Classical Information Theory

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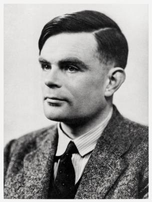
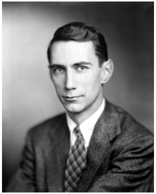
Quantum Information Seminar

13 March 2020

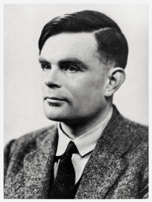
Overview

1. Turing
   1. Automata Theory
   2. Turing Machines
   3. Complexity Classes
2. Shannon
   1. Entropy
   2. Channels





Part 1: Turing



- 1930: Before invention of computers: Turing machines
- Goals:
  - What can Turing machines do and what not → Decidability
  - What can Turing machines do efficiently → Intractability

Ingredients

- Σ the alphabet: finite set of symbols
  $\{a, \ldots, z\}, \{0, 1\}$
- w a word: a finite sequence of symbols in Σ
  $hello, 01101$
- ε the empty word
- $\Sigma^\star$ the Kleene star: set of all possible words with symbols in Σ. More formally:

$$\Sigma^0 = \{\varepsilon\}$$
$$\Sigma^1 = \Sigma$$
$$\Sigma^{i+1} = \{ab \mid a \in \Sigma^i, b \in \Sigma\}$$
$$\Sigma^\star = \cup_{i \geq 0} \Sigma^i$$

- $\mathcal{L} \subseteq \Sigma^\star$ a language: set of words
  $\emptyset, \Sigma^\star$, english

Definition (Deterministic Finite Automaton (DFA))

A deterministic finite automaton A is a tuple $(\Sigma, Q, \delta, q_0, F)$, where

- Σ *is an alphabet*
- Q *is a finite set of states*
- $\delta : Q \times \Sigma \to Q$ *is a transition function*
- $q_0 \in Q$ *is the initial state*
- $F \subseteq Q$ *is the set of final states*

Example

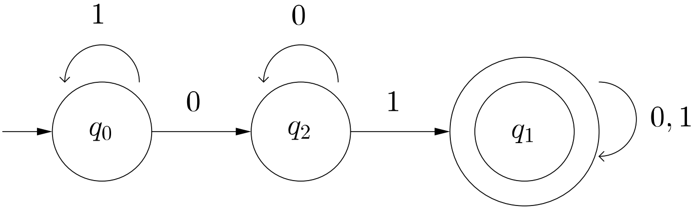
- $\Sigma = \{0, 1\}$
- $Q = \{q_0, q_1, q_2\}$
- transition table

δ	0	1
q_0	q_2	q_0
q_1	q_1	q_1
q_2	q_2	q_1

- $F = \{q_1\}$

Example

transition diagram:



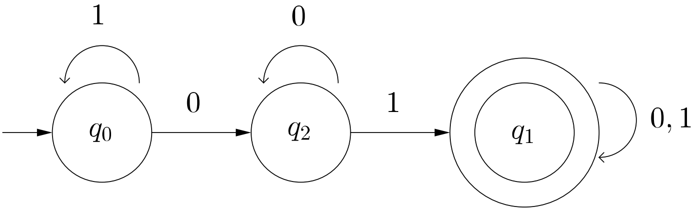
We can define the transition map for words, inductively as follows

$$\begin{aligned}\hat{\delta} : Q \times \Sigma^\star &\to Q \\ (q, wa) &\mapsto \delta(\hat{\delta}(q, w), a).\end{aligned}$$

Notions:

- An *execution of a word* $w \in \Sigma^\star$ *by* A is $\hat{\delta}(q_0, w)$.
- A word $w \in \Sigma^\star$ is *accepted by* A, if $\hat{\delta}(q_0, w) \in F$.

1100 is not accepted by A



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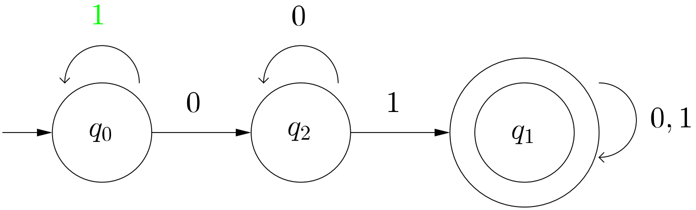
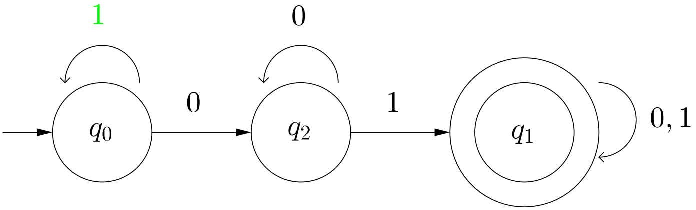
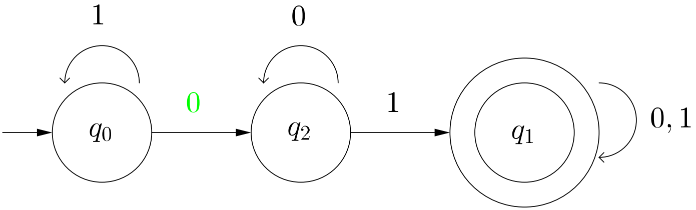


$1{\color{green}1}00$ **is not accepted by** A



$11\color{green}{0}0$ **is not accepted by** A



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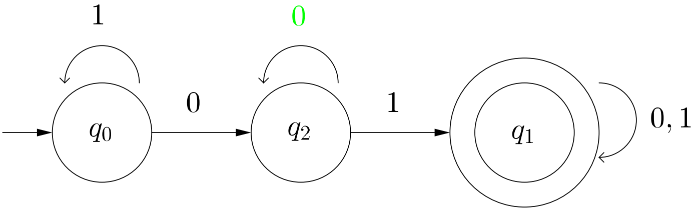
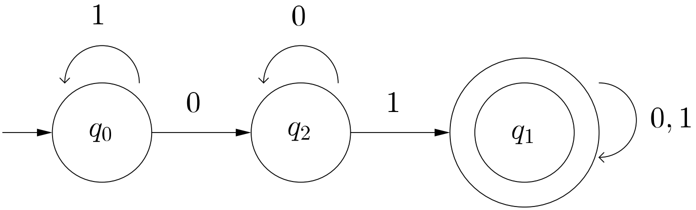


