Growth-fragmentation models, random and deterministic

Alex Watson$^1$

Joint work with Jean Bertoin

$^1$University of Zurich
The fragmentation equation

\[ \partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, \int_{[\frac{1}{2},1]} \{ f(xy) + f(x(1-y)) - f(x) \} K(dy) \right\rangle, \]

\[ f \in C^\infty_c(0, \infty), \]

\[ \mu_0 = \delta_1 \]
The fragmentation equation

\[ \partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, \int_{[\frac{1}{2}, 1]} \left\{ f(xy) + f(x(1-y)) - f(x) \right\} K(dy) \right\rangle, \]

\[ f \in \mathcal{C}_c^\infty(0, \infty), \]

\[ \mu_0 = \delta_1 \]

- Require \( \int (1-y) K(dy) < \infty \)
The growth-fragmentation equation

\[ \partial_t \langle \mu_t, f \rangle \]
\[ = \left\langle \mu_t, \int_{[\frac{1}{2},1]} \left\{ f(xy) + f(x(1 - y)) - f(x) \right\} K(dy) \right\rangle, \]
The growth-fragmentation equation

\[ \partial_t \langle \mu_t, f \rangle = \langle \mu_t, axf'(x) \rangle \]

\[ + \int_{[\frac{1}{2},1)} \{ f(xy) + f(x(1 - y)) - f(x) \} K(dy) \], \]

Require only \( \int (1 - y)^2 K(dy) < \infty \).

'Solutions' \( (\mu_t) \) are measures on \( (0, \infty) \) and vaguely continuous in time.

Probabilistic approach: Haas, Banasiak

Analytic/applied approaches: Doumic, Escobedo, Gabriel, ...

Alex Watson

Growth-fragmentation models
The growth-fragmentation equation

\[ \partial_t \langle \mu_t, f \rangle = \langle \mu_t, axf'(x) \rangle + \int_{[\frac{1}{2},1]} \left\{ f(xy) + f(x(1 - y)) - f(x) + (1 - y)xf'(x) \right\} K(dy) \]
The growth-fragmentation equation

\[ \partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, axf'(x) \right. \]
\[ + \int_{[\frac{1}{2},1)} \{ f(xy) + f(x(1-y)) - f(x) + (1-y)xf'(x) \} K(dy) \left\rangle, \right. \]

\[ a \in \mathbb{R} \]
The growth-fragmentation equation

\[
\partial_t \langle \mu_t, f \rangle = \langle \mu_t, axf'(x) \rangle + \int_{[\frac{1}{2},1)} \{ f(xy) + f(x(1-y)) - f(x) + (1-y)xf'(x) \} K(dy) \]

- \( a \in \mathbb{R} \)
- Require only \( \int (1-y)^2 K(dy) < \infty \).
The growth-fragmentation equation

\[ \partial_t \langle \mu_t, f \rangle = \langle \mu_t, a x f'(x) \rangle + \int_{[\frac{1}{2}, 1]} \left\{ f(xy) + f(x(1 - y)) - f(x) + (1 - y)xf'(x) \right\} K(dy), \]

- \( a \in \mathbb{R} \)
- Require only \( \int (1 - y)^2 K(dy) < \infty \).
- ‘Solutions’ (\( \mu_t \)) are measures on \((0, \infty)\) and vaguely continuous in time.
The growth-fragmentation equation

\[ \partial_t \langle \mu_t, f \rangle = \left( \mu_t, axf'(x) \right) + \int_{[\frac{1}{2}, 1]} \{ f(xy) + f(x(1 - y)) - f(x) + (1 - y)xf'(x) \} K(dy) \], \]

- \( a \in \mathbb{R} \)
- Require only \( \int (1 - y)^2 K(dy) < \infty \).
- ‘Solutions’ \((\mu_t)\) are measures on \((0, \infty)\) and vaguely continuous in time
- Probabilistic approach: Haas, Banasiak
The growth-fragmentation equation

\[ \partial_t \langle \mu_t, f \rangle = \langle \mu_t, a f'(x) \rangle + \int_{[\frac{1}{2},1]} \left\{ f(xy) + f(x(1-y)) - f(x) + (1-y)xf'(x) \right\} K(dy) \],

