Algorithmique de base

Master 1, Université de Rennes

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Exercise 1. Let $A(T) = \frac{1-T^2}{1+T^2}$ and $B(T) = \frac{2T}{1+T^2}$. Show that calculating the resultant in T of the numerators of $X - A(T)$ and $Y - B(T)$ gives that the curve $(A(T), B(T))$ has the equation

$$X^2 + Y^2 = 1.$$

Exercise 2. Let $f = 4x^2 + 5x + 6$ and $g = 3x + 2 \in \mathbb{Z}[x]$.

1. Using the extended Euclidean algorithm, determine S and T in $\mathbb{Q}[x]$ such that $Sf + Tg = 1$.
2. Determine $r = \mathrm{res}(f, g)$.
3. Let M be the Sylvester matrix associated with (f, g) in the bases $\{(1, 0), (0, x), (0, 1)\}$ and $\{x^2, x, 1\}$. Solve the linear system

$$MX = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

   for the unknown $X \in \mathbb{Q}^3$.
4. Deduce U and V in $\mathbb{Q}[x]$ such that $Uf + Vg = r$.
5. Compare (S, T) and (U, V).

Exercise 3. Let $f(x, y) = x^2y + 3x - 1$ and $g(x, y) = 6x^2 + y^2 - 4$.

1. Determine $h(y) = \mathrm{res}_x(f(x, y), g(x, y))$ and then $h(0)$.
2. Determine $r = \mathrm{res}_x(f(x, 0), g(x, 0))$.
3. Compare r and $h(0)$. Comment the result.

Exercise 4. Let $f(X) = X^3 + pX + q \in \mathbb{C}[X]$. Using a calculation of the resultant, determine a necessary and sufficient condition for f to have a double root.

Exercise 5. Solve the system in $\mathbb{C}^2$:

$$\begin{cases} x^2y + x = 1, \\ x^2y + x + y^2 = 4. \end{cases}$$