1010 **is accepted by** A



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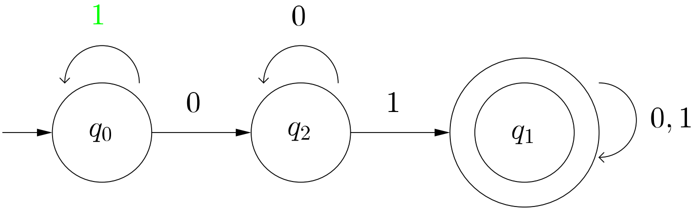

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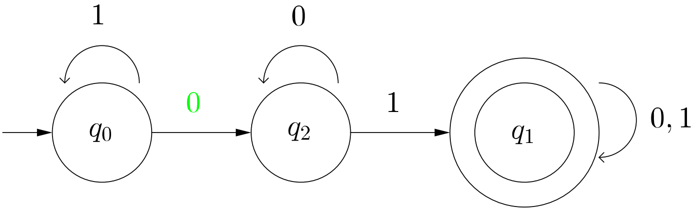

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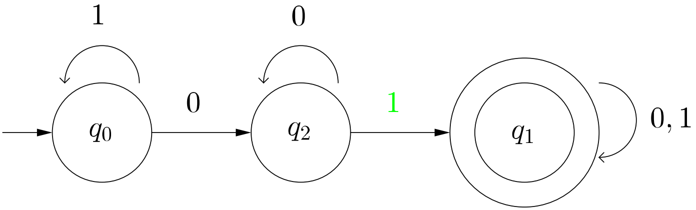
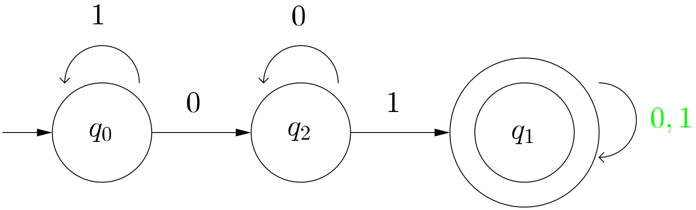


$101\textcolor{green}{0}$ **is accepted by** A



Definition

The language accepted by A is

$$\mathcal{L}(A) = \{w \mid \hat{\delta}(q_0, w) \in F\}.$$

Definition

We call a language $\mathcal{L}$ regular, if there exists a deterministic finite automaton A, such that $\mathcal{L} = \mathcal{L}(A)$.

In our example the language accepted by the automaton is all binary words containing 01.
Notation: $\mathcal{L} = (0+1)^\star 01(0+1)^\star$.
Homework Give a deterministic finite automaton accepting all binary words ending in 00.

Definition (Nondeterministic finite automaton (NFA))

A nondeterministic finite automaton A is a tuple $(\Sigma, Q, \delta, q_0, F)$, *where*

- Σ *is an alphabet*
- Q *is a finite set of states*
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ *is a transition function*
- $q_0 \in Q$ *is the initial state*
- $F \subset Q$ *are the final states*

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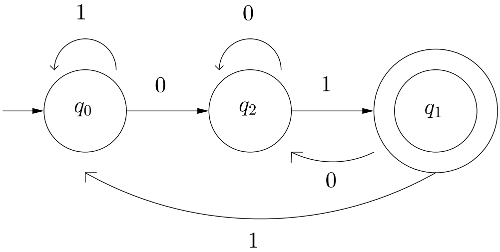
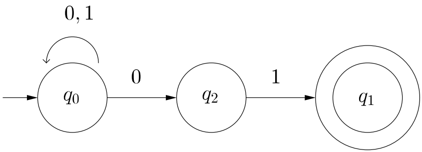
Automata Theory

DFA accepting words ending in 01



NFA accepting words ending in 01



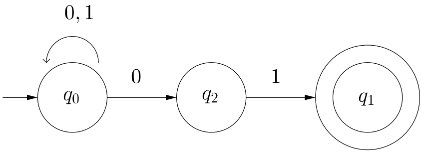
Transition table for the NFA

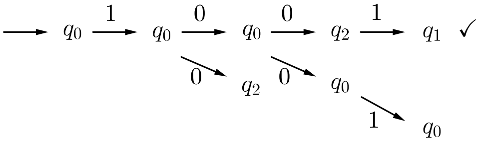
δ	0	1
q_0	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\emptyset$	$\emptyset$
q_2	$\emptyset$	$\{q_1\}$

and the language accepted by an NFA is

$$\mathcal{L}(A) = \{w \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset\}.$$

Example 1001 is accepted





Theorem (Cool Fact)

If A_N is an NFA, then there exists an A_D a DFA, such that

$$\mathcal{L}(A_N) = \mathcal{L}(A_D).$$

Homework Give a nondeterministic finite automaton accepting all binary words containing 01 or ending in 00.