- \( a \in \mathbb{R} \)
- Require only \( \int (1-y)^2 K(dy) < \infty \).
- ‘Solutions’ \( (\mu_t) \) are measures on \((0, \infty)\) and vaguely continuous in time
- Probabilistic approach: Haas, Banasiak
- Analytic/applied approaches: Doumic, Escobedo, Gabriel, . . .
Questions

- Existence and representation
- Uniqueness
- Asymptotics
- Non-existence
Questions

- Existence and representation
- Uniqueness
- Asymptotics
- Non-existence
Questions

- Existence and representation
- Uniqueness
- Asymptotics
- Non-existence
Questions

- Existence and representation
- Uniqueness
- Asymptotics
- Non-existence
Suppose that $K[1/2, 1] = \lambda < \infty$.

- Start with a single ‘fragment’ of size $x$
- After an $\text{Exp}(\lambda)$ clock rings, it dies and produces offspring
- With probability $K(dy)/\lambda$, create new fragments, of size $xy$ and $x(1-y)$
- They evolve independently
Suppose that $K[1/2, 1] = \lambda < \infty$.

- Start with a single ‘fragment’ of size $x$
- After an $\text{Exp}(\lambda)$ clock rings, it dies and produces offspring
- With probability $K(dy)/\lambda$, create new fragments, of size $xy$ and $x(1-y)$
- They evolve independently
Suppose that $K[1/2, 1) = \lambda < \infty$.

- Start with a single ‘fragment’ of size $x$
- After an $\text{Exp}(\lambda)$ clock rings, it dies and produces offspring
- With probability $K(dy)/\lambda$, create new fragments, of size $xy$ and $x(1 - y)$
- They evolve independently
Suppose that $K[1/2, 1] = \lambda < \infty$.

- Start with a single ‘fragment’ of size $x$
- After an $\text{Exp}(\lambda)$ clock rings, it dies and produces offspring
- With probability $K(dy)/\lambda$, create new fragments, of size $xy$ and $x(1 - y)$
- They evolve independently
Suppose that $K[1/2, 1) = \lambda < \infty$.

- Start with a single ‘fragment’ of size $x$
- After an Exp($\lambda$) clock rings, it dies and produces offspring
- With probability $K(dy)/\lambda$, create new fragments, of size $xy$ and $x(1 - y)$
- They evolve independently

See this as a point process:

$$\mathcal{Y}(t) = \sum_{u \text{ fragments}} \delta_{\text{size}(u)} \mathbb{1}\{u \text{ alive at time } t\}.$$
Point process perspective

\[ Y(t) = \sum_{u \text{ fragments}} \delta_{\text{size}(u)} \mathbb{1}\{u \text{ alive at time } t\} \]

\[ Z(t) = Y(t) \circ \log^{-1} = \sum_{u \text{ fragments}} \delta_{\log(\text{size}(u))} \mathbb{1}\{u \text{ alive at time } t\} \]

\[ \mathbb{Z} \]

0 \quad \text{Exp}(\lambda)

\begin{align*}
\frac{1}{\lambda} K(dy) \\
0 + \log y \\
0 + \log(1 - y)
\end{align*}

\text{Exp}(\lambda)
Point process perspective

\[ \mathcal{Y}(t) = \sum_{u \text{ fragments}} \delta_{\text{size}(u)} \mathbb{I}\{u \text{ alive at time } t\} \]

\[ \mathcal{Z}(t) = \mathcal{Y}(t) \circ \log^{-1} = \sum_{u \text{ fragments}} \delta_{\log(\text{size}(u))} \mathbb{I}\{u \text{ alive at time } t\} \]

is a compound Poisson process with immigration
Compensated fragmentation processes, \( \int (1 - y)^2 K(dy) < \infty \)

- Generalise \( Z \)
- Create a Lévy process whose Lévy measure is the image of \( K(dy) \) under \( \log \)
- It will have no positive jumps but it will not be decreasing
- Again, at every jump of size \( \log y \), immigrate a new particle at relative position \( \log(1 - y) \)
- Define \( Y = Z \circ \exp^{-1} \).
Compensated fragmentation processes, $\int (1 - y)^2 K(dy) < \infty$

