,-1\}$, $H_0(M)$ is 0-motivic, and $H_{-1}(M)$ is 0-connected. The homology functors with respect to the 1-motivic t-structure is denoted by ${}^1H_i$. In ${}^1\mathcal{H}^{\mathcal{M}}(k)$ we have special objects called 1-*motives*. They are defined as follows.

0.7. **Definition.** *An object* $M \in \mathbf{DM}_{\text{eff}}(k)$ *is a* 1-motive *if* $H_i(M) = 0$ *for* $i \notin \{0,-1\}$*,* $H_0(M)$ *is a* 0*-motivic sheaf and* $H_{-1}(M)$ *is a* 0*-connected* 1*-motivic sheaf.*

It is easy to see the link between our definition and Deligne's classical definition of 1-motives. Moreover, it can be shown that $\mathbf{M}_1(k) \subset {}^1\mathcal{H}^{\mathcal{M}}(k)$ is a thick abelian subcategory and that the inclusion has a left adjoint $Alb : {}^1\mathcal{H}^{\mathcal{M}}(k) \to \mathbf{M}_1(k)$. Thus, the following definition makes sense.

0.8. **Definition.** *The* 2-motivic *t-structure* $({}^2\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^2\mathcal{T}^{\mathcal{M}}_{\leq 0}(k))$ *is obtained by perverting the* 1*-motivic t-structure using the subcategory* $\mathbf{M}_1(k) \subset {}^1\mathcal{H}^{\mathcal{M}}(k)$. *The heart of the* 2*-motivic t-structure is denoted by* ${}^2\mathcal{H}^{\mathcal{M}}(k)$ *and its objects are called* $(2,\mathcal{H})$-sheaves*.*

It is now a matter of unrolling the definitions to see that a $(2,\mathcal{H})$-sheaf is an object $M \in \mathbf{DM}_{\text{eff}}(k)$ satisfying all the conditions of Definition 0.1 with the exception of $H_{-2}(M)$ being a 2-motivic sheaf. Theorem 0.3 follows then quite easily from this.

References

[1] J. Ayoub, *The n-motivic t-structure for $n = 0$, 1 and* 2, Advances in Mathematics **226** (2011), 111–138.

[2] J. Ayoub and L. Barbieri-Viale, 1-*Motivic sheaves and the Albanese functor*, Journal of Pure and Applied Algebra **213** (2009), 809–839.

[3] C. Mazza, V. Voevodsky and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, Volume 2.

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