Theorem (Properties of Regular Languages)

Let $\mathcal{L}, \mathcal{M} \subseteq \Sigma^\star$ be regular languages, then

- $\mathcal{L}^\star$ *is a regular language.*
- $\mathcal{L}\mathcal{M}$ *is a regular language.*
- $\mathcal{L} \cap \mathcal{M}$ *is a regular language.*
- $\mathcal{L} \cup \mathcal{M}$ *is a regular language.*
- $\mathcal{L}^R$ *is a regular language.*
- $\overline{\mathcal{L}}$ *is a regular language.*

Proof of $\overline{\mathcal{L}}$ is a regular language.

$$\overline{\mathcal{L}} = \{w \in \Sigma^\star \mid w \notin L\} = \Sigma^\star \setminus \mathcal{L}.$$

Let $A = (\Sigma, Q, \delta, q_0, F)$ be a DFA accepting $\mathcal{L}$. Define the DFA B to be $(\Sigma, Q, \delta, q_0, Q \setminus F)$. We claim that $\mathcal{L}(B) = \overline{\mathcal{L}}$:

$$w \in \mathcal{L}(B) \Leftrightarrow \hat{\delta}(q_0, w) \in Q \setminus F \Leftrightarrow w \notin \mathcal{L}.$$

$\square$

Homework: Prove that if $\mathcal{L}$ is a regular language, then

$$\mathcal{L}_{pre} = \{w \mid \exists a \in \Sigma \text{ with } wa \in \mathcal{L}\}$$

is a regular language.

Definition (Deterministic Turing Machine (DTM))

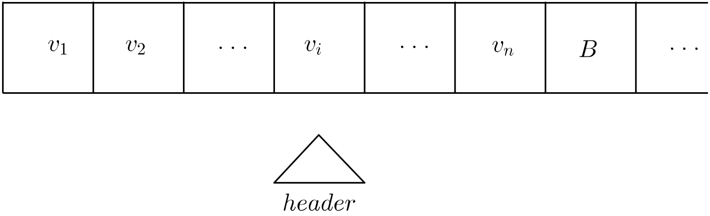
A deterministic Turing machine M is a tuple $(\Sigma, \Gamma, B, Q, q_0, F, \delta)$, where

- Σ *is an alphabet, called input alphabet*
- $\Gamma \supset \Sigma$ *is an alphabet, called tape alphabet*
- $B \in \Gamma \setminus \Sigma$ *is the blank symbol*
- Q *is a finite set of states*
- $q_0 \in Q$ *is the initial state*
- $F \subseteq Q$ *is the set of final states*
- δ *is a partial function*

$$\delta : Q \times \Gamma \to Q \times \{L, R, S\} \times \Gamma$$
$$(q, s) \mapsto (q', D, s')$$

Different notation: $-1 = L$ left, $1 = R$ right, $0 = S$ stay.

Turing Machines



- The tape is bounded on the left.
- The tape is infinite on the right.
- The tape is divided into cells.
- Each cell carries a symbol from Γ.
- The header can read and write.

Definition (Configuration)

A configuration of a DTM $M = (\Sigma, \Gamma, B, Q, q_0, F, \delta)$ *is* (q, i, v), *where*

- $q \in Q$ *is the state in which* M *is in*
- $i \in \mathbb{N}$ *is the cell number to which the header is pointing*
- $v \in \Gamma^\star$ *is the word written on the tape from the first to the last non-blank symbol*

Definition

A configuration $c' = (q', i', v')$ *is derived in one step from* $c = (q, i, v)$ *in the DTM M, if*

- $\delta(q, v_i) = (q', D, a)$,
- $i' = \begin{cases} i+1 & \text{if } D = R \\ i & \text{if } D = S \\ i-1 & \text{if } D = L \end{cases}$,
- $v' = v$, *except that* $v'_i = a$.

Notation: $c \vdash c'$

Definition

A configuration c' is derived from c in the DTM M, if there exists a sequence of configurations $c_1, \ldots, c_k$, such that

$$c \vdash c_1 \vdash \cdots \vdash c_k \vdash c'.$$

Notation: $c \vdash^{\star} c'$.

Notions

- The *inital configuration* of M on the input w is $(q_0, 1, w)$.
- The *execution* of M on the input w is the sequence of configurations $(c_0, \ldots)$, where c_0 is the initial configuration and $c_i \vdash c_{i+1} \forall i$.
- The *final configuration* is a configuration (q, i, v), such that $\delta(q, v_i)$ is not defined.
- The DTM M *stops* on the input w, if the execution of M on the input w reaches a final configuration.
- If the DTM M stops on the input w, then the *computation* $M(w)$ of M on the input w is the word written on the tape, when the final configuration is reached.

- The DTM M *accepts* w, if the execution of M on the input w reaches a final configuration (q, i, v), with $q \in F$.
- The DTM M *rejects* w, if the execution of M on the input w reaches a final configuration (q, i, v), with $q \notin F$.
- The *language accepted by* M is the set of words w, such that M accepts w.
- The *function computed by* M is the partial function, that associated $M(w)$ to w, for all w, such that M stops on w.
- The language $\mathcal{L}$ is *derived* by M, if $\mathcal{L}$ is accepted by M and M always stops.

Example

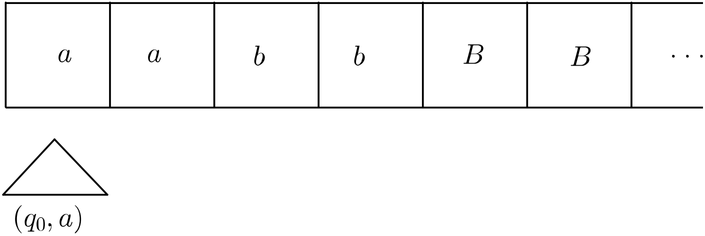
A DTM accepting $\mathcal{L} = \{a^n b^n \mid n \geq 0\}$ is given by

- $\Sigma = \{a, b\}$,
- $\Gamma = \{a, b, D_a, D_b, B\}$,
- $Q = \{q_0, q_{wb}, q_{sa}, q_{fa}, q_e, q_f, q_r\}$,
- $F = \{q_f\}$

and

δ	a	b	D_a	D_b	B
q_0	(q_{wb}, R, D_a)	(q_r, S, b)			(q_f, S, B)
q_{wb}	(q_{wb}, R, a)	(q_{sa}, L, D_b)		(q_{wb}, R, D_b)	(q_r, S, B)
q_{sa}	(q_{sa}, L, a)		(q_{fa}, R, D_a)	(q_{sa}, L, D_b)	
q_{fa}	(q_{wb}, R, D_a)			(q_e, S, D_b)	
q_e				(q_e, R, D_b)	(q_f, S, B)