- Generalise $Z$
- Create a Lévy process whose Lévy measure is the image of $K(dy)$ under log
- It will have no positive jumps but it will not be decreasing
- Again, at every jump of size $\log y$, immigrate a new particle at relative position $\log(1 - y)$
- Define $\mathcal{Y} = Z \circ \exp^{-1}$.
Compensated fragmentation processes, $\int (1 - y^2) K(dy) < \infty$

- Generalise $Z$
- Create a Lévy process whose Lévy measure is the image of $K(dy)$ under $\log$
- It will have no positive jumps but it will not be decreasing
- Again, at every jump of size $\log y$, immigrate a new particle at relative position $\log(1 - y)$
- Define $\mathcal{Y} = Z \circ \exp^{-1}$.
Compensated fragmentation processes, $\int (1 - y)^2 K(dy) < \infty$

- Generalise $\mathcal{Z}$
- Create a Lévy process whose Lévy measure is the image of $K(dy)$ under log
- It will have no positive jumps but it will not be decreasing
- Again, at every jump of size $\log y$, immigrate a new particle at relative position $\log(1 - y)$
- Define $\mathcal{Y} = \mathcal{Z} \circ \exp^{-1}$. 

\[ Z \]

\[ \text{Diagram showing the process } Z \text{ with jumps at relative positions } \log(1 - y). \]
Compensated fragmentation processes, $\int (1 - y)^2 K(dy) < \infty$

- Generalise $\mathcal{Z}$
- Create a Lévy process whose Lévy measure is the image of $K(dy)$ under log
- It will have no positive jumps but it will not be decreasing
- Again, at every jump of size $\log y$, immigrate a new particle at relative position $\log(1 - y)$
- Define $\mathcal{Y} = \mathcal{Z} \circ \exp^{-1}$. 

\[ \int (1 - y)^2 K(dy) < \infty \]
Compensated fragmentation processes, $\int (1 - y)^2 K(dy) < \infty$

- Generalise $\mathcal{Z}$
- Create a Lévy process whose Lévy measure is the image of $K(dy)$ under log
- It will have no positive jumps but it will not be decreasing
- Again, at every jump of size $\log y$, immigrate a new particle at relative position $\log(1 - y)$
- Define $\mathcal{Y} = \mathcal{Z} \circ \exp^{-1}$.

If $\int (1 - y) K(dy) < \infty$, it is an ‘exchangeable fragmentation’ with growth/erosion.
We pick out a single trajectory from the fragmentation process $\mathcal{Y}$.

- Cut the process at time $t$ and examine the particles
- Pick a particle $u$ with probability $\propto \mathcal{Y}_u(t)^\omega = \exp\{\omega Z_u(t)\}$ (with any $\omega \geq 2$)
- Trace its trajectory back
- It forms an exponential Lévy process; call it $\xi$
Spines and solutions

We pick out a single trajectory from the fragmentation process $\mathcal{Y}$.

- Cut the process at time $t$ and examine the particles
- Pick a particle $u$ with probability $\propto \mathcal{Y}_u(t)^\omega = \exp\{\omega \mathcal{Z}_u(t)\}$ (with any $\omega \geq 2$)
- Trace its trajectory back
- It forms an exponential Lévy process; call it $\xi$
Spines and solutions

We pick out a single trajectory from the fragmentation process $\mathcal{Y}$.

- Cut the process at time $t$ and examine the particles
- Pick a particle $u$ with probability $\propto \mathcal{Y}_u(t) = \exp\{\omega Z_u(t)\}$ (with any $\omega \geq 2$)
- Trace its trajectory back
- It forms an exponential Lévy process; call it $\xi$
Spines and solutions

We pick out a single trajectory from the fragmentation process $\mathcal{Y}$.

- Cut the process at time $t$ and examine the particles
- Pick a particle $u$ with probability $\propto \mathcal{Y}_u(t)^\omega = \exp\{\omega \mathcal{Z}_u(t)\}$ (with any $\omega \geq 2$)
- Trace its trajectory back
- It forms an exponential Lévy process; call it $\xi$

[Diagram showing trajectories $\mathcal{Z}$, $\mathcal{Z}_{u_1}(t)$, $\mathcal{Z}_{u_2}(t)$, $\mathcal{Z}_{u_3}(t)$, $\mathcal{Z}_{u_4}(t)$]
Spines and solutions

We pick out a single trajectory from the fragmentation process $\mathcal{Y}$.

