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Master Thesis

A construction of the smash product of spectra via $(\infty, 1)$-categories

Nino Scalbi

supervised by
PD Dr. Alessandro Valentino

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Introduction

The concept of stabilization in algebraic topology has its origin in the Suspension Theorem by Hans Freudenthal. This important concept led to the definition of stable homotopy groups, creating a whole new field in algebraic topology: stable homotopy theory. In the 1960s, topological spectra appeared on the scene of stable homotopy theory, due to the work of Atiyah on bordisms [Ati61] and Whitehead on generalized homology theories [Whi62]. Since then, spectra have been used successfully in algebraic topology and geometry, providing many new insights. From an early stage, it was clear that the stable category of spectra should allow a symmetric monoidal structure,such as the smash product on pointed topological spaces. This structure, called the *smash product of spectra* was first introduced by Boardman [Boa65] and Vogt [Vog70]. There have been various approaches to define the smash product of spectra, for example via Γ-spaces by Lydakis [Lyd99], or via the topological Day convolution in [MMSS01], which is the approach we are going to follow in the present thesis.

More precisely, we define the category of spectra via module objects over the sphere spectrum in the enriched functor category $[\mathcal{C}, \mathbf{Top}^*_{cg}]$. Here the category $\mathcal{C}$ carries the structural properties of the desired spectrum. That is, for a sequential spectrum consisting only of a sequence of pointed topological spaces X_n and structure maps $\Sigma X_n \to X_{n+1}$, we consider the category **Seq** with objects given by natural numbers. This approach allows us to construct sequential, symmetric and orthogonal spectra in a very efficient way. To have the possibility to define module objects in the enriched functor category, we need to endow it with a symmetric monoidal structure. Such a structure is given by the *Day convolution.* One of the main advantages of the Day convolution is that it can be defined in a very general setting, allowing us to treat many different cases with only one construction. In the first part, we follow the notes "Introduction to Stable Homotopy Theory" [nLa19], which are based on the approach of [MMSS01] and [HSS98]. We define the categories of sequential, symmetric and orthogonal spectra and then we construct the smash product of structured spectra. Then we endow these categories with the projective model structure, which leads to the definition of the strict model structure of spectra. However, the strict model structure does not capture the concept of stabilization. Therefore, we need to introduce the stable model structure of spectra, providing a convenient framework to perform stable homotopy theory.

One of the major recent applications of the theory of spectra is the classification of Topological Quantum Field Theories (TQFT). That is, deformation classes of invertible Topological Field Theories are in one-to-one correspondence with the torsion subgroup of the abelian group given by homotopy classes of maps of certain spectra. This result is due to the work of Freed, Hopkins and Teleman in [FHT10] and [FH16], which is based on the description of the homotopy type of the cobordism category using spectra, by [GMTW09]. A TQFT being a symmetric monoidal functor between (∞, n)-categories is said to be *invertible* if it factors trough the corresponding higher Picard groupoid. Therefore, the classification of such invertible field theories goes hand in hand with the classification of functors between Picard groupoids. However, this classification is based on the fact that any Picard groupoid defines a spectrum. To understand how Picard groupoids and spectra are related we need to introduce Γ-spaces and Γ-categories. In Section 4, we show how one can associate a Γ-category to any Picard groupoid, which itself defines a spectrum. The relation between Γ-spaces and spectra was first studied by Segal in [Seg74]. However, we use a different construction following the lecture notes of Boyarchenko on Picard groupoids and spectra [Boy19]. The combination of these two approaches allows us to construct a pair of functors

$$\{\text{Picard groupoids}\} \rightleftarrows \{\text{connective spectra}\}$$

which unfortunately do *not* define an adjunction. One approach to handle this problem is to focus on the functors between the corresponding simplicial localizations. Indeed, we can endow the category of Picard groupoids with a model structure and, by considering the stable model structure on the category of spectra, we get a pair of functors between their simplicial localizations. This pair of simplicial functors should eventually become an adjunction and capture the relation between Picard groupoids and spectra. Since the study of simplicial localizations is strongly related with the study of $(\infty, 1)$-categories, we will follow Lurie's approach

[Lur09] and introduce ∞-categories (weak Kan complexes) as a convenient model for $(\infty, 1)$-categories. With this framework, we should eventually be able to restate the relation between Picard groupoids and spectra in the setting of ∞-categories. Current research in this area investigating the connection between Waldhausen categories and spectra in the ∞-categorical setting can be found for example in [Fio13].

One of the advantages of considering the ∞-category of Picard groupoids and the ∞-category of spectra is that they capture higher coherence data. As an example, consider the category of spectra. By construction it is a simplicial category. The consequences of having a symmetric monoidal structure on the category of spectra is the ability to define algebraic objects, such as monoid objects and module objects within this category. Therefore, the smash product of spectra leads to the possibility to construct an *E_∞-ring*. An E_∞-ring is an algebraic object characterized by the property that associativity is only satisfied up to homotopy. However, these homotopies themselves should also satisfy some associativity relations up to higher homotopies, and so on. The idea of trying to define algebraic objects satisfying relations in a homotopy coherent way agrees with the motivation behind higher category theory. This connection between spectra and higher category theory is investigated in the second part of the thesis. More precisely, we first define two models for $(\infty, 1)$-categories, simplicial categories and ∞-categories. We then endow these categories with suitable model structures, and see how the nerve-realization construction $N_\Delta \dashv \mathfrak{C}$ provides a Quillen equivalence between these model categories. As a main result, we establish the connection between symmetric monoidal structures on simplicial categories and coCartesian fibrations on ∞-categories. More precisely, we show how a symmetric monoidal structure $(\mathcal{C}, \otimes, 1)$ on a simplicial category can be encoded into a single enriched functor $p : \mathcal{C}^\otimes \to \mathbf{FinSet}^*$. This construction is well known for ordinary categories and the resulting functors are called *Grothendieck opfibrations*. We show that there exists an enriched analogue construction, called *simplicial Grothendieck opfibration*.

Grothendieck opfibrations, as mentioned before, define symmetric monoidal structures. Therefore, we show that its corresponding type of maps in the ∞-categorical setting is that of a *coCartesian fibration*. A symmetric monoidal structure on an ∞-category is therefore given by a certain coCartesian fibration. Whereas it is relatively straightforward to show that a non-enriched Grothendieck opfibration between ordinary categories induces a coCartesian fibration via the regular nerve functor, it is more challenging to show that simplicial Grothendieck opfibrations also induce coCartesian fibrations via the coherent nerve functor. The latter allows us to show that any simplicial symmetric monoidal structure defines a corresponding structure on its coherent nerve. The previous observation corresponds to the results of [BW18a] which were proven by using a different approach. This agreement established while writing the present thesis. Moreover, it was shown by Nikolaus and Sagave in [NS15] that any symmetric monoidal structure on a presentable ∞-category comes from a monoidal structure on a model category. Now we can apply this result to the Day convolution of enriched functors, which provides a different approach to define the Day convolution on the ∞-category of enriched functors. The standard approach to define ∞-categorical Day convolution is using the theory of operads, such as for example in [Lur12]. However, these two constructions should agree up to weak equivalence.

The lack of results concerning the compatibility between the two notions of Day convolution was one of the main motivations for the present thesis, and remains unknown to the author until this day. Another interesting question is the possibility to construct the ∞-category of spectra and its smash product using the Day convolution. To answer this question, one needs to check if the same techniques used to define the simplicial category of spectra also work when dealing with ∞-categories. The fact that the coherent nerve functor has nice preservation properties concerning monoidal structures gives hope for an affirmative answer.

The present work starts with a short introduction to stable homotopy theory.
In Section 1 we introduce stable homotopy groups and generalized homology theories and show how spectra are related to them via the Brown Representability Theorem.
In Section 2 we construct the smash product of spectra following the notes on [nLa19].
In Section 3 we use the definition of the category of spectra via diagram categories to define the strict and stable model structure.

In Section 4 we show how Picard groupoids, Γ-spaces and spectra are related to each other. In particular this motivates the use of ∞-categories.
In Section 5 we first introduce the Bergner model structure on the category of small simplicial categories. Then following [Lur09] and [Gro10] we introduce ∞-categories and show how symmetric monoidal structures are being preserved via the coherent nerve functor. Finally we can apply the results on symmetric monoidal ∞-categories to the Day convolution of enriched functors. The Day convolution being a key ingredient in the construction of the smash product, motivates further investigation concerning the ∞-category of spectra and its relation to the ∞-category of enriched functors. We conclude by giving a short sketch of possible developments in this particular area.

Notation

In the following we denote with

- **sSet** the category of simplicial sets.
- $\mathbf{sSet}^*$ the category of pointed simplicial sets.
- $\mathbf{Top}_{cg}$ the category of compactly generated topological spaces and continuous maps.
- $\mathbf{Top}^*_{cg}$ the category of pointed compactly generated topological spaces and base point preserving continuous maps.
- $\mathbf{Top}_{CW}$ the category of topological spaces admitting a CW-complex structure.
- $\mathbf{Top}^*_{cg,fin}$ the category of pointed compactly generated spaces admitting a structure of a finite CW-complex.
- Δ the simplex category.
- **Cat** the category of small categories and functors.
- **sSet-Cat** the category of small simplicial categories and simplicial functors.
- $\mathrm{Ho}\,\mathcal{M}$ the homotopy category of a model category $\mathcal{M}$.
- $\mathrm{h}\mathcal{C}$ the category of components of a simplicial category $\mathcal{C}$.

Contents

1 Stable homotopy theory

The philosophy behind stable homotopy theory is the stability of certain topological invariants under taking the suspension. Such invariants are for example homology and cohomology theories or, as showed by Freudenthal, also homotopy theory. The study of such invariants, satisfying this stability axiom, led to the definition of generalized homology and cohomology theories and their connections to spectra. Since then stable homotopy theory has been frequently used to reduce hard geometric problems to more accessible problems in stable homotopy theory. This was done for example in the work of Thom [Tho54], where he introduced the Thom spectrum and showed how the homotopy groups of this spectrum are in relation with the cobordism ring.

In this section we follow [AGP08] and [Swi02].

1.1 Freudenthal suspension theorem

The foundation of stable homotopy theory was laid by the Freudenthal suspension theorem in 1937. It demonstrates how the suspension operation acts on higher homotopy groups. That is, given a pointed space X we define the suspension homomorphism

$$\begin{aligned} \Sigma : \pi_q(X) &\longrightarrow \pi_{q+1}(\Sigma X) \\ [f] &\longmapsto [\Sigma f] \end{aligned}$$

where for a map $f : S^q \to X$ we define $\Sigma f : S^q \wedge S^1 \xrightarrow{f \wedge \mathrm{id}} X \wedge S^1$.

Theorem 1.1. (Freudenthal suspension, [AGP08]) *Let X be an n-connected pointed space. Then the suspension homomorphism*

$$\Sigma : \pi_q(X) \to \pi_{q+1}(\Sigma X)$$

is an isomorphism for $q < 2n + 1$ and a surjection for $q = 2n + 1$.

If we now consider an n-connected space X, we obtain by iteration of the suspension homomorphism a sequence of maps.

$$\pi_q(X) \to \pi_{q+1}(\Sigma X) \to \pi_{q+2}(\Sigma^2 X) \to \cdots$$

By the Freudenthal suspension theorem, we would guess that this sequence eventually stabilizes. Indeed, it can be shown that in the sequence

$$\cdots \to \pi_{q+k}(\Sigma^k X) \to \pi_{q+k+1}(\Sigma^{k+1} X) \to \cdots$$

all morphisms are isomorphisms for $k > q - 1 - 2n$. Therefore, taking the colimit yields the stable homotopy group of X.

Definition 1.1. Let X be a pointed space. Then we define the **q-th stable homotopy group** of X as the colimit over the above sequence

$$\pi_q^S(X) := \operatorname*{colim}_k \pi_{q+k}(\Sigma^k X)$$

To understand the structure behind stable homotopy theory, we consider a pointed space X and denote with $X_k = \Sigma^k X$ the k-th suspension of X. This yields a sequence of pointed spaces $(X_k)_{k \in \mathbb{N}}$ together with structure maps

$$\Sigma X_k \to X_{k+1}$$

This structure maps are the canonical homeomorphisms, which do not carry any information. Therefore, we want to generalize the above definition, such that the general structure is preserved, but allowing the structure maps to be of any kind. This leads to the definition of a topological spectrum.

1.2 Sequential spectra

Definition 1.2. A **sequential spectrum** in the category $\mathbf{Top}^*_{cg}$ is a sequence of pointed compactly generated topological spaces $X = (X_n)_{n\in\mathbb{N}}$ together with pointed continuous maps

$$\sigma_n : S^1 \wedge X_n \to X_{n+1}$$

for all $n \in \mathbb{N}$, called the structure maps.

A morphism of sequential spectra $f : X \to Y$ is a collection of base point preserving continuous maps $f_n : X_n \to Y_n$ such that all diagrams of the form

$$\begin{array}{ccc} S^1 \wedge X_n & \xrightarrow{\sigma_n^X} & X_{n+1} \\ {\scriptstyle \mathrm{id}\wedge f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ S^1 \wedge Y_n & \xrightarrow[\sigma_n^Y]{} & Y_{n+1} \end{array}$$

commute. The corresponding category of sequential spectra with component spaces in the category $\mathbf{Top}^*_{cg}$ is denoted with $\mathrm{SeqSpec}(\mathbf{Top}^*_{\mathrm{cg}})$.

Remark 1.1. A first example of a sequential spectrum is given by iterating the suspension of a pointed space. This yields the suspension spectrum of a pointed space X, denoted by $\Sigma^\infty X$. In the case where we take $X = S^0$, the corresponding suspension spectrum is called the sequential sphere spectrum $\mathbb{S}_{\mathrm{seq}} = \Sigma^\infty S^0$.

In the spirit of Definition 1.1 we now define the homotopy groups of sequential spectra.

Definition 1.3. Let $X = (X_n)$ be a sequential spectrum. Then we define the q-th homotopy group of X as

$$\pi_q(X) := \operatorname*{colim}_k \, \pi_{q+k}(X_k) \quad \text{for } q \in \mathbb{Z}.$$

Remark 1.2. Notice that by construction

$$\pi_q(\Sigma^\infty X) = \pi_q^S(X)$$

the q-th homotopy group of the suspension spectrum of X equals the q-th stable homotopy group of the space X. Moreover, the above definition extends to a functor

$$\pi_\bullet : \mathrm{SeqSpec}(\mathbf{Top}^*_{\mathrm{cg}}) \to \mathbf{Ab}^{\mathbb{Z}}$$

from the category of sequential spectra to the category of $\mathbb{Z}$-graded abelian groups. Indeed, an application of the Eckmann-Hilton argument shows, that the homotopy groups of sequential spectra are abelian groups.

Example 1.1. ([Fre13]) As already mentioned, in stable homotopy theory spectra can be used to break down hard geometric problems to algebraic ones. Consider for example the abelian group Ω_n^{SO} of oriented cobordism classes of n-dimensional manifolds. This cobordism groups are in general hard to compute. However, they are related to the stable homotopy groups of a certain spectrum via the Pontrjagin-Thom isomorphism. In the following, we are going to construct this spectrum, called the Thom spectrum.

To do so, we first need to recall an important property of the Thom space of a real vector bundle. Suppose we are given a real vector bundle $E \to B$, then there is a homeomorphism

$$\mathrm{Th}(\underline{\mathbb{R}}^n \oplus E) \simeq S^n \wedge \mathrm{Th}(E) = \Sigma^n \mathrm{Th}(E)$$

Let now $ESO(n) \to BSO(n)$ denote the universal principal $SO(n)$-bundle. Then the associated fiber bundle

$$ESO(n) \times_{SO(n)} \mathbb{R}^n =: S(n)$$

is called the universal vector bundle or the tautological bundle. It can be shown that the tautological bundle satisfies the following stability property.

$$\begin{array}{ccc} \underline{\mathbb{R}} \oplus S(n) & \xrightarrow{f} & S(n+1) \\ \downarrow & \lrcorner & \downarrow \\ BSO(n) & \xrightarrow[i]{} & BSO(n+1) \end{array}$$

That is, there is an isomorphism between the pullback of the tautological bundle $S(n+1)$ along the inclusion i and the bundle $\underline{\mathbb{R}} \oplus S(n)$.

$$\underline{\mathbb{R}} \oplus S(n) \cong i^*(S(n+1))$$

Now we define the (oriented) **Thom spectrum** to be the following sequential spectrum.

$$(MSO)_n := \mathrm{Th}(S(n))$$

The structure maps are constructed as follows.

$$\sigma_n : S^1 \wedge \mathrm{Th}(S(n)) \xrightarrow{\simeq} \mathrm{Th}(\underline{\mathbb{R}} \oplus S(n)) \xrightarrow{\mathrm{Th}(f)} \mathrm{Th}(S(n+1))$$

This shows that $(MSO)_n$ indeed defines a sequential spectrum. The reason why the Thom spectrum plays an important role in stable homotopy theory is due to the following fact, known as the Pontrjagin-Thom isomorphism. Namely, there is an isomorphism

$$\pi_n(MSO) \cong \Omega_n^{SO}$$

from the n-th stable homotopy group of the Thom spectrum to the abelian group of oriented cobordism classes of n-dimensional manifolds.

1.3 Generalized homology and cohomology

Definition 1.4. A **reduced generalized cohomology theory** h is a collection of functors

$$h^q : \left(\mathbf{Top}^*_{cg}\right)^{\mathrm{op}} \to \mathbf{Ab}$$

and natural isomorphisms

$$\sigma^q : h^q \circ \Sigma \to h^{q-1}$$

indexed by $q \in \mathbb{Z}$, satisfying the following axioms.

(i) (HOMOTOPY INVARIANCE) If $f, g : X \to Y$ are two base point preserving maps, such that there is a base point preserving homotopy $f \sim g$ between them, then the induced maps

$$f^* = g^* : h^q(Y) \to h^q(X)$$

are equal for all $q \in \mathbb{Z}$.

(ii) (EXACTNESS) For every inclusion $i : A \hookrightarrow X$ of pointed spaces, there is an exact sequence of abelian groups for all $q \in \mathbb{Z}$

$$h^q(\mathrm{Cone}(i)) \xrightarrow{j^*} h^q(X) \xrightarrow{i^*} h^q(A)$$

where $j : X \to \mathrm{Cone}(i)$ is the canonical inclusion into the cone of i.

Example 1.2. An example of a reduced generalized cohomology theory is given by the ordinary reduced singular cohomology $\widetilde{H}^q(-, G)$ with coefficients in an abelian group G. Notice that "generalized" in this context means that we also want to consider cohomology theories which do not need to satisfy the Dimension axiom, being part of the Eilenberg-Steenrod axioms.

Similarly there is also an axiomatic definition of a **reduced generalized homology theory**, being a collection of functors and natural isomorphisms

$$h_q : \mathbf{Top}^*_{cg} \to \mathbf{Ab} \qquad\qquad \sigma_q : h_q \to h_{q+1} \circ \Sigma$$

indexed by $q \in \mathbb{Z}$, satisfying the corresponding dualized axioms given by Definition 1.4.

We want to show that to any sequential spectrum $E = \{E_n\}$ we can associate a reduced generalized homology resp. cohomology theory. Indeed, let X be a pointed CW-complex and define the spectrum $E \wedge X$ by taking the smash product component-wise. This means that $(E \wedge X)_n = E_n \wedge X$. Given a sequential spectrum E, we denote with $\Sigma^n E$ the spectrum shifted by n. The component spaces of the shifted spectrum are then just given by $(\Sigma^n E)_m = E_{m+n}$. Now we can make the following definition.

Definition 1.5. Let $E = \{E_n\}$ be a sequential spectrum. Then the **associated homology** resp. **associated cohomology theory** is given by

$$h_n^E(X) := \pi_n(E \wedge X)$$
$$h^n_E(X) := [\Sigma^\infty X, \Sigma^n E]$$

Remark 1.3. Notice that in the above definition, the notation $[-,-]$ denotes the set of equivalence classes of maps of spectra up to homotopy. Here a homotopy between two maps of spectra $f_0, f_1 : E \to E'$ is given by a map $H : E \wedge I^+ \to E'$ such that for the two inclusions

$$i_0 : E \to E \wedge I^+ \qquad \text{and} \qquad i_1 : E \to E \wedge I^+$$

we have that $H \circ i_0 = f_0$ and $H \circ i_1 = f_1$. Moreover, it can be shown that for two sequential spectra E, E' the set $[E, E']$ is an abelian group.

Example 1.3. Taking the sphere spectrum $\mathbb{S}$, we notice that the associated homology theory is given by

$$h_n^{\mathbb{S}}(X) = \pi_n(\mathbb{S} \wedge X) = \pi_n(\Sigma^\infty X) = \pi_n^S(X)$$

the stable homotopy groups.

Since a cohomology theory is just a collection of functors satisfying certain axioms, we want to investigate the properties of such functors. Especially the Brown Representability Theorem plays an important role, connecting generalized cohomology theories with spectra. Brown showed that a certain class of functors called Brown functors are representable. These functors need to satisfy certain axioms, such as allowing a Mayer-Vietoris argument and preserving (co)products. Therefore, it is clear that the axioms are inspired by the properties of the cohomology functors.

Remark 1.4. Notice that in the following definition the homotopy category of pointed CW-complexes is not to be confused with the homotopy category induced by a model structure. In this case $\mathrm{h}\mathbf{Top}^*_{CW}$ denotes the category with the same objects as in $\mathbf{Top}^*_{CW}$ and morphisms given by homotopy classes of maps. That is, for X, Y two CW-complexes we have

$$\mathrm{h}\mathbf{Top}^*_{CW}(X, Y) = \pi_0\left(\mathbf{Maps}\,[X, Y]\right)$$

This category is also called the **category of components** and will be defined with more generality in Definition 5.6.

Definition 1.6. Consider a contravariant functor T from the category of components of pointed CW-complexes to the category of pointed sets.

$$T : \mathrm{h}\mathbf{Top}^*_{CW} \to \mathbf{Set}^*$$

Then T is called a **Brown functor** if it fulfills the following two axioms

(i) (Wedge Axiom) Let $\{X_\alpha\}$ be a family of pointed spaces and consider $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ the inclusion. Then the map induced by the inclusions i_α

$$T\left(\bigvee_\alpha X_\alpha\right) \to \prod_\alpha T(X_\alpha)$$

is an isomorphism of sets.

(ii) (Mayer-Vietoris) Let $(X; A, B)$ be an excisive triad. Then for any $u \in T(A)$ and $v \in T(B)$ such that $u|_{T(A\cap B)} = v|_{T(A\cap B)}$, there exists an element $z \in T(X)$ such that $z|_{T(A)} = u$ and $z|_{T(B)} = v$.

Example 1.4. It can be shown that the reduced cohomology functors $\widetilde{H}^q(-, G)$ satisfy both axioms and are therefore Brown functors for all $q \in \mathbb{N}$.

Proposition 1.1. ([Swi02]) *For E a sequential spectrum the associated homology and cohomology theories h_*^E and h_E^* are indeed reduced generalized homology/cohomology theories in the sense of Definition 1.4 and moreover they satisfy the wedge axiom of Definition 1.6.*

Remark 1.5. Notice that a reduced homology/cohomology theory satisfying the wedge axiom is called an **additive homology theory**.

Theorem 1.2. (Brown representability, [AGP08]) *Every Brown functor T is representable in the category of path-connected pointed CW-complexes, i.e. there is a pointed CW-complex Y, unique up to homotopy equivalence, and a natural isomorphism*

$$\Phi : [-, Y] \longrightarrow T$$

This theorem has an important consequence, namely consider some h^* to be a reduced generalized cohomology theory satisfying the wedge axiom. It can be shown that the Mayer-Vietoris axiom follows from the axioms of a reduced cohomology theory. Hence for all $q \in \mathbb{Z}$ the functor

$$h^q : \mathbf{Top}^*_{CW} \to \mathbf{Ab}$$

is a Brown functor and therefore by the Brown Representability Theorem there exists a family of CW-complexes Y_q, and natural isomorphisms

$$[Z, Y_q] \xrightarrow{\sim} h^q(Z)$$

for all connected pointed CW-complexes Z and all $q \in \mathbb{Z}$. Now by defining $E_q := \Omega Y_{q+1}$ it follows that for any CW-complex X , not necessarily connected, its suspension ΣX is connected and therefore

$$h^q(X) \cong h^{q+1}(\Sigma X) \cong [\Sigma X, Y_q] \cong [X, \Omega Y_q] = [X, E_q]$$

Hence to any reduced cohomology theory we can associate a family of CW-complexes $\{Y_q\}$ unique up to homotopy, such that there are natural isomorphisms

$$[-, Y_q] \xrightarrow{\sim} h^q$$

Since h^q is a reduced cohomology theory, it follows that for all spaces X there are natural isomorphisms

$$[X, Y_q] \cong h^q(X) \cong h^{q+1}(\Sigma X) \cong [\Sigma X, Y_{q+1}]$$

and therefore we can deduce that

$$[X, Y_q] \xrightarrow{\sim} [X, \Omega Y_{q+1}]$$

for all CW-complexes X. This shows that there is a family of homotopy equivalences

$$\tilde{\sigma}_q : Y_q \to \Omega Y_{q+1}$$

We conclude that any reduced cohomology theory gives rise to a sequential spectrum having the property, that the dualized structure maps $\tilde{\sigma}_q$ are in fact homotopy equivalences. Such a spectrum is then called an Ω-spectrum.

Definition 1.7. An **Ω-spectrum** in the category $\mathbf{Top}^*_{cg}$ is a sequential spectrum $\{X_n\}$, such that the adjoint structure maps

$$\tilde{\sigma}_n : X_n \to \mathbf{Maps}\left[S^1, X_{n+1}\right] = \Omega X_{n+1}$$

are weak homotopy equivalences for all $n \in \mathbb{Z}$.

This definition allows us to state the following beautiful theorem, being one of the main consequences of Theorem 1.2.

Theorem 1.3. ([AGP08]) *Each additive reduced cohomology theory h^* on the category $\mathbf{Top}^*_{cg}$ determines an Ω-spectrum $E = \{E_n\}$ such that there are natural isomorphisms*

$$h^n \xrightarrow{\sim} [-, E_n]$$

Conversely for any Ω-spectrum $E = \{E_n\}$ there is a natural isomorphism

$$h^n_E(-) \cong [-, E_n]$$

to the associated additive reduced cohomology theory h^*_E.

Example 1.5. An important example is given by the reduced singular cohomology with coefficients in an abelian group G. It is clear that $H^*(-, G)$ defines an additive reduced cohomology theory. Hence it follows that there exists an Ω-spectrum $K(G, n)$ such that

$$\widetilde{H}^n(X, G) \cong [X, K(G, n)]$$

This spectrum is called the Eilenberg-MacLane spectrum with component spaces given by the unique (up to homotopy) CW-complexes $K(G, n)$ such that for $q \geq 1$

$$\pi_q(K(G, n)) = \begin{cases} G & \text{if } q = n \\ 0 & \text{else} \end{cases}$$

There is a similar result on the representability of reduced generalized homology theories satisfying the wedge axiom.

Theorem 1.4. ([Swi02]) *Each additive reduced generalized homology theory h_* on the category $\mathbf{Top}^*_{cg}$ determines an Ω-spectrum $E = \{E_n\}$ representing the homology theory. This means that there are natural isomorphisms*

$$h_n \xrightarrow{\sim} h^E_n$$

This short introduction to stable homotopy theory should be seen as a motivation to the following sections. In particular we have seen that spectra and homology/cohomology theories are closely related. Therefore, we want to examine the category of spectra and define a smash product, making it into a symmetric monoidal category. This provides an interesting framework, for example allowing the definition of dualizable objects, having applications in the classification of Topological Quantum Field Theories. The construction of a smash product of structured spectra was done by Boardman, Vogt, Puppe and Adams in the 1960s and 1970s. The approach that we are going to use is making more use of category theory and is based on the Day convolution of enriched functors, developed by Day [Day70] in 1970. The concept of diagram categories to define spectra was used in [MMSS01] and [HSS98] which we use together with [nLa19] as main references.

2 The smash product of spectra

Why do we want to consider a smash product of spectra? To answer this question we look at the following example. Suppose we are given the monoidal category of pointed topological spaces endowed with the smash product. Then a monoid object in this category is given by a space X together with a multiplication

$$\mu : X \wedge X \to X$$

which satisfies the associativity and the unitality axiom. However, this construction seems too strict, as we wish to have objects satisfying weaker versions of associativity and unitality. Namely, we want to look at spaces endowed with a structure being associative and unital only up to coherent homotopy. The need of having a good definition of such homotopy coherent structures motivates the definition of the smash product of spectra. In particular it is possible to pass certain structures on categories, such as Day convolution and smash product of spectra, to their analogue structures on $(\infty, 1)$-categories. This allows us to define so called E_∞-rings, being commutative monoids in the stable $(\infty, 1)$-category of spectra. These results also led to the notion of "Brave New Algebra", focusing on the properties of structured ring spectra.

In this section we follow the lecture notes "Introduction to Stable Homotopy Theory" [nLa19], providing a modern approach to stable homotopy theory.

2.1 Categorical algebra

In the following we will mostly work with topologically enriched categories. Therefore, we will call a topologically enriched category just a topological category, in the hope to keep the content more legible. Similarly a topologically enriched functor will be just called a topological functor.

Definition 2.1. A **(pointed) topological monoidal category** is a (pointed) topologically enriched category $\mathcal{C}$ equipped with

1. A (pointed) topologically enriched functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

called the tensor product.

2. An object $1_\mathcal{C}$ called the unit object.
3. A natural isomorphism

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

called the associator.

4. Natural isomorphisms

$$l_X : 1_\mathcal{C} \otimes X \to X$$

$$r_X : X \otimes 1_\mathcal{C} \to X$$

called left and right unitor.

Such that the triangle and pentagon axioms are satisfied.