Example $aabb$



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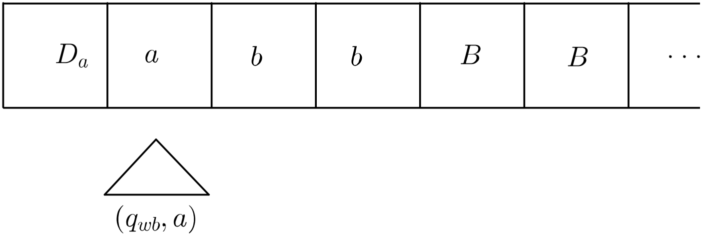

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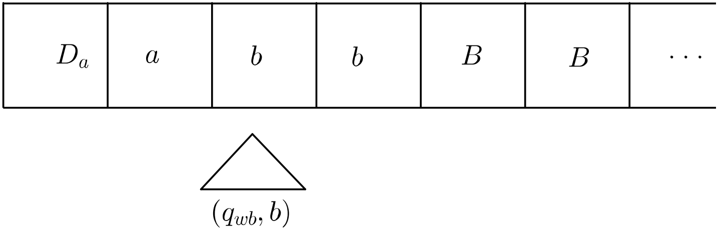

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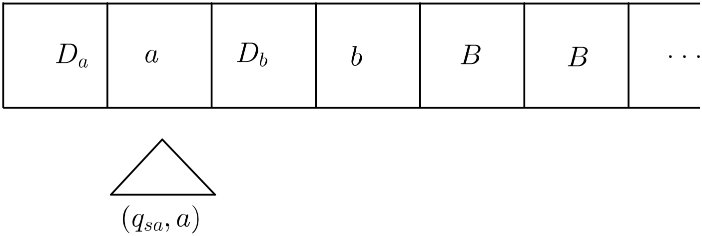

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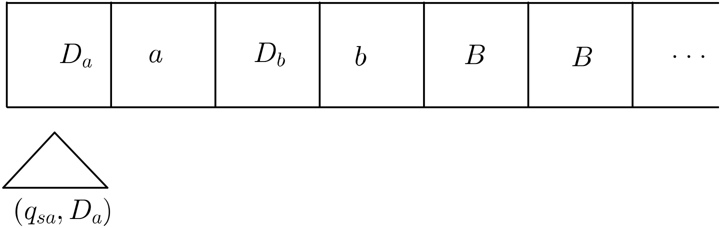

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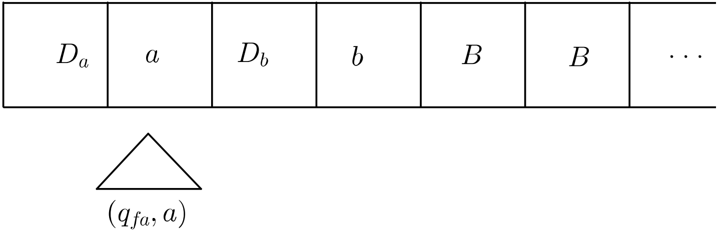

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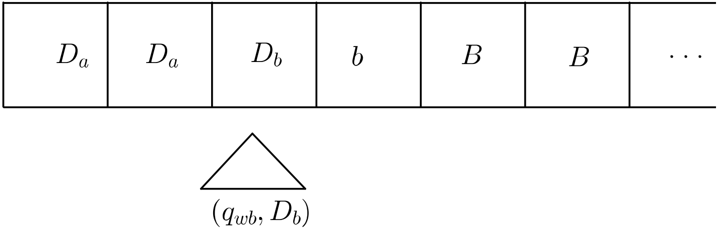

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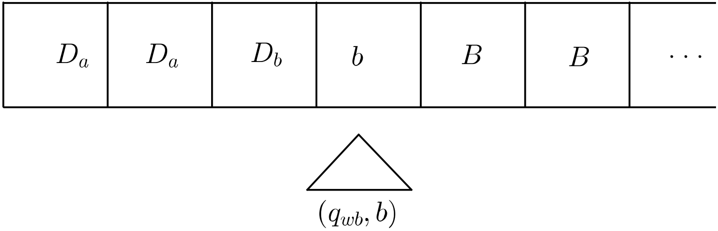

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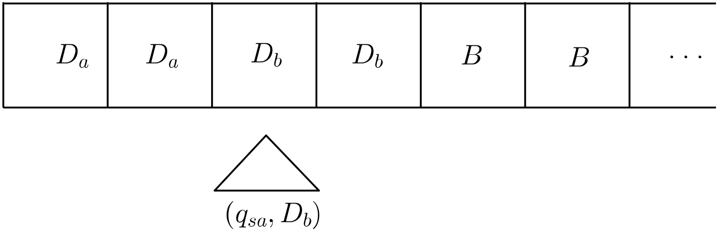

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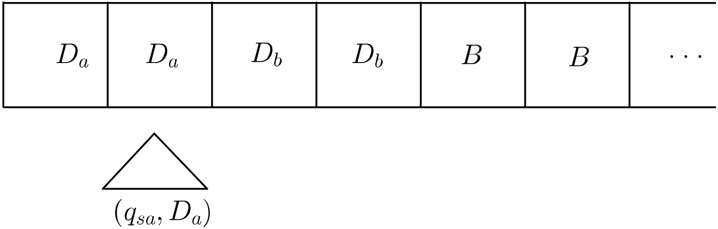