- Cut the process at time $t$ and examine the particles
- Pick a particle $u$ with probability $\propto \mathcal{Y}_u(t)^\omega = \exp\{\omega \mathcal{Z}_u(t)\}$ (with any $\omega \geq 2$)
- Trace its trajectory back
- It forms an exponential Lévy process; call it $\xi$

\[ \mathcal{Z} \]

$\mathcal{Z}_{u_1}(t)$

$\mathcal{Z}_{u_2}(t)$

$\mathcal{Z}_{u_3}(t)$

$\mathcal{Z}_{u_4}(t)$

$t$
Spines and solutions

\[ \partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, a x f'(x) \right\rangle + \int_{[\frac{1}{2}, 1]} \left\{ f(xy) + f(x(1-y)) - f(x) + (1-y)xf'(x) \right\} K(dy) \].

- fragmentation process \( \mathcal{Y} \)
- ‘spine’ \( \xi \) (with weighting \( \omega \))
Spines and solutions

\[ \partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, axf'(x) \right\rangle \\
+ \int_{[\frac{1}{2},1]} \left\{ f(xy) + f(x(1-y)) - f(x) + (1-y)xf'(x) \right\} K(dy) \].

- fragmentation process \( \mathcal{Y} \)
- ‘spine’ \( \xi \) (with weighting \( \omega \))

**Theorem**

*Let*

\[ \langle \mu_t, f \rangle = \mathbb{E}_{\delta_1} \left[ \sum_u f(\mathcal{Y}_u(t)) \right] = e^{ct} \mathbb{E}_1 [\xi(t)^{-\omega} f(\xi(t))]. \]

*This is the unique solution of the above equation with \( \mu_0 = \delta_1 \) and domain \( f \in C_c^\infty(0, \infty) \).*
Self-similarity

A generalisation:

\[ \partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, x^\alpha \right[ a x f'(x) \\
+ \int_{[\frac{1}{2},1]} \{ f(xy) + f(x(1-y)) - f(x) + (1-y)xf'(x) \} K(dy) \right], \]

Things get more interesting!
The role of the spine is played by a positive, self-similar Markov process with index $-\alpha$. Take the old spine $\xi$ and apply the Lamperti transform:

Let $T(s) = \int_0^s \xi(u)^{-\alpha} \, du$, and write $S$ for its inverse. The self-similar spine is the process $X(t) = \xi(S(t))$. 

Alex Watson
Growth-fragmentation models
The role of the spine is played by a positive, self-similar Markov process with index $-\alpha$.

Take the old spine $\xi$ and apply the Lamperti transform:

Let

$$T(s) = \int_0^s \xi(u)^{-\alpha} \, du,$$

and write $S$ for its inverse.

The self-similar spine is the process $X(t) = \xi(S(t))$. 

Self-similar spines
Self-similar spines

- The role of the spine is played by a positive, self-similar Markov process with index $-\alpha$.
- Take the old spine $\xi$ and apply the Lamperti transform:
  
  Let
  
  $$T(s) = \int_0^s \xi(u)^{-\alpha} \, du,$$

  and write $S$ for its inverse.
- The self-similar spine is the process $X(t) = \xi(S(t))$.
Self-similar spines

- The role of the spine is played by a positive, self-similar Markov process with index $-\alpha$.
- Take the old spine $\xi$ and apply the Lamperti transform:
  
  Let
  
  $$T(s) = \int_0^s \xi(u)^{-\alpha} \, du,$$

  and write $S$ for its inverse.
- The self-similar spine is the process $X(t) = \xi(S(t))$. 

Alex Watson
Growth-fragmentation models
Solutions, $\alpha < 0$

$$\kappa(q) = aq + \int_{[\frac{1}{2},1)} \{y^q + (1 - y)^q - 1 + (1 - y)q\} K(dy)$$

- Assume there exists $\omega \in \mathbb{R}$ with $\kappa(\omega) = 0$ and $\kappa'(\omega) > 0$.
- $X$ is the self-similar spine with weighting $\omega$
Solutions, $\alpha < 0$

$$\kappa(q) = aq + \int_{[\frac{1}{2}, 1)} \{y^q + (1 - y)^q - 1 + (1 - y)q\} K(dy)$$