A1 (Triangle axiom) For all objects $X, Y \in \mathcal{C}$ the following diagram commutes.

$$\begin{array}{ccc} (X \otimes 1_\mathcal{C}) \otimes Y & \xrightarrow{a_{X,1_\mathcal{C},Y}} & X \otimes (1_\mathcal{C} \otimes Y) \\ & {\scriptstyle r_X \otimes \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes \mathrm{l}_Y} & \\ & X \otimes Y & \end{array}$$

A2 (Pentagon axiom) For all objects $W, X, Y, Z \in \mathcal{C}$ the following diagram commutes.

$$\begin{array}{ccccc} & & (W \otimes X) \otimes (Y \otimes Z) & & \\ & \overset{a_{W\otimes X,Y,Z}}{\nearrow} & & \overset{a_{W,X,Y\otimes Z}}{\searrow} & \\ ((W \otimes X) \otimes Y) \otimes Z & & & & W \otimes (X \otimes (Y \otimes Z)) \\ \downarrow{\scriptstyle a_{W,X,Y}\otimes \mathrm{id}_Z} & & & & \uparrow{\scriptstyle \mathrm{id}_W \otimes a_{X,Y,Z}} \\ (W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow[a_{W,X\otimes Y,Z}]{} & & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

Example 2.1. The category of pointed compactly generated topological spaces $\left(\mathbf{Top}^*_{cg}, \wedge, S^0\right)$ endowed with the smash product is a pointed topological monoidal category. The unit object is given by S^0. Moreover, we notice that for two topological spaces $X, Y \in \mathbf{Top}^*_{cg}$ there is a canonical homeomorphism

$$X \wedge Y \simeq Y \wedge X$$

In fact many monoidal categories carry this kind of symmetry, motivating the definition of braided and symmetric monoidal structures.

Definition 2.2. A **topological braided monoidal category** is a topological monoidal category $(\mathcal{C}, \otimes, 1)$ equipped with a family of natural isomorphisms

$$\tau_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

called the braiding, such that the following diagrams commute for all $X, Y, Z \in \mathcal{C}$.

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{\tau_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X \\ \downarrow{\scriptstyle \tau_{X,Y}\otimes \mathrm{id}} & & & & \downarrow{\scriptstyle a_{Y,Z,X}} \\ (Y \otimes X) \otimes Z & \xrightarrow[a_{Y,X,Z}]{} & Y \otimes (X \otimes Z) & \xrightarrow[\mathrm{id}\otimes \tau_{X,Z}]{} & Y \otimes (Z \otimes X) \end{array}$$

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{a^{-1}_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\tau_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) \\ \downarrow{\scriptstyle \mathrm{id}\otimes \tau_{Y,Z}} & & & & \downarrow{\scriptstyle a^{-1}_{Z,X,Y}} \\ X \otimes (Z \otimes Y) & \xrightarrow[a^{-1}_{X,Z,Y}]{} & (X \otimes Z) \otimes Y & \xrightarrow[\tau_{X,Z}\otimes \mathrm{id}]{} & (Z \otimes X) \otimes Y \end{array}$$

The commutativity of these two diagrams is sometimes denoted as the hexagon identities.

Example 2.2. A nice example of a braided monoidal category also motivating the term "braided" is given by the braid category $\mathbb{B}$. The objects in $\mathbb{B}$ are given by natural numbers $0, 1, 2, \ldots$ and the morphisms are given by

$$\mathbb{B}(n, m) = \begin{cases} \mathbb{B}_n & \text{if } n = m \\ \emptyset & \text{if } n \neq m \end{cases}$$

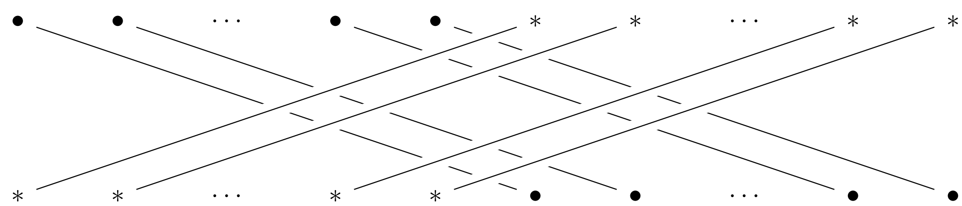
where $\mathbb{B}_n$ denotes the braid group on n strings. The braid category can be endowed with a monoidal product as follows.

$$\begin{aligned} + : \mathbb{B} \times \mathbb{B} &\longrightarrow \mathbb{B} \\ (n, m) &\longmapsto n + m \end{aligned}$$

Now we equip the monoidal category $\mathbb{B}$ with a braiding. That is, a family of natural isomorphisms

$$\tau_{n,m} : n + m \to m + n$$

given by the following braid.



Definition 2.3. A **topological symmetric monoidal category** is a topological braided monoidal category $(\mathcal{C}, \otimes, 1, \tau)$ such that the braiding satisfies the following condition:

$$\tau_{Y,X} \circ \tau_{X,Y} = \mathrm{id}_{X \otimes Y}$$

for all objects $X, Y \in \mathcal{C}$.

Remark 2.1. Notice that the braiding on the braid category $\mathbb{B}$ does not satisfy the above condition.

$$\tau_{m,n} \circ \tau_{n,m} \neq \mathrm{id}_{n+m}$$

Hence the braided monoidal category $\mathbb{B}$ is not symmetric.

Definition 2.4. Given a (pointed) topological symmetric monoidal category $\mathcal{C}$ with tensor product $\otimes$, we call $\mathcal{C}$ **closed monoidal** if for each object $Y \in \mathcal{C}$ there is a pair of adjoint functors

$$\mathcal{C} \underset{\mathrm{hom}(Y,-)}{\overset{Y \otimes (-)}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}$$

For all $X, Y \in \mathcal{C}$ the object $\mathrm{hom}(Y, X) \in \mathcal{C}$ is called the **internal hom** of X and Y. In particular for all objects X, Y, Z there are natural isomorphisms

$$\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(X, \mathrm{hom}(Y, Z))$$

Remark 2.2. The notation for the internal hom introduced above for a general symmetric monoidal category is not going to be used in the specific cases appearing in the present thesis. We then use the "standard" notation. That is,

- in the category $\mathbf{Top}_{cg}^*$ we denote the inner hom with respect to the smash product by $\mathbf{Maps}[-,-]$
- in the category **sSet** we denote the inner hom with respect to the Cartesian product by $[-,-]$

The strategy is to construct the category of spectra as a category of certain module objects in an enriched functor category. Therefore, we first need to specify what we mean by an enriched functor and how we enrich the category of such functors. To do so we need the notion of enriched ends, which we will introduce here in the specific case working over the category $\mathbf{Top}_{cg}^*$. In Section 5.1 we will generalize this construction and see how the definition of an enriched end via equalizers agrees with the classical definition using dinatural transformations.

In the following we will work mostly with pointed topological categories. Hence let $\mathcal{C}, \mathcal{D}$ be two such categories. Then we denote with $\mathcal{C} \times \mathcal{D}$ their pointed topological product category. It follows that

$$(\mathcal{C} \times \mathcal{D})\,((c_1, d_1), (c_2, d_2)) = \mathcal{C}(c_1, c_2) \wedge \mathcal{D}(d_1, d_2)$$

Now given a bifunctor

$$F : \mathcal{C} \times \mathcal{D} \to \mathbf{Top}_{cg}^*$$

it follows that for all objects $(c_1, d_1), (c_2, d_2)$ there are maps

$$F_{(c_1,d_1),(c_2,d_2)} : \mathcal{C}(c_1, c_2) \wedge \mathcal{D}(d_1, d_2) \to \mathbf{Top}_{cg}^*\,(F(c_1, d_1), F(c_2, d_2))$$

Using the closed model structure on $\mathbf{Top}^*_{cg}$ with internal hom denoted by $\mathbf{Maps}[-,-]$ we have that

$$\mathbf{Maps}\left[\mathcal{C}(c_1,c_2), \mathbf{Maps}\left[F(c_1,d),F(c_2,d)\right]\right] \cong \mathbf{Maps}\left[\mathcal{C}(c_1,c_2)\wedge F(c_1,d),F(c_2,d)\right]$$
$$(\theta_{c_1,c_2}(d) : f \mapsto F(f,\mathrm{id}_d)) \mapsto \rho_{c_1,c_2,d}$$

and similarly

$$\mathbf{Maps}\left[\mathcal{D}(d_1,d_2), \mathbf{Maps}\left[F(c,d_1),F(c,d_2)\right]\right] \cong \mathbf{Maps}\left[\mathcal{D}(d_1,d_2)\wedge F(c,d_1),F(c,d_2)\right]$$
$$\left(\theta'_{d_1,d_2}(c) : g \mapsto F(\mathrm{id}_c,g)\right) \mapsto \lambda_{d_1,d_2,c}$$

The two maps constructed via the internal hom andjunction are denoted with

$$\rho_{c_1,c_2,d} : \mathcal{C}(c_1,c_2)\wedge F(c_1,d) \to F(c_2,d)$$
$$\lambda_{d_1,d_2,c} : \mathcal{D}(d_1,d_2)\wedge F(c,d_1) \to F(c,d_2)$$

Considering the special case where $\mathcal{D} = \mathcal{C}^{\mathrm{op}}$ we obtain the following two actions.

$\rho_{c_1,c_2,d} : \mathcal{C}(c_1,c_2)\wedge F(c_2,d) \to F(c_1,d)$ given by the pullback on the first variable.
$\lambda_{d_1,d_2,c} : \mathcal{C}(d_1,d_2)\wedge F(c,d_1) \to F(c,d_2)$ given by the pushforward on the second variable.

Hence we are ready to define topological enriched ends and coends of topological bifunctors.

Definition 2.5. Let $\mathcal{C}$ be a small pointed topological category and let

$$F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Top}^*_{cg}$$

be a pointed topological functor. Then we define

(i) the **enriched coend** of F, denoted $\int^{c\in\mathcal{C}} F(c,c)$, is the coequalizer in $\mathbf{Top}^*_{cg}$ of the two actions

$$\coprod_{c,d\in\mathcal{C}} \mathcal{C}(c,d)\wedge F(d,c) \underset{\coprod \lambda_{d,c,d}}{\overset{\coprod \rho_{c,d,c}}{\rightrightarrows}} \coprod_{c\in\mathcal{C}} F(c,c) \xrightarrow{coeq.} \int^{c\in\mathcal{C}} F(c,c)$$

which are given by

$$\rho_{c,d,c} : \mathcal{C}(c,d)\wedge F(d,c) \to F(c,c)$$
$$\lambda_{d,c,d} : \mathcal{C}(c,d)\wedge F(d,c) \to F(d,d)$$

(ii) the **enriched end** of F, denoted $\int_{c\in\mathcal{C}} F(c,c)$, is the equalizer in $\mathbf{Top}^*_{cg}$ of the two actions

$$\int_{c\in\mathcal{C}} F(c,c) \xrightarrow{eq.} \prod_{c\in\mathcal{C}} F(c,c) \underset{\coprod \theta'_{c,d}(c)}{\overset{\coprod \theta_{d,c}(d)}{\rightrightarrows}} \prod_{c,d\in\mathcal{C}} \mathbf{Maps}\left[\mathcal{C}(c,d),F(c,d)\right]$$

which are given by

$$\theta_{d,c}(d) : F(d,d) \to \mathbf{Maps}\left[\mathcal{C}(c,d),F(c,d)\right]$$
$$\theta'_{c,d}(c) : F(c,c) \to \mathbf{Maps}\left[\mathcal{C}(c,d),F(c,d)\right]$$

Lemma 2.1. *Let $\mathcal{C}$ be a small pointed topological category. Then for $F, G : \mathcal{C} \to \mathbf{Top}^*_{cg}$ two pointed topological functors there is a bifunctor*

$$\mathbf{Maps}\,[F(-), G(-)] : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Top}^*_{cg}$$

whose end is a topological space

$$\int_{c \in \mathcal{C}} \mathbf{Maps}\,[F(c), G(c)] \in \mathbf{Top}^*_{cg}$$

such that the underlying pointed set is isomorphic to the set of natural transformations between F and G. Hence for the forgetful functor $U : \mathbf{Top}^*_{cg} \to \mathbf{Set}^*$, there is an isomorphism

$$U\left(\int_{c \in \mathcal{C}} \mathbf{Maps}\,[F(c), G(c)]\right) \cong [\mathcal{C}, \mathbf{Top}^*_{cg}](F, G)$$

Proof. Using the fact that the forgetful functor U preserves all limits, there is an equalizer diagram in $\mathbf{Set}^*$

$$U\left(\int_{c \in \mathcal{C}} \mathbf{Maps}\,[F(c), G(c)]\right) \to \prod_{c \in \mathcal{C}} \mathbf{Top}^*_{cg}(F(c), G(c)) \rightrightarrows \prod_{c,d \in \mathcal{C}} \mathbf{Top}^*_{cg}\left(\mathcal{C}(c,d), \mathbf{Maps}\,[F(c), G(c)]\right)$$

Now we notice that

$$\prod_{c \in \mathcal{C}} \mathbf{Top}^*_{cg}(F(c), G(c)) \cong \{\eta_c : F(c) \to G(c)\}$$

Hence the left hand side is equivalent to the collection of morphisms in $\mathbf{Top}^*_{cg}$ indexed on objects of $\mathcal{C}$. To compute the equalizer we have to look at the actions. They are given by

$$\rho_{c_2,c_1,d} : \mathbf{Maps}\,[F(c_2), G(d)] \longrightarrow \mathbf{Maps}\,[\mathcal{C}(c_1, c_2), \mathbf{Maps}\,[F(c_1), G(d)]]$$
$$f : F(c_2) \to G(d) \mapsto (g : c_1 \to c_2 \mapsto f \circ F(g))$$

which is essentially precomposing with $F(g)$ and

$$\rho'_{d_1,d_2,c} : \mathbf{Maps}\,[F(c), G(d_1)] \longrightarrow \mathbf{Maps}\,[\mathcal{C}(d_1, d_2), \mathbf{Maps}\,[F(c), G(d_2)]]$$
$$h : F(c) \to G(d_1) \mapsto (k : d_1 \to d_2 \mapsto G(k) \circ h)$$

which is postcomposing with $G(k)$. Hence taking the equalizer means that in the set $\{\eta_c : F(c) \to G(c)\}$ for all morphisms $k : c \to d$ the action of precomposing η_d by $F(k)$ is equal to the action of postcomposing η_c by $G(k)$. This means that the following diagram is commutative.

$$\begin{array}{ccc} F(c) & \xrightarrow{\eta_c} & G(c) \\ {\scriptstyle F(k)}\downarrow & & \downarrow{\scriptstyle G(k)} \\ F(d) & \xrightarrow[\eta_d]{} & G(d) \end{array}$$

This means that $\eta : F \to G$ is a natural transformation. □

The above lemma motivates the following definition of a topological enrichment of the category $[\mathcal{C}, \mathbf{Top}^*_{cg}]$ of pointed topological functors.

Definition 2.6. Let $\mathcal{C}$ be a small pointed topological category. Then we define a topological enrichment on $[\mathcal{C}, \mathbf{Top}^*_{cg}]$ by defining

$$[\mathcal{C}, \mathbf{Top}^*_{cg}](F, G) := \int_{c \in \mathcal{C}} \mathbf{Maps}\,[F(c), G(c)]$$

via the end construction from Lemma 2.1.

Remark 2.3. The composition operation of the topological enrichment is defined as follows.

$$\begin{array}{ccc}
\left(\int_{c\in\mathcal{C}} \mathbf{Maps}\,[F(c),G(c)]\right) \wedge \left(\int_{c\in\mathcal{C}} \mathbf{Maps}\,[G(c),H(c)]\right) & \longrightarrow & \prod_{c\in\mathcal{C}} \mathbf{Maps}\,[F(c),G(c)] \wedge \mathbf{Maps}\,[G(c),H(c)] \\
& \searrow^{m} & \downarrow \\
& & \prod_{c\in\mathcal{C}} \mathbf{Maps}\,[F(c),H(c)]
\end{array}$$

Notice that the horizontal arrow is given by the smash product of the following maps, being part of the definition of the two ends.

$$\int_{c\in\mathcal{C}} \mathbf{Maps}\,[F(c),G(c)] \to \prod_{c\in\mathcal{C}} \mathbf{Maps}\,[F(c),G(c)]$$

$$\int_{c\in\mathcal{C}} \mathbf{Maps}\,[G(c),H(c)] \to \prod_{c\in\mathcal{C}} \mathbf{Maps}\,[G(c),H(c)]$$

The vertical arrow is given by the composition in $\mathbf{Top}^*_{cg}$.

$$\begin{array}{ccccc}
\int_{c\in\mathcal{C}} \mathbf{Maps}\,[F(c),H(c)] & \longrightarrow & \prod_{c\in\mathcal{C}} \mathbf{Maps}\,[F(c),H(c)] & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & \cdots \\
& \nwarrow^{c} & \uparrow m & & \\
& & \left(\int_{c\in\mathcal{C}} \mathbf{Maps}\,[F(c),G(c)]\right) \wedge \left(\int_{c\in\mathcal{C}} \mathbf{Maps}\,[G(c),H(c)]\right) & &
\end{array}$$

By construction the composition m satisfies $\pi_1 \circ m = \pi_2 \circ m$. Hence by the universal property of the equalizer there is a map

$$c : [\mathcal{C}, \mathbf{Top}^*_{cg}](F,G) \wedge [\mathcal{C}, \mathbf{Top}^*_{cg}](G,H) \to [\mathcal{C}, \mathbf{Top}^*_{cg}](F,H)$$

which defines the desired composition.

The following two propositions are the topologically enriched analogues of the Yoneda Lemma resp. the Co-Yoneda Lemma.

Proposition 2.1. ([Kel05]) *Let $\mathcal{C}$ be a small pointed topological category and $F : \mathcal{C} \to \mathbf{Top}^*_{cg}$ a pointed topological functor. Then for any $c \in \mathcal{C}$ there is a natural isomorphism*

$$[\mathcal{C}, \mathbf{Top}^*_{cg}](\mathcal{C}(c,-),F) \cong F(c)$$

Remark 2.4. Using Definition 2.6 there is an isomorphism

$$\int_{d\in\mathcal{C}} \mathbf{Maps}\,[\mathcal{C}(c,d),F(d)] \cong F(c)$$

Proposition 2.2. *Let $\mathcal{C}$ and $F : \mathcal{C} \to \mathbf{Top}^*_{cg}$ be as above. Then for every $c \in \mathcal{C}$ there is a natural isomorphism*

$$\int^{d\in\mathcal{C}} \mathcal{C}(d,c) \wedge F(d) \cong F(c)$$

Proof. This is the dual argument of the above proposition. □

The aim will be to endow the category $[\mathcal{C}, \mathbf{Top}_{cg}^*]$ with a suitable monoidal product that we can later use to define the smash product of spectra. Since $(\mathbf{Top}_{cg}^*, \wedge, S^0)$ is already monoidal, it motivates the following proposition.

Remark 2.5. Notice that given a pointed topological functor $X \in [\mathcal{C}, \mathbf{Top}_{cg}^*]$ and a pointed space $K \in \mathbf{Top}_{cg}^*$, the "smash product" of X and K defines a pointed topological functor as follows.

$$\begin{aligned} X \wedge K : \mathcal{C} &\longrightarrow \mathbf{Top}_{cg}^* \\ c &\longmapsto X(c) \wedge K \end{aligned}$$

This construction extends to a functor denoted by

$$(-) \wedge K : [\mathcal{C}, \mathbf{Top}_{cg}^*] \to [\mathcal{C}, \mathbf{Top}_{cg}^*]$$

Similarly we define for $K \in \mathbf{Top}_{cg}^*$ and $X \in [\mathcal{C}, \mathbf{Top}_{cg}^*]$ a pointed topological functor by

$$\begin{aligned} \mathbf{Maps}\,[K, X] : \mathcal{C} &\to \mathbf{Top}_{cg}^* \\ c &\mapsto \mathbf{Maps}\,[K, X(c)] \end{aligned}$$

Also this construction extends to a functor

$$\mathbf{Maps}\,[K, -] : [\mathcal{C}, \mathbf{Top}_{cg}^*] \to [\mathcal{C}, \mathbf{Top}_{cg}^*]$$

Proposition 2.3. *Let $\mathcal{C}$ be as above and $X, Y \in [\mathcal{C}, \mathbf{Top}_{cg}^*]$ and $K \in \mathbf{Top}_{cg}^*$. Then there are natural isomorphisms*

(i) $[\mathcal{C}, \mathbf{Top}_{cg}^*](X \wedge K, Y) \cong \mathbf{Maps}\left[K, [\mathcal{C}, \mathbf{Top}_{cg}^*](X, Y)\right]$

(ii) $[\mathcal{C}, \mathbf{Top}_{cg}^*](X, \mathbf{Maps}\,[K, Y(-)]) \cong \mathbf{Maps}\left[K, [\mathcal{C}, \mathbf{Top}_{cg}^*](X, Y)\right]$

Hence there is a pair of adjoint functors

$$[\mathcal{C}, \mathbf{Top}_{cg}^*] \underset{\mathbf{Maps}[K,-]}{\overset{(-)\wedge K}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} [\mathcal{C}, \mathbf{Top}_{cg}^*]$$

Proof. (i) By definition one has

$$[\mathcal{C}, \mathbf{Top}_{cg}^*](X \wedge K, Y) = \int_{c \in \mathcal{C}} \mathbf{Maps}\,[X(c) \wedge K, Y(c)]$$

Now using that $\mathbf{Top}_{cg}^*$ is closed monoidal we obtain

$$\begin{aligned} \int_{c \in \mathcal{C}} \mathbf{Maps}\,[X(c) \wedge K, Y(c)] &\cong \int_{c \in \mathcal{C}} \mathbf{Maps}\,[K, \mathbf{Maps}\,[X(c), Y(c)]] \\ &\cong \mathbf{Maps}\left[K, \int_{c \in \mathcal{C}} \mathbf{Maps}\,[X(c), Y(c)]\right] \end{aligned}$$

where the second isomorphism comes from the fact that the functor $\mathbf{Maps}\,[K, -]$ preserves ends. Again by Definition 2.6 one obtains

$$\mathbf{Maps}\left[K, \int_{c \in \mathcal{C}} \mathbf{Maps}\,[X(c), Y(c)]\right] = \mathbf{Maps}\left[K, [\mathcal{C}, \mathbf{Top}_{cg}^*](X, Y)\right]$$

(ii) By definition one has

$$[\mathcal{C}, \mathbf{Top}^*_{cg}](X, \mathbf{Maps}\,[K, Y,)] = \int_{c\in\mathcal{C}} \mathbf{Maps}\,[X(c), \mathbf{Maps}\,[K, Y(c)]]$$

Then by the closed monoidal structure in $\mathbf{Top}^*_{cg}$ it follows that

$$\begin{aligned}\int_{c\in\mathcal{C}} \mathbf{Maps}\,[X(c), \mathbf{Maps}\,[K, Y(c)]] &\cong \int_{c\in\mathcal{C}} \mathbf{Maps}\,[X(c)\wedge K, Y(c)] \\ &\cong \int_{c\in\mathcal{C}} \mathbf{Maps}\,[K, \mathbf{Maps}\,[X(c), Y(c)]] \\ &\cong \mathbf{Maps}\left[K, \int_{c\in\mathcal{C}} \mathbf{Maps}\,[X(c), Y(c)]\right] \\ &= \mathbf{Maps}\left[K, [\mathcal{C}, \mathbf{Top}^*_{cg}](X, Y)\right]\end{aligned}$$

Combining the isomorphisms from (i) and (ii) we get that the two functors indeed form an adjoint pair. □

2.2 Topological Day convolution

The aim of this section will be to define a suitable closed monoidal structure on the functor category $[\mathcal{C}, \mathbf{Top}^*_{cg}]$, which will be essential in the construction of the smash product of spectra. In the following, let $(\mathcal{C}, \otimes, 1_\mathcal{C})$ be a small pointed topological monoidal category.

Definition 2.7. The **topological Day convolution** tensor product on $[\mathcal{C}, \mathbf{Top}^*_{cg}]$ is given by the bifunctor

$$\begin{aligned}\underset{\text{Day}}{\otimes} : [\mathcal{C}, \mathbf{Top}^*_{cg}] \times [\mathcal{C}, \mathbf{Top}^*_{cg}] &\to [\mathcal{C}, \mathbf{Top}^*_{cg}] \\ (X, Y) &\mapsto X \underset{\text{Day}}{\otimes} Y\end{aligned}$$

where we define

$$\left(X \underset{\text{Day}}{\otimes} Y\right)(c) := \int^{c_1, c_2 \in \mathcal{C}\times\mathcal{C}} \mathcal{C}(c_1 \otimes c_2, c) \wedge X(c_1) \wedge Y(c_2)$$

As the categories $\mathcal{C}$ and $\mathbf{Top}^*_{cg}$ are both endowed with a monoidal structure, we want to investigate the relation between these structures and the topological Day convolution. To do so, we need the following definition, which arises naturally from the monoidal structure on $\mathbf{Top}^*_{cg}$.

Definition 2.8. The **external tensor product** on $[\mathcal{C}, \mathbf{Top}^*_{cg}]$ is given by the bifunctor

$$\bar{\wedge} : [\mathcal{C}, \mathbf{Top}^*_{cg}] \times [\mathcal{C}, \mathbf{Top}^*_{cg}] \to [\mathcal{C}\times\mathcal{C}, \mathbf{Top}^*_{cg}]$$

which is defined as follows.

$$(X \,\bar{\wedge}\, Y)(c_1, c_2) = X(c_1) \wedge Y(c_2)$$

The next proposition relates the external tensor product and the topological Day convolution via Kan extensions.

Proposition 2.4. *The Day convolution tensor product of two functors $X, Y \in [\mathcal{C}, \mathbf{Top}^*_{cg}]$ is isomorphic to the left Kan extension of the external tensor product along the monoidal product on $\mathcal{C}$. Hence we have that there is a natural isomorphism*

$$X \underset{\text{Day}}{\otimes} Y \cong \text{Lan}_{\otimes}(X \mathbin{\bar{\wedge}} Y)$$

where

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\ {\scriptstyle X \bar{\wedge} Y}\downarrow & \swarrow {\scriptstyle \text{Lan}_{\otimes}(X \bar{\wedge} Y)} & \\ \mathbf{Top}^*_{cg} & & \end{array}$$

Proof. Using the coend formula in Remark 5.4, left Kan extensions can be written as coends, i.e

$$\text{Lan}_{\otimes}(X \mathbin{\bar{\wedge}} Y)(c) = \int^{(m_1,m_2)\in\mathcal{C}\times\mathcal{C}} \mathcal{C}(m_1 \otimes m_2, c) \bullet (X(m_1) \wedge Y(m_2))$$

where $\bullet$ denotes the copowering in $\mathbf{Top}^*_{cg}$ which is given by the smash product. Hence we can write

$$\text{Lan}_{\otimes}(X \mathbin{\bar{\wedge}} Y)(c) = \int^{(m_1,m_2)\in\mathcal{C}\times\mathcal{C}} \mathcal{C}(m_1 \otimes m_2, c) \wedge X(m_1) \wedge Y(m_2)$$

Now using Definition 2.7 we immediately have that

$$\text{Lan}_{\otimes}(X \mathbin{\bar{\wedge}} Y)(c) = \left(X \underset{\text{Day}}{\otimes} Y\right)(c)$$

□

Since we can describe the Day convolution as a Kan extension, it has the following universal property.

Corollary 2.1. *Using the characterization of the Day convolution as a left Kan extension, it inherits a universal property. That is, for every $X, Y, Z \in [\mathcal{C}, \mathbf{Top}^*_{cg}]$*

$$[\mathcal{C}, \mathbf{Top}^*_{cg}]\left(X \underset{\text{Day}}{\otimes} Y, Z\right) \cong [\mathcal{C} \times \mathcal{C}, \mathbf{Top}^*_{cg}]\left(X \mathbin{\bar{\wedge}} Y, Z \circ \otimes\right)$$

Proof. This follows immediately from the universal property of the left Kan extension of $X \mathbin{\bar{\wedge}} Y$ along $\otimes$. □

Now we can finally endow the functor category with the desired monoidal product given by the Day convolution tensor product.

Proposition 2.5. *The Day convolution tensor product makes $[\mathcal{C}, \mathbf{Top}^*_{cg}]$ into a pointed topological monoidal category with tensor unit $\mathcal{C}(1_{\mathcal{C}}, -)$. Moreover, if the category $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ is equipped with a symmetric braiding τ, then so is $\left([\mathcal{C}, \mathbf{Top}^*_{cg}], \underset{\text{Day}}{\otimes}, \mathcal{C}(1, -)\right)$.*

Proof. (i) Associativity We need to show that for all $X, Y, Z \in [\mathcal{C}, \mathbf{Top}^*_{cg}]$ there are natural isomorphisms

$$a_{X,Y,Z} : \left(X \underset{\text{Day}}{\otimes} Y\right) \underset{\text{Day}}{\otimes} Z \xrightarrow{\cong} X \underset{\text{Day}}{\otimes} \left(Y \underset{\text{Day}}{\otimes} Z\right)$$

Indeed, we have

$$X \underset{\text{Day}}{\otimes} \left(Y \underset{\text{Day}}{\otimes} Z\right)(c) = \int^{c_1,c_2} \mathcal{C}(c_1 \otimes c_2, c) \wedge X(c_1) \wedge \left(\int^{d_1,d_2} \mathcal{C}(d_1 \otimes d_2, c_2) \wedge Y(d_1) \wedge Z(d_2)\right)$$

Now using the Fubini theorem for coends we obtain

$$\int^{c_1,c_2} \mathcal{C}(c_1 \otimes c_2, c) \wedge X(c_1) \wedge \left(\int^{d_1,d_2} \mathcal{C}(d_1 \otimes d_2, c_2) \wedge Y(d_1) \wedge Z(d_2) \right)$$
$$\cong \int^{c_1,d_1,d_2} \left(\int^{c_2} \mathcal{C}(c_1 \otimes c_2, c) \wedge \mathcal{C}(d_1 \otimes d_2, c_2) \right) \wedge X(c_1) \wedge Y(d_1) \wedge Z(d_2)$$

Using the Co-Yoneda Lemma 2.2 we get

$$\int^{c_2} \mathcal{C}(c_1 \otimes c_2, c) \wedge \mathcal{C}(d_1 \otimes d_2, c_2) \cong \mathcal{C}(c_1 \otimes (d_1 \otimes d_2), c)$$

and therefore the above equation simplifies to

$$\cong \int^{c_1,d_1,d_2} \mathcal{C}(c_1 \otimes (d_1 \otimes d_2), c) \wedge X(c_1) \wedge Y(d_1) \wedge Z(d_2)$$

On the other side we have similarly (using Fubini and the Co-Yoneda Lemma)

$$\left(X \underset{\text{Day}}{\otimes} Y \right) \underset{\text{Day}}{\otimes} Z(c) = \int^{c_1,c_2} \mathcal{C}(c_1 \otimes c_2, c) \wedge \left(\int^{d_1,d_2} \mathcal{C}(d_1 \otimes d_2, c_1) \wedge X(d_1) \wedge Y(d_2) \right) \wedge Z(c_2)$$
$$\cong \int^{c_2,d_1,d_2} \left(\int^{c_1} \mathcal{C}(c_1 \otimes c_2, c) \wedge \mathcal{C}(d_1 \otimes d_2, c_1) \right) \wedge X(d_1) \wedge Y(d_2) \wedge Z(c_2)$$
$$\cong \int^{c_2,d_1,d_2} \mathcal{C}((d_1 \otimes d_2) \otimes c_2, c) \wedge X(d_1) \wedge Y(d_2) \wedge Z(c_2)$$

Now using the associator in $\mathcal{C}$ in the first variable

$$a_{x,y,z} : x \otimes (y \otimes z) \xrightarrow{\sim} (x \otimes y) \otimes z$$

we get the induced natural isomorphisms, defining the associator in $[\mathcal{C}, \mathbf{Top}^*_{cg}]$. As the associator in $\mathcal{C}$ satisfies the pentagon axiom, the induced diagram in $[\mathcal{C}, \mathbf{Top}^*_{cg}]$ also commutes and therefore the pentagon axiom holds true.