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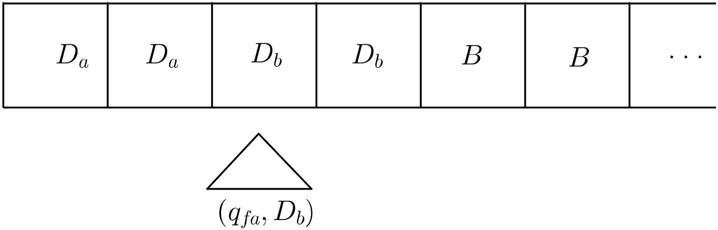

Turing Machines

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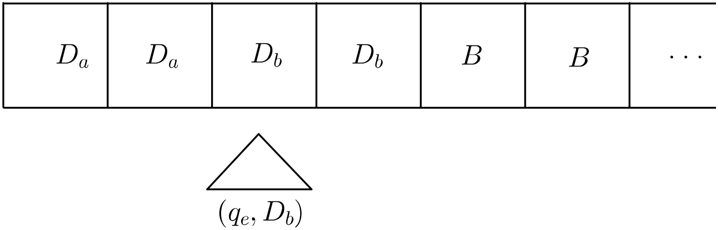

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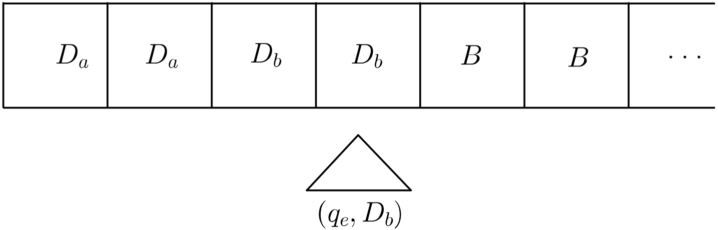

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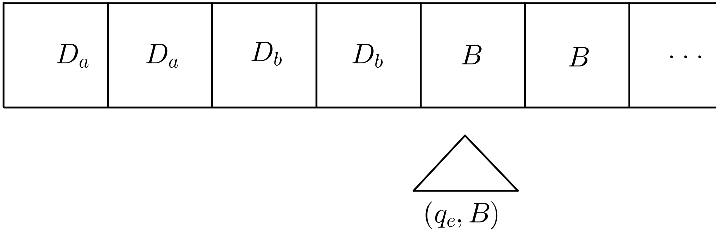

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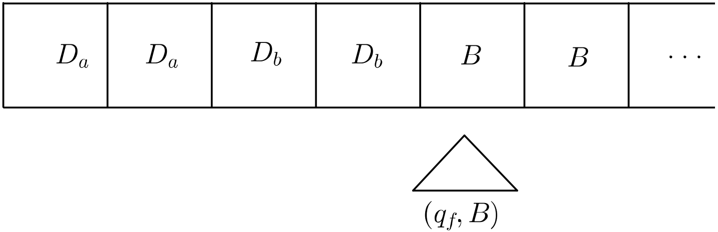

Homework Describe formally a DTM that accepts the binary encodings of even numbers.

Difference to Automaton

- An Automaton is without memory, whereas a TM has a memory in from of the tape.
- The TM can change the word written on the tape.
- An Automaton is a TM, that never changes the direction, nor changes the symbols on the tape.
- TMs accept more languages: recursively enumerable languages

A nondeterministic TM (NTM) is a TM, where the partial function δ has multiple outputs and the TM can choose one.

An NTM M accepts an input w, if there is any sequence of configurations on w that reaches a final configuration.

Theorem

If M_N is an NTM, then there exists M_D a DTM, such that

$$\mathcal{L}(M_n) = \mathcal{L}(M_D).$$

BUT the DTM may take exponentially more time than the NTM.

A nondeterministic TM (NTM) is a TM, where the partial function δ has multiple outputs and the TM can choose one.

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BUT the DTM may take exponentially more time than the NTM.

What is the difference between a TM and a classical computer?

- A computer can simulate a TM.
- A TM can simulate a computer (if n is the number of steps of a computer, then the TM needs at most a polnomial in n number of steps)
- They accept the same language

We solved the question of what computers can do. What is it that computers cannot do?

Definition (Decidable)

A language $\mathcal{L}$ is decidable, if there exists a TM M, such that $\mathcal{L} = \mathcal{L}(M)$ and M always stops.

Equivalently, we can ask, are there undecidable languages/problems?

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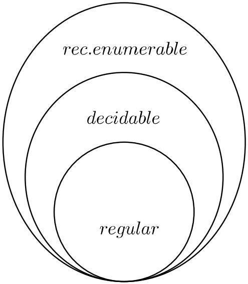


Examples

- Regular language: Binary words containing 01
- Decidable, but not regular: $\mathcal{L} = \{a^n b^n \mid n \geq 0\}$
- Recursively enumerable but not decidable: the Halting problem: $H(M) = \{w \mid M \text{ halts on input } w\}$, $\mathcal{L} = \{(M, w) \mid w \in H(M)\}$.
- No recursively enumerable $\mathcal{L} = \{M \mid \mathcal{L}(M) = \emptyset\}$.

What can be solved efficiently?

Definition (Running Time)

A TM M is said to have running time/ time complexity $T(n)$, if, whenever M is given an input w of length n, M halts after at most $T(n)$ moves.

Definition (P)

A problem $\mathcal{P}$ is in P, if it can be solved by a DTM in polynomial time.

Examples

- Multiplication: Given $a, b, k \in \mathbb{N}$ encoded in binary, is the kth bit of $a \cdot b$ equal to 1?
- Paths: Given a graph $\mathcal{G}$ and s, t vertices, is there a path from s to t?
- Given $n \in \mathbb{N}$, is n a prime?

Complexity Classes

Definition (NP)

A problem $\mathcal{P}$ is in NP, if it can be solved by a NTM in polynomial time.

or equivalently

Definition

A problem $\mathcal{P}$ is in NP, if a candidate for a solution can be checked by a DTM in polynomial time.

Clearly $P \subseteq NP$ but it remains one of the hardest problems to prove or disprove if $P = NP$.