- Assume there exists $\omega \in \mathbb{R}$ with $\kappa(\omega) = 0$ and $\kappa'(\omega) > 0$.
- $X$ is the self-similar spine with weighting $\omega$

**Theorem ($\alpha < 0$)**

- There exists a solution $(\mu_t)$ to the self-similar growth-fragmentation equation, such that $\langle \mu_t, x^\omega \rangle \equiv 1$ and $\mu_0 = \delta_1$. It is given by $\langle \mu_t, f \rangle = \mathbb{E}_1[X(t)^{-\omega} f(X_t)]$. 
Solutions, $\alpha < 0$

\[ \kappa(q) = aq + \int_{[\frac{1}{2}, 1]} \{y^q + (1 - y)^q - 1 + (1 - y)q\} K(dy) \]

- Assume there exists $\omega \in \mathbb{R}$ with $\kappa(\omega) = 0$ and $\kappa'(\omega) > 0$.
- $X$ is the self-similar spine with weighting $\omega$.

Theorem ($\alpha < 0$)

- There exists a solution $(\mu_t)$ to the self-similar growth-fragmentation equation, such that $\langle \mu_t, x^\omega \rangle \equiv 1$ and $\mu_0 = \delta_1$. It is given by $\langle \mu_t, f \rangle = \mathbb{E}_1[X(t)^{-\omega} f(X_t)]$.
- There exists another solution $(\gamma_t)$, such that $\langle \gamma_t, x^\omega \rangle \equiv 1$ for $t > 0$ but $\gamma_0 = 0$. It is given by $\langle \gamma_t, f \rangle = \mathbb{E}_0[X(t)^{-\omega} f(X_t)]$. 

Alex Watson  
Growth-fragmentation models
Solutions, $\alpha < 0$

$$\kappa(q) = aq + \int_{[\frac{1}{2},1)} \{y^q + (1 - y)^q - 1 + (1 - y)q\} K(dy)$$

- Assume there exists $\omega \in \mathbb{R}$ with $\kappa(\omega) = 0$ and $\kappa'(\omega) > 0$.
- $X$ is the self-similar spine with weighting $\omega$

**Corollary (to the proof)**

*If we require the growth-fragmentation equation to hold for all functions $x^q$, $q \in \mathbb{R}$, then $(\mu_t)$ is the unique solution with $\mu_0 = \delta_1$.***
Proposition

For $f \in C_b(0, \infty)$,

$$\int f(t^{-1/|\alpha|} x)x^\omega \mu_t(dx) \to \int f(x)x^\omega \gamma_1(dx), \quad t \to \infty.$$
Asymptotics, $\alpha < 0$

Suppose there exist $\rho < \omega$ such that $\kappa(\rho) = \kappa(\omega) = 0$.

**Proposition**

For $f \in C_0(0, \infty)$,

$$\int \frac{f(t^{-1}x|\alpha|)x^\rho \mu_t(dx)}{g(t)} \rightarrow \int f(x)\nu(dx), \quad t \rightarrow \infty,$$

where $g \in \text{RV}(-\sigma)$, $\sigma = (\omega - \rho)/|\alpha|$, and $\nu$ is related to factorisations of the exponential functional; cf. Haas–Rivero.
Suppose there exist $\rho < \omega$ such that $\kappa(\rho) = \kappa(\omega) = 0$.

**Proposition**

For $f \in C_0(0, \infty)$,

\[
\int \frac{f(t^{-1}x|\alpha|)x^\rho \mu_t(dx)}{g(t)} \rightarrow \int f(x)\nu(dx), \quad t \rightarrow \infty,
\]

where $g \in RV(-\sigma)$, $\sigma = (\omega - \rho)/|\alpha|$, and $\nu$ is related to factorisations of the exponential functional; cf. Haas–Rivero.
Assume that there exist $\rho < \omega$ such that $\kappa(\rho) = \kappa(\omega) = 0$. $X$ is the self-similar spine with weighting $\rho$.

**Theorem ($\alpha > 0$)**

There exists a solution $(\mu_t)$ to the self-similar growth-fragmentation equation, such that $\langle \mu_t, x^\rho \rangle \equiv 1$ and $\mu_0 = \delta_1$. It is given by $\langle \mu_t, f \rangle = \mathbb{E}_1 [X(t)^{-\rho}f(X_t)]$. 