(ii) Tensor unit We need to show that $\mathcal{C}(1, -)$ is the tensor unit, i.e there are natural isomorphisms

$$r_X : X \underset{\text{Day}}{\otimes} \mathcal{C}(1, -) \xrightarrow{\cong} X$$
$$l_X : \mathcal{C}(1, -) \underset{\text{Day}}{\otimes} X \xrightarrow{\cong} X$$

Indeed, by definition

$$X \underset{\text{Day}}{\otimes} \mathcal{C}(1, -) = \int^{c_1,c_2} \mathcal{C}(c_1 \otimes c_2, -) \wedge X(c_1) \wedge \mathcal{C}(1, c_2)$$
$$\cong \int^{c_1} \left(\int^{c_2} \mathcal{C}(c_1 \otimes c_2, -) \wedge \mathcal{C}(1, c_2) \right) \wedge X(c_1)$$

Now again by the Co-Yoneda Lemma and the fact that 1 is the tensor unit in $\mathcal{C}$ we get

$$\int^{c_2} \mathcal{C}(c_1 \otimes c_2, -) \wedge \mathcal{C}(1, c_2) \cong \mathcal{C}(c_1, -)$$

Hence we obtain

$$X \underset{\text{Day}}{\otimes} \mathcal{C}(1, -) \cong \int^{c_1} \mathcal{C}(c_1, -) \wedge X(c_1) \cong X$$

The left unitor isomorphism is constructed in a similar way. Similarly the triangle axiom is induced by the triangle axiom given by the monoidal structure on $\mathcal{C}$.

(iii) Braiding Using the same strategy as above, write the Day convolution product as a coend and use the braiding in $\mathcal{C}$ to define an induced braiding in $[\mathcal{C}, \mathbf{Top}^*_{cg}]$.

□

It turns out that the monoidal structure on $[\mathcal{C}, \mathbf{Top}^*_{cg}]$ is closed, which is the statement of the next proposition.

Proposition 2.6. *The category* $\left([\mathcal{C}, \mathbf{Top}^*_{cg}], \underset{\text{Day}}{\otimes}, \mathcal{C}(1,-)\right)$ *is closed monoidal with internal hom given by*

$$[X,Y]_{\text{Day}}(c) = \int_{c_1,c_2} \mathbf{Maps}\left[\mathcal{C}(c\otimes c_1, c_2), \mathbf{Maps}\left[X(c_1), Y(c_2)\right]\right]$$

Proof. We need to show that there is a pair of adjoint functors

$$[\mathcal{C}, \mathbf{Top}^*_{cg}] \underset{Y \underset{\text{Day}}{\otimes} (-)}{\overset{[Y,-]_{\text{Day}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} [\mathcal{C}, \mathbf{Top}^*_{cg}]$$

Let $X, Y, Z \in [\mathcal{C}, \mathbf{Top}^*_{cg}]$. Then by definition

$$\begin{aligned}
[\mathcal{C}, \mathbf{Top}^*_{cg}](X, [Y,Z]_{\text{Day}}) &= \int_c \mathbf{Maps}\left[X(c), [Y,Z]_{\text{Day}}(c)\right] \\
&\cong \int_c \int_{c_1,c_2} \mathbf{Maps}\left[X(c), \mathbf{Maps}\left[\mathcal{C}(c\otimes c_1, c_2), \mathbf{Maps}\left[Y(c_1), Z(c_2)\right]\right]\right]
\end{aligned}$$

Here we used the definition of the internal hom and the fact that $\mathbf{Maps}[X(c), -]$ preserves ends. Now using the closed monoidal structure on $\mathbf{Top}^*_{cg}$ we get

$$\mathbf{Maps}\left[X(c), \mathbf{Maps}\left[\mathcal{C}(c\otimes c_1, c_2), \mathbf{Maps}\left[Y(c_1), Z(c_2)\right]\right]\right] \cong \mathbf{Maps}\left[X(c)\wedge \mathcal{C}(c\otimes c_1, c_2)\wedge Y(c_1), Z(c_2)\right]$$

Hence it follows

$$\begin{aligned}
[\mathcal{C}, \mathbf{Top}^*_{cg}](X, [Y,Z]_{\text{Day}}) &\cong \int_c \int_{c_1,c_2} \mathbf{Maps}\left[\mathcal{C}(c\otimes c_1, c_2)\wedge X(c)\wedge Y(c_1), Z(c_2)\right] \\
&\cong \int_{c_2} \mathbf{Maps}\left[\int^{c,c_1} \mathcal{C}(c\otimes c_1, c_2)\wedge X(c)\wedge Y(c_1), Z(c)\right] \\
&= \int_{c_2} \mathbf{Maps}\left[(X \underset{\text{Day}}{\otimes} Y)(c_2), Z(c_2)\right] \\
&= [\mathcal{C}, \mathbf{Top}^*_{cg}](X \underset{\text{Day}}{\otimes} Y, Z)
\end{aligned}$$

where we used Fubini and again the fact that $\mathbf{Maps}[-,-]$ "preserves" ends resp. coends. □

Proposition 2.7. (Yoneda Embedding) *The Yoneda Embedding*

$$\begin{aligned}
\mathcal{Y} : (\mathcal{C}^{\text{op}}, \otimes, 1) &\longrightarrow \left([\mathcal{C}, \mathbf{Top}^*_{cg}], \underset{\text{Day}}{\otimes}, \mathcal{C}(1,-)\right) \\
c &\longmapsto \mathcal{C}(c,-)
\end{aligned}$$

defines a strong monoidal functor, i.e the embedding preserves the tensor product and the tensor unit up to isomorphism.

Remark 2.6. To avoid messy notation we define $y_c(-) := \mathcal{C}(c,-)$

Proof. The fact that the tensor unit is preserved follows from Proposition 2.5 where we show that y_1 is indeed the tensor unit with respect to the Day convolution. Hence what is left to show is the preservation of the tensor product.

$$\begin{aligned}
(y_{c_1} \underset{\text{Day}}{\otimes} y_{c_2})(c) &\cong \int^{d_1,d_2} \mathcal{C}(d_1 \otimes d_2, c) \wedge \mathcal{C}(c_1, d_1) \wedge \mathcal{C}(c_2, d_2) \\
\text{(Fubini and 2x Co-Yoneda Lemma)} \quad &\cong \mathcal{C}(c_1 \otimes c_2, c) \\
&= y_{c_1 \otimes c_2}(c)
\end{aligned}$$

□

2.3 $\mathbb{S}$-modules

In this section we want to describe spectra as module objects over a monoid in the category $[\mathcal{C}, \mathbf{Top}^*_{cg}]$. This will later allow us to define the smash product in a proper way and deduce some of its most important properties. To do so, we need to introduce some categorical constructions, such as monoid objects and module objects.

Definition 2.9. Given a monoidal category $(\mathcal{C}, \otimes, 1)$ we call a triple (A, μ, ε) a **monoid object** in $\mathcal{C}$ if $A \in \mathcal{C}$ is an object and

$$\begin{aligned}
\mu &: A \otimes A \to A \\
\varepsilon &: 1 \to A
\end{aligned}$$

are two maps called multiplication and unit, satisfying the following axioms.

(i) (Associativity)

$$\begin{array}{ccccc}
(A \otimes A) \otimes A & \xrightarrow{a_{(A,A,A)}} & A \otimes (A \otimes A) & \xrightarrow{\mathrm{id} \otimes \mu} & A \otimes A \\
\downarrow{\scriptstyle \mu \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \mu} \\
A \otimes A & & \xrightarrow{\quad\quad \mu \quad\quad} & & A
\end{array}$$

(ii) (Unitality)

$$\begin{array}{ccccc}
1 \otimes A & \xrightarrow{\varepsilon \otimes \mathrm{id}} & A \otimes A & \xleftarrow{\mathrm{id} \otimes \varepsilon} & A \otimes 1 \\
& {\scriptstyle l_A} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle r_A} & \\
& & A & &
\end{array}$$

Where r_A and l_A are the right and left unitor isomorphisms in $\mathcal{C}$.

(iii) (Commutativity) Moreover, if $(\mathcal{C}, \otimes, 1)$ has the structure of a symmetric monoidal category with braiding τ, then a monoid (A, μ, ε) is called a **commutative monoid**, if the following diagram commutes.

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{\tau_{A,A}} & A \otimes A \\
& {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu} & \\
& A &
\end{array}$$

Remark 2.7. We write $\mathrm{Mon}(\mathcal{C}, \otimes, 1)$ for the category of monoids in $\mathcal{C}$ and $\mathrm{CMon}(\mathcal{C}, \otimes, 1)$ for its subcategory of commutative monoids.

Definition 2.10. Given a monoid (A, μ, ε) in $\mathcal{C}$ a **left-module object** in $\mathcal{C}$ over (A, μ, ε) consists of an object $N \in \mathcal{C}$ and a morphism $\rho : A \otimes N \to N$, called the action, such that the following axioms are satisfied.

(i) (Unitality)

$$\begin{array}{ccc} 1 \otimes N & \xrightarrow{\varepsilon \otimes \mathrm{id}} & A \otimes N \\ & \searrow^{l_N} & \downarrow \rho \\ & & N \end{array}$$

(ii) (Action property)

$$\begin{array}{ccccc} (A \otimes A) \otimes N & \xrightarrow{a_{(A,A,N)}} & A \otimes (A \otimes N) & \xrightarrow{\mathrm{id} \otimes \rho} & A \otimes N \\ \mu \otimes \mathrm{id} \downarrow & & & & \downarrow \rho \\ A \otimes N & & \xrightarrow{\quad \rho \quad} & & N \end{array}$$

Remark 2.8. Given a monoidal category $(\mathcal{C}, \otimes, 1)$ and a monoid (A, μ, ε) we denote with A-Mod$(\mathcal{C}, \otimes, 1)$ the category of module objects in $\mathcal{C}$ over A.

Lemma 2.2. *There is an equivalence of categories*

$$[\mathcal{C}, \mathbf{Top}^*_{cg}] \simeq y_1\text{-Mod}([\mathcal{C}, \mathbf{Top}^*_{cg}])$$

Proof. This follows immediately from the fact that in a monoidal category every object can be regarded as a left module over the tensor unit in a canonical way. □

Now we want to investigate the (commutative) monoid objects in the functor category $[\mathcal{C}, \mathbf{Top}^*_{cg}]$. To do so we first need the following definitions.

Definition 2.11. Let $(\mathcal{C}, \otimes, 1_\mathcal{C})$ and $(\mathcal{D}, \otimes, 1_\mathcal{D})$ be two topological monoidal categories. A **topological lax monoidal functor** $F : \mathcal{C} \to \mathcal{D}$ consists of the following data.

1. A topological functor $F : \mathcal{C} \to \mathcal{D}$.
2. A morphism $e : 1_\mathcal{D} \to F(1_\mathcal{C})$.
3. A natural transformation for all $x, y \in \mathcal{C}$

$$\mu_{x,y} : F(x) \otimes F(y) \to F(x \otimes y)$$

satisfying the following axioms.

(i) (Associativity) For all objects $x, y, z \in \mathcal{C}$ the diagram commutes

$$\begin{array}{ccc} (F(x) \otimes_\mathcal{D} F(y)) \otimes_\mathcal{D} F(z) & \xrightarrow{a^\mathcal{D}_{F(x),F(y),F(z)}} & F(x) \otimes_\mathcal{D} (F(y) \otimes_\mathcal{D} F(z)) \\ \mu_{x,y} \otimes \mathrm{id}_{F(z)} \downarrow & & \downarrow \mathrm{id}_{F(x)} \otimes \mu_{y,z} \\ F(x \otimes_\mathcal{C} y) \otimes_\mathcal{D} F(z) & & F(x) \otimes_\mathcal{D} F(y \otimes_\mathcal{C} z) \\ \mu_{x \otimes y, z} \downarrow & & \downarrow \mu_{x, y \otimes z} \\ F((x \otimes_\mathcal{C} y) \otimes_\mathcal{C} z) & \xrightarrow{F(a^\mathcal{C}_{x,y,z})} & F(x \otimes_\mathcal{C} (y \otimes_\mathcal{C} z)) \end{array}$$

(ii) (Unitality) For all $x \in \mathcal{C}$ the following diagrams commute

$$\begin{array}{ccc} 1_{\mathcal{D}} \otimes_{\mathcal{D}} F(x) & \xrightarrow{e \otimes \mathrm{id}_{F(x)}} & F(1_{\mathcal{C}}) \otimes_{\mathcal{D}} F(x) \\ {\scriptstyle l^{\mathcal{D}}_{F(x)}} \downarrow & & \downarrow {\scriptstyle \mu_{1,x}} \\ F(x) & \xleftarrow[F(l^{\mathcal{C}}_x)]{} & F(1_{\mathcal{C}} \otimes_{\mathcal{C}} x) \end{array} \qquad \begin{array}{ccc} F(x) \otimes_{\mathcal{D}} 1_{\mathcal{D}} & \xrightarrow{\mathrm{id}_{F(x)} \otimes e} & F(x) \otimes_{\mathcal{D}} F(1_{\mathcal{C}}) \\ {\scriptstyle r^{\mathcal{D}}_{F(x)}} \downarrow & & \downarrow {\scriptstyle \mu_{x,1}} \\ F(x) & \xleftarrow[F(r^{\mathcal{C}}_x)]{} & F(x \otimes_{\mathcal{C}} 1_{\mathcal{C}}) \end{array}$$

(iii) (Commutativity) If moreover the categories $\mathcal{C}$ and $\mathcal{D}$ are equipped with a symmetric braiding τ and σ, then the lax monoidal functor F is called **braided monoidal**, if the following diagram commutes.

$$\begin{array}{ccc} F(x) \otimes F(y) & \xrightarrow{\sigma_{F(x),F(y)}} & F(y) \otimes F(x) \\ {\scriptstyle \mu_{x,y}} \downarrow & & \downarrow {\scriptstyle \mu_{y,x}} \\ F(x \otimes y) & \xrightarrow[F(\tau_{x,y})]{} & F(y \otimes x) \end{array}$$

Remark 2.9. Morphisms of such lax monoidal functors are natural transformations which are compatible with the product and the unit, and satisfy certain diagram axioms. Such natural transformations are called monoidal natural transformations. Hence we can write $\mathrm{MonFunc}(\mathcal{C}, \mathcal{D})$ for the category of lax monoidal functors and monoidal natural transformations and $\mathrm{SymMonFunc}(\mathcal{C}, \mathcal{D})$ for the category of symmetric lax monoidal functors and symmetric monoidal natural transformations. A lax monoidal functor is called a (strong) monoidal functor, if the morphisms μ and e are isomorphisms.

Proposition 2.8. ([MMSS01]) *Let $(\mathcal{C}, \otimes, 1)$ be a pointed topological (symmetric) monoidal category and regard $(\mathbf{Top}^*_{cg}, \wedge, S^0)$ as a pointed topological symmetric monoidal category. Then monoids resp. commutative monoids in $\left([\mathcal{C}, \mathbf{Top}^*_{cg}], \underset{\mathrm{Day}}{\otimes}, y_1\right)$ are equivalent to lax monoidal resp. symmetric lax monoidal functors of the form*

$$F : (\mathcal{C}, \otimes, 1) \to (\mathbf{Top}^*_{cg}, \wedge, S^0)$$

This means that there are equivalences of categories

$$\mathrm{Mon}\left([\mathcal{C}, \mathbf{Top}^*_{cg}], \underset{\mathrm{Day}}{\otimes}, y_1\right) \simeq \mathrm{MonFunc}(\mathcal{C}, \mathbf{Top}^*_{cg})$$
$$\mathrm{CMon}\left([\mathcal{C}, \mathbf{Top}^*_{cg}], \underset{\mathrm{Day}}{\otimes}, y_1\right) \simeq \mathrm{SymMonFunc}(\mathcal{C}, \mathbf{Top}^*_{cg})$$

To define spectra as module objects, we need to define the corresponding source categories, which are **Seq**, **Sym** and **Orth**.

Definition 2.12. We define the following pointed topological symmetric monoidal categories

(i) The category **Seq** whose objects are natural numbers and whose morphisms are given by

$$\mathbf{Seq}(n, m) = \begin{cases} S^0 & \text{if } n = m \\ * & \text{if } n \neq m \end{cases}$$

The tensor product is given by addition and the tensor unit is 0. Therefore, **Seq** is symmetric monoidal.

(ii) The category **Sym** whose objects are finite sets

$$\overline{n} = \{1, ..., n\}$$
$$\overline{0} = \emptyset$$

and whose morphisms are automorphisms on those sets, i.e

$$\mathbf{Sym}(\overline{n}, \overline{m}) = \begin{cases} (S_n)_+ & \text{if } \overline{n} = \overline{m} \\ * & \text{if } \overline{n} \neq \overline{m} \end{cases}$$

The tensor product is given by the disjoint union of finite sets and the tensor unit is the empty set $\overline{0}$. **Sym** is a symmetric monoidal category with braiding

$$\tau_{n,m} : \overline{n} \amalg \overline{m} \to \overline{m} \amalg \overline{n}$$

which is the canonical morphism in S_{n+m} that shuffles the first n elements past the remaining m elements.

(iii) The category **Orth** whose objects are finite dimensional real inner product spaces $(V, \langle -, - \rangle)$ and whose morphisms are given by linear isometric isomorphisms, i.e.

$$\mathbf{Orth}(V, W) = \begin{cases} O(V)_+ & \text{if } \dim(V) = \dim(W) \\ * & \text{else} \end{cases}$$

The tensor product is given by the direct sum and the tensor unit by the zero vector space. **Orth** is a symmetric monoidal category with braiding

$$\tau_{V,W} : V \oplus W \to W \oplus V$$

given by the canonical orthogonal transformation.

(iv) The full subcategory $\mathbf{Top}^*_{cg,fin} \hookrightarrow \mathbf{Top}^*_{cg}$ with objects given by pointed compactly generated spaces admitting a structure of a finite CW-complex. The symmetric monoidal structure is given by the ordinary smash product of pointed topological spaces, with unit object S^0 and braiding

$$\tau_{X,Y} : X \wedge Y \xrightarrow{\cong} Y \wedge X$$

given by the canonical homeomorphism.

Remark 2.10. There is a sequence of faithful subcategory inclusions

$$\begin{array}{ccccccc} \mathbf{Seq} & \hookrightarrow & \mathbf{Sym} & \hookrightarrow & \mathbf{Orth} & \hookrightarrow & \mathbf{Top}^*_{cg,fin} \\ n & \longmapsto & \overline{n} & \longmapsto & \mathbb{R}^n & \longmapsto & S^n \end{array}$$

where the last inclusion is given by the one-point compactification, normally denoted by $V \mapsto S^V$.
As these categories are all possible source categories for the functor category $[\mathcal{C}, \mathbf{Top}^*_{cg}]$, this sequence of inclusions should induce a sequence of functors on those functor categories.

Proposition 2.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a lax monoidal functor between pointed topological monoidal categories. Then the induced functor*

$$\begin{aligned} F^* : [\mathcal{D}, \mathbf{Top}^*_{cg}] &\longrightarrow [\mathcal{C}, \mathbf{Top}^*_{cg}] \\ X &\longmapsto X \circ F \end{aligned}$$

preserves monoid objects under the Day convolution tensor product. Hence there is a functor

$$F^* : \mathrm{Mon}([\mathcal{D}, \mathbf{Top}^*_{cg}]) \to \mathrm{Mon}([\mathcal{C}, \mathbf{Top}^*_{cg}])$$

Moreover, for any fixed monoid object $A \in \mathrm{Mon}([\mathcal{D}, \mathbf{Top}^*_{cg}])$ there is a functor

$$F^* : A\text{-Mod}([\mathcal{D}, \mathbf{Top}^*_{cg}]) \longrightarrow F^*(A)\text{-Mod}([\mathcal{C}, \mathbf{Top}^*_{cg}])$$

Proof. Using Proposition 2.8 we have that

$$\mathrm{Mon}\left([\mathcal{D},\mathbf{Top}^*_{cg}], \underset{\mathrm{Day}}{\otimes}, y_1\right) \simeq \mathrm{MonFunc}(\mathcal{D},\mathbf{Top}^*_{cg})$$

the category of monoids is equivalent to the corresponding category of lax monoidal functors. Since the induced functor F^* is just composition with F, where F is lax monoidal by assumption. We now use the fact that the composition of lax monoidal functors yields again a lax monoidal functor. Hence monoid objects are sent to monoid objects. The same arguments also show that for a fixed monoid object A, the functor F^* "preserves" A-module objects. □

Using the above proposition we obtain a sequence of restriction functors

$$\mathrm{Exc}(\mathbf{Top}^*_{cg}) \xrightarrow{\mathrm{orth}^*} [\mathbf{Orth},\mathbf{Top}^*_{cg}] \xrightarrow{\mathrm{sym}^*} [\mathbf{Sym},\mathbf{Top}^*_{cg}] \xrightarrow{\mathrm{seq}^*} [\mathbf{Seq},\mathbf{Top}^*_{cg}]$$

where we denote with $\mathrm{Exc}(\mathbf{Top}^*_{cg}) = [\mathbf{Top}^*_{cg,fin},\mathbf{Top}^*_{cg}]$ the category of **pre-excisive functors**. The categories $[\mathbf{Orth},\mathbf{Top}^*_{cg}]$, $[\mathbf{Sym},\mathbf{Top}^*_{cg}]$ and $[\mathbf{Seq},\mathbf{Top}^*_{cg}]$ are called the categories of orthogonal, symmetric and sequential sequences. Having in mind that we want to model spectra using these functor categories, we define the corresponding sphere spectra as follows.

$$\begin{aligned}\mathbb{S}_{\mathrm{exc}} &:= \mathbf{Top}^*_{cg,fin}(S^0,-)\\ \mathbb{S}_{\mathrm{orth}} &:= \mathrm{orth}^*\left(\mathbb{S}_{\mathrm{exc}}\right)\\ \mathbb{S}_{\mathrm{sym}} &:= \mathrm{sym}^*\left(\mathbb{S}_{\mathrm{orth}}\right)\\ \mathbb{S}_{\mathrm{seq}} &:= \mathrm{seq}^*\left(\mathbb{S}_{\mathrm{sym}}\right)\end{aligned}$$

Notice that by definition the excisive sphere spectrum $\mathbb{S}_{\mathrm{exc}}$ is the tensor unit, whereas $\mathbb{S}_{\mathrm{orth}}$, $\mathbb{S}_{\mathrm{sym}}$ and $\mathbb{S}_{\mathrm{seq}}$ are *not* the tensor units in their corresponding sequence categories.

Proposition 2.10. ([nLa19]) *The functors* seq, sym *and* orth *are strong monoidal functors when equipped with the following canonical isomorphisms*

(i) $\mathrm{seq}(n) \amalg \mathrm{seq}(m) = \{1,...,n\} \amalg \{1,...,m\} \cong \{1,...,m+n\} = \mathrm{seq}(n+m)$

(ii) $\mathrm{sym}(\overline{n}) \oplus \mathrm{sym}(\overline{m}) = \mathbb{R}^n \oplus \mathbb{R}^m \cong \mathbb{R}^{n+m} = \mathrm{sym}(\overline{n} \amalg \overline{m})$

(iii) $\mathrm{orth}(V) \wedge \mathrm{orth}(W) = S^V \wedge S^W \simeq S^{V\oplus W} = \mathrm{orth}(V \oplus W)$

Moreover, the functors sym *and* orth *are braided monoidal, whereas the functor* seq *is not braided monoidal.*

Now using the fact that $\mathbb{S}_{exc}$ is the tensor unit we have by Lemma 2.2 that there is an equivalence of categories

$$\mathrm{Exc}(\mathbf{Top}^*_{cg}) \simeq \mathbb{S}_{\mathrm{exc}}\text{-Mod}(\mathrm{Exc}(\mathbf{Top}^*_{cg}))$$

By Proposition 2.10 seq, sym and orth are strong monoidal, hence by applying Proposition 2.9 we get induced functors

$$\begin{aligned}\mathrm{orth}^* &: \mathbb{S}_{\mathrm{exc}}\text{-Mod} \to \mathbb{S}_{\mathrm{orth}}\text{-Mod}\\ \mathrm{sym}^* &: \mathbb{S}_{\mathrm{orth}}\text{-Mod} \to \mathbb{S}_{\mathrm{sym}}\text{-Mod}\\ \mathrm{seq}^* &: \mathbb{S}_{\mathrm{sym}}\text{-Mod} \to \mathbb{S}_{\mathrm{seq}}\text{-Mod}\end{aligned}$$

Notice that the sphere spectra are all monoid objects in their corresponding sequence categories. Hence it makes sense taking module objects over them. Moreover, it follows that $\mathbb{S}_{\mathrm{exc}}$, $\mathbb{S}_{\mathrm{orth}}$ and $\mathbb{S}_{\mathrm{sym}}$ are commutative monoids since the corresponding subcategory inclusions are braided monoidal.

Now we want to use the above categories of module objects to formalize spectra.

Lemma 2.3. *Using the identification of Proposition 2.8 the monoid objects* $\mathbb{S}_{\text{seq}}$*,* $\mathbb{S}_{\text{sym}}$ *and* $\mathbb{S}_{\text{orth}}$ *have the following representation as monoidal functors*

$$\begin{aligned}\mathbb{S}_{\text{seq}} : \mathbf{Seq} &\longrightarrow \mathbf{Top}^*_{cg} \\ n &\longmapsto S^n\end{aligned} \qquad \begin{aligned}\mathbb{S}_{\text{sym}} : \mathbf{Sym} &\longrightarrow \mathbf{Top}^*_{cg} \\ \overline{n} &\longmapsto S^n\end{aligned} \qquad \begin{aligned}\mathbb{S}_{\text{orth}} : \mathbf{Orth} &\longrightarrow \mathbf{Top}^*_{cg} \\ V &\longmapsto S^V\end{aligned}$$

Proof. Follow the identification of monoid objects with monoidal functors given by Proposition 2.8. □

Recall the definition of the category SeqSpec($\mathbf{Top}^*_{cg}$) given in Definition 1.2. In the following proposition we show that sequential spectra are indeed module objects over the sequential sphere spectrum in the category of sequential sequences $[\mathbf{Seq}, \mathbf{Top}^*_{cg}]$.

Proposition 2.11. *There is an equivalence of categories*

$$(-)^{\text{seq}} : \mathbb{S}_{\text{seq}}\text{-Mod} \simeq \text{SeqSpec}(\mathbf{Top}^*_{cg})$$

Proof. On objects the functor $(-)^{\text{seq}}$ is defined as

$$X \mapsto X^{\text{seq}}$$

where the component spaces are given by

$$(X^{\text{seq}})_n = X(n)$$

Since X is a module object over $\mathbb{S}_{\text{seq}}$ there is an action on X given by

$$\rho : \mathbb{S}_{\text{seq}} \underset{\text{Day}}{\otimes} X \to X$$

Using the universal property of the Day convolution from Corollary 2.1 we obtain

$$\begin{gathered}[\mathbf{Seq}, \mathbf{Top}^*_{cg}](\mathbb{S}_{\text{seq}} \underset{\text{Day}}{\otimes} X, X) \cong [\mathbf{Seq} \times \mathbf{Seq}, \mathbf{Top}^*_{cg}](\mathbb{S}_{\text{seq}} \bar{\wedge} X, X \circ +) \\ \rho \longmapsto \tilde{\rho}\end{gathered}$$

Therefore, given a pair $(m, n) \in \mathbf{Seq} \times \mathbf{Seq}$ there are maps

$$\mathbb{S}_{\text{seq}}(m) \wedge X(n) \to X(m+n)$$

Taking $m = 1$ and applying Lemma 2.3, we obtain the desired structure maps.

$$S^1 \wedge X(n) \to X(n+1)$$

On the other hand we now need to show that any $\mathbb{S}_{\text{seq}}$-action arises from structure maps of the form

$$S^1 \wedge X(n) \to X(n+1)$$

Therefore, assume we are given structure maps

$$\sigma_n : S^1 \wedge X(n) \to X(n+1)$$

Then by the following diagram

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X(n) & \xrightarrow{\text{id} \wedge \sigma_n} & S^1 \wedge X(n+1) \\ \simeq \downarrow & & \downarrow \sigma_{n+1} \\ S^2 \wedge X(n) & \overset{}{\underset{\rho_{(2,n)}}{\dashrightarrow}} & X(n+2) \end{array}$$

there is a unique morphism $\rho_{(2,n)}$ such that the diagram commutes. By an inductive argument there is a unique family $\rho_{(m,n)} : S^m \wedge X(n) \to X(n+m)$ of morphisms such that the following diagram commutes

$$\begin{array}{ccc} S^{n_1} \wedge S^{n_2} \wedge X(n_3) & \xrightarrow{\mathrm{id}\wedge\rho_{(n_2,n_3)}} & S^{n_1} \wedge X(n_2+n_3) \\ \simeq\downarrow & & \downarrow\rho_{(n_1,n_2+n_3)} \\ S^{n_1+n_2} \wedge X(n_3) & \xrightarrow[\rho_{(n_1+n_2,n_3)}]{} & X(n_1+n_2+n_3) \end{array}$$

Using the characterization of the sphere spectrum as a monoidal functor, i.e $\mathbb{S}_{\mathrm{seq}}(m) = S^m$, the above constructed family $\{\rho_{(m,n)}\}$ induces a morphism $\rho \in [\mathbf{Seq} \times \mathbf{Seq}, \mathbf{Top}^*_{cg}](\mathbb{S}_{\mathrm{seq}} \bar{\wedge} X, X \circ +)$ which then gives rise to an action

$$\rho : \mathbb{S}_{\mathrm{seq}} \underset{\mathrm{Day}}{\otimes} X \to X$$

using the isomorphism of Corollary 2.1. The fact that the above diagram commutes shows that (X, ρ) satisfies the action property and is therefore a module object over the sphere spectrum.

□

This proposition motivates the following definition.

Definition 2.13. We define the following categories.