Examples

- Knapsack: Given $(p_1, \ldots, p_k) \in \mathbb{Z}^k$ and $t \in \mathbb{Z}$, is there a subset $S \subset \{1, \ldots, k\}$, such that $\sum_{i \in S} p_i = t$?
- Clique: Given a graph $\mathcal{G}$ and $k \in \mathbb{N}$, does $\mathcal{G}$ contain a clique of size k, i.e. a set S of k vertices, such that $\forall u, v \in S : (u, v)$ is an edge of $\mathcal{G}$?

Definition (Polynomial Time Reduction)

Given two problems $\mathcal{P}_1$ and $\mathcal{P}_2$, we can reduce $\mathcal{P}_1$ to $\mathcal{P}_2$ in polynomial time, if

- *any instance of $\mathcal{P}_1$ can be transformed in polynomial time to an instance of $\mathcal{P}_2$,*
- *assuming a polynomial time oracle that solves $\mathcal{P}_2$, we get a solution of this instance,*
- *we can transform the solution of $\mathcal{P}_2$ is polynomial time to a solution of $\mathcal{P}_1$.*

$\mathcal{P}_1$ is at least as hard as $\mathcal{P}_2$.

Definition (NP-hard)

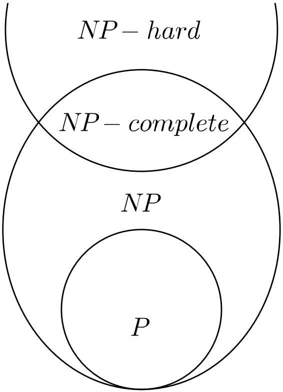
A problem $\mathcal{P}$ is called NP-hard, if any problem in NP can be reduced in polynomial time to $\mathcal{P}$.

Consequences

- Solving an NP-hard problem in polynomial time, means *any* problem in NP can be solved in polynomial time.
- To prove $P = NP$, it is enough to find a polynomial time algorithm for *one* NP-hard problem.
- To prove a new problem is NP-hard, it is enough to find a polynomial time reduction of *one* NP-hard problem to this new problem.

Definition (NP-complete)

A problem $\mathcal{P}$ is called NP-complete, if $\mathcal{P}$ in NP-hard and in NP.



Examples of *NP*-complete problems:

- Knapsack problem
- Clique problem

Examples of problems in *NP*, that are not *NP*-hard:

- Integer factorization: Given $n = p \cdot q \in \mathbb{N}$, where p, q are primes find p and q.
- Discrete logarithm problem: Given $n \in \mathbb{N}$ and $x, y \in \mathbb{Z}/n\mathbb{Z}$, find $k \in \mathbb{N}$, such that $y = x^k \mod n$.

Examples of *NP*-hard problems, that are not in *NP*:

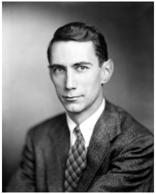
- Halting Problem
- Towers of Hanoi

There are many more complexity classes:
google "Complexity Zoo" to find a list of over 500 classes.

Important Examples

- $PSPACE$: Problems that can be solved by a DTM using polynomial space
- EXP: Problems that can be solved by a DTM in exponential time
- $CO-NP$: the complement of all languages that are in NP.

Part 2: Shannon



- 1948: Father of Information Theory with the article "A mathematical theory of communication"
- Goals:
  - What is "information" $\rightarrow$ Entropy
  - How can we provide information efficiently and reliably? $\rightarrow$ Channels, Codes

Before Shannon in 1928: **Hartley**

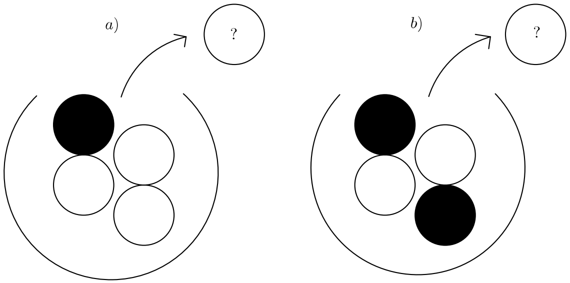
- Information is the value of a random variable
- Also suggested a measure of information

Hartley's measure of the amount of information by observing a discrete random variable X

$$I(X) = \log_b(L),$$

where L is the number of possible values of X.

But there is a problem

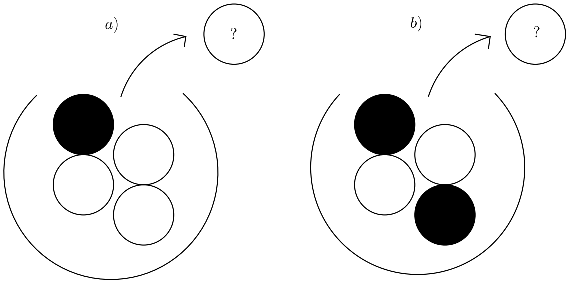


Since $L = 2$, in both examples $I(X) = 1$.

But in a) a white ball is worth less information

Hartley ignores the probabilities of the values

But there is a problem



Since $L = 2$, in both examples $I(X) = 1$.
But in a) a white ball is worth less information
Hartley ignores the probabilities of the values

What should Hartley have done instead?

In a) there is 1 chance out of 4 of choosing a black ball:

$$\log_2\left(\frac{4}{1}\right) = 2$$

and there are 3 chances out of 4 of choosing a white ball:

$$\log_2\left(\frac{4}{3}\right) = 0.415$$

Weight them by their probabilities of occurence:

$$\frac{1}{4}\cdot 2 + \frac{3}{4}\cdot 0.415 = 0.811.$$

Or equivalently

$$-\frac{1}{4}\log_2\left(\frac{1}{4}\right) - \frac{3}{4}\log_2\left(\frac{3}{4}\right) = 0.811.$$

Entropy

In general, if the ith possible value of X has probability p_i, then the amount of information provided by X is

$$-\sum_{i=1}^{L} p_i \log(p_i).$$

What if $p_i = 0$?