Alex Watson  Growth-fragmentation models
Assume that there exist $\rho < \omega$ such that $\kappa(\rho) = \kappa(\omega) = 0$.

$X$ is the self-similar spine with weighting $\rho$

**Theorem ($\alpha > 0$)**

There exists a solution $(\mu_t)$ to the self-similar growth-fragmentation equation, such that $\langle \mu_t, x^\rho \rangle \equiv 1$ and $\mu_0 = \delta_1$. It is given by $\langle \mu_t, f \rangle = \mathbb{E}_1[X(t)^{-\rho} f(X_t)]$. 
Solutions, $\alpha > 0$

- Assume that there exist $\rho < \omega$ such that $\kappa(\rho) = \kappa(\omega) = 0$.
- $X$ is the self-similar spine with weighting $\rho$

**Theorem ($\alpha > 0$)**

- There exists a solution $(\mu_t)$ to the self-similar growth-fragmentation equation, such that $\langle \mu_t, x^\rho \rangle \equiv 1$ and $\mu_0 = \delta_1$. It is given by $\langle \mu_t, f \rangle = \mathbb{E}_1[X(t)^{-\rho}f(X_t)]$. 
Solutions, $\alpha > 0$

- Assume that there exist $\rho < \omega$ such that $\kappa(\rho) = \kappa(\omega) = 0$.
- $X$ is the self-similar spine with weighting $\rho$

**Theorem ($\alpha > 0$)**

- There exists a solution $(\mu_t)$ to the self-similar growth-fragmentation equation, such that $\langle \mu_t, x^\rho \rangle \equiv 1$ and $\mu_0 = \delta_1$. It is given by $\langle \mu_t, f \rangle = \mathbb{E}_1[X(t)^{-\rho}f(X_t)]$.
- There exists another solution $(\gamma_t)$, such that $\langle \gamma_t, x^\rho \rangle \equiv 1$ for $t > 0$ but $\gamma_0 = 0$. It is given by $\langle \gamma_t, f \rangle = \mathbb{E}_{+\infty}[X(t)^{-\rho}f(X_t)]$. 
‘Explosion’: self-similar fragmentations ($\alpha < 0$)

What goes wrong when there is no $\omega$ with $\kappa(\omega) = 0$?
‘Explosion’: self-similar fragmentations ($\alpha < 0$)

For a ‘ray’ $\nu$,

$$\lambda_{\nu}(t) = \int_0^t e^{\alpha Z_{\nu}(s)} ds,$$

and write $L_{\nu}$ for its inverse. This is a ‘stopping line’ time-change. The self-similar fragmentation process is then

$$Y(\alpha)(t) = Y(L(t)) = Z(L(t)) \circ \exp^{-1}.$$

Alex Watson

Growth-fragmentation models
For a ‘ray’ \( v \), we define a functional

\[
\lambda_v(t) = \int_0^t e^{\alpha Z_v(s)} \, ds,
\]

and write \( L_v \) for its inverse. This is a ‘stopping line’ time-change.
‘Explosion’: self-similar fragmentations ($\alpha < 0$)

For a ‘ray’ $\nu$, we define a functional

$$\lambda_\nu(t) = \int_0^t e^{\alpha Z_\nu(s)} \, ds,$$

and write $L_\nu$ for its inverse. This is a ‘stopping line’ time-change.

The self-similar fragmentation process is then

$$\mathcal{Y}(t) = \mathcal{Y}(L(t)) = \mathcal{Z}(L(t)) \circ \exp^{-1}.$$
Explosion, $\alpha < 0$

Even in the simplest case (finite fragmentation), we have:

**Proposition**

If $\kappa = 0$ has no solutions then, for any $b > 0$, there exists a random time $S$ such that $\#\{u : \mathcal{Y}_u^{(\alpha)}(S) \in [1, 1 + b]\} = \infty$
Open questions

- Biased mass functions ($\alpha \neq 0$)
- Strengthen non-existence result
- Process variant of ‘starting from zero’
Further reading

J. Bertoin
Compensated fragmentation processes and limits of dilated fragmentations

J. Bertoin, A. R. Watson
Probabilistic aspects of critical growth-fragmentation equations
Thank you!