(i) The category of **orthogonal spectra** given by $\mathrm{OrthSpec}(\mathbf{Top}^*_{cg}) := \mathbb{S}_{\mathrm{orth}}\text{-Mod}$.

(ii) The category of **symmetric spectra** given by $\mathrm{SymSpec}(\mathbf{Top}^*_{cg}) := \mathbb{S}_{\mathrm{sym}}\text{-Mod}$.

This abstract characterization of spectra allows us to define the smash product of orthogonal and symmetric spectra.

Let (A, μ, ε) be a commutative monoid in the closed symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ and consider $(N_1, \rho_1), (N_2, \rho_2) \in A\text{-Mod}(\mathcal{C})$ two module objects over A. Then there are maps

$$\begin{array}{ccccc} N_1 \otimes A \otimes N_2 & & \xrightarrow{\mathrm{id}\otimes\rho_2} & & N_1 \otimes N_2 \\ & \searrow_{\tau_{(N_1,A)}\otimes\mathrm{id}} & & \nearrow_{\rho_1\otimes\mathrm{id}} & \\ & & A \otimes N_1 \otimes N_2 & & \end{array}$$

We define the tensor product of N_1 and N_2 over A as the coequalizer of

$$N_1 \otimes A \otimes N_2 \underset{(\rho_1\circ\tau_{(N_1,A)})\otimes\mathrm{id}}{\overset{\mathrm{id}\otimes\rho_2}{\rightrightarrows}} N_1 \otimes N_2 \xrightarrow{coeq.} N_1 \otimes_A N_2$$

Proposition 2.12. ([HSS98]) *Let $\mathcal{C}$ and A be as above, then if all coequalizers exist in $\mathcal{C}$ the bifunctor*

$$\otimes_A : A\text{-Mod}(\mathcal{C}) \times A\text{-Mod}(\mathcal{C}) \to A\text{-Mod}(\mathcal{C})$$

makes A-Mod($\mathcal{C}$) into a symmetric monoidal category with tensor unit A. Moreover, if all equalizers exist in $\mathcal{C}$, then the monoidal structure is closed, with internal hom given by $\hom_A$

Remark 2.11. Recall that the monoidal category $\mathcal{C}$ is closed with internal hom given by $\hom(x, y)$. Then the internal hom in the category A-Mod($\mathcal{C}$) is defined as the equalizer

$$\hom_A(N_1, N_2) \xrightarrow{eq.} \hom(N_1, N_2) \rightrightarrows \hom(A \otimes N_1, N_2)$$

where the upper morphism is given by

$$\hom(\rho_1, N_2) : \hom(N_1, N_2) \to \hom(A \otimes N_1, N_2)$$

and the lower is given by the composition

$$\hom(N_1, N_2) \xrightarrow{\hom(N_1,\varphi)} \hom(N_1, \hom(A, A \otimes N_2)) \xrightarrow{\sim} \hom(A \otimes N_1, A \otimes N_2) \xrightarrow{\hom(A\otimes N_1,\rho_2)} \hom(A \otimes N_1, N_2)$$

where the map φ is the image under the adjunction isomorphism of the identity map $\mathrm{id}_{A\otimes N_2}$.

Using the fact that $\mathbb{S}_{\mathrm{sym}}$ and $\mathbb{S}_{\mathrm{orth}}$ are both commutative monoids, Proposition 2.12 implies that the categories $\mathrm{OrthSpec}(\mathbf{Top}^*_{cg})$ and $\mathrm{SymSpec}(\mathbf{Top}^*_{cg})$ are both equipped with a closed symmetric monoidal tensor product. The unit object is then given by the corresponding sphere spectrum. This tensor product will be called the smash product of spectra.

Definition 2.14. The categories $\mathrm{OrthSpec}(\mathbf{Top}^*_{cg})$ and $\mathrm{SymSpec}(\mathbf{Top}^*_{cg})$ both carry a closed symmetric monoidal tensor product, called the **smash product of spectra**, denoted by

$$\begin{aligned} \wedge : \mathrm{SymSpec}(\mathbf{Top}^*_{cg}) \times \mathrm{SymSpec}(\mathbf{Top}^*_{cg}) &\longrightarrow \mathrm{SymSpec}(\mathbf{Top}^*_{cg}) \\ (X, Y) &\longmapsto X \otimes_{\mathbb{S}_{\mathrm{sym}}} Y \end{aligned}$$

$$\begin{aligned} \wedge : \mathrm{OrthSpec}(\mathbf{Top}^*_{cg}) \times \mathrm{OrthSpec}(\mathbf{Top}^*_{cg}) &\longrightarrow \mathrm{OrthSpec}(\mathbf{Top}^*_{cg}) \\ (X, Y) &\longmapsto X \otimes_{\mathbb{S}_{\mathrm{orth}}} Y \end{aligned}$$

3 The strict and stable model category of spectra

In this chapter we continue following the lecture notes "Introduction to Stable Homotopy Theory" [nLa19] in order to endow the category of spectra with the strict and the stable model structure.

3.1 The strict model category of spectra

Following Definition 1.2 there is a natural way to endow the category $\mathrm{SeqSpec}(\mathbf{Top}^*_{cg})$ with a model structure. On the other hand there is also a natural way to endow the enriched functor category $[\mathcal{C}, \mathbf{Top}^*_{cg}]$ with a model structure, called the projective model structure. The aim of this section is to show that these two model structures agree under the equivalence established in Proposition 2.11.

Definition 3.1. Let $f \in \mathrm{SeqSpec}(\mathbf{Top}^*_{cg})$ be a morphism of spectra. Then we say that f is

(i) a **strict weak equivalence** if each component $f_n : X_n \to Y_n$ is a weak equivalence in the classical model structure on $\mathbf{Top}^*_{cg}$.

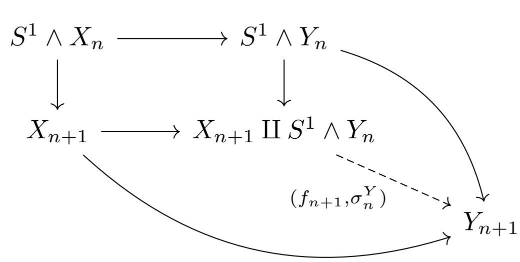
(ii) a **strict fibration** if each component f_n is a fibration in the classical model structure on $\mathbf{Top}^*_{cg}$.

(iii) a **strict cofibration** if the maps

$$\begin{aligned} f_0 : X_0 &\longrightarrow Y_0 \\ (f_{n+1}, \sigma^Y_n) : X_{n+1} \amalg (S^1 \wedge Y_n) &\longrightarrow Y_{n+1} \end{aligned}$$

are cofibrations in the classical model structure on $\mathbf{Top}^*_{cg}$.

Remark 3.1. The map (f_{n+1}, σ^Y_n) is given by the universal property of the pushout



Theorem 3.1. *The classes of morphisms* W_{strict}*,* Fib_{strict} *and* Cof_{strict} *endow the category* $\mathrm{SeqSpec}(\mathbf{Top}^*_{cg})$ *with a model structure, which will be called the* ***strict model structure on sequential spectra****.*

Proof. See Theorem 3.3 □

Proposition 3.1. *A sequential spectrum* $X \in \mathrm{SeqSpec}(\mathbf{Top}^*_{cg})$ *is cofibrant in the strict model structure precisely if*

(i) X_0 *is cofibrant in* $\mathbf{Top}^*_{cg}$

(ii) *for all* $n \in \mathbb{N}$ *the map* $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ *is a cofibration in* $\mathbf{Top}^*_{cg}$

In particular CW-Spectra are cofibrant.

Proof. This follows directly from Definition 3.1. □

Lemma 3.1. *In the strict model structure on* $\mathrm{SeqSpec}(\mathbf{Top}^*_{cg})$ *every object* X *is fibrant.*

Proof. This follows directly from Definition 3.1. □

By Proposition 2.11 we can identify sequential spectra as module objects in the category $[\mathbf{Seq}, \mathbf{Top}^*_{cg}]$. For functor categories whose target categories are endowed with a model structure there are two canonical model structures, the projective and the injective model structure. Hence, it seems straightforward, that we endow the category $\mathrm{SeqSpec}(\mathbf{Top}^*_{cg})$ with such a model structure. The question that arises at this point is, if it will be equivalent to the strict model structure defined in Theorem 3.1.

Remark 3.2. Recall that an **excellent model category** $\mathcal{S}$ is a model category equipped with a symmetric monoidal structure satisfying the following conditions.

1. The model category $\mathcal{S}$ is combinatorial.

2. Every monomorphism in $\mathcal{S}$ is a cofibration, and the collection of cofibrations is stable under products.

3. The collection of weak equivalences in $\mathcal{S}$ is stable under filtered colimits.

4. The symmetric monoidal structure
$$\otimes : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$$
is a Quillen bifunctor.

5. The model category $\mathcal{S}$ satisfies the invertibility hypothesis (see [Lur09]).

Notice that we introduce the notion of an excellent model category because of technical reasons. That is, we only use the fact that the category **sSet** is an excellent model category, such that we can show the existence of the projective model structure on enriched diagrams in **sSet**.

Definition 3.2. Let $\mathcal{S}$ be an excellent model category, $\mathcal{A}$ an $\mathcal{S}$-enriched cofibrantly generated model category and $\mathcal{C}$ a small $\mathcal{S}$-enriched category. A natural transformation $\eta : F \to G$ in the enriched functor category $[\mathcal{C}, \mathcal{A}]$ is said to be a

(i) **projective weak equivalence** if the induced map $F(x) \to G(x)$ is a weak equivalence in $\mathcal{A}$ for all $x \in \mathcal{C}$.

(ii) **projective fibration** if the induced map $F(x) \to G(x)$ is a fibration in $\mathcal{A}$ for all $x \in \mathcal{C}$.

(iii) **projective cofibration** if it has the left lifting property with respect to every morphism α in $[\mathcal{C}, \mathcal{A}]$ which is simultaneously a weak equivalence and a projective fibration.

Proposition 3.2. ([Lur09]) *Let $\mathcal{S}$, $\mathcal{A}$ and $\mathcal{C}$ be as in Definition 3.2. Then there is a model structure on $[\mathcal{C}, \mathcal{A}]$ which is determined by the weak equivalences, projective fibrations and projective cofibrations. This model structure is called the* ***projective model structure*** *on $[\mathcal{C}, \mathcal{A}]$.*

In particular we have that the category **sSet** endowed with the standard model structure and the Cartesian product is an excellent model category. Hence it follows that also the model category $\mathbf{sSet}^*$ of pointed simplicial sets endowed with the smash product defines an excellent model category. On the other hand we have that the Quillen model structure on $\mathbf{Top}^*_{cg}$ does not have the property of a combinatorial model structure. Nevertheless, also for diagrams in $\mathbf{Top}^*_{cg}$ there exists a projective model structure.

Theorem 3.2. ([Pia91]) *Let $\mathcal{C}$ be a small category enriched over $\mathbf{Top}^*_{cg}$ and consider the category of topological functors $[\mathcal{C}, \mathbf{Top}^*_{cg}]$. Then the projective model structure on $[\mathcal{C}, \mathbf{Top}^*_{cg}]$ exists. More precisely, the classes of weak equivalences, projective fibrations and projective cofibrations given by Definition 3.2 define a model structure.*

To use Proposition 3.2 to define a model structure on SeqSpec($\mathbf{Top}^*_{cg}$) we need to identify the category of module objects over the sphere spectrum with a functor category.

Proposition 3.3. *Let (A, μ, ε) be a monoid in $(\mathcal{C}, \otimes, 1)$. Then A canonically becomes a left module over itself by taking $\rho = \mu$. More generally, for $X \in \mathcal{C}$ any object, we have that $A \otimes X$ naturally becomes a left A-module object by setting*

$$\rho : A \otimes (A \otimes X) \xrightarrow{\alpha^{-1}_{(A,A,X)}} (A \otimes A) \otimes X \xrightarrow{\mu \otimes \mathrm{id}} A \otimes X$$

The A-modules of this form are called ***free modules*** *and the corresponding free functor $F : \mathcal{C} \to A\text{-Mod}(\mathcal{C})$ is left adjoint to the forgetful functor U, i.e.*

$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} A\text{-Mod}(\mathcal{C}) \quad (\bot)$$

Proof. We need to show that for any object $X \in \mathcal{C}$ and any A-module object N, there is a natural isomorphism

$$A\text{-Mod}(\mathcal{C})(F(X), N) \cong \mathcal{C}(X, U(N))$$

Hence we consider a morphism f out of a free A-module. That is, a morphism in $\mathcal{C}$

$$f : A \otimes X \to N$$

such that the following diagram commutes.

$$\begin{array}{ccc} A \otimes A \otimes X & \xrightarrow{\mathrm{id} \otimes f} & A \otimes N \\ {\scriptstyle \mu \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \rho} \\ A \otimes X & \xrightarrow[f]{} & N \end{array}$$

To obtain a map $\tilde{f} : X \to N$ consider the following composition.

$$\tilde{f} : X \xrightarrow{l_X} 1 \otimes X \xrightarrow{\varepsilon \otimes \mathrm{id}} A \otimes X \xrightarrow{f} N$$

This composition then defines the natural map

$$\begin{aligned} \Psi : A\text{-Mod}(\mathcal{C})(F(X), N) &\to \mathcal{C}(X, U(N)) \\ f : A \otimes X \to N &\mapsto \tilde{f} : X \to N \end{aligned}$$

By definition the map $\tilde{f}$ fits into the following commutative diagram.

$$
\begin{array}{ccc}
A \otimes X & \xrightarrow{\mathrm{id}\otimes \tilde{f}} & A \otimes N \\
{\scriptstyle \mathrm{id}\otimes l_X}\big\downarrow & & \big\downarrow \\
A \otimes 1 \otimes X & & = \\
{\scriptstyle \mathrm{id}\otimes\varepsilon\otimes\mathrm{id}}\big\downarrow & & \big\downarrow \\
A \otimes A \otimes X & \xrightarrow[\mathrm{id}\otimes f]{} & A \otimes N
\end{array}
$$

Pasting both diagrams on top of each other yields the following diagram.

$$
\begin{array}{ccc}
A \otimes X & \xrightarrow{\mathrm{id}\otimes \tilde{f}} & A \otimes N \\
\big\downarrow & & \big\downarrow{\scriptstyle =} \\
A \otimes A \otimes X & \xrightarrow{\mathrm{id}\otimes f} & A \otimes N \\
{\scriptstyle \mu\otimes\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\
A \otimes X & \xrightarrow[f]{} & N
\end{array}
$$

By the unit law it follows that the left vertical composition is the identity. Therefore, we conclude by the commutativity of the diagram

$$f = \rho \circ (\mathrm{id}_A \otimes \tilde{f})$$

that f is uniquely determined by $\tilde{f}$. Therefore, the natural map Ψ is indeed an isomorphism. □

Remark 3.3. Recall that by Proposition 2.7 there is an embedding

$$(\mathcal{C}^{\mathrm{op}}, \otimes, 1) \to \left([\mathcal{C}, \mathbf{Top}^*_{cg}], \underset{\mathrm{Day}}{\otimes}, y_1 \right)$$

Hence for $A \in [\mathcal{C}, \mathbf{Top}^*_{cg}]$ a monoid object we can define the category A-FreeMod$_{\mathcal{C}}$ of free modules over A on objects in $\mathcal{C}$ under the Yoneda embedding. That is, the category with free left A modules of the form $A \underset{\mathrm{Day}}{\otimes} y_c$ for $c \in \mathcal{C}$ as objects and $A\text{-FreeMod}_{\mathcal{C}}(c_1, c_2) = A\text{-Mod}\left(A \underset{\mathrm{Day}}{\otimes} y_{c_1}, A \underset{\mathrm{Day}}{\otimes} y_{c_2} \right)$ as morphisms.

Proposition 3.4. ([MMSS01]) *Let $A \in [\mathcal{C}, \mathbf{Top}^*_{cg}]$ be a monoid object. Then the category of left module objects over A in $[\mathcal{C}, \mathbf{Top}^*_{cg}]$ is equivalent to the enriched functor category with source category A-FreeMod$_{\mathcal{C}}$*

$$A\text{-Mod}([\mathcal{C}, \mathbf{Top}^*_{cg}]) \simeq \left[A\text{-FreeMod}_{\mathcal{C}}, \mathbf{Top}^*_{cg}\right]$$

Using the above proposition we can now represent the categories $\mathrm{SeqSpec}(\mathbf{Top}^*_{cg}), \mathrm{SymSpec}(\mathbf{Top}^*_{cg})$ and $\mathrm{OrthSpec}(\mathbf{Top}^*_{cg})$ as enriched functor categories. Therefore, we use Proposition 3.2 in the simplicial setting to endow them with a model structure, whereas in the topological setting we use Theorem 3.2. Indeed we have the following equivalences of categories

$$
\begin{aligned}
\mathrm{SeqSpec}(\mathbf{Top}^*_{cg}) &= \mathbb{S}_{\mathrm{seq}}\text{-Mod} \simeq \left[\mathbb{S}_{\mathrm{seq}}\text{-FreeMod}_{\mathbf{Seq}}, \mathbf{Top}^*_{cg}\right] \\
\mathrm{SymSpec}(\mathbf{Top}^*_{cg}) &= \mathbb{S}_{\mathrm{sym}}\text{-Mod} \simeq \left[\mathbb{S}_{\mathrm{sym}}\text{-FreeMod}_{\mathbf{Sym}}, \mathbf{Top}^*_{cg}\right] \\
\mathrm{OrthSpec}(\mathbf{Top}^*_{cg}) &= \mathbb{S}_{\mathrm{orth}}\text{-Mod} \simeq \left[\mathbb{S}_{\mathrm{orth}}\text{-FreeMod}_{\mathbf{Orth}}, \mathbf{Top}^*_{cg}\right]
\end{aligned}
$$

To apply Proposition 3.2 and Theorem 3.2 to these enriched functor categories respectively, we need to check that $\mathbb{S}_\star$-FreeMod$_\star$ are small $\mathbf{sSet}^*$ resp. $\mathbf{Top}^*_{cg}$-enriched categories for $\star \in \{\mathrm{Seq}, \mathrm{Sym}, \mathrm{Orth}\}$. Indeed, the categories $\mathbb{S}_\star$-FreeMod$_\star$ are $\mathbf{Top}^*_{cg}$-enriched by definition and small, since the objects are identified with those in the corresponding small categories **Seq**, **Sym** and **Orth**. In the simplicial setting, the categories **Seq**, **Sym** and **Orth** can be enriched accordingly.

Theorem 3.3. *The categories of*

(i) *pre-excisive functors* $\mathrm{Exc}(\mathbf{Top}^*_{cg})$ *resp.* $\mathrm{Exc}(\mathbf{sSet}^*)$

(ii) *orthogonal spectra* $\mathrm{OrthSpec}(\mathbf{Top}^*_{cg})$ *resp.* $\mathrm{OrthSpec}(\mathbf{sSet}^*)$

(iii) *symmetric spectra* $\mathrm{SymSpec}(\mathbf{Top}^*_{cg})$ *resp.* $\mathrm{SymSpec}(\mathbf{sSet}^*)$

(iv) *sequential spectra* $\mathrm{SeqSpec}(\mathbf{Top}^*_{cg})$ *resp.* $\mathrm{SeqSpec}(\mathbf{sSet}^*)$

each admit a projective model category structure whose weak equivalences and fibrations are those morphisms who induce weak equivalences or fibrations on all component spaces in the classical model structure on $\mathbf{Top}^*_{cg}$ *or on* $\mathbf{sSet}^*$ *respectively. These model structures are called the* ***strict model structures of spectra****. Moreover, there is a diagram of Quillen pairs, given by the adjoint pairs* $\mathrm{orth}! \dashv \mathrm{orth}^*$, $\mathrm{sym}! \dashv \mathrm{sym}^*$ *and* $\mathrm{seq}! \dashv \mathrm{seq}^*$.

$$\mathbb{S}_{\mathrm{exc}}\text{-}\mathrm{Mod}_{\mathrm{strict}} \underset{\mathrm{orth}!}{\overset{\mathrm{orth}^*}{\rightleftarrows}} \mathbb{S}_{\mathrm{orth}}\text{-}\mathrm{Mod}_{\mathrm{strict}} \underset{\mathrm{sym}!}{\overset{\mathrm{sym}^*}{\rightleftarrows}} \mathbb{S}_{\mathrm{sym}}\text{-}\mathrm{Mod}_{\mathrm{strict}} \underset{\mathrm{seq}!}{\overset{\mathrm{seq}^*}{\rightleftarrows}} \mathbb{S}_{\mathrm{seq}}\text{-}\mathrm{Mod}_{\mathrm{strict}}$$

Proof. In the case where we consider spectra in pointed simplicial sets, the existence of the model structures is given in by Proposition 3.2 using that $\mathbf{sSet}^*$ is an excellent model category and the fact that there are equivalences of categories

$$\mathbb{S}_\star\text{-Mod} \simeq [\mathbb{S}_\star\text{-FreeMod}_\star, \mathbf{sSet}^*]$$

for $\star$ in $\{\mathrm{Seq}, \mathrm{Sym}, \mathrm{Orth}, \mathrm{Exc}\}$. In the case where we consider spectra with component spaces in $\mathbf{Top}^*_{cg}$, the existence of the projective model structure is given by Theorem 3.2 and the fact that there are equivalences of categories

$$\mathbb{S}_\star\text{-Mod} \simeq \left[\mathbb{S}_\star\text{-FreeMod}_\star, \mathbf{Top}^*_{cg}\right]$$

for $\star$ in $\{\mathrm{Seq}, \mathrm{Sym}, \mathrm{Orth}, \mathrm{Exc}\}$. Then by Definition 3.2, the fibrations and weak equivalences are those morphisms who induce fibrations and weak equivalences on component spaces in the model structure on $\mathbf{Top}^*_{cg}$. The existence of the adjoint functors seq!, sym! and orth! is given by topological/simplicial left Kan extension, i.e. for an object $X \in \mathbb{S}_{\mathrm{seq}}$ we have that $X : \mathbf{Seq} \to \mathbf{Top}^*_{cg}$ is a topological functor. Then consider the left Kan extension of X along $\mathrm{seq} : \mathbf{Seq} \to \mathbf{Sym}$

$$\begin{array}{ccc} \mathbf{Seq} & \xrightarrow{X} & \mathbf{Top}^*_{cg} \\ {\scriptstyle \mathrm{seq}}\downarrow & \nearrow_{\mathrm{Lan}_{\mathrm{seq}}(X)} & \\ \mathbf{Sym} & & \end{array}$$

Hence we can define the adjoint functor.

$$\begin{aligned} \mathrm{seq}! : \mathbb{S}_{\mathrm{seq}}\text{-Mod} &\longrightarrow \mathbb{S}_{\mathrm{sym}}\text{-Mod} \\ X &\longmapsto \mathrm{Lan}_{\mathrm{seq}}(X) \end{aligned}$$

Similarly we define the functors sym! and orth!. These functors are Quillen pairs, since the three right adjoint restriction functors are defined along the inclusions of Remark 2.10 and therefore preserve weak equivalences and fibrations. Then by Definition A.9 and Lemma 1.3.4 in [Hov99] it follows that the adjunction pairs are indeed Quillen pairs. □

Similarly as the Quillen model structure on $\mathbf{Top}^*_{cg}$ determines classical homotopy theory, we want to endow the categories of spectra with a suitable model structure. This model structure should then determine stable homotopy theory. More precisely, we want to have a notion of weak equivalences between spectra, having the property that the induced maps on stable homotopy groups are isomorphisms. Such a model structure is then called a stable model structure. In particular the suspension and loop space functors Σ and Ω induce equivalences on the stable homotopy category of spectra, motivating its name. It becomes clear that the strict model structure, only providing Quillen equivalences on the component spaces, does not have this property. Therefore, we consider the following definition.

Definition 3.3. Let $E \in \mathbb{S}_\star\text{-Mod}$ be a spectrum for $\star$ in $\{\mathrm{Seq}, \mathrm{Sym}, \mathrm{Orth}, \mathrm{Exc}\}$. Then define the **stable homotopy groups of** E to be

$$\pi_*(E) := \pi_*^S(\mathrm{seq}^*(E))$$

the stable homotopy groups of E regarded as a sequential spectrum. A map of spectra $f : E \to E'$ is then called a **stable weak equivalence**, if the induced maps on the stable homotopy groups are isomorphisms

$$\pi(f)_* : \pi(E)_* \xrightarrow{\cong} \pi(E')_*$$

Theorem 3.4. *The category of sequential spectra* $\mathrm{SeqSpec}(\mathbf{Top}^*_{cg})$ *admits a model structure with*

(i) *weak equivalences given by stable weak equivalences.*

(ii) *cofibrations given by the cofibrations in the strict model structure.*

(iii) *fibrant objects are given by* Ω*-spectra.*

This model structure is called the stable model structure on sequential spectra.

Proof. One way to show that such a model structure exists, is to take the left Bousfield localization of the strict model structure $\mathrm{SeqSpec}(\mathbf{Top}^*_{cg})_{\mathrm{strict}}$ with respect to the class of stable weak equivalences. The resulting model structure has precisely as weak equivalences the stable weak equivalences, and as cofibrations the cofibrations in the strict model structure. What is then left to show is that the fibrant objects are given by the Ω-spectra. The existence of the left Bousfield localization is given by Proposition 4.1.8. and Theorem 4.1.1. in [Hir03]. □

Remark 3.4. Similarly as for sequential spectra we can also endow the categories of structured spectra $\mathrm{SymSpec}(\mathbf{Top}^*_{cg})$ and $\mathrm{OrthSpec}(\mathbf{Top}^*_{cg})$ with a stable model structure. Also in this model structure the weak equivalences are given by the stable weak equivalences and the fibrant objects are given by the Ω-spectra. Moreover, we have that the homotopy theory of structured spectra is equivalent to the homotopy theory of sequential spectra. More precisely, there are Quillen equivalences

$$\mathrm{SeqSpec}(\mathbf{Top}^*_{cg})_{\mathrm{stable}} \underset{\mathrm{seq}^*}{\overset{\mathrm{seq}!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{OrthSpec}(\mathbf{Top}^*_{cg})_{\mathrm{stable}}$$

$$\mathrm{SeqSpec}(\mathbf{Top}^*_{cg})_{\mathrm{stable}} \underset{\mathrm{seq}^*}{\overset{\mathrm{seq}!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{SymSpec}(\mathbf{Top}^*_{cg})_{\mathrm{stable}}$$

A detailed discussion about the relations between the stable model categories of structured spectra can be found in [MMSS01].

4 Γ-spaces and the K-theory functor

To motivate the relation between spectra, Γ-spaces and Picard groupoids, we recall an important result by [FHT10] on the classification of invertible quantum field theories. Freed and Hopkins define in [FH16] an extended n-dimensional Topological Quantum Field Theory (TQFT) to be a symmetric monoidal functor from the (∞, n)-category of bordisms $\mathrm{Bord}_n(H_n)$ to a symmetric monoidal (∞, n)-category $\mathcal{C}$.

$$F : \mathrm{Bord}_n(H_n) \to \mathcal{C}$$

Here H_n denotes the symmetry group of the bordism category. We will not go into further details on the symmetry group or on the structure of the bordism category, as it should remain only a motivating example. A TQFT is said to be invertible, if it factors trough the higher Picard groupoids of the corresponding categories. First notice that both (∞, n)-categories are endowed with a symmetric monoidal structure. The category $\mathcal{C}$ by assumption and $\mathrm{Bord}_n(H_n)$ by disjoint union of manifolds. Hence an invertible TQFT allows the following decomposition

$$\begin{array}{ccc} \mathrm{Bord}_n(H_n) & \xrightarrow{F} & \mathcal{C} \\ \downarrow & & \uparrow \\ \overline{\mathrm{Bord}_n(H_n)} & \xrightarrow[\widetilde{F}]{} & \mathcal{C}^{\times} \end{array}$$

where $\overline{\mathrm{Bord}_n(H_n)}$ is the Picard groupoid obtained by adding formal inverses and $\mathcal{C}^{\times}$ the Picard groupoid obtained by removing the non-invertible morphisms. Therefore, classifying invertible TQFTs is equivalent to the classification of symmetric monoidal functors of Picard groupoids.

To classify such functors, it is convenient to first pass via the K-theory functor to Γ-categories and then pass to connective spectra via Segal's construction. Hence the classification problem of invertible TQFTs turns into a classification problem of maps of spectra, which can be approached with stable homotopy theory. Keeping this example in mind, we now show how we can pass from Picard groupoids to spectra via Γ-spaces and vice versa.

4.1 Γ-spaces

Following [Seg74] and [Boy19] we introduce Γ-spaces and show how they are related to Spectra. Whereas the approach of Boyarchenko is based on categorical constructions, such as Kan extensions, the approach of Segal is more of a constructive nature.

Definition 4.1. Let Γ be the category whose objects are finite sets and whose morphisms are are defined as follows. For $S, T \in \Gamma$, a morphism $S \to T$ is a map of sets $\theta : S \to P(T)$ such that $\theta(x) \cap \theta(y) = \emptyset$ whenever $x, y \in S$ with $x \neq y$.

This category is also called Segal's category.

Definition 4.2. Let $\mathcal{C}$ be a category. Then a **Γ-object** of $\mathcal{C}$ is a functor

$$\Gamma^{\mathrm{op}} \to \mathcal{C}$$

The **category of Γ-objects** of $\mathcal{C}$ is given by the functor category $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{C})$ and will be denoted by $\Gamma\mathcal{C}$.

Remark 4.1. While working with Γ-objects one is only interested in the category Γ^{op}, which is equivalent to the category of finite pointed sets, denoted by $\mathbf{FinSet}^*$. Hence Γ-objects can be regarded as functors

$$\mathbf{FinSet}^* \to \mathcal{C}$$

The definition of Γ-objects seems similar to the one of simplicial objects, which show to be a slightly weaker version of Γ-objects. Indeed, any Γ-object yields a simplicial object as follows. Consider the faithful functor

$$\begin{aligned} \iota : \Delta &\longrightarrow \Gamma \\ [m] &\longmapsto \underline{m} \end{aligned}$$

where $[m] = \underline{m} = \{1, ..., m\}$ and for $f : [m] \to [n]$ morphism in Δ the morphism $\iota(f) : \underline{m} \to \underline{n}$ is defined by

$$\begin{aligned} \theta : \underline{m} &\longrightarrow P(\underline{n}) \\ i &\longmapsto \theta(i) = \{j \mid f(i-1) < j \leq f(i)\} \end{aligned}$$

Then for $X : \Gamma^{\text{op}} \to \mathcal{C}$ we define its associated simplicial object as the composition

$$X^{\text{simp}} : \Delta^{\text{op}} \xrightarrow{\iota^{\text{op}}} \Gamma^{\text{op}} \xrightarrow{X} \mathcal{C}$$

Remark 4.2. In the following we will mostly look at Γ-objects in the categories $\mathbf{Top}^*_{cg}$, **sSet** and **Cat**.