Notation:

- If f is a real valued function, then Supp(f) is the subset of its domain, where f takes non-zero values.
- P_X is the probability distribution for the discrete r.v. X

Definition (Uncertainty/Entropy)

The uncertainty or entropy of a discrete random variable X is

$$H(X) = -\sum_{x \in Supp(P_X)} P_X(x) \log_b(P_X(x)).$$

Entropy

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Remark

$$H(X) = E[-\log(P_X(X))].$$

Also works for discrete random vectors:

Remark

$$H(X, Y) = E[-\log(P_{X,Y}(X, Y))].$$

Example:
X has two possible values x_1 and x_2 with $P_X(x_1) = p$ and $P_X(x_2) = 1 - p$, for some $0 < p < 1$, then the uncertainty of X in bits is the *binary entropy function*

$$H(X) = -p\log_2(p) - (1-p)\log_2(1-p) = h(p).$$

Entropy

Theorem (Information Theory inequality)

For a positive real number r

$$\log(r) \leq (r-1)\log(e).$$

With equality if and only if $r = 1$.

Theorem

If the discrete random variable X *has* L *possible values, then*

$$0 \leq H(X) \leq \log(L),$$

with equality on the left side, if $P_X(x) = 1$ *for some* x*, and equality on the right side, if* $P_X(x) = \frac{1}{L}$ *for all* x*.*

Definition (Conditional Uncertainty)

The conditional uncertainty/entropy of the discrete random variable X given the event $Y = y$ occurs is

$$H(X \mid Y = y) = -\sum_{x \in Supp(P_{X|Y}(.|y))} P_{X|Y} \log(P_{X|Y}(x \mid y)).$$

Remark

$$H(X \mid Y = y) = E[-\log(P_{X|Y}(X \mid Y)) \mid Y = y].$$

Corollary

If the discrete random variable X has L possible values, then

$$0 \leq H(X \mid Y = y) \leq \log(L),$$

with equality on the left side, if $P_{X|Y}(x \mid y) = 1$ for some x, and equality on the right side, if $P_{X|Y}(x \mid y) = \frac{1}{L}$ for all x.

Definition (Conditional Uncertainty)

The conditional uncertainty of the discrete random variable X given the discrete random variable Y is

$$H(X \mid Y) = \sum_{y \in Supp(P_Y)} P_Y(y) H(X \mid Y = y).$$

Remark

$$H(X \mid Y) = E[-\log(P_{X|Y}(X \mid Y))].$$

Corollary

If the discrete random variable X has L possible values then

$$0 \leq H(X \mid Y) \leq \log(L),$$

with equaltiy on the left side, if for all $y \in Supp(P_Y) : P_{X|Y}(x \mid y) = 1$ for some x, i.e. Y essentially determines X, and equality on the right side, if for all $y \in Supp(P_Y) : P_{X|Y}(x \mid y) = \frac{1}{L}$ for all x.

Definition (Information Divergence/ Relative Entropy)

If X and $\tilde{X}$ are discrete random variables with the same set of possible values, then the information divergence between P_X and $P_{\tilde{X}}$ is

$$D(P_X \mid\mid P_{\tilde{X}}) = \sum_{x \in Supp(P_X)} P_X(x) \log\left(\frac{P_X(x)}{P_{\tilde{X}}(x)}\right).$$

Note:

- If there is a $x \in \mathrm{Supp}(P_X)$ but not in $\mathrm{Supp}(P_{\tilde{X}})$, i.e. $P_X(x) \neq 0$ and $P_{\tilde{X}}(x) = 0$, then $D(P_X \mid\mid P_{\tilde{X}}) = \infty$.
- In general: $D(P_X \mid\mid P_{\tilde{X}}) \neq D(P_{\tilde{X}} \mid\mid P_X)$.

Remark

$$D(P_X \,||\, P_{\tilde{X}}) = E\left[\log\left(\frac{P_X(x)}{P_{\tilde{X}}(x)}\right)\right].$$

Theorem (Divergence Inequality)

$$D(P_X \,||\, P_{\tilde{X}}) \geq 0,$$

with equality if and only if $P_X = P_{\tilde{X}}$.

Knowing Y reduces our uncertainty about X

Theorem (2. Entropy Inequality)

For any two discrete random variables X, Y

$$H(X \mid Y) \leq H(X),$$

with equality if and only if X *and* Y *are independent.*

Theorem (The Chain Rule for Uncertainty)

$$H(X_1, \ldots, X_N) = H(X_1) + H(X_2 \mid X_1) + \cdots + H(X_N \mid X_1, \ldots, X_{N-1}).$$

But wait, what is information now?

Shannon: "Information is the difference between uncertainties."
How much information does the random variable Y give about the random variable X?
Shannon: "The amount by which Y reduces the uncertainty about X."

Definition (Mutual Information)

The mutual information between the discrete random variable X and Y is

$$I(X;Y) = H(X) - H(X \mid Y).$$

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Definition (Mutual Information)

The mutual information between the discrete random variable X and Y is

$$I(X;Y) = H(X) - H(X \mid Y).$$

Why mutual?

$$
\begin{aligned}
H(X, Y) &= H(X) + H(Y \mid X) \\
&= H(Y) + H(X \mid Y)
\end{aligned}
$$

Hence

$$H(X) - H(X \mid Y) = H(Y) - H(Y \mid X)$$

That is

$$I(X; Y) = I(Y; X).$$

Definition (Conditional Mutual Information)

The conditional mutual information between the discrete random variable X and Y given the event $Z = z$ occurs is

$$I(X; Y \mid Z = z) = H(X \mid Z = z) - H(X \mid Y, Z = z).$$

Definition (Conditional Mutual Information)

The conditional mutual information between the discrete random variable X and Y given the discrete random variable Z is

$$I(X; Y \mid Z) = H(X \mid Z) - H(X \mid Y, Z).$$

Theorem

For any two discrete random variables X, Y

$$0 \leq I(X;Y) \leq \min\{H(X), H(Y)\},$$

with equailty on the left side, if X and Y are independent, and equality on the right side, if Y essentially determines X or X essentially determines Y.

Channels

Now we have solved the question of what is information.

How can we transmit information efficiently and reliably from its source to the destination?

- The source can choose the signal.
- The channel specifies the conditional probabilites of the signals that can be received.

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- The channel specifies the conditional probabilites of the signals that can be received.

Channels

We will only consider time-discrete channels, such that the channel input and output can be described as sequences of random variables:

- Input sequence: $X_1, \ldots$
- Output sequence: $Y_1, \ldots$

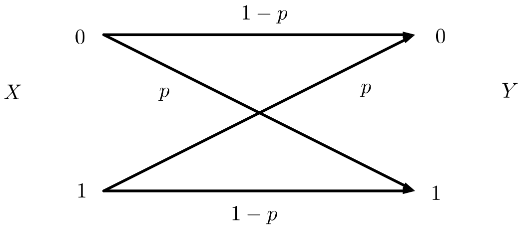
Definition (Discrete Memoryless Channel (DMC))

A discrete memoryless channel (DMC) consists of

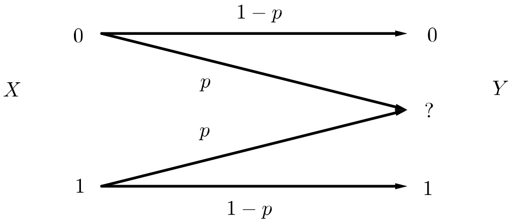
- *A the input alphabet: its symbols represent one of the signals the sender chan choose*
- *B the output alphabet: its symbols represent one of the output signals*
- *$P_{Y|X}(. \mid x)$ the conditional probability distribution over B for all $x \in A$, which governs the channel behaviour, such that*

$$P(y_n \mid x_1, \ldots, x_n, y_1, \ldots, y_{n-1}) = P_{Y|X}(y_n \mid x_n).$$

Example: Binary Symmetric Channel (BSC)



Example: Binary Erasure Channel (BEC)



Channels

Definition (DMC without Feedback)

We call a DMC to be without feedback, if

$$P(x_n \mid x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}) = P(x_n \mid x_1, \ldots, x_{n-1}),$$

i.e., we are not using the past output digits to choose new inputs.

Theorem

When a DMC is used without feedback, then

$$P(y_1, \ldots, y_n \mid x_1, \ldots, x_n) = \prod_{i=1}^{n} P_{Y|X}(y_i \mid x_i).$$

Recall: The DMC specifies the conditional probability distribution, but the sender is free to choose the input probability distribution.

Definition (Capacity)

The capacity of a channel is

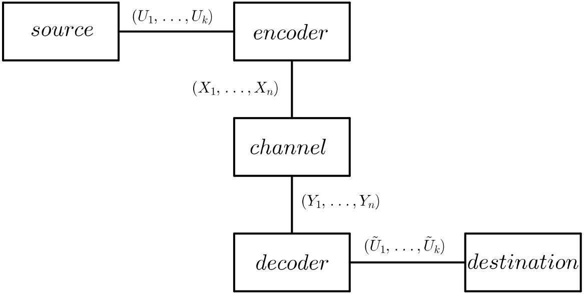
$$C = \max_{P_X} \{ I(X;Y) \}.$$

Example:

- BSC: $C = 1 - h(p)$
- BEC: $C = 1 - p$

How to reliably transmit information through a DMC?

We use k *information bits* to encode a message into n *channel digits*.
This has a rate of $R = \frac{k}{n}$ bits per use.
The channel is noisy, i.e., it enters some errors in what we send:
We encode $U_1, \ldots, U_k$ and send this to a receiver, while the receiver might decode $\tilde{U}_1, \ldots, \tilde{U}_k$.



Definition (Bit Error Probability)

The fraction of the digits that are in error is the bit error probability

$$P_b = \frac{1}{k}\sum_{i=1}^{k} p_{ei},$$

where

$$p_{ei} = P(\tilde{U}_i \neq U_i).$$

Definition (Block Error Probability)

The block error probability

$$P_B = P((\tilde{U}_1, \ldots, \tilde{U}_k) \neq (U_1, \ldots, U_k)).$$

Clearly

$$P_b \leq P_B \leq kP_b.$$

Theorem

If the information bits are sent at rate R via a DMC of capacity $C < R$ without feedback, then the bit error probability at the destination satisfies

$$P_b \geq h^{-1}\left(1 - \frac{C}{R}\right),$$

where h is the binary entropy function, and

$$h^{-1}(x) = \min\{p \mid h(p) = x\}.$$

Thus P_b cannot be very small when $R > C$.

Theorem (Noisy Coding Theorem for DMC)

Consider a transmission of information bits at rate $R = \frac{k}{n}$ *via a DMC of capacity* $C > R$ *without feedback, then given any* $\varepsilon > 0$ *one can always achieve*

$$P_B < \varepsilon$$

by choosing n *large enough.*

This was the bombshell of Shannons 1948 paper:

If $R < C$ *one can get reliability.*

Summary

- What computers can do:
  - Automata theory are memoryless Turing machines, accepting regular languages
  - Turing machines are basically classical computers
- How efficiently they can do it:
  - Complexity classes
- What is information:
  - the difference of uncertainty
- How to transmit information reliably:
  - through channels
  - using coding theory

The End

References

- Part 1: Turing
  - My memory on Mathilde Bouvels lecture "Computability and Complexity Theory"
  - "An Introduction to Automata Theory, Languages and Computation" by John Hopcroft
- Part 2: Shannon
  - Lecture notes on "Applied Digital Information Theory" by James L. Massey