Ex 6. Let $x_1, \dots, x_n \in \mathbb{K}$. We define for $k = 1, \dots, k$

$$\boxed{P_k = x_1^k + \dots + x_n^k}$$

the k-th Newton sum.

1. Write a pseudocode that takes as input $x \in \mathbb{K}$ and an integer n and returns the list of x^k for k power of 2, less than or equal to n in $O(\log n)$.

Pseudocode: Power(x, n)
Input: $x \in \mathbb{K}$, $n \in \mathbb{Z}$
Output: $x^0, x^1, x^2, x^4, \dots, x^{2^t}$, where $2^t \leq n$

$t = \lfloor \log_2 n \rfloor$

$L[0] = 1$, $L[1] = x$

for $i = 2, \dots, t$ do

$x = x^2$

$L[i] = x$

end for

The cost is $O(\log n)$ arithmetic operations since the for is iterated $\log n$ times.

2. Given $x_1, \dots, x_m$ returns P_K in $O(m \log m)$ arithmetic operations in $\mathbb{K}$.

NewtonSum($m, x_1, \dots, x_m$)

Input: m, $L = [x_1, \dots, x_m]$

Output: $x_1^k + \dots + x_m^k$, for $k \le m$, k power of 2

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Sums = [0, ..., 0];   Powers = [ ];
for i in [1, #L] do
    Powers[i] = [Powers(x, m)]
end for

for j = 1, ..., log_2 m do
    for k = 1, ..., m do
        Sums[j] += Powers[k][j]
    end for
end for
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This will cost $O(m \log m)$ operations

3. Prove that $\frac{x}{1-x} = x + x^2 + x^3 + \dots$

If this is true then $\frac{1}{1-x} = 1 + x + x^2 + \dots$

Call $\frac{1}{1-x} = S$.

Then $S - Sx = 1$ hence

$S = 1 + Sx$

$= 1 + (1+Sx)x = 1 + x + Sx^2$

$= 1 + x + (1+Sx)x^2 = 1 + x + x^2 + Sx^3$

$= \cdots$ so on $\Rightarrow S = 1 + x + x^2 + \cdots$

4. Deduce that $\displaystyle\sum_{i=1}^{n} \frac{x_i X}{1 - x_i X} = \sum_{k \geq 1} P_k X^k$

To avoid confusion, instead of x_i write α_i

$$\frac{\alpha_i X}{1 - \alpha_i X} = (\alpha_i X) + (\alpha_i X)^2 + (\alpha_i X)^3 + \cdots$$

Hence $\displaystyle\sum_{i=1}^{n} \frac{\alpha_i X}{1 - \alpha_i X} = \sum_{i=1}^{n} \left(\alpha_i X + (\alpha_i X)^2 + \cdots \right)$

$$= \sum_{i=1}^{n} \left(\sum_{k \geq 1} (\alpha_i X)^k \right) =$$

$$= \sum_{k \geq 1} \left(\sum_{i=1}^{n} (\alpha_i)^k \right) X^k = \sum_{k \geq 1} P_k X^k$$

5. Compute P_k for $k \leq n$ in $O(M(n) \log n)$

Since $\displaystyle\sum_{k \geq 1} P_k X^k = \sum_{i=1}^{n} \frac{\alpha_i X}{1 - \alpha_i X}$

we compute the sums of $\frac{A_i(x)}{B_i(x)}$ where

$A_i(x) = \alpha_i x$ and $B_i(x) = 1 - \alpha_i(x)$

with a divide and conquer approach.