Definition 4.3. Let $\mathcal{C}$ be a model category. Then a Γ-object $A : \Gamma^{\text{op}} \to \mathcal{C}$ is said to be a **special Γ-object** if for each $n \geq 0$ the morphisms

$$A(\underline{n}) \to \overbrace{A(\underline{1}) \times \ldots \times A(\underline{1})}^{n\text{-times}}$$

induced by the maps $A(\theta_k) : A(\underline{n}) \to A(\underline{1})$ where

$$\begin{aligned} \theta_k : \underline{1} &\to P(\underline{n}) \\ 1 &\to \{k\} \end{aligned}$$

are weak equivalences in the model structure on $\mathcal{C}$.

Remark 4.3. In the case $n = 0$ notice that for A to be a special Γ-space it is required that the map $A(\underline{0}) \to *$ is a weak equivalence, where $*$ is the terminal object in $\mathcal{C}$.

By endowing the category **Cat** with the canonical model structure we obtain the following definition.

Definition 4.4. A Γ-category $\mathcal{C} : \Gamma^{\text{op}} \to \mathbf{Cat}$ is said to be **special**, if

(i) $\mathcal{C}(\underline{0})$ is equivalent to the category with one object and one morphism

(ii) for each $n \geq 0$ the functor

$$P_n : \mathcal{C}(\underline{n}) \to \mathcal{C}(\underline{1}) \times \ldots \times \mathcal{C}(1)$$

is an equivalence of categories.

Corollary 4.1. ([Seg74]) *Let $\mathcal{C}$ be a special Γ-category. Then its classifying space $|\mathcal{C}|$ is a special Γ-space*

$$\begin{aligned} |\mathcal{C}| : \Gamma^{\text{op}} &\to \mathbf{Top} \\ S &\mapsto |N(\mathcal{C}(S))| \end{aligned}$$

Lemma 4.1. *Let $A : \Gamma^{\text{op}} \to \mathbf{Top}$ be a special Γ-space. Then the space $A(\underline{1})$ has an H-space structure.*

Proof. Since A is special we have that the map $A(\underline{2}) \to A(\underline{1}) \times A(\underline{1})$ is a homotopy equivalence, hence we define the product on $A(\underline{1})$ as follows.

$$A(\underline{1}) \times A(\underline{1}) \to A(\underline{2}) \to A(\underline{1})$$

Notice that the right map is induced by $\underline{1} \to \underline{2}$ in Γ given by $\theta : 1 \mapsto \{1, 2\}$. The neutral element is given by the unique map $\underline{0}_+ \to \underline{1}_+$ in the category $\mathbf{FinSet}^*$ which then induces an inclusion $A(\underline{0}) \to A(\underline{1})$. By hypothesis $A(\underline{0}) \simeq *$ hence we take the neutral element in $A(\underline{1})$ to be the image of the base point $*$ under the above inclusion. □

Remark 4.4. Let X be a topological H-space. Then we say that X allows a weak homotopy inverse if $\pi_0(X)$ has a group structure induced by the H-space structure.

Definition 4.5.

(i) A topological Γ-space A is said to be **very special**, if it is special and the H-space structure on $A(\underline{1})$ admits a weak homotopy inverse.

(ii) A Γ-space $A : \Gamma^{\text{op}} \to \mathbf{sSet}^*$ is said to be **very special**, if it is special and its geometric realization $|A|$ is very special, i.e. the H-space structure on $|A(\underline{1})|$ admits a weak homotopy inverse.

(iii) A Γ-category $\mathcal{C} : \Gamma^{\text{op}} \to \mathbf{Cat}$ is said to be **very special**, if it is special and its classifying space $|\mathcal{C}|$ is very special, i.e. the H-space structure on $|N(\mathcal{C}(\underline{1}))|$ admits a weak homotopy inverse.

Now that we have established the basic properties of Γ-spaces we want to investigate how Γ-spaces and sequential spectra are related. The aim will be to construct a pair of adjoint functors $\text{Hom}(\mathbb{S}, -)$ and $\mathbb{S} \wedge -$ between the categories $\text{SeqSpec}(\mathbf{sSet}^*)$ and $\Gamma\mathbf{sSet}^*$.

Let $E, F \in \text{SeqSpec}(\mathbf{sSet}^*)$ and consider the enriched hom space $\text{SeqSpec}(\mathbf{sSet}^*)(E, F) \in \mathbf{sSet}^*$ which is defined in the same way as in Definition 2.6. We want to extend this hom space to a Γ-space. Therefore, we define

$$\text{Hom}(E, F)(\underline{n}) := \text{SeqSpec}(\mathbf{sSet}^*)(\overbrace{E \times \ldots \times E}^{n\text{-times}}, F)$$

Moreover, for any $f : \underline{m} \to \underline{n}$ we need to define structure maps

$$\text{Hom}(E, F)(\underline{n}) \to \text{Hom}(E, F)(\underline{m})$$

Hence let f be given by the corresponding map $\theta : \underline{m} \to P(\underline{n})$. Then define the map $\tilde{f} : E^m \to E^n$ as follows. For any $1 \leq i \leq m$ there are diagonal maps

$$\Delta(i) : E \to \overbrace{E \times \ldots \times E}^{\theta(i)\text{-times}}$$

hence there is a family of maps

$$\tilde{f}_i : E \xrightarrow{\Delta(i)} E \times \ldots \times E \hookrightarrow E^n$$

which then induce a unique morphism $\tilde{f} : E^m \to E^n$. Notice that in the case where $\theta(i) = \emptyset$, we define the diagonal map $\Delta(i) : E \to *$ to be the unique map to the terminal object in $\text{SeqSpec}(\mathbf{sSet}^*)$. Now the map $\tilde{f}$ induces a map

$$\tilde{f}_* : \text{SeqSpec}(\mathbf{sSet}^*)(E^n, F) \to \text{SeqSpec}(\mathbf{sSet}^*)(E^m, F)$$

which is the desired structure map.

Lemma 4.2. *Considering the stable model structure on* $\text{SeqSpec}(\mathbf{sSet}^*)$ *and that F is an Ω-spectrum, then the Γ-space* $\text{Hom}(E, F)$ *is very special.*

Proof. First we need to show that the Γ-space $\text{Hom}(E, F)$ is special, i.e. that the maps

$$P_n : \text{Hom}(E, F)(\underline{n}) \to \text{SeqSpec}(\mathbf{sSet}^*)(E, F) \times \ldots \times \text{SeqSpec}(\mathbf{sSet}^*)(E, F)$$

are weak equivalences of simplicial sets for any $n \geq 0$. Indeed, notice that the inclusion

$$E \vee W \hookrightarrow E \times E$$

is a stable homotopy equivalence. Therefore, the induced maps

$$\begin{array}{ccc}
\mathrm{SeqSpec}(\mathbf{sSet}^*)(E \vee \ldots \vee E, F) & \searrow & \\
\uparrow & & \\
\mathrm{SeqSpec}(\mathbf{sSet}^*)(E \times \ldots \times E, F) & \overset{P_n}{\dashrightarrow} & \mathrm{SeqSpec}(\mathbf{sSet}^*)(E, F) \times \ldots \times \mathrm{SeqSpec}(\mathbf{sSet}^*)
\end{array}$$

give the desired homotopy equivalences P_n and thus the Γ-space $\mathrm{Hom}(E, F)$ is special. To show that $\mathrm{SeqSpec}(\mathbf{sSet}^*)$ is very special we need to look at the component space $\mathrm{SeqSpec}(\mathbf{sSet}^*)(E, F)$. By definition we have

$$\pi_0\left(\mathrm{SeqSpec}(\mathbf{sSet}^*)(E, F)\right) = \left[S^0, \mathrm{SeqSpec}(\mathbf{sSet}^*)(E, F)\right]$$

Now using Proposition 2.3 it follows that

$$\left[S^0, \mathrm{SeqSpec}(\mathbf{sSet}^*)(E, F)\right] \cong [E \wedge S^0, F] \cong [E, F]$$

Now using that F is an Ω-spectrum implies that $[E, F]$ is an abelian group. Hence the Γ-space is very special. $\square$

Proposition 4.1. *Let* $\mathbb{S} \in \mathrm{SeqSpec}(\mathbf{sSet}^*)$ *be the sphere spectrum. Then we have an adjunction*

$$\mathbb{S} \wedge - : \Gamma\mathbf{sSet}^* \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{SeqSpec}(\mathbf{sSet}^*) : \mathrm{Hom}(\mathbb{S}, -)$$

First we give the construction of the functor $\mathbb{S} \wedge -$ and after we show that they indeed form a pair of adjoint functors. To construct the functor $\mathbb{S} \wedge -$ we need the notion of simplicial spheres.

Definition 4.6. Let $[n] \in \Delta$, then we define the simplicial set $\Delta[n]$ as follows

$$\begin{aligned}
\Delta[n] : \Delta^{\mathrm{op}} &\to \mathbf{Set}^* \\
[q] &\mapsto \Delta([q], [n])
\end{aligned}$$

Now define $S^0 = * \amalg *$ and S^1 as the pushout in $\mathbf{sSet}^*$

$$\begin{array}{ccc}
S^0 & \longrightarrow & * \\
\downarrow & & \downarrow \\
\Delta[1] & \longrightarrow & S^1
\end{array}$$

For $n \geq 1$ define $S^n = \overbrace{S^1 \wedge \ldots \wedge S^1}^{n\text{-times}}$. These simplicial sets are called **simplicial spheres**.

Definition 4.7. The inclusion

$$\begin{aligned}
S : \Gamma^{\mathrm{op}} &\to \mathbf{sSet}^* \\
\underline{n} &\mapsto S^n
\end{aligned}$$

is called the **sphere inclusion**.

Now let $A : \Gamma^{\mathrm{op}} \to \mathbf{sSet}^*$ be a Γ-space and consider its left Kan extension along the sphere inclusion S.

$$\begin{array}{ccc}
\Gamma^{\mathrm{op}} & \xrightarrow{A} & \mathbf{sSet}^* \\
{\scriptstyle S}\downarrow & \nearrow_{\mathrm{Lan}_S(A)} & \\
\mathbf{sSet}^* & &
\end{array}$$

The functor $\mathrm{Lan}_S(A) : \mathbf{sSet}^* \to \mathbf{sSet}^*$ induces a functor

$$\begin{aligned} \mathrm{Lan}_S(A)_* : \mathrm{SeqSpec}(\mathbf{sSet}^*) &\to \mathrm{SeqSpec}(\mathbf{sSet}^*) \\ E &\mapsto \mathrm{Lan}_S(A) \circ E \end{aligned}$$

Then we define the functor $\mathbb{S} \wedge -$ as

$$\begin{aligned} \mathbb{S} \wedge - : \Gamma\mathbf{sSet}^* &\to \mathrm{SeqSpec}(\mathbf{sSet}^*) \\ A &\mapsto \mathrm{Lan}_S(A)_*(\mathbb{S}) \end{aligned}$$

where $\mathbb{S}$ denotes the sphere spectrum in $\mathrm{SeqSpec}(\mathbf{sSet}^*)$.

Proof. We need to show that the functors $\mathrm{Hom}(\mathbb{S}, -)$ and $\mathbb{S}\wedge-$ form an adjunction. Therefore, let $A \in \Gamma\mathbf{sSet}^*$ and $E \in \mathrm{SeqSpec}(\mathbf{sSet}^*)$. Then look at the set of morphisms

$$\Gamma\mathbf{sSet}^*(A, \mathrm{Hom}(\mathbb{S}, E)) \cong \int_{\underline{k}\in\Gamma^{\mathrm{op}}} \mathbf{sSet}^* \left(A(\underline{k}), \mathrm{SeqSpec}(\mathbf{sSet}^*) \left(\mathbb{S}^{\times k}, E\right)\right)$$

Now using the fact that for any $K \in \mathbf{sSet}^*$ the functor

$$K \wedge - : \mathrm{SeqSpec}(\mathbf{sSet}^*) \to \mathrm{SeqSpec}(\mathbf{sSet}^*)$$

is the left adjoint of the functor

$$\begin{aligned} \mathrm{SeqSpec}(\mathbf{sSet}^*)(K, -) : \mathrm{SeqSpec}(\mathbf{sSet}^*) &\longrightarrow \mathrm{SeqSpec}(\mathbf{sSet}^*) \\ E &\longmapsto (n \mapsto [K, E_n]) \end{aligned}$$

(which is the analogue of Proposition 2.3 for $\mathbf{sSet}^*$), we obtain

$$\mathbf{sSet}^* \left(A(\underline{k}), \mathrm{SeqSpec}(\mathbf{sSet}^*) \left(\mathbb{S}^{\times k}, E\right)\right) \cong \mathrm{SeqSpec}(\mathbf{sSet}) \left(A(\underline{k}) \wedge \mathbb{S}^{\times k}, E\right)$$

and therefore

$$\begin{aligned} \Gamma\mathbf{sSet}^*(A, \mathrm{Hom}(\mathbb{S}, E)) &\cong \int_{\underline{k}\in\Gamma^{\mathrm{op}}} \mathrm{SeqSpec}(\mathbf{sSet}) \left(A(\underline{k}) \wedge \mathbb{S}^{\times k}, E\right) \\ &\cong \mathrm{SeqSpec}(\mathbf{sSet}^*) \left(\int^{\underline{k}\in\Gamma^{\mathrm{op}}} A(\underline{k}) \wedge \mathbb{S}^{\times k}, E\right) \end{aligned}$$

On the other hand we have by definition and Lemma 4.3 that

$$(\mathbb{S} \wedge A)_n \cong \int^{\underline{k}\in\Gamma^{\mathrm{op}}} (S^n)^{\times k} \wedge A(\underline{k})$$

Hence we get that

$$\begin{aligned} \mathbb{S} \wedge A &\cong \int^{n\in\mathbf{Seq}} \mathbf{Seq}(n, -) \wedge (\mathbb{S} \wedge A)_n \\ &\cong \int^{n\in\mathbf{Seq}} \int^{\underline{k}\in\Gamma^{\mathrm{op}}} \mathbf{Seq}(n, -) \wedge (S^n)^{\times k} \wedge A(\underline{k}) \\ &\cong \int^{\underline{k}\in\Gamma^{\mathrm{op}}} A(\underline{k}) \wedge \left(\int^{n\in\mathbf{Seq}} \mathbf{Seq}(n, -) \wedge (S^n)^{\times k}\right) \\ &\cong \int^{\underline{k}\in\Gamma^{\mathrm{op}}} A(\underline{k}) \wedge \mathbb{S}^{\times k} \end{aligned}$$

and now it follows that

$$\Gamma\mathbf{sSet}^*(A, \mathrm{Hom}(\mathbb{S}, E)) \cong \mathrm{SeqSpec}(\mathbf{sSet}^*) \left(\mathbb{S} \wedge A, E\right)$$

By construction this isomorphism is natural and therefore the functors form an adjoint pair. □

Lemma 4.3. ([DGM12]) *For a* Γ*-space* A *there is a natural isomorphism*

$$(\mathbb{S} \wedge A)_n \cong \int^{\underline{k} \in \Gamma^{\mathrm{op}}} (S^n)^{\times k} \wedge A(\underline{k})$$

for any $n \in \mathbb{N}$.

Proof. This follows form the fact that the left Kan extension $\mathrm{Lan}_S(A)$ can be written as a coend given by the so called coend formula in Remark 5.4. Therefore, we have

$$\mathrm{Lan}_S(A) \cong \int^{\underline{k} \in \Gamma^{\mathrm{op}}} \mathbf{sSet}^*(S^k, -) \wedge A(\underline{k})$$

Hence it follows that

$$(\mathbb{S} \wedge A)_n = (\mathrm{Lan}_S(A) \circ \mathbb{S})_n \cong \int^{\underline{k} \in \Gamma^{\mathrm{op}}} \mathbf{sSet}^*(S^k, S^n) \wedge A(\underline{k})$$

which concludes the proof. □

Since the construction of the functor $\mathbb{S} \wedge -$ is rather abstract, we would like to have an explicit construction such that computations can be done more easily. In [Seg74], Segal provides such a construction being an explicit description of this functor.

Definition 4.8. Let $A \in \Gamma\mathbf{sSet}^*$ be a special Γ-space and $X \in \mathbf{sSet}^*$ a pointed simplicial set. Then we define an associated functor $X : \Gamma \to \mathbf{sSet}^*$ as follows. For any $\underline{n} \in \Gamma$ put $X(\underline{n}) := X^{\times n}$ and for any map $\underline{m} \to \underline{n}$ which is given by $\theta : \underline{m} \to P(\underline{n})$ define

$$\begin{aligned} X(\theta) : X^{\times m} &\to X^{\times n} \\ (x_1, ..., x_m) &\mapsto (x'_1, ..., x'_n) \end{aligned}$$

where $x'_j = x_i$ for $j \in \theta(i)$ and $x'_j = *$ otherwise. Then we define the Γ-space $X \otimes_\Gamma A$

$$\begin{aligned} X \otimes_\Gamma A : \Gamma^{\mathrm{op}} &\to \mathbf{sSet}^* \\ \underline{k} &\mapsto \left(\coprod_{n \geq 0} X(\underline{n}) \times A(\underline{n} \times \underline{k}) \right) / \sim \end{aligned}$$

where the quotient is given by the equivalence relation

$$X^m \times A(\underline{m} \times \underline{k}) \ni (x_1, ..., x_m, A(\theta \times \mathrm{id}_{\underline{k}})(a)) \sim (X(\theta)(x_1, ..., x_m), a) \in X^n \times A(\underline{n} \times \underline{k})$$

for all $x_j \in X$ and all $a \in A(\underline{n} \times \underline{k})$ and any map $\theta : \underline{m} \to \underline{n}$. Notice that with $\theta \times \mathrm{id}_{\underline{k}}$ we mean the morphism in Γ given by

$$\begin{aligned} \underline{m} \times \underline{k} &\to P(\underline{n} \times \underline{k}) \\ (i, j) &\mapsto \theta(i) \times \{j\} \end{aligned}$$

Definition 4.9. Let $A \in \Gamma\mathbf{sSet}^*$ be a special Γ-space. Then we define its n**-th classifying space** $B^n A$ as the Γ-space

$$B^n A = S^n \otimes_\Gamma A$$

The sequence $A(\underline{1}), BA(\underline{1}), B^2 A(\underline{1}), ...$ then defines a sequential spectra denoted by $\mathbf{B}A$. In fact there is a functor

$$\mathbf{B} : \Gamma\mathbf{sSet}^* \to \mathrm{SeqSpec}(\mathbf{sSet}^*)$$

Proposition 4.2. ([Seg74]) *There is a pair of adjoint functors*

$$\mathbf{B} : \Gamma\mathbf{sSet}^* \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathrm{SeqSpec}(\mathrm{sSet}^*) : \mathrm{Hom}(\mathbb{S}, -)$$

4.2 Picard groupoids and spectra

In the following let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category. Then we define

$$\pi_0(\mathcal{C}) := \{\ [X] \mid X \in \mathcal{C}\}$$

as the set of isomorphism classes of objects in $\mathcal{C}$. Notice that $\pi_0(\mathcal{C})$ is an abelian monoid under the tensor product, i.e. there is an action given by

$$\begin{aligned} \pi_0(\mathcal{C}) \times \pi_0(\mathcal{C}) &\to \pi_0(\mathcal{C}) \\ ([X],[Y]) &\mapsto [X \otimes Y] \end{aligned}$$

with identity element [1].

Definition 4.10. A category $\mathcal{C}$ is said to be a **groupoid** if it is a small category and every morphism is an isomorphism.

Definition 4.11. A symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ is a **Picard groupoid** if the underlying category is a groupoid and $\pi_0(\mathcal{C})$ is an abelian group.

Remark 4.5. For a Picard groupoid $\mathcal{C}$ there is an autoequivalence for any fixed object $X \in \mathcal{C}$ given by the functor

$$\begin{aligned} X \otimes - : \mathcal{C} &\to \mathcal{C} \\ Y &\mapsto X \otimes Y \end{aligned}$$

This functor induces an isomorphism on the following groups of automorphisms in $\mathcal{C}$.

$$\begin{aligned} \psi_X : \mathrm{Aut}_{\mathcal{C}}(1) &\to \mathrm{Aut}_{\mathcal{C}}(X) \\ \varphi : 1 \to 1 &\mapsto X \otimes \varphi : X \to X \end{aligned}$$

Now define the following group homomorphism

$$\begin{aligned} \pi_0(\mathcal{C}) &\to \mathrm{Aut}_{\mathcal{C}}(1) \\ [X] &\mapsto c_X = \psi^{-1}_{X \otimes X}(\tau_{X,X}) \end{aligned}$$

where $\tau_{X,X} \in \mathrm{Aut}_{\mathcal{C}}(X \otimes X)$ is the braiding of the symmetric monoidal category.

Definition 4.12. A Picard groupoid $\mathcal{C}$ is said to be **strictly commutative** or just **strict** if for every $X \in \mathcal{C}$ we have $c_X = \mathrm{id}_1$.

Now we want to construct a functor

$$K : \{\text{strict symmetric monoidal categories }\} \to \Gamma\mathbf{Cat}$$

called the K-theory functor.

Let $\mathcal{P}$ be a strict symmetric monoidal category. Then for every finite set $S \in \Gamma$ define the category $K(\mathcal{P})(S)$ whose objects are collections $\{X_U\}_{U \subset S}$ of objects $X_U \in \mathcal{P}$ indexed by subsets $U \subset S$ together with isomorphisms

$$\psi_{U,V} : X_U \otimes X_V \xrightarrow{\cong} X_{U \cup V}$$

for all pairs of disjoint subsets $U, V \subset S$ such that they are compatible with the symmetric braiding.

$$\begin{array}{ccc} X_U \otimes X_V & \xrightarrow{\psi_{U,V}} & X_{U \cup V} \\ {\scriptstyle \tau_{X_U, X_V}} \downarrow & & \| \\ X_V \otimes X_U & \xrightarrow[\psi_{V,U}]{} & X_{V \cup U} \end{array}$$

We set $X_\emptyset = 1_\mathcal{P}$ and for all $U \subset S$ the map $\psi_{\emptyset,U} : 1_\mathcal{P} \otimes X_U \to X_U$ is the counit constraint and $\psi_{U,\emptyset} : X_U \otimes 1_\mathcal{P} \to X_U$ is the unit constraint. A morphism in $K(\mathcal{P})(S)$

$$f : (\{X_U\}, \{\psi_{U,V}\})_{U,V \subset S} \to (\{Y_U\}, \{\widetilde{\psi}_{U,V}\})_{U,V \subset S}$$

is given by a collection $\{f_U\}_{U \subset S}$ of morphisms $f_U : X_U \to Y_U$ in $\mathcal{P}$ indexed by $U \subset S$, such that they are compatible with the maps $\psi_{U,V}$ and $\widetilde{\psi}_{U,V}$ and $f_\emptyset = \mathrm{id}_{1_\mathcal{P}}$.
In the following, we want to show that $K(\mathcal{P})$ can be equipped with the structure of a Γ-category, i.e. $K(\mathcal{P}) : \Gamma^{\mathrm{op}} \to \mathbf{Cat}$ defines a functor. Hence let $f : S \to T$ be a morphism in Γ given by $\theta : S \to P(T)$. Then there is an induced functor

$$\begin{aligned} K(\mathcal{P})[f] : K(\mathcal{P})(T) &\to K(\mathcal{P})(S) \\ \{X_U\}_{U \subset T} &\mapsto \{X_{\theta^{-1}(U)}\} \end{aligned}$$

which maps a morphism $g : X \to Y$ in $K(\mathcal{P})(T)$ to the morphism

$$K(\mathcal{P})[f](g) : \{X_{\theta^{-1}(U)}\} \to \{Y_{\theta^{-1}(U)}\}$$

which is indexed by $g_{\theta^{-1}(U)} : X_{\theta^{-1}(U)} \to Y_{\theta^{-1}(U)}$.

Example 4.1. Let $S = \underline{2}$ and $T = \underline{3}$ and let $f : \underline{2} \to \underline{3}$ be the morphism given by

$$\begin{aligned} \theta : \{1,2\} &\to P(\{1,2\}) \\ 1 &\mapsto \{1,2\} \\ 2 &\mapsto \{3\} \end{aligned}$$

Now let $\mathcal{P}$ be a strict symmetric monoidal category as above. Objects of $K(\mathcal{P})(\underline{3})$ are given by $X = (1, X_{\{1\}}, X_{\{2\}}, X_{\{3\}}, X_{\{1,2\}}, X_{\{1,3\}}, X_{\{2,3\}}, X_{\{1,2,3\}})$. Then the induced functor $K(\mathcal{P})[f]$ maps the object X to $K(\mathcal{P})[f](X) = (1, Z_{\{1\}}, Z_{\{2\}}, Z_{\{1,2\}})$ where we have

$$\begin{aligned} Z_{\{1\}} &= X_U \text{ such that } \theta^{-1}(U) = \{1\} \text{ hence } Z_{\{1\}} = X_{\{1,2\}} \\ Z_{\{2\}} &= X_U \text{ such that } \theta^{-1}(U) = \{2\} \text{ hence } Z_{\{2\}} = X_{\{3\}} \\ Z_{\{1,2\}} &= X_U \text{ such that } \theta^{-1}(U) = \{1,2\} \text{ hence } Z_{\{1,2\}} = 1 \end{aligned}$$

therefore $K(\mathcal{P})[f](X) = (1, X_{\{1,2\}}, X_{\{3\}}, 1)$.

This structure turns $K(\mathcal{P})$ into a Γ-category.

Lemma 4.4. *For every strict symmetric monoidal category $\mathcal{P}$ the associated Γ-category $K(\mathcal{P})$ is special and if $\pi_0(\mathcal{P})$ is an abelian group, then $K(\mathcal{P})$ is very special.*

Proof. Let $\mathcal{P}$ be a strict symmetric monoidal category. We want to show that $K(\mathcal{P})$ is a special Γ-category, i.e. by Definition 4.4 we need to show that $K(\mathcal{P})(\underline{0})$ is equivalent to the category with one object and one morphism and that for each $n \geq 0$ there are equivalences of categories

$$P_n : K(\mathcal{P})(\underline{n}) \to K(\mathcal{P})(\underline{1}) \times \cdots \times K(\mathcal{P})(\underline{1}) \tag{1}$$

By definition the category $K(\mathcal{P})(\underline{0})$ has one object $X_\emptyset = 1_\mathcal{P}$ and one morphism id_1. Moreover, the functor (1) is by definition

$$\begin{aligned} P_n : K(\mathcal{P})(\underline{n}) &\longrightarrow K(\mathcal{P})(\underline{1}) \times \cdots \times K(\mathcal{P})(\underline{1}) \\ \{X_U\} &\longmapsto \big(\{1, X_{\{1\}}\}, \{1, X_{\{2\}}\}, \dots, \{1, X_{\{n\}}\}\big) \end{aligned}$$

Consider now the inverse functor

$$\begin{aligned} Q_n : K(\mathcal{P})(\underline{1}) \times \cdots \times K(\mathcal{P})(\underline{1}) &\longrightarrow K(\mathcal{P})(\underline{n}) \\ \big(\{1, Z^1\}, \dots, \{1, Z^n\}\big) &\longmapsto \{X_U\}_{U \subset \underline{n}} \end{aligned}$$

where

$$X_U = \bigotimes_{j \in U} Z^j$$

Looking at the composition of those functors we have

$$\begin{aligned} P_n \circ Q_n : K(\mathcal{P})(\underline{n}) &\longrightarrow K(\mathcal{P})(\underline{n}) \\ \{X_U\} &\longmapsto \left\{ \bigotimes_{j \in U} X_{\{j\}} \right\} \end{aligned}$$

Using the fact that there are isomorphisms $X_{U \cup V} \xrightarrow{\cong} X_U \otimes X_V$ for all disjoint U, V it follows that there are natural isomorphisms

$$\bigotimes_{j \in U} X_{\{j\}} \xrightarrow{\cong} X_U$$

and therefore $P_n \circ Q_n$ is naturally isomorphic to the identity functor on $K(\mathcal{P})(\underline{n})$. On the other side consider the composition

$$\begin{aligned} Q_n \circ P_n : K(\mathcal{P})(\underline{1})^{\times n} &\longrightarrow K(\mathcal{P})(\underline{n}) \longrightarrow K(\mathcal{P})(\underline{1})^{\times n} \\ \big(\{1, Z^1\}, \dots, \{1, Z^n\}\big) &\longmapsto \left\{ \bigotimes_{j \in U} Z^j \right\} \longmapsto \big(\{1, Z^1\}, \dots, \{1, Z^n\}\big) \end{aligned}$$

which is equal to the identity functor on $K(\mathcal{P})(\underline{1})^{\times n}$. Therefore, P_n is an equivalence of categories for all $n \geq 0$.

Now let $\pi_0(\mathcal{P})$ be an abelian group. To show that $K(\mathcal{P})$ is very special, we need to show that the H-space structure on $K(\mathcal{P})(\underline{1})$ admits a weak inverse. First we notice that the H-space structure on $K(\mathcal{P})(\underline{1})$ is given by the monoidal product, which also induces the group structure on $\pi_0(\mathcal{P})$. Indeed, the product on $K(\mathcal{P})(\underline{1})$ is given by

$$\begin{aligned} K(\mathcal{P})(\underline{1}) \times K(\mathcal{P})(\underline{1}) &\longrightarrow K(\mathcal{P})(\underline{2}) \longrightarrow K(\mathcal{P})(\underline{1}) \\ \big(\{1, Z^1\}, \{1, Z^2\}\big) &\longmapsto \{1, Z^1, Z^2, Z^1 \otimes Z^2\} \longmapsto \{1, Z^1 \otimes Z^2\} \end{aligned}$$

Now we notice that $K(\mathcal{P})(\underline{1}) \cong \mathcal{P}$ hence there is a natural isomorphism of groups

$$\pi_0(\mathcal{P}) \xrightarrow{\cong} \pi_0(K(\mathcal{P})(\underline{1}))$$

which shows that the H-space structure on $K(\mathcal{P})(\underline{1})$ admits a weak inverse. This shows that $K(\mathcal{P})$ is a very special Γ-category. □

Remark 4.6. Lemma 4.4 implies that if we restrict the functor K to the category of strict Picard groupoids, we get a functor

$$K : \{\text{strict Picard groupoids}\} \to \Gamma_{\text{vs}}\mathbf{Cat}$$

where $\Gamma_{\text{vs}}\mathbf{Cat}$ denotes the subcategory of very special Γ-categories.

On the other hand we can associate a Picard groupoid to a very special Γ-space $A : \Gamma^{\text{op}} \to \mathbf{sSet}^*$ in the following way.

Definition 4.13. Let X be a topological space. Then define the **fundamental groupoid** $\Pi(X)$ to be the category with objects given by the points of X and with morphisms $x \to y$ given by the equivalence classes of paths from x to y up to homotopy.

Now let A be a very special Γ-space. Define $\Pi(A) := \Pi(|A(\underline{1})|)$ to be the fundamental groupoid of the geometric realization of $A(\underline{1})$. The H-space structure on $A(\underline{1})$ induces a bifunctor denoted by

$$\otimes : \Pi(A) \times \Pi(A) \to \Pi(A)$$

Lemma 4.5. ([Pat12]) *The bifunctor* $\otimes : \Pi(A) \times \Pi(A) \to \Pi(A)$ *induced by the H-space structure on* $A(\underline{1})$ *gives* $\Pi(A)$ *the structure of a symmetric monoidal category with tensor unit given by the base point of* $A(\underline{1})$*. Moreover, if* A *is very special, then* $\Pi(A)$ *is a Picard groupoid.*

To investigate how the functors K and Π are related we need the following lemma.

Lemma 4.6. ([DGM12]) *Let* $N : \mathbf{Cat} \to \mathbf{sSet}$ *be the nerve functor. Then for* $\mathcal{C} : \Gamma^{\mathrm{op}} \to \mathbf{Cat}$ *a* Γ*-category consider the composition*

$$\begin{aligned} N \circ \mathcal{C} : \Gamma^{\mathrm{op}} &\to \mathbf{Cat} \to \mathbf{sSet}^* \\ \underline{n} &\mapsto \mathcal{C}(\underline{n}) \mapsto N\left(\mathcal{C}(\underline{n})\right)_+ \end{aligned}$$

which defines naturally a Γ*-space. If* $\mathcal{C}$ *is a special resp. very special* Γ*-category then* $N \circ \mathcal{C}$ *is special resp. very special.*

Using the above lemma we get a pair of functors

$$\{\text{strict Picard groupoids}\} \xrightarrow{K} \Gamma_{\mathrm{vs}}\mathbf{Cat} \xrightarrow{N\circ -} \Gamma_{\mathrm{vs}}\mathbf{sSet}^* \quad \text{with } \Pi : \Gamma_{\mathrm{vs}}\mathbf{sSet}^* \to \{\text{strict Picard groupoids}\}$$

Composing this functor pair with the corresponding functors defined in Proposition 4.1 yields the following pair of functors,

$$\{\text{strict Picard groupoids}\} \xrightarrow{K} \Gamma_{\mathrm{vs}}\mathbf{Cat} \xrightarrow{N\circ -} \Gamma_{\mathrm{vs}}\mathbf{sSet}^* \xrightarrow{\mathbb{S}\wedge -} \mathrm{SeqSpec}(\mathbf{sSet}^*)$$

$$\{\text{strict Picard groupoids}\} \xleftarrow{\quad\Pi\quad} \Gamma_{\mathrm{vs}}\mathbf{sSet}^* \xleftarrow{\mathrm{Hom}(\mathbb{S},\text{ - })} \Omega\text{-}\mathrm{SeqSpec}(\mathbf{sSet}^*)$$

which show how to pass from strict Picard groupoids to sequential spectra and vice versa. Since the functors $\mathbb{S} \wedge -$ and $\mathrm{Hom}(\mathbb{S}, -)$ form an adjunction pair, the question arises if the functors Π and $(N \circ -) \circ K$ are also adjoint. To give an answer to this question we need to understand the fundamental groupoid functor $\Pi : \mathbf{sSet}^* \to \mathbf{Grpd}$.

4.3 The fundamental groupoid functor

Proposition 4.3. *There exists an adjunction*

$$\tau : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Cat} : N$$

Proof. Since **Cat** is locally small and co-complete, we can apply the nerve-realization machinery of Proposition 5.4 which then proves the result. ☐

Definition 4.14. The forgetful functor $U : \mathbf{Grpd} \to \mathbf{Cat}$ which sends a groupoid to its underlying category has a left adjoint, denoted by $G : \mathbf{Cat} \to \mathbf{Grpd}$ and is called the **free groupoid functor**.

Theorem 4.1. ([GJ09]) *Let Z be a simplicial set. Then the groupoids $(G \circ \tau)(Z)$ and $\Pi(|Z|)$ are naturally equivalent as categories.*

Corollary 4.2. *There is a pair of adjoint functors*

$$\mathbf{sSet} \underset{N}{\overset{G\circ\tau}{\rightleftarrows}} \mathbf{Grpd}$$

Proof. The adjunction pair follows from the fact that the composition of adjoint functors is again adjoint. In this case we use Proposition 4.3 and compose the two adjoint pairs

$$\mathbf{Cat} \underset{U}{\overset{G}{\rightleftarrows}} \mathbf{Grpd} \qquad\qquad \mathbf{sSet} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{Cat}$$

□

These adjunction pairs lift naturally to adjunction pairs of Γ-objects.

Lemma 4.7. *There is a pair of adjoint functors*

$$\Gamma\mathbf{Grpd} \underset{(G\circ\tau)\circ(-)}{\overset{N\circ(-)}{\rightleftarrows}} \Gamma\mathbf{sSet}$$

Proof. We need to show that for any $A \in \Gamma\mathbf{sSet}$ and $\mathcal{G} \in \Gamma\mathbf{Grpd}$ there are natural isomorphisms

$$\Gamma\mathbf{Grpd}((G \circ \tau) \circ A, G) \cong \Gamma\mathbf{sSet}(A, N \circ \mathcal{G}) \tag{2}$$

Using the Co-Yoneda Lemma we obtain

$$\Gamma\mathbf{Grpd}((G \circ \tau) \circ A, \mathcal{G}) \cong \int_{\underline{k}\in\Gamma^{\mathrm{op}}} \mathbf{Grpd}\Big((G \circ \tau)(A(\underline{k})), \mathcal{G}(\underline{k})\Big)$$
$$\Gamma\mathbf{sSet}(A, N \circ \mathcal{G}) \cong \int_{\underline{k}\in\Gamma^{\mathrm{op}}} \mathbf{sSet}\Big(A(\underline{k}), N(\mathcal{G}(\underline{k}))\Big)$$

Now we use Corollary 4.2 to obtain natural isomorphisms for any $\underline{k} \in \Gamma^{\mathrm{op}}$

$$\mathbf{sSet}\Big(A(\underline{k}), N(\mathcal{G}(\underline{k}))\Big) \cong \mathbf{Grpd}\Big((G \circ \tau)(A(\underline{k})), \mathcal{G}(\underline{k})\Big)$$

which then induce the desired natural isomorphisms of (2). □

Now that we have established all the necessary adjunctions, we put the pieces together and try to understand how they correspond to each other. Theorem 4.1 allows us to write Π as the composition of $G \circ \tau$ and $\mathrm{EV}_{\underline{1}}$ up to equivalence of categories, as indicated in the diagram below. Notice that $\mathrm{EV}_{\underline{1}}$ denotes the evaluation functor at the object $\underline{1} \in \Gamma^{\mathrm{op}}$.

$$\begin{array}{ccc}
\{\text{strict Picard groupoids}\} & \xrightarrow{K} & \Gamma_{\mathrm{vs}}\mathbf{Cat} \\
{\scriptstyle G\circ\tau}\uparrow & \overset{\Pi}{\nwarrow} & \downarrow{\scriptstyle N\circ(-)} \\
\mathbf{sSet} & \xleftarrow[\mathrm{Ev}_{\underline{1}}]{} & \Gamma_{\mathrm{vs}}\mathbf{sSet}
\end{array}$$

This diagram is not commutative, since the lower left triangle just commutes up to equivalence of categories. One approach to handle this problem is to endow the category of strict Picard groupoids with a model structure, such that the weak equivalences are given by equivalences of groupoids, see for example [Hol08]. Then by endowing **sSet** with its standard model structure, we obtain a diagram of model categories. By simply passing to the homotopy categories of the corresponding model categories, a lot of information would be lost. Therefore, we should rather consider the diagram of their corresponding simplicial localizations resp. their associated ∞-categories. This approach is a motivation for the following chapter, where we introduce ∞-categories and see how we can endow them with symmetric monoidal structures.

5 Higher categories

Heuristically a higher category is a structure given by a class of objects called 0-cells, a class of morphisms between the objects called 1-cells, a class of morphisms between the 1-cells called 2-cells, and so on. These higher morphisms or higher cells can be composed and need to satisfy certain coherence properties. Moreover, by asking that all k-cells are isomorphisms for all $k > n$, it arises the motivation behind an (∞, n)-category. Imposing these invertibility properties on higher morphisms provides a generalization of ∞-groupoids, where we consider all higher morphisms to be invertible. There exist many models for higher categories, each of which carries its advantages and disadvantages. The reason why we need to find structures being models for higher categories is the fact that there is currently no axiomatic definition of a higher category. For certain higher categories however, an axiomatization is possible, such as given by the work of Toën in [Toe04]. One example of such a model are topological categories. Indeed, let $\mathcal{C}$ be a topological category. Then the 0-cells are the objects of $\mathcal{C}$ and the 1-cells are the morphisms of $\mathcal{C}$. Having two 1-cells, say $f : x \to y$ and $g : x \to y$, a 2-cell is then given by a path $\gamma : f \Rightarrow g$ in the topological space $\mathcal{C}(x, y)$, i.e. $\gamma : I \to \mathcal{C}(x, y)$. Given two 2-cells, say $\gamma : f \Rightarrow g$ and $\delta : f \Rightarrow g$, a 3-cell is then given by a homotopy between the paths γ and δ, i.e. a 3-cell is a map $H : I \times I \to \mathcal{C}(x, y)$. In this manner we can define cells in any dimension, leading to a model of a higher category, namely an $(\infty, 1)$-category.

An important property of $(\infty, 1)$-categories is that they each represent a homotopy theory. Indeed, it was shown by Dwyer and Kan in [DK80], that any category with weak equivalences $\mathcal{D}$ provides a simplicial category $L\mathcal{D}$, the simplicial localization of $\mathcal{D}$. This localization still carries a coherent homotopical structure, in contrast to the regular localization, where one just formally inverts the weak equivalences ([Ber10]). On the other hand one can show, that for any simplicial category $\mathcal{C}$, there is a category with weak equivalences $\mathcal{D}$, such that $\mathcal{C}$ is weakly equivalent to the simplicial localization $L\mathcal{D}$ of $\mathcal{D}$. This notion of weak equivalence between simplicial categories, which we define in Section 5.2, is called Dwyer-Kan equivalence and provides a model structure on the category **sSet-Cat** of small simplicial categories. This notion of weak equivalence has the property that if two model categories $\mathcal{M}$ and $\mathcal{N}$ are Quillen equivalent, then their simplicial localizations $L\mathcal{M} \simeq L\mathcal{N}$ are weakly equivalent. Hence the model structure on simplicial categories is one approach to construct a homotopy theory of homotopy theories.

The previously established theory to describe the category of structured spectra using the Day convolution relies heavily on topological categories. Hence the question of whether we can define the Day convolution in a homotopy coherent way arises. The motivation behind this approach comes from the fact that topological or simplicial categories provide a model for $(\infty, 1)$-categories. Therefore, we want to investigate how algebraic structures, such as symmetric monoidal products can be defined within this model. There have been various approaches to provide models for $(\infty, 1)$-categories and it turned out that most of these models are Quillen equivalent. As for example discussed in [Ber10]. Since we defined spectra via enriched diagram categories, it seems straightforward to consider the category of small topological or simplicial categories as our preferred model. Therefore, we first need to establish some basic theory on enriched categories, especially on topological and on simplicial categories.

5.1 Enriched categories

In the following we will focus on simplicial categories. The same statements also hold for topological categories using the change of base Quillen equivalence induced by the geometric realization adjunction. As already mentioned before, we will denote the category of small simplicial categories by **sSet-Cat**. To construct the topological Day convolution, we defined an enrichment on the functor category $[\mathcal{C}, \mathbf{Top}^*_{cg}]$. This enrichment is given by Definition 2.6 and can be generalized to give an enrichment of the functor category $[\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}$ where $\mathcal{C}$ and $\mathcal{D}$ are simplicial categories. This construction allows us to eventually prove that the category **sSet-Cat** is closed cartesian. To do so we first recall the notion of enriched ends.

For ordinary categories and functors, ends are introduced using dinatural transformations, such as in [ML98].

Definition 5.1. Given two ordinary categories $\mathcal{C}, \mathcal{D}$ and two ordinary functors $S, T : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ a **dinatural transformation** $\alpha : S \Rightarrow T$ is a function which assigns to each object $c \in \mathcal{C}$ a morphism

$\alpha_c : S(c,c) \to T(c,c)$ in $\mathcal{D}$, called the component of α at c, in such a way that for every $f : c \to c'$ in $\mathcal{C}$ the following diagram commutes.

$$\begin{array}{ccccc} & S(c,c) & \xrightarrow{\alpha_c} & T(c,c) & \\ {}^{S(f,\mathrm{id})}\nearrow & & & & \searrow^{T(\mathrm{id},f)} \\ S(c',c) & & & & T(c,c') \\ {}_{S(\mathrm{id},f)}\searrow & & & & \nearrow_{T(f,\mathrm{id})} \\ & S(c',c') & \xrightarrow[\alpha_{c'}]{} & T(c',c') & \end{array}$$

Using the language of dinatural transformations, we can give a concise definition of an end of a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ between ordinary categories.

Definition 5.2. An **end** of a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a pair $\langle e, \pi \rangle$ where e is an object of $\mathcal{D}$ and $\pi : e \Rightarrow F$ is a dinatural transformation with the property that for every dinatural transformation $\beta : d \Rightarrow F$ there is a unique arrow $h : d \to e$ in $\mathcal{D}$, such that for all $c \in \mathcal{C}$ we have $\beta_c = \pi_c \circ h$. The end of the functor F is usually denoted with $e = \int_{c \in \mathcal{C}} F(c,c)$.

To define ends in the enriched setting, we first need to establish a suitable analogue for dinatural transformations, which we will call simplicial dinatural transformations.

Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{sSet}$ be a simplicial functor. Then we notice that there are two "actions" of $\mathcal{C}$ on F which are given component wise by

$$\lambda_{c_1,c_2,c_3} : F(c_1,c_2) \times \mathcal{C}(c_2,c_3) \to F(c_1,c_3)$$
$$\rho_{c_1,c_2,c_3} : F(c_2,c_3) \times \mathcal{C}(c_1,c_2) \to F(c_1,c_3)$$

These actions arise from the fact that $\mathbf{sSet}$ is closed cartesian. More precisely, for any c_1, c_2, c_3 objects in $\mathcal{C}$ there are maps

$$F(c_1,-)_{c_2,c_3} : \mathcal{C}(c_2,c_3) \to \mathbf{sSet}(F(c_1,c_2), F(c_1,c_3))$$
$$F(-,c_3)_{c_1,c_2} : \mathcal{C}(c_1,c_2) \to \mathbf{sSet}(F(c_2,c_3), F(c_1,c_3))$$

Then the fact that $\mathbf{sSet}$ is closed cartesian shows that there are isomorphisms

$$\mathbf{sSet}\left[\mathcal{C}(c_2,c_3), \mathbf{sSet}(F(c_1,c_2), F(c_1,c_3))\right] \xrightarrow{\cong} \mathbf{sSet}\left[F(c_1,c_2) \times \mathcal{C}(c_2,c_3), F(c_1,c_3)\right]$$
$$F(c_1,-)_{c_2,c_3} \longmapsto \lambda_{c_1,c_2,c_3}$$

$$\mathbf{sSet}\left[\mathcal{C}(c_1,c_2), \mathbf{sSet}(F(c_2,c_3), F(c_1,c_3))\right] \xrightarrow{\cong} \mathbf{sSet}\left[F(c_2,c_3) \times \mathcal{C}(c_1,c_2), F(c_1,c_3)\right]$$
$$F(-,c_3)_{c_1,c_2} \longmapsto \rho_{c_1,c_2,c_3}$$

Using this special property of simplicial functors F, we give the following definition.

Definition 5.3. Let $\mathcal{C}$ be a simplicial category and $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{sSet}$ a simplicial functor. Then for an object $s \in \mathbf{sSet}$ a **simplicial dinatural transformation** $\alpha : s \Rightarrow F$ from s to F is a family of maps $\alpha_c : v \to F(c,c)$ in $\mathbf{sSet}$ indexed over objects $c \in \mathcal{C}$, such that for all pairs of objects c, c' in $\mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc} s \times \mathcal{C}(c,c') & \xrightarrow{\alpha_c \times \mathrm{id}} & F(c,c) \times \mathcal{C}(c,c') \\ {\scriptstyle \alpha_{c'} \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \lambda_{c,c,c'}} \\ F(c',c') \times \mathcal{C}(c,c') & \xrightarrow[\rho_{c,c',c'}]{} & F(c,c') \end{array}$$

Hence we can give now the definition of a simplicial end, which is the exact analogue of Definition 5.2 using simplicial dinatural transformations.

Definition 5.4. A **simplicial end** of a simplicial functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{sSet}$ is a pair $\langle e, \pi \rangle$ where e is a simplicial set and $\pi : e \Rightarrow F$ is a simplicial dinatural transformation with the property that for every simplicial dinatural transformation $\beta : s \Rightarrow F$ there is a unique map of simplicial sets $h : s \to e$, such that for all $c \in \mathcal{C}$ we have $\beta_c = \pi_c \circ h$. The simplicial end of the simplicial functor F is usually denoted with $e = \int_{c \in \mathcal{C}} F(c, c)$.

Remark 5.1. A different approach to define enriched ends and coedns can be given using equalizers and coequalizers. Indeed, let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{sSet}$ be a simplicial functor and $\alpha_c : e \to F(c, c)$ a family of maps in $\mathbf{sSet}$. Asking that $\alpha : e \Rightarrow F$ defines a simplicial dinatural transformation is equivalent to the statement that e equalizes the following diagram,

$$e = \int_{c \in \mathcal{C}} F(c, c) \xrightarrow{\prod \alpha_c} \prod_{c \in \mathcal{C}} F(c, c) \underset{\prod \rho^*_{c,d,d}}{\overset{\prod \lambda^*_{c,c,d}}{\rightrightarrows}} \prod_{c,d \in \mathcal{C}} [\mathcal{C}(c, d), F(c, d)]$$

where

$$\lambda^*_{c_1,c_2,c_3} : F(c_1, c_2) \to [\mathcal{C}(c_2, c_3), F(c_1, c_3)]$$
$$\rho^*_{c_1,c_2,c_3} : F(c_2, c_3) \to [\mathcal{C}(c_2, c_3), F(c_1, c_3)]$$

are the adjoint maps of λ_{c_1,c_2,c_3} and ρ_{c_1,c_2,c_3}. Then the universal property of the end $\langle e, \alpha \rangle$ translates into the universal property showing that e is the equalizer of the above diagram. Similarly coends can be described by coequalizers.

$$\coprod_{c,d \in \mathcal{C}} \mathcal{C}(c, d) \times F(d, c) \underset{\coprod \rho_{c,d,c}}{\overset{\coprod \lambda_{d,c,d}}{\rightrightarrows}} \coprod_{c \in \mathcal{C}} F(c, c) \xrightarrow{coeq.} \int^{c \in \mathcal{C}} F(c, c)$$

Hence existence of enriched ends and coends is equivalent to the existence of ends and coends in the given category.

Now we are ready to give the general construction of the enrichment on the category of simplicial functors $[\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}$.

Definition 5.5. Let $\mathcal{C}, \mathcal{D}$ be two simplicial categories and consider the category of simplicial functors from $\mathcal{C}$ to $\mathcal{D}$ denoted by $[\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}$. Let F, G be two simplicial functors, then there is a bifunctor

$$\mathcal{D}(F(-), G(-)) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{sSet}$$

whose end defines the simplicial enrichment on $[\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}$, which is denoted by

$$[\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}(F, G) := \int_{c \in \mathcal{C}} \mathcal{D}(F(c), G(c))$$

Remark 5.2. The end always exists, since the category $\mathbf{sSet}$ is complete and cocomplete, i.e. has all small limits and all small colimits respectively. What is left to show is that the above definition defines indeed a simplicial enrichment on the functor category $[\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}$. That is, we need to show that there is a composition, given by a simplicial map

$$[\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}(F, G) \times [\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}(G, H) \to [\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}(F, H)$$

Lemma 5.1. *The category of simplicial functors* $[\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}$ *with hom sets given by the above definition is a small simplicial category.*

Proof. What is left to show is that there is a composition in $[\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}$. Hence let F, G, H be simplicial functors from $\mathcal{C}$ to $\mathcal{D}$. Then by definition there are dinatural transformations with component maps

$$\begin{aligned}
\alpha_c &: \int_{c \in \mathcal{C}} \mathcal{D}(F(c), G(c)) \to \mathcal{D}(F(c), G(c)) \\
\beta_c &: \int_{c \in \mathcal{C}} \mathcal{D}(G(c), H(c)) \to \mathcal{D}(G(c), H(c)) \\
\gamma_c &: \int_{c \in \mathcal{C}} \mathcal{D}(F(c), H(c)) \to \mathcal{D}(F(c), H(c))
\end{aligned}$$

Hence by taking their product we get a family of maps

$$\alpha_c \times \beta_c : \int_{c \in \mathcal{C}} \mathcal{D}(F(c), G(c)) \times \int_{c \in \mathcal{C}} \mathcal{D}(G(c), H(c)) \to \mathcal{D}(F(c), G(c)) \times \mathcal{D}(G(c), H(c))$$

Then using composition in $\mathcal{D}$ we obtain a family of simplicial maps

$$\mathrm{comp}^{\mathcal{D}}_{F(c),G(c),H(c)} \circ (\alpha_c \times \beta_c) : \int_{c \in \mathcal{C}} \mathcal{D}(F(c), G(c)) \times \int_{c \in \mathcal{C}} \mathcal{D}(G(c), H(c)) \to \mathcal{D}(F(c), H(c))$$

Now we want to use the universal property of the simplicial dinatural transformation γ to show that there exists a unique map

$$\int_{c \in \mathcal{C}} \mathcal{D}(F(c), G(c)) \times \int_{c \in \mathcal{C}} \mathcal{D}(G(c), H(c)) \to \int_{c \in \mathcal{C}} \mathcal{D}(F(c), H(c))$$

Hence we need to show that $\mathrm{comp}^{\mathcal{D}}_{F(c),G(c),H(c)} \circ (\alpha_c \times \beta_c)$ defines a simplicial dinatural transformation. Indeed, let c, c' be two objects of $\mathcal{C}$, then we need to show that the following diagram commutes.

$$\begin{array}{ccc}
\int_{c \in \mathcal{C}} \mathcal{D}(F(c), G(c)) \times \int_{c \in \mathcal{C}} \mathcal{D}(G(c), H(c)) \times \mathcal{C}(c, c') & \xrightarrow{\alpha_c \times \beta_c \times \mathrm{id}} & \mathcal{D}(F(c), G(c)) \times \mathcal{D}(G(c), H(c)) \times \mathcal{C}(c, c') \\
\Big\downarrow {\scriptstyle \alpha_{c'} \times \beta_{c'} \times \mathrm{id}} & & \Big\downarrow {\scriptstyle \mathrm{comp}^{\mathcal{D}} \times \mathrm{id}} \\
\mathcal{D}(F(c'), G(c')) \times \mathcal{D}(G(c'), H(c')) \times \mathcal{C}(c, c') & & \mathcal{D}(F(c), H(c)) \times \mathcal{C}(c, c') \\
\Big\downarrow {\scriptstyle \mathrm{comp}^{\mathcal{D}} \times \mathrm{id}} & & \Big\downarrow {\scriptstyle \lambda^{F,H}_{c,c,c'}} \\
\mathcal{D}(F(c'), H(c')) \times \mathcal{C}(c, c') & \xrightarrow[\rho^{F,H}_{c,c'c'}]{} & \mathcal{D}(F(c), G(c'))
\end{array}$$

To prove this, we notice that the following diagrams commute by construction.

$$\begin{array}{ccc}
\int_{c \in \mathcal{C}} \mathcal{D}(F(c), G(c)) \times \mathcal{C}(c, c') & \xrightarrow{\alpha_c \times \mathrm{id}} & \mathcal{D}(F(c), G(c)) \times \mathcal{C}(c, c') \\
\Big\downarrow {\scriptstyle \alpha_{c'} \times \mathrm{id}} & & \Big\downarrow {\scriptstyle \lambda^{F,G}_{c,c,c'}} \\
\mathcal{D}(F(c), G(c)) \times \mathcal{C}(c, c') & \xrightarrow[\rho^{F,G}_{c,c',c'}]{} & \mathcal{D}(F(c), G(c'))
\end{array}$$

$$\begin{array}{ccc}
\int_{c \in \mathcal{C}} \mathcal{D}(G(c), H(c)) \times \mathcal{C}(c, c') & \xrightarrow{\beta_c \times \mathrm{id}} & \mathcal{D}(G(c), H(c)) \times \mathcal{C}(c, c') \\
\Big\downarrow {\scriptstyle \beta_{c'} \times \mathrm{id}} & & \Big\downarrow {\scriptstyle \lambda^{G,H}_{c,c,c'}} \\
\mathcal{D}(G(c), H(c)) \times \mathcal{C}(c, c') & \xrightarrow[\rho^{G,H}_{c,c',c'}]{} & \mathcal{D}(G(c), H(c'))
\end{array}$$

Hence the above diagrams show, that for all $x \in \int_{c\in\mathcal{C}} \mathcal{D}(F(c), G(c))$ and all $y \in \int_{c\in\mathcal{C}} \mathcal{D}(G(c), H(c))$ and all $f \in \mathcal{C}(c, c')$ the following morphisms agree

$$\alpha_{c'}(x) \circ F(f) = G(f) \circ \alpha_c(x)$$
$$\beta_{c'}(y) \circ G(f) = H(f) \circ \beta_c(y)$$

Then the upper right path of the first diagram sends (x, f) to the morphism $H(f) \circ \beta_c(y) \circ \alpha_c(x)$ and the lower left path sends (x, f) to the morphism $\beta_{c'}(y) \circ \alpha_{c'}(x) \circ F(f)$. Then using the above identities we have

$$\begin{aligned}\beta_{c'}(y) \circ \alpha_{c'}(x) \circ F(f) &= \beta_{c'}(y) \circ G(f) \circ \alpha_c(x)\\ &= H(f) \circ \beta_c(y) \circ \alpha_c(x)\end{aligned}$$

Hence the two morphisms agree, which shows that the first diagram commutes. This shows that $\mathrm{comp}^{\mathcal{D}}_{F(c),G(c),H(c)} \circ (\alpha_c \times \beta_c)$ defines indeed a simplicial dinatural transformation. Then by the universal property of ends, there is a unique map of simplicial sets

$$\Gamma : \int_{c\in\mathcal{C}} \mathcal{D}(F(c), G(c)) \times \int_{c\in\mathcal{C}} \mathcal{D}(G(c), H(c)) \to \int_{c\in\mathcal{C}} \mathcal{D}(F(c), H(c))$$

such that for all $c \in \mathcal{C}$ the following diagram commutes.

$$\begin{array}{ccc} \int_{c\in\mathcal{C}} \mathcal{D}(F(c), G(c)) \times \int_{c\in\mathcal{C}} \mathcal{D}(G(c), H(c)) & \xrightarrow{\Gamma} & \int_{c\in\mathcal{C}} \mathcal{D}(F(c), H(c)) \\ \downarrow{\scriptstyle \alpha_c \times \beta_c} & & \downarrow{\scriptstyle \gamma_c} \\ \mathcal{D}(F(c), G(c)) \times \mathcal{D}(G(c), H(c)) & \xrightarrow[\mathrm{comp}^{\mathcal{D}}_{F(c),G(c),H(c)}]{} & \mathcal{D}(F(c), H(c)) \end{array}$$

Then Γ defines the desired composition in the category $[\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}$ which makes it a simplicial category. □

By the above lemma we have that $[\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}$ defines a simplicial category. Therefore, we can now prove the following important proposition, which states that the enriched functor category defines an inner hom object.

Proposition 5.1. *Let $\mathcal{C}$ be any simplicial category. Then the following functors form an adjoint pair.*

$$\mathbf{sSet\text{-}Cat} \underset{[\mathcal{C},-]_{\mathbf{sSet}}}{\overset{\mathcal{C}\times(-)}{\rightleftarrows}} \mathbf{sSet\text{-}Cat} \quad (\perp)$$

Proof. We need to show that for any two simplicial categories $\mathcal{A}$ and $\mathcal{D}$ there are natural isomorphisms

$$\mathbf{sSet\text{-}Cat}(\mathcal{A} \times \mathcal{C}, \mathcal{D}) \cong \mathbf{sSet\text{-}Cat}(\mathcal{A}, [\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}})$$

Hence we first define a map

$$\varphi : \mathbf{sSet\text{-}Cat}(\mathcal{A} \times \mathcal{C}, \mathcal{D}) \to \mathbf{sSet\text{-}Cat}(\mathcal{A}, [\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}})$$

which associates to every simplicial functor $F : \mathcal{A} \times \mathcal{C} \to \mathcal{D}$ a corresponding simplicial functor $\varphi(F)$. Given such a simplicial functor F, we define

$$\begin{aligned}\varphi(F) : \mathcal{A} &\to [\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}\\ a &\mapsto F(a, -)\end{aligned}$$

Notice that $F(a, -)$ is indeed a simplicial functor, as by assumption we have that for all $c, c' \in \mathcal{C}$, there is a simplicial map

$$F_{(a,c),(a,c')} : \mathcal{A}(a, a) \times \mathcal{C}(c, c') \to \mathcal{D}(F(a, c), F(a, c'))$$

which induces the map

$$F_{(a,c),(a,c')}(\mathrm{id}_a,-) : \mathcal{C}(c,c') \to \mathcal{D}(F(a,c),F(a,c'))$$

This shows that $F(a,-)$ is indeed an object of $[\mathcal{C},\mathcal{D}]_{\mathbf{sSet}}$. Next, we need to show that $\varphi(F)$ is a simplicial functor. That is, for a,a' objects in $\mathcal{A}$ there need to be simplicial maps of the form

$$\varphi(F)_{a,a'} : \mathcal{A}(a,a') \to [\mathcal{C},\mathcal{D}]_{\mathbf{sSet}}(F(a,-),F(a',-))$$

preserving composition and identities. Using the fact that F is simplicial we have for any two fixed objects $a,a' \in \mathcal{A}$ a family of simplicial maps indexed by objects $c \in \mathcal{C}$

$$\alpha_c = F_{(a,c),(a',c)}(-,\mathrm{id}_c) : \mathcal{A}(a,a') \to \mathcal{D}(F(a,c),F(a',c))$$

We want to show that α_c are the component maps of a simplicial dinatural transformation $\alpha : \mathcal{A}(a,a') \Rightarrow \mathcal{D}(F(a,-),F(a',-))$. That is, we need to show that the following diagram of simplicial sets commutes for all c,c' objects in $\mathcal{C}$.

$$\begin{array}{ccc}
\mathcal{A}(a,a') \times \mathcal{C}(c,c') & \xrightarrow{\alpha_c \times \mathrm{id}} & \mathcal{D}(F(a,c),F(a',c)) \times \mathcal{C}(c,c') \\
{\scriptstyle \alpha_{c'} \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \lambda_{c,c,c'}} \\
\mathcal{D}(F(a,c'),F(a',c')) \times \mathcal{C}(c,c') & \xrightarrow[\rho_{c,c',c'}]{} & \mathcal{D}(F(a,c),F(a',c'))
\end{array}$$

Hence we need to show that for all $f : a \to a'$ and all $g : c \to c'$ the following morphisms in $\mathcal{D}$ agree.

$$F(a',g) \circ F(f,c) = F(f,c') \circ F(a,g)$$

This property is clearly satisfied since F is assumed to be a simplicial functor from $\mathcal{A} \times \mathcal{C}$ to $\mathcal{D}$. Therefore, α defines a simplicial dinatural transformation. Then the universal property for simplicial ends implies that there is a simplicial map

$$\varphi(F)_{a,a'} : \mathcal{A}(a,a') \to \int_{c \in \mathcal{C}} \mathcal{D}(F(a,c),F(a',c)) = [\mathcal{C},\mathcal{D}]_{\mathbf{sSet}}\left(\varphi(F)(a),\varphi(F)(a')\right)$$

such that for all $c \in \mathcal{C}$ the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{A}(a,a') & \xrightarrow{\varphi(F)_{a,a'}} & \int_{c \in \mathcal{C}} \mathcal{D}(F(a,c),F(a',c)) \\
 & {\scriptstyle \alpha_c}\searrow & \downarrow{\scriptstyle \pi_c} \\
 & & \mathcal{D}(F(a,c),F(a',c))
\end{array}$$

What is left to show is that the simplicial map $\varphi(F)_{a,a'}$ preserves composition and identities, i.e. we need to show that

$$\begin{aligned}
\varphi(F)_{a,a}(\mathrm{id}_a) &= \mathrm{id}_{F(a,-)} \\
\varphi(F)_{a,a''}(g \circ f) &= \varphi(F)_{a',a''}(g) \circ \varphi(F)_{a,a'}(f)
\end{aligned}$$

Using the projection π_c we can easily see that for all $c \in \mathcal{C}$ we have

$$\begin{aligned}
\alpha_c(\mathrm{id_a}) &= F_{(a,c),(a,c)}(\mathrm{id_a},\mathrm{id}_c) \\
&= \mathrm{id}_{F(a,c)} \\
&= \pi_c(\varphi(F)_{a,a}(\mathrm{id}_a))
\end{aligned}$$

which implies that $\varphi(F)_{a,a}(\mathrm{id}_a) = \mathrm{id}_{F(a,-)}$.

To show the second property we consider objects a,a',a'' in $\mathcal{A}$ and the fact that the following diagram is commutative for all $c \in \mathcal{C}$.

$$\begin{array}{ccc} \mathcal{A}(a,a')\times\mathcal{A}(a',a'') & \xrightarrow{\mathrm{comp}^{\mathcal{A}}_{a,a',a''}} & \mathcal{A}(a,a'') \\ \downarrow & \overset{\beta_c}{\dashrightarrow} & \downarrow \\ \mathcal{D}(F(a,c),F(a',c))\times\mathcal{D}(F(a',c),F(a'',c)) & \xrightarrow[\mathrm{comp}^{\mathcal{D}}]{} & \mathcal{D}(F(a,c),F(a'',c)) \end{array}$$

This is again due to the fact, that F is a simplicial functor. Hence we get a family of simplicial maps β_c which again are the component maps of a simplicial dinatural transformation.

$$\beta:\mathcal{A}(a,a')\times\mathcal{A}(a',a'')\Rightarrow\mathcal{D}(F(a,-),F(a'',-))$$

Indeed, as above we check the commutativity of the following diagram of simplicial sets for all c,c' in $\mathcal{C}$.

$$\begin{array}{ccc} \mathcal{A}(a,a')\times\mathcal{A}(a',a'')\times\mathcal{C}(c,c') & \xrightarrow{\beta_c\times\mathrm{id}} & \mathcal{D}(F(a,c),F(a'',c))\times\mathcal{C}(c,c') \\ {\scriptstyle \beta_{c'}\times\mathrm{id}}\downarrow & & \downarrow \\ \mathcal{D}(F(a,c'),F(a'',c'))\times\mathcal{C}(c,c') & \longrightarrow & \mathcal{D}(F(a,c),F(a'',c)) \end{array}$$

Again the commutativity of this diagram is a direct consequence of the fact that F is simplicial. Then by the universal property of simplicial ends there is a simplicial map

$$\kappa:\mathcal{A}(a,a')\times\mathcal{A}(a',a'')\to\int_{c\in\mathcal{C}}\mathcal{D}(F(a,c),F(a'',c'))$$

such that $\beta_c=\pi''_c\circ\kappa$ for all c in $\mathcal{C}$. Notice that π''_c is the component map given by

$$\pi''_c:\int_{c\in\mathcal{C}}\mathcal{D}(F(a,c),F(a'',c'))\to\mathcal{D}(F(a,c),F(a'',c'))$$

Using the construction above we know that the maps

$$\begin{aligned} \alpha_c&:\mathcal{A}(a,a')\to\mathcal{D}(F(a,c),F(a',c)) \\ \alpha'_c&:\mathcal{A}(a',a'')\to\mathcal{D}(F(a',c),F(a'',c)) \end{aligned}$$

define simplicial natural transformations, which then show that there are unique maps of simplicial sets

$$\begin{aligned} \varphi(F)_{a,a'}&:\mathcal{A}(a,a')\to\int_{c\in\mathcal{C}}\mathcal{D}(F(a,c),F(a',c)) \\ \varphi(F)_{a',a''}&:\mathcal{A}(a',a'')\to\int_{c\in\mathcal{C}}\mathcal{D}(F(a',c),F(a'',c)) \end{aligned}$$

Then by definition we have that $\beta_c=\mathrm{comp}^{\mathcal{D}}\circ(\alpha' c\times\alpha''_c)$, which shows that the upper triangle of solid arrows commutes for all $c\in\mathcal{C}$.

$$\begin{array}{ccc} \mathcal{A}(a,a')\times\mathcal{A}(a',a'') & \xrightarrow{\varphi(F)_{a,a'}\times\varphi(F)_{a',a''}} & \int_{c\in\mathcal{C}}\mathcal{D}(F(a,c),F(a',c))\times\int_{c\in\mathcal{C}}\mathcal{D}(F(a',c),F(a'',c)) \\ {\scriptstyle \alpha_c\times\alpha'_c}\downarrow & \overset{\pi_c\times\pi'_c}{\swarrow} & \downarrow{\scriptstyle\Gamma} \\ \mathcal{D}(F(a,c),F(a',c))\times\mathcal{D}(F(a',c),F(a'',c)) & & \int_{c\in\mathcal{C}}\mathcal{D}(F(a,c),F(a'',c)) \\ & \underset{\mathrm{comp}^{\mathcal{D}}}{\searrow} & \downarrow{\scriptstyle\pi''_c} \\ & & \mathcal{D}(F(a,c),F(a'',c)) \end{array}$$

By construction of the composition Γ we know that the lower right diagram commutes. Hence the whole diagram commutes, which shows that $\kappa=\Gamma\circ(\varphi(F)_{a,a'}\times\varphi(F)_{a',a''})$ by uniqueness of these maps. Hence for any two maps $f:a\to a'$ and $g:a'\to a''$, we have that $\kappa(f,g)=\varphi(F)(g\circ f)=\Gamma(\varphi(F)(f),\varphi(F)(g))=\varphi(F)(g)\circ\varphi(F)(f)$. This finally shows that $\varphi(F):\mathcal{A}\to[\mathcal{C},\mathcal{D}]_{\mathbf{sSet}}$ is indeed a simplicial functor.

Conversely we want to define a map

$$\psi : \mathbf{sSet\text{-}Cat}(\mathcal{A}, [\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}) \to \mathbf{sSet\text{-}Cat}(\mathcal{A} \times \mathcal{C}, \mathcal{D})$$

Therefore, let $G : \mathcal{A} \to [\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}}$ be a simplicial functor. Then define the functor $\psi(G)$ on objects as follows

$$\begin{aligned} \psi(G) : \mathcal{A} \times \mathcal{C} &\to \mathcal{D} \\ (a, c) &\mapsto G(a)[c] \end{aligned}$$

where $G(a) : \mathcal{C} \to \mathcal{D}$ is a simplicial functor by definition. For $f : a \to a'$ an arrow in $\mathcal{A}$ we have a simplicial natural transformation.

$$\psi(G)(f) : G(a) \to G(a')$$

That is, for all $c \in \mathcal{C}$ there are arrows

$$\psi(G)(f)_c : G(a)[c] \to G(a')[c]$$

such that for any arrow $h : c \to c'$ in $\mathcal{C}$ the following diagram commutes.

$$\begin{array}{ccc} G(a)[c] & \xrightarrow{\psi(G)(f)_c} & G(a')[c] \\ {\scriptstyle G(a)[h]}\downarrow & & \downarrow{\scriptstyle G(a')[h]} \\ G(a)[c'] & \xrightarrow[\psi(G)(f)_{c'}]{} & G(a')[c'] \end{array}$$

This shows that $\psi(G)(f, h) : G(a)[c] \to G(a')[c']$ is well defined and that $\psi(G)$ is indeed a simplicial functor.

By construction φ and ψ are inverse to each other and therefore we have established the required isomorphism

$$\mathbf{sSet\text{-}Cat}(\mathcal{A} \times \mathcal{C}, \mathcal{D}) \cong \mathbf{sSet\text{-}Cat}(\mathcal{A}, [\mathcal{C}, \mathcal{D}]_{\mathbf{sSet}})$$

which is natural by construction. □

Using that the cartesian product on **sSet-Cat** defines a symmetric monoidal structure, it follows as an immediate consequence of Proposition 5.1 that the monoidal structure is closed.

Proposition 5.2. *The category* **sSet**-**Cat** *endowed with the cartesian product has the structure of a closed symmetric monoidal category. The unit is given by the category with one object and one morphism* $[0]$ *and the inner hom is given by the simplicial functor category* $[-,-]_{\mathbf{sSet}}$.

Proof. This follows from Proposition 5.1 and the fact that $(\mathbf{sSet\text{-}Cat}, \times, [0])$ is a symmetric monoidal category. □

5.2 The Bergner model structure

Similarly as for ordinary category theory, the notion of an isomorphism of simplicial categories is way too strong. Intuitively we consider two simplicial categories to be weak equivalent if there exists a simplicial functor $F : \mathcal{C} \to \mathcal{D}$ which is essentially surjective up to simplicial homotopy and such that for all $x, y \in \mathcal{C}$ the induced map of simplicial sets

$$F_{x,y} : \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$$

is a Quillen weak equivalence of simplicial sets. Therefore, we want to define a model structure on **sSet-Cat**, which has as weak equivalences exactly these weaker versions of equivalences of simplicial categories. This weak equivalences were first considered by Kan and Dwyer in their work on simplicial localizations [DK80] and later it was shown by Bergner in [Ber04] that they define a model category structure on **sSet-Cat**.

Definition 5.6. Given a simplicial category $\mathcal{C}$ we define its homotopy category or its **category of components** $\mathrm{h}\mathcal{C}$ to be the category with the same objects, and morphisms given by

$$\mathrm{h}\mathcal{C}(x,y) = \pi_0\mathcal{C}(x,y)$$

The composition is induced by the composition of $\mathcal{C}$ via the connected components functor π_0.

Remark 5.3. It is shown in [Col06] that for suitable simplicial categories $\mathcal{C}$ there exists a model structure on $\mathcal{C}$, such that the classical homotopy category $\mathrm{Ho}\,\mathcal{C}$ is equivalent to the category of components $\mathrm{h}\mathcal{C}$.

Definition 5.7. A simplicial functor $F : \mathcal{C} \to \mathcal{D}$ is

(i) a weak equivalence or **Dwyer-Kan equivalence** if

  (a) the induced functor $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is an equivalence of categories

  (b) for any $x, y \in \mathcal{C}$, the induced map

$$F_{x,y} : \mathcal{C}(x,y) \to \mathcal{D}(F(x), F(y))$$

  is a Quillen weak equivalence.

(ii) a fibration if

  (a) for any objects $x, y \in \mathcal{C}$, the induced map

$$F_x, y : \mathcal{C}(x,y) \to \mathcal{D}(F(x), F(y))$$

  is a Kan fibration of simplicial sets.

  (b) the induced functor $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is an isofibration.

Having defined the classes of weak equivalences and fibrations, we can state the following theorem.

Theorem 5.1. ([Ber04]) *There is a cofibrantly generated model category structure on* **sSet-Cat** *with weak equivalences given by the Dwyer-Kan equivalences and fibrations given by Definition 5.7. This model structure is called the Bergner model structure.*

Proposition 5.3. ([Ber04]) *The Bergner model structure on* **sSet-Cat** *is right proper.*

Since fibrations in the Bergner model structure are explicitly defined and not only given by lifting properties, it turns out that the fibrant objects have a nice characterization, given only by local properties.

Definition 5.8. A simplicial category $\mathcal{C}$ is said to be **locally fibrant**, if for all $X, Y \in \mathcal{C}$ the corresponding simplicial hom set $\mathcal{C}(X,Y)$ is a Kan complex.

Corollary 5.1. *A simplicial category $\mathcal{C}$ is fibrant in the Bergner model structure, if and only if it is locally fibrant.*

Proof. Let $\mathcal{C}$ be a simplicial category. Then it is fibrant if the map $\mathcal{C} \to [0]$ is a fibration. That is, for all objects X, Y in $\mathcal{C}$ the induced map

$$\mathcal{C}(X,Y) \to *$$

is a Kan fibration. Therefore, $\mathcal{C}(X,Y)$ is a Kan complex for all objects X, Y and hence $\mathcal{C}$ is locally fibrant. Conversely, let $\mathcal{C}$ be a locally fibrant category. Then to deduce that $\mathcal{C}$ is fibrant, we only need to show that the functor induced by $F : \mathcal{C} \to [0]$ on the categories of components

$$\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}[0]$$

is an isofibration. But since $\mathrm{h}[0] = [0]$ contains only one object, this holds trivially. Therefore we conclude that $\mathcal{C}$ is fibrant. □

An important observation is that even though we are given a closed symmetric monoidal structure on **sSet-Cat**, the model structure is not compatible with this product. More precisely, the category **sSet-Cat** endowed with the Bergner model structure and the cartesian product is not a monoidal model category. Indeed, consider the simplicial category $[1]$ with two objects and only one non-trivial morphism. Denote with $\partial[1]$ the simplicial category with only two objects and only identity morphisms. Then the functor $i : \partial[1] \to [1]$ is a cofibration, since it is one of the generating cofibrations. Now consider the functor

$$i \Box i : ([1] \times \partial[1]) \coprod_{\partial[1] \times \partial[1]} (\partial[1] \times [1]) \to [1] \times [1]$$

Then the left hand side is the discrete simplicial category given by

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \uparrow & & \uparrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

Now let $\mathcal{C}$ denote this category. Then we see that $\mathcal{C}((0,0),(1,1)) = \partial\Delta^1$. But for the category on the right side we have $([1] \times [1])\,((0,0),(1,1)) = \Delta^0 \times \Delta^0 = \Delta^0$. Hence the induced map

$$\partial\Delta^1 \to \Delta^0$$

is not a monomorphism and therefore not a cofibration in the Quillen model structure on simplicial sets. This shows that $i \Box i$ is not a cofibration. Here we use the fact that a cofibration $F : \mathcal{C} \to \mathcal{D}$ in the Bergner model structure induces cofibrations in the Quillen model structure on the simplicial hom sets.

$$F_{x,y} : \mathcal{C}(x,y) \to \mathcal{D}(F(x), F(y))$$

This shows in particular that the cartesian product is not a Quillen bifunctor, hence the model structure can not be monoidal.

5.3 Nerve and realization

The theory of enriched categories leads to an important general categorical construction, which is called the nerve-realization construction.

Definition 5.9. Let $\mathcal{V}$ be a category and consider a category $\mathcal{C}$ which is enriched over $\mathcal{V}$. Then we say that $\mathcal{C}$ is **tensored over** $\mathcal{V}$ if there exists a functor

$$\otimes : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$$

such that for every $v \in \mathcal{V}$ and $x, y \in \mathcal{C}$ there is a natural isomorphism

$$\mathcal{C}(v \otimes x, y) \cong \mathcal{V}(v, \mathcal{C}(x,y))$$

Remark 5.4. Enriched categories allowing a tensoring can be used to give a construction for Kan extensions. Suppose we are given functors $G : \mathcal{C} \to \mathcal{A}$ and $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ and $\mathcal{A}$ are $\mathcal{V}$-enriched categories. Assuming that $\mathcal{A}$ is tensored over $\mathcal{V}$, it follows that the left Kan extension of G along F is given by the coend

$$\mathrm{Lan}_F(G) \cong \int^{c \in \mathcal{C}} \mathcal{D}(F(c), -) \otimes G(c)$$

This presentation is sometimes called the "coend formula". For more details you may consider [Lor15].

Definition 5.10. Let $\mathcal{C}$ be a $\mathcal{V}$-enriched small category. Then the **enriched Yoneda embedding** is the functor given by

$$\begin{aligned} \mathcal{Y} : \mathcal{C} &\to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}) \\ c &\mapsto \mathcal{C}(-, c) \end{aligned}$$

Now assume we are given a functor $i : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ is a small category and $\mathcal{D}$ is tensored over $\mathcal{V}$. Then we can define the $\mathcal{D}$**-coherent nerve functor** as

$$\begin{aligned} N : \mathcal{D} &\to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}) \\ d &\mapsto \mathcal{D}(i(-), d) \end{aligned}$$

Moreover, we can define the corresponding realization functor $\tau : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}) \to \mathcal{D}$ as left Kan extension of i along the enriched Yoneda embedding $\mathcal{Y}$.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad i \quad} & \mathcal{D} \\ {\scriptstyle \mathcal{Y}}\big\downarrow & \nearrow_{\mathrm{Lan}_{\mathcal{Y}}(i)} & \\ \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}) & & \end{array}$$

Proposition 5.4. *Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be a locally small $\mathcal{V}$-enriched cocomplete category tensored over $\mathcal{V}$. Then for every functor $i : \mathcal{C} \to \mathcal{D}$ the $\mathcal{D}$-coherent nerve N is a right adjoint to the realization functor τ, i.e.*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}) \underset{N}{\overset{\tau}{\rightleftarrows}} \mathcal{D}$$

Proof. Let $K \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V})$ and $d \in \mathcal{D}$. Then we have by the coend formula

$$\begin{aligned} \mathcal{D}\left(\tau(K), d\right) &\cong \int_{c \in \mathcal{C}} \mathcal{D}\left(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V})(\mathcal{Y}(c), K) \otimes i(c), d\right) \\ \text{(Yoneda Lemma)} \quad &\cong \int_{c \in \mathcal{C}} \mathcal{D}(K(c) \otimes i(c), d) \\ \text{(tensoring)} \quad &\cong \int_{c \in \mathcal{C}} \mathcal{V}\left(K(c), \mathcal{D}(i(c), d)\right) \\ &= \int_{c \in \mathcal{C}} \mathcal{V}\left(K(c), N(d)(c)\right) \\ &\cong \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V})\left(K, N(d)\right) \end{aligned}$$

Since all the isomorphisms are natural, the statement follows immediately. ☐

Example 5.1. Let T be a topological space and denote with $\mathcal{O}(T)$ the category of open subsets of T. Then define the functor

$$\begin{aligned} i : \mathcal{O}(T) &\longrightarrow \mathbf{Top}_{/T} \\ U &\longmapsto U \hookrightarrow T \end{aligned}$$

Since the category **Top** is locally small, it is enriched over **Set**. Moreover, **Top** is cocomplete, hence it is tensored over **Set**. The same holds then for $\mathbf{Top}_{/T}$ and we can apply the nerve-realization machinery, which induces the following adjunction.

$$\mathrm{Fun}(\mathcal{O}(T)^{\mathrm{op}}, \mathbf{Set}) \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{Top}_{/T}$$

Now notice that $\mathrm{Fun}(\mathcal{O}(T)^{\mathrm{op}}, \mathbf{Set}) = \mathbf{PSh}(T)$ is just the category of presheaves on the topological space T. Hence given a presheaf $\mathcal{F}$, its sheafification is given by $\mathcal{F}^* = (\tau \circ N)(\mathcal{F})$. Therefore, one could define the category of sheaves on T as the essential image of the functor $\tau \circ N$. Similarly, given a space $U \hookrightarrow T$, its étalification is given by $(N \circ \tau)(U)$, hence one could define the category of étale-spaces over T to be the essential image of the functor $N \circ \tau$. Then it follows that there is an equivalence of categories

$$\tau : \mathbf{Sh}(T) \rightleftarrows \mathbf{Et}(T) : N$$

This nice example of a nerve-realization construction is due to [MML92].

5.4 ∞-categories

In the previous section we established a model structure on simplicial categories, providing a homotopy theory of $(\infty,1)$-categories. The advantage of topological categories as a model for $(\infty,1)$-categories is that these objects are relatively easy to define. On the other hand they are hard to work with and the model structure on simplicial categories does not behave well, such as not being a cartesian model structure. Therefore, a more suitable model of $(\infty,1)$-categories is needed. Such a new model is given by simplicial sets, in particular weak Kan complexes. First denoted as quasi-categories they were introduced by Boardman and Vogt in [BV06] and then studied by Joyal in [Joy02]. Moreover, there is a homotopy theory of quasi-categories, which turns out to be Quillen equivalent to the homotopy theory of simplicial categories. Following [Lur09] and [Gro10] we will define an ∞-category to be a weak Kan complex. That is, a simplicial set with certain lifting properties. Then we will state the connection between this new model and the previous one given by simplicial categories. To do so we need to endow **sSet** with a new model structure, called the Joyal model structure.

Definition 5.11. Let $K \in \mathbf{sSet}$ be a simplicial set, then K is a **Kan complex** if for any $0 \leq i \leq n$ and any morphism $\Lambda^n_i \to K$ in **sSet** there is an extension $\Delta^n \to K$ such that the following diagram commutes

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

Remark 5.5. Here Λ^n_i denotes the i-th horn, obtained from the standard n-simplex Δ^n by deleting the interior and the face opposite the i-th vertex.
Equivalently one can define Kan complexes as the fibrant objects in the standard model category structure on **sSet**.

Corollary 5.2. *Let $X \in \mathbf{Top}_{cg}$ be a topological space. Then the associated singular simplicial complex* $\mathrm{Sing}(X)$ *is a Kan complex.*

Proof. Since in the standard model structure on $\mathbf{Top}_{cg}$ every object is fibrant and since by definition the right Quillen functor $\mathrm{Sing} : \mathbf{Top}_{cg} \to \mathbf{sSet}$ preserves fibrations, it follows that $\mathrm{Sing}(X)$ is fibrant in **sSet** for every topological space X. Therefore, $\mathrm{Sing}(X)$ is a Kan complex. □

Remark 5.6. Similarly as one defines the Bergner model structure on **sSet-Cat** it is possible to endow the category of small topological categories $\mathbf{Top}_{cg}$**-Cat** with a model structure, such that the following adjunction

$$\mathbf{sSet\text{-}Cat} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Top}_{cg}\mathbf{\text{-}Cat}$$

defines a Quillen equivalence. Such an adjunction is usually called a change of base adjunction, as we change the base of enrichment. This follows from the general discussion about model structures on enriched categories in section A.3.2. in [Lur09].

Now we notice that given a topological category $\mathcal{C}$ it follows by definition that the corresponding simplicial category $\mathrm{Sing}\,\mathcal{C}$ is fibrant in the Bergner model structure. Indeed, the simplicial category $\mathrm{Sing}\,\mathcal{C}$ has hom spaces given by

$$\mathrm{Sing}\,\mathcal{C}(x,y) = \mathrm{Sing}\left(\mathcal{C}(x,y)\right)$$

Since by Corollary 5.2 the singular complex of a topological space is a Kan complex, it is an immediate consequence of Proposition 5.1 that $\mathrm{Sing}\,\mathcal{C}$ is a fibrant object in the Bergner model structure. We formulate this statement as a corollary.

Corollary 5.3. *For any topological category $\mathcal{C}$, its corresponding simplicial category* $\mathrm{Sing}\,\mathcal{C}$ *is fibrant in the Bergner model structure.*

Definition 5.12. Let $\mathcal{C}$ be an ordinary category. Then define the nerve of $\mathcal{C}$ to be the simplicial set

$$\begin{aligned} N(\mathcal{C}) : \Delta^{\mathrm{op}} &\to \mathbf{Set} \\ [n] &\mapsto \mathrm{Fun}([n], \mathcal{C}) \end{aligned}$$

Here $\mathrm{Fun}([n], \mathcal{C})$ denotes the set of all functors $[n] \to \mathcal{C}$, where $[n]$ is the category with

- $\mathrm{Ob}([n]) = [n] = \{0, 1, ..., n\}$
- $[n](i,j) = \begin{cases} \mathrm{id}_i \text{ if } i = j \\ i \to j \text{ if } i < j \\ \emptyset \text{ if } i > j \end{cases}$

Remark 5.7. The definition of the nerve follows from the generalized nerve-realization construction given by Proposition 5.4. The realization functor then defines a left adjoint to the nerve functor and is constructed as a left Kan extension. In this particular case, we define the realization functor $\tau : \mathbf{sSet} \to \mathbf{Cat}$ as follows

$$\begin{array}{ccc} \Delta & \xrightarrow{i} & \mathbf{Cat} \\ {\scriptstyle \mathcal{Y}}\downarrow & \nearrow_{\mathrm{Lan}_{\mathcal{Y}}(i)} & \\ \mathbf{sSet} & & \end{array}$$

Then the realization is given by $\tau = \mathrm{Lan}_{\mathcal{Y}}(i)$ the left Kan extension of i along the Yoneda embedding $\mathcal{Y}$.

Proposition 5.5. *The following functors form an adjoint pair*

$$\mathbf{sSet} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{Cat} \quad (\tau \dashv N)$$

Proof. Since **Cat** is locally small and cocomplete, we can apply the nerve-realization machinery of Proposition 5.4 which then proves the result. ☐

We now want to investigate what kind of simplicial sets arise from applying the nerve functor to categories.

Proposition 5.6. ([Lur09]) *Let K be a simplicial set. Then the following are equivalent.*

(i) $\exists$ *a small category $\mathcal{C}$ such that $K \cong N(\mathcal{C})$*

(ii) *For any $0 < i < n$ and any morphism $\Lambda^n_i \to K$ there is a unique extension making the diagram commute*

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \big\downarrow & \overset{\exists!}{\nearrow} & \\ \Delta^n & & \end{array}$$

Definition 5.13. An ∞**-category** is a simplicial set K which fulfills the following lifting property. For any $0 < i < n$ and any morphism $\Lambda^n_i \to K$ there is an extension $\Delta^n \to K$ such that the following diagram commutes

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

Example 5.2. By Proposition 5.6 it follows that a first class of examples of ∞-categories is given by the nerves $N(\mathcal{C})$ of categories.

Now that we have established the basic definition of an ∞-category, we want to define analogous structures such as the opposite category of an ∞-category or over and under ∞-categories.

Definition 5.14. Let S be a simplicial set. Then define S^{op} to be the simplicial set given by $S^{\text{op}}_n = S_n$ and with face and degeneracy maps given by

$$d_i^{\text{op}} : S_n^{\text{op}} \to S_{n-1}^{\text{op}} = d_{n-i} : S_n \to S_{n-1}$$
$$s_i^{\text{op}} : S_n^{\text{op}} \to S_{n+1}^{\text{op}} = s_{n-i} : S_n \to S_{n+1}$$

This simplicial set is called the **opposite simplicial set**. If K is an ∞-category, then we define its **opposite ∞-category** to be K^{op}, which is clearly an ∞-category.

To define over and under ∞-categories, we try to establish a universal property, characterizing over and under categories for ordinary categories. Then by passing the universal property to **sSet** we can give analogue definitions for ∞-categories.

Definition 5.15. For $\mathcal{C}, \mathcal{D}$ two categories, we define the **join** $\mathcal{C} * \mathcal{D}$ to be the category with

- objects $\text{Ob}(\mathcal{C} * \mathcal{D}) = \text{Ob}(\mathcal{C}) \amalg \text{Ob}(\mathcal{D})$
- and for $X, Y \in \mathcal{C} * \mathcal{D}$ morphisms given by

$$\mathcal{C} * \mathcal{D}(X, Y) = \begin{cases} \mathcal{C}(X, Y) \text{ if } X, Y \in \mathcal{C} \\ \mathcal{D}(X, Y) \text{ if } X, Y \in \mathcal{D} \\ \emptyset \text{ if } X \in \mathcal{D} \text{ and } Y \in \mathcal{C} \\ * \text{ if } X \in \mathcal{C} \text{ and } Y \in \mathcal{D} \end{cases}$$

Now recall that for a category $\mathcal{C}$ and $X \in \mathcal{C}$, the over-category $\mathcal{C}_{/X}$ is the category with objects given by morphisms $C \to X$ in $\mathcal{C}$, and morphisms given by commutative triangles of the form.

$$\begin{array}{ccc} C & \longrightarrow & C' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Notice that specifying an object $X \in \mathcal{C}$ is equivalent to give a functor $X : [0] \to \mathcal{C}$. Then the over category is described using the universal property, that for any other category $\mathcal{D}$ there is a bijection

$$\mathbf{Cat}(\mathcal{D}, \mathcal{C}_{/X}) \cong \text{Hom}_X(\mathcal{D} * [0], \mathcal{C})$$

where $\text{Hom}_X(\mathcal{D} * [0], \mathcal{C})$ is the subset of $\mathbf{Cat}(\mathcal{D} * [0], \mathcal{C})$ consisting of all those functors F such that $F|_{[0]} = X : [0] \to \mathcal{C}$. We now define an analogous construction called the join of simplicial sets. The simplicial join allows us to define over and under ∞-categories by using the same universal property.

Definition 5.16. Let S, K be two simplicial sets. Then define the **simplicial join** $S * K$ to be the simplicial set given by

$$(S * K)[n] = S[n] \cup K[n] \cup \left(\bigcup_{i+j=n-1} S[i] \times K[j] \right)$$

Remark 5.8. It can be shown that for two ∞-categories S, K their join $S * K$ is also an ∞-category (Proposition 1.2.8.3. in [Lur09]).

Definition 5.17. Let S, K be two simplicial sets and $p : K \to S$ any morphism. Then define the simplicial set $S_{/p}$ as follows.

$$\begin{aligned} S_{/p} : \Delta^{\mathrm{op}} &\to \mathbf{Set} \\ [n] &\mapsto \mathrm{Hom}_p(\Delta^n * K, S) \end{aligned}$$

Here $\mathrm{Hom}_p(\Delta^n * K, S) = \{f \in \mathbf{sSet}(\Delta^n * K, S) \mid f_{|K} = p\}$.

Proposition 5.7. *Let K, S and p be as above. Then the simplicial set $S_{/p}$ has the universal property that for any other $Y \in \mathbf{sSet}$ there is a bijection*

$$\mathbf{sSet}(Y, S_{/p}) \cong \mathrm{Hom}_p(Y * K, S)$$

Hence $S_{/p}$ is the analogue of the over-category in the simplicial setting and is called the ***simplicial set over*** *p. Moreover, if S is an ∞-category, then $S_{/p}$ is also an ∞-category.*

Remark 5.9. The proof of Proposition 5.7 uses the notions of inner and left fibrations which are introduced later in Definition 5.22 and Definition 5.23. Therefore, the reader may skip the proof in a first round.

Proof. Notice that for $Y = \Delta^n$ a standard simplex, the universal property is fulfilled by definition, since we have

$$\mathbf{sSet}(\Delta^n, S_{/p}) \cong (S_{/p})_n = \mathrm{Hom}_p(\Delta^n * K, S)$$

Now recall that any simplicial set can be obtained from standard simplices by gluing. That is, for a simplicial set Y we can define the diagram

$$\begin{aligned} F_Y : \Delta_{/Y} &\longrightarrow \mathbf{sSet} \\ \Delta^n \to Y &\longmapsto \Delta^n \end{aligned}$$

such that colim $F_Y \cong Y$. Therefore, we only need to check that the corresponding sides are compatible under taking colimits. Since for all $X \in \mathbf{sSet}$ the hom functor

$$\mathbf{sSet}(-, X) : \mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}$$

sends colimits in **sSet** to limits, it follows that

$$\mathbf{sSet}(Y, S_{/p}) \cong \mathbf{sSet}(\mathrm{colim}\ F_Y, S_{/p}) \cong \lim \mathbf{sSet}(F_Y, S_{/p})$$

Similarly we obtain for the left hand side

$$\mathrm{Hom}_p(Y * K, S) \cong \mathrm{Hom}_p((\mathrm{colim}\ F_Y) * K, S)$$

Now using that the functor $- * K$ preserves colimits (Remark 1.8.8.2 in [Lur09]) it follows that

$$\mathrm{Hom}_p(Y * K, S) \cong \lim \mathrm{Hom}_p(F_Y * K, S)$$

Since the functor F_Y sends objects to standard simplices, it follows from the above discussion that

$$\lim \mathbf{sSet}(F_Y, S_{/p}) \cong \lim \mathrm{Hom}_p(F_Y * K, S)$$

which concludes the proof of the first part. Now let S be an ∞-category. Then by Proposition 5.13, where we consider the diagram

$$\emptyset \subset K \xrightarrow{p} S \xrightarrow{q} \Delta^0$$

it follows under the assumption that S is an ∞-category that the map q is an inner fibration. Thus we deduce that the induced map

$$S_{/p} \to S \underset{\Delta^0}{\times} (\Delta^0)_{/r} \cong S$$

where $r = q \circ p$, is a right fibration. Then using that right fibrations have the right lifting property with respect to all horn inclusions for $0 < i \leq n$ it follows in particular, that for every map $\Lambda^n_i \to S_{/p}$ there is a lift in the following diagram for $0 < i < n$.

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & S_{/p} \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & S \end{array}$$

This shows that $S_{/p}$ is an ∞-category. A similar proof using the dual statement of Proposition 5.13 on left fibrations shows that, under the same assumptions, $S_{p/}$ is also an ∞-category. □

Remark 5.10. Given an ∞-category $\mathcal{C}$, specifying objects $x \in \mathcal{C}$ is equivalent to give morphisms $x : \Delta^0 \to \mathcal{C}$. Hence we define the over ∞-category as $\mathcal{C}_{/x}$ using Definition 5.17. Similarly to give an arrow $f : x \to y$ in $\mathcal{C}$ it is the same as to give a morphism $f : \Delta^1 \to \mathcal{C}$, hence the over ∞-category $\mathcal{C}_{/f}$ is defined in the same way.

The aim will now be to show that ∞-categories can either be modeled by simplicial categories. More precisely there will be a Quillen equivalence between the category of simplicial sets endowed with a new model structure and the category simplicial categories endowed with the Bergner model structure. To do so we fist need to establish a pair of adjoint functors between those categories.

Definition 5.18. Let J be a finite non-empty ordered set. Then define the simplicial category $\mathfrak{C}\left[\Delta^J\right]$ to be the category with

- objects $\operatorname{Ob}\left(\mathfrak{C}\left[\Delta^J\right]\right) = J$
- and for $i, j \in J$ the morphisms are given by

$$\mathfrak{C}\left[\Delta^J\right](i,j) = \begin{cases} \emptyset \text{ if } j < i \\ N(P_{i,j}) \text{ if } i \leq j \end{cases}$$

  where $P_{i,j}$ denotes the partially ordered set

$$P_{i,j} := \{I \subset J \,|\, i, j \in I \text{ and for all } k \in I \,:\, i \leq k \leq j\}$$

  Then regarding $P_{i,j}$ as a category, its nerve $N(P_{i,j})$ gives the desired simplicial enrichment.
- For $i_0 \leq i_1 \leq \cdots \leq i_n$ in J, the composition

$$\mathfrak{C}\left[\Delta^J\right](i_0, i_1) \times \cdots \times \mathfrak{C}\left[\Delta^J\right](i_{n-1}, i_n) \to \mathfrak{C}\left[\Delta^J\right](i_0, i_n)$$

  is induced by the following map of partially ordered sets.

$$\begin{aligned} P_{i_0,i_1} \times \cdots \times P_{i_{n-1},i_n} &\to P_{i_0,i_n} \\ (I_1, \cdots, I_n) &\mapsto I_1 \cup \cdots \cup I_n \end{aligned}$$

It turns out that the above construction is functorial, i.e. that the above construction defines a functor.

Definition 5.19. Let $f : J \to L$ be a morphism in Δ between two partially ordered finite sets. Then define the simplicial functor on objects

$$\begin{aligned} \mathfrak{C}[f] : \mathfrak{C}\left[\Delta^J\right] &\to \mathfrak{C}\left[\Delta^L\right] \\ i &\mapsto f(i) \end{aligned}$$

and on morphisms for $i \leq j$ in J

$$\mathfrak{C}\left[\Delta^J\right](i,j) \to \mathfrak{C}\left[\Delta^L\right](f(i), f(j))$$

by applying the nerve functor to the map

$$\begin{aligned} P_{i,j} &\to P_{f(i),f(j)} \\ I &\mapsto f(I) \end{aligned}$$

This then defines a functor

$$\begin{aligned} \mathfrak{C} : \Delta &\to \mathbf{sSet\text{-}Cat} \\ [n] &\mapsto \mathfrak{C}\left[\Delta^n\right] \end{aligned}$$

Definition 5.20.

(i) Let $\mathcal{C}$ be a simplicial category. Then the **simplicial nerve** $N_\Delta(\mathcal{C})$ is the simplicial set given by

$$\begin{aligned} N_\Delta(\mathcal{C}) : \Delta^{\mathrm{op}} &\to \mathbf{Set} \\ [n] &\mapsto \mathbf{sSet\text{-}Cat}\left(\mathfrak{C}\left[\Delta^n\right], \mathcal{C}\right) \end{aligned}$$

(ii) Let $\mathcal{C}$ be a topological category. Then the **topological nerve** $N_{\mathrm{T}}(\mathcal{C})$ is the simplicial set given by

$$\begin{aligned} N_T(\mathcal{C}) : \Delta^{\mathrm{op}} &\to \mathbf{Set} \\ [n] &\mapsto \mathbf{sSet\text{-}Cat}\left(\mathfrak{C}\left[\Delta^n\right], \mathrm{Sing}(\mathcal{C})\right) \end{aligned}$$

Remark 5.11. Notice that in general the simplicial resp. topological nerve of a simplicial resp. topological category does not coincide with the nerve of the underlying ordinary category.

Proposition 5.8. ([Lur09])

(i) *Let $\mathcal{C}$ be a simplicial category such that for every pair $X, Y \in \mathcal{C}$ the simplicial set $\mathcal{C}(X,Y)$ is a Kan complex, then the simplicial nerve $N_\Delta(\mathcal{C})$ is an ∞-category.*

(ii) *Let $\mathcal{C}$ be a topological category. Then the topological nerve $N_{\mathrm{T}}(\mathcal{C})$ is an ∞-category.*

Now that we have constructed the coherent nerve functor, we want to define its left adjoint, the realization functor. Consider the Yoneda embedding

$$\begin{aligned} \mathcal{Y} : \Delta &\to \mathbf{sSet} \\ [n] &\mapsto \Delta(-,[n]) \end{aligned}$$

Then since the category of simplicial categories has all small limits, the left Kan extension of $\mathfrak{C} : \Delta \to \mathbf{sSet\text{-}Cat}$ along the Yoneda embedding exists.

$$\begin{array}{ccc} \Delta & \xrightarrow{\mathfrak{C}} & \mathbf{sSet\text{-}Cat} \\ \mathcal{Y}\big\downarrow & \nearrow_{\mathrm{Lan}_{\mathcal{Y}}(\mathfrak{C})} & \\ \mathbf{sSet} & & \end{array}$$

Definition 5.21. The left Kan extension $\mathrm{Lan}_{\mathcal{Y}}(\mathfrak{C})$ constructed above is called the **realization functor** and denoted by

$$\mathfrak{C} : \mathbf{sSet} \to \mathbf{sSet\text{-}Cat}$$

Proposition 5.9. *The realization functor $\mathfrak{C} : \mathbf{sSet} \to \mathbf{sSet\text{-}Cat}$ is left adjoint to the coherent nerve functor N_Δ.*

Remark 5.12. Following Remark 5.4, the extension $\mathfrak{C}$ can be described as a coend

$$\begin{aligned}\mathfrak{C}(K) &= \int^{[m]\in\Delta} \mathbf{sSet}(\Delta^m, K) \times \mathfrak{C}[\Delta^m] \\ &\cong \int^{[m]\in\Delta} K_m \times \mathfrak{C}[\Delta^m]\end{aligned}$$

where we consider the set K_m as a simplicial category with objects given by the set K_m and only identity morphisms.

Proof. Since the category **sSet-Cat** is locally small and co-complete, we can apply the nerve-realization machinery of Proposition 5.4 which proves the statement. □

Combining the above proposition with the adjunction leads to the following corollary.

Corollary 5.4. *The following functors form an adjoint pair.*

$$\mathbf{sSet} \underset{N_T}{\overset{|\mathfrak{C}|}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Top}_{cg}\text{-}\mathbf{Cat}$$

Proof. By definition the topological nerve is given by the composition $N_T = N_\Delta \circ \mathrm{Sing}$ and similarly we have that $|\mathfrak{C}| = |\cdot| \circ \mathfrak{C}$. Having that $\mathfrak{C} \dashv N_\Delta$ and $|\cdot| \dashv \mathrm{Sing}$ it follows that $|\mathfrak{C}| \dashv N_T$. □

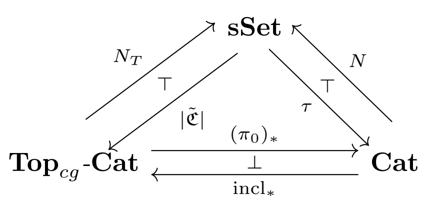
Suppose we are given an ordinary category $\mathcal{C}$, then it can be regarded as a topological category by endowing its hom spaces with the discrete topology. Similarly, it can also be regarded as a simplicial category by endowing its hom spaces with the constant simplicial structure. Applying the coherent nerve on a "trivially" enriched category should agree with applying the regular nerve on the ordinary base category $\mathcal{C}$. To show that this is indeed the case, we need the following proposition.

Proposition 5.10. ([Rie14]) *Let* $\pi_0 : \mathbf{Top}_{cg} \to \mathbf{Set}$ *denote the path component functor. Then* π_0 *is left adjoint to the discrete inclusion functor* $\mathrm{incl} : \mathbf{Set} \to \mathbf{Top}_{cg}$ *which endows any set with the discrete topology.*

Since both functors preserve products, the above adjunction induces a change of base adjunction of categories enriched over $\mathbf{Top}_{cg}$ and **Set**.

$$\mathbf{Top}_{cg}\text{-}\mathbf{Cat} \underset{\mathrm{incl}_*}{\overset{(\pi_0)_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Set}\text{-}\mathbf{Cat} = \mathbf{Cat}$$

Proposition 5.11. ([Rie14]) *The following diagram of adjoint pairs is commutative. That is, the inner and the outer triangle commute.*



Remark 5.13. Using similar arguments, the same also holds for the simplicial nerve-realization adjunction $\mathfrak{C} \dashv N_\Delta$ and the induced change of base adjunction $(\pi_0)_* \dashv \mathrm{const}_*$.

5.5 The Joyal model structure

We now have established a coherent adjunction pair $\mathfrak{C} \dashv N_\Delta$ using the nerve-realization process and a model structure on the category of simplicial categories. Eventually we want to show that the above adjunction is in fact a Quillen equivalence. The Joyal model structure on **sSet** is now the only missing piece in this picture, and shall be discussed in this section.

Theorem 5.2. ([Lur09]) *There exists a combinatorial model structure on the category* **sSet** *where*

(i) *a morphism $p : S \to S'$ of simplicial sets is a cofibration if and only if it is a monomorphism.*

(ii) *a morphism $p : S \to S'$ of simplicial sets is a weak equivalence or* ***Joyal equivalence*** *if and only if the induced simplicial functor*

$$\mathfrak{C}[p] : \mathfrak{C}[S] \to \mathfrak{C}[S']$$

is a Dwyer-Kan equivalence of simplicial categories.

This model structure is called the ***Joyal model structure*** *on* **sSet** *and we have that the adjunction*

$$\mathbf{sSet}_{\text{Joyal}} \underset{N_\Delta}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{sSet}\text{-}\mathbf{Cat}$$

is a Quillen equivalence.

Since fibrations are defined only up to lifting properties, the characterization of fibrant objects can not be obtained as easily as in the Bergner model structure. On the other hand, the motivation behind the Joyal model structure is only revealed by its fibrant objects, which are precisely the ∞-categories.

Theorem 5.3. ([Lur09]) *In the Joyal model structure on* **sSet** *a simplicial set X is fibrant if and only if X is an ∞-category.*

Remark 5.14. Notice that this agrees with Proposition 5.8 in the following sense. We have seen that a simplicial category $\mathcal{C}$ is fibrant if and only if it is locally fibrant. That is, all hom sets are Kan complexes. Now using that the coherent nerve is a right Quillen functor it follows that $N_\Delta(\mathcal{C})$ is a fibrant object in the Joyal model structure, hence is an ∞-category.

We have already seen in the definition of ∞-categories that lifting properties are in general not always given for all horn inclusions. Hence we need a classification of maps having the right lifting properties with respect to certain horn inclusions. This leads to the definition of inner, left and right fibrations, which are due to Joyal [Joy02].

Definition 5.22. Let $p : S \to T$ be a morphism of simplicial sets. Then we call p an **inner fibration** if it has the right lifting property with respect to all inner horn inclusions, i.e. for all commutative diagrams $0 < i < n$

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & S \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & T \end{array}$$

there exists a unique morphism $\Delta^n \to S$ making the diagram commute.

Remark 5.15. Inner fibrations normally arise as induced maps of the form $N_\Delta(\mathcal{C}) \to N_\Delta(\mathcal{D})$ where $\mathcal{C} \to \mathcal{D}$ are fibrations in the Bergner model structure. In fact we can even weaken this assumption. Namely, we can consider maps only satisfying the property that the induced simplicial maps on the hom spaces are Kan fibrations. Most morphisms that we will consider between ∞-categories will be of this form.

Definition 5.23. A morphism of simplicial sets $f : X \to S$ is

(i) a **left fibration** if f has the right lifting property with respect to all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for all $0 \leq i < n$.

(ii) a **right fibration** if f has the right lifting property with respect to all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for all $0 < i \leq n$.

Remark 5.16. The reason why we consider left fibrations is that they characterize simplicial sets fibered over Kan complexes. That is, given a left fibration $f : X \to S$ and any vertex $s \in S$, the fiber $X_s = X \underset{S}{\times} \{s\}$ is a Kan complex and for any edge $f : s \to s'$ in S there is an induced map $f_! : X_s \to X_{s'}$ on the fibers.

Proposition 5.12. ([Lur09]) *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between simplicial categories. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are fibrant, and that for every pair of objects c, c' in $\mathcal{C}$, the associated map*

$$\mathcal{C}(c, c') \to \mathcal{D}(F(c), F(c'))$$

is a Kan fibration. Then the induced map $N_\Delta(F) : N_\Delta(\mathcal{C}) \to N_\Delta(\mathcal{D})$ is an inner fibration between ∞-categories.

Remark 5.17. Left and right fibrations will turn out to play an important role in characterizing so called coCartesian fibrations, which incorporate the notion of symmetric monoidal structures on ∞-categories. Moreover, they also set the necessary framework to state the ∞-categorical Grothendieck construction, being one of the Key ingredients in the proof of Theorem 5.2, which can be found in [Lur09]. In our case the ∞-categorical Grothendieck construction is used to prove the important Proposition 5.17. Therefore, we want to investigate some important stability properties of left fibrations.

Proposition 5.13. ([Joy02]) *Suppose we are given a diagram of simplicial sets*

$$A \subseteq B \xrightarrow{p} X \xrightarrow{q} S$$

where q is an inner fibration and let $r = q \circ p : B \to S$, $p_0 = p_{|A}$, and $r_0 = r_{|A}$. Then the following holds.

(i) *The induced map $X_{p/} \to X_{p_0/} \underset{S_{r_0/}}{\times} S_{r/}$ is a left fibration.*

(ii) *The induced map $X_{/p} \to X_{/p_0} \underset{S_{/r_0}}{\times} S_{/r}$ is a right fibration.*

Lemma 5.2. *Suppose that we are given the following diagram*

$$\emptyset \subseteq \Delta^0 \xrightarrow{c} X \xrightarrow{q} S$$

where q is a Joyal fibration. Then the induced map $X_{c/} \to S_{d/}$ is a Joyal fibration, where $d = q(c)$ is a vertex in S.

Proof. Since a Joyal fibration is in particular also an inner fibration, it follows by Proposition 5.13 that the following map is a left fibration.

$$X_{c/} \to X \underset{S}{\times} S_{d/}$$

Looking at the pullback diagram

$$\begin{array}{ccc} X \underset{S}{\times} S_{d/} & \longrightarrow & S_{d/} \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

we notice that given the Joyal fibration $q : X \to S$, the map $X \underset{S}{\times} S_{d/} \to S_{d/}$ is also a Joyal fibration since fibrations are preserved under pullbacks. Hence taking the composition, we obtain the desired map

$$X_{c/} \to X \underset{S}{\times} S_{d/} \to S_{d/}$$

which is then clearly a Joyal fibration. □

As mentioned before, the proof of Theorem 5.2 relies heavily on the ∞-categorical Grothendieck construction, which we will introduce shortly. The Grothendieck construction allows us to detect left fibrations as fibrant objects in a certain model structure on the category $\mathbf{sSet}_{/B}$ of simplicial sets over B. Moreover, we will establish a Quillen equivalence to the category of simplicial diagrams endowed with the projective model structure, such that fibrant objects correspond to diagrams with component spaces given by Kan complexes.

Remark 5.18. Notice that for a simplicial set X, the left cone over X is given by $X^{\triangleleft} := \Delta^0 * X$ and the right cone by $X^{\triangleright} := X * \Delta^0$. Moreover, for a simplicial set $g : X \to B$ the cone of g is given by the following pushout.

$$\begin{array}{ccc} X & \longrightarrow & X^{\triangleleft} \\ \big\downarrow{\scriptstyle g} & & \big\downarrow \\ B & \longrightarrow & X^{\triangleleft} \underset{X}{\amalg} B \end{array}$$

Definition 5.24. Let B be a simplicial set and consider a morphism $f : X \to Y$ in $\mathbf{sSet}_{/B}$. Then f is a **covariant equivalence** if the induced morphism of cones

$$X^{\triangleleft} \underset{X}{\amalg} B \to Y^{\triangleleft} \underset{Y}{\amalg} B$$

is a Joyal equivalence.

Theorem 5.4. ([Lur09]). *There is a structure of a left proper model category on* $\mathbf{sSet}_{/B}$ *such that a morphism* $f : X \to Y$ *is a*

(i) *cofibration if it is monomorphism*

(ii) *weak equivalence if it is a covariant equivalence*

and such that the fibrant objects are given by the left fibrations over B*. This model structure is called the* ***covariant model structure*** *on* $\mathbf{sSet}_{/B}$.

The ∞-categorical Grothendieck construction establishes an equivalence between simplicial maps $f : X \to B$ having the property that the fibers X_b are Kan complexes for all vertices $b \in B$, and simplicial functors $\mathfrak{C}(B) \to \mathbf{sSet}$ taking values in Kan complexes. The covariant model structure then defines a homotopy theory of such fibered simplicial maps, which detects them as fibrant objects. Therefore, we endow the category of simplicial functors $[\mathfrak{C}(B), \mathbf{sSet}]$ with the projective model structure, such that fibrant objects are given by functors taking values in Kan complexes.

Remark 5.19. Recall that in the projective model structure on $[\mathfrak{C}(B), \mathbf{sSet}]$ a natural transformation between simplicial functors $\eta : F \to G$ is

(i) a weak equivalence if for all objects $c \in \mathfrak{C}(B)$ the induced maps $\eta_c : F(c) \to G(c)$ are Quillen equivalences.

(ii) a fibration if for all objects $c \in \mathfrak{C}(B)$ the induced maps $\eta_c : F(c) \to G(c)$ are Kan fibrations.

(iii) a cofibration if it has the left lifting property with respect to every natural transformation τ between objects in $[\mathfrak{C}(B), \mathbf{sSet}]$ being a weak equivalence and a fibration.

Theorem 5.5. ([Ste15]) *Let B be a simplicial set. Then there is a Quillen equivalence*

$$\mathbf{sSet}_{/B} \underset{\mathrm{Un}_B}{\overset{\mathrm{St}_B}{\rightleftarrows}} [\mathfrak{C}(B), \mathbf{sSet}]$$

between the covariant model structure on $\mathbf{sSet}_{/B}$ and the projective model structure on $[\mathfrak{C}(B), \mathbf{sSet}]$.

Remark 5.20. This result, also known as the straightening-unstraightening Theorem, is the core of the ∞-categorical Grothendieck construction. Given a morphism $f : X \to B$ over B, the straightening functor St_B associates to f a simplicial functor

$$\mathrm{St}_B(f) : \mathfrak{C}(B) \to \mathbf{sSet}$$

Notice that f also induces a functor of simplicial categories $\mathfrak{C}(X) \to \mathfrak{C}(B)$. Then taking the pushout in **sSet-Cat** we obtain the simplicial category

$$\begin{array}{ccc} \mathfrak{C}(X) & \longrightarrow & \mathfrak{C}(B) \\ \downarrow & & \downarrow \\ \mathfrak{C}(X^{\triangleleft}) & \longrightarrow & \mathfrak{C}(X^{\triangleleft}) \coprod_{\mathfrak{C}(X)} \mathfrak{C}(B) \end{array}$$

which we denote by $\mathrm{St}_B(X)$. Then the functor $\mathrm{St}_B(f)$ is given by

$$\mathrm{St}_B(f) : \mathfrak{C}(B) \to \mathrm{St}_B(X) \xrightarrow{\mathrm{St}_B(X)(v,-)} \mathbf{sSet}$$

where v denotes the cone point of $X^{\triangleleft}$. The existence of the right adjoint functor Un_B can be shown formally using the adjoint functor theorem. The idea behind the unstraightening functor is that given a simplicial functor $F : \mathfrak{C}(B) \to \mathbf{sSet}$, we can think of it as an assignment of a simplicial set F_b to every vertex b of B, and a simplicial map $F_b \to F_{b'}$ to every edge from b to b', together with coherence data taking care of the higher dimensional cells of B. The simplicial sets F_b can now be identified with fibers of a morphism $f : X \to B$ over B. Hence the unstraightening functor associates to the functor F a morphism f over B, such that for all vertices b in B, the fibers X_b are homotopy equivalent to $F(b)$.

In the case where the simplicial set B is given by the coherent nerve $N_\Delta(\mathcal{C})$ of a simplicial category, the Grothendieck construction can be slightly modified.

Corollary 5.5. ([Ste15]) *The following pair of functors is a Quillen equivalence*

$$\mathbf{sSet}_{/N_\Delta(\mathcal{C})} \underset{\mathrm{Un}_{N_\Delta(\mathcal{C})}}{\overset{\mathrm{St}_{N_\Delta(\mathcal{C})}}{\rightleftarrows}} [\mathcal{C}, \mathbf{sSet}]$$

Remark 5.21. The following statement on left fibrations can be proven using the Grothendieck construction introduced above. Similarly the dual statements regarding right fibrations hold, which can be found in [Lur09]. Notice that in this case one should consider the contravariant model structure on the category $\mathbf{sSet}_{/\mathbf{B}}$, where fibrant objects are given by right fibrations.

Proposition 5.14. ([Lur09]) *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {}_{p}\searrow \quad \swarrow_{q} & \\ & S & \end{array}$$

where p and q are left fibrations. Then the following are equivalent.

(i) *For each vertex $s \in S$ the induced map $X_s \to Y_s$ is a homotopy equivalence of Kan complexes.*

(ii) *The map f is a Joyal equivalence.*

By definition it is clear, that a trivial Kan fibration is also a left fibration. The converse is not true in general, but if the fibers of the left fibration satisfy certain properties, we can deduce the following important statement.

Proposition 5.15. ([Lur09]) *Let $p : S \to T$ be a left fibration of simplicial sets. Suppose that for every vertex $t \in T$, the fiber S_t is contractible. Then p is a trivial Kan fibration.*

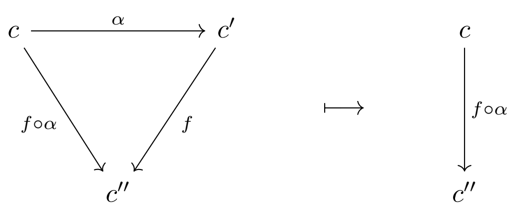
Having established the basic properties of inner, left and right fibrations, we now want to concentrate on the "regular" fibrations in the Joyal model structure. Since Joyal fibrations are defined in Theorem 5.2 only via lifting properties, we want to establish a different characterization. We can do so, if the Joyal fibrations are morphisms between fibrant objects.

Proposition 5.16. ([Lur09]) *Let $p : C \to D$ be a simplicial map where D is a fibrant object in the Joyal model structure. Then p is a Joyal fibration if and only if the following two conditions are satisfied.*

(i) *p is an inner fibration.*

(ii) *For every equivalence $f : d \to d'$ in D and every vertex $c \in C$ with $p(c) = d$, there exists an equivalence $f' : c \to c'$ in C with $p(f') = f$.*

An important application of the Grothendieck construction is the characterization of left fibrations arising from simplicial functors. In particular, we want to show that the forgetful functor $\mathcal{C}_{\alpha/} \to \mathcal{C}_{c/}$ from the category under the morphism $\alpha : c \to c'$, projecting to the left morphism



induces a left fibration $N_\Delta(\mathcal{C}_{\alpha/}) \to N_\Delta(\mathcal{C}_{c/})$.

Proposition 5.17. *Let $\mathcal{C}$ be a simplicial category and $\alpha \in \mathcal{C}$ a morphism. Suppose $\alpha : c \to c'$, then the simplicial map*

$$N_\Delta(\mathcal{C}_{\alpha/}) \to N_\Delta(\mathcal{C}_{/c})$$

induced by the forgetful functor sending a commutative triangle under α to the left morphism, is a left fibration.

Proof. Consider the simplicial functor

$$\begin{aligned} U : \mathcal{C}_{c/} &\to \mathbf{sSet} \\ s &\mapsto N_\Delta(\mathcal{C}_{\alpha/})_s \end{aligned}$$

where $N_\Delta(\mathcal{C}_{\alpha/})_s$ is the fiber of the simplicial map $u : N_\Delta(\mathcal{C}_{\alpha/}) \to N_\Delta(\mathcal{C}_{c/})$ over the vertex $s : c \to c''$. Then by definition, the Grothendieck construction implies that the associated simplicial map $\mathrm{Un}_{N_\Delta(\mathcal{C}_{c/})}(U) \cong u$. Notice that this follows from the property that the fibers of the simplicial map u are given by discrete simplicial sets (as shown below), hence the coherence datum is trivial. In general, the given construction does not create an isomorphism, but rather a weak equivalence. Since Corollary 5.5 implies that the unstraightening functor is a right Quillen functor, it preserves fibrations. We recall that left fibrations are the fibrant objects in the covariant model structure, hence it is left to show that the functor U is fibrant in the projective model structure. This means, we need to show that for all objects $s \in \mathcal{C}_{c/}$, the simplicial set $N_\Delta(\mathcal{C}_{\alpha/})_s$ is a Kan complex. By definition we have that the fiber is given by the pullback $N_\Delta(\mathcal{C}_{\alpha/}) \underset{N_\Delta(\mathcal{C}_{c/})}{\times} *$ which is isomorphic to $N_\Delta\left(\mathcal{C}_{\alpha/} \underset{\mathcal{C}_{c/}}{\times} [0]\right)$ since the coherent nerve functor preserves limits. Looking at the pullback in **sSet-Cat** we notice

$$\begin{array}{ccc} \mathcal{C}_{\alpha/} \underset{\mathcal{C}_{c/}}{\times} [0] & \longrightarrow & [0] \\ \downarrow & & \downarrow s \\ \mathcal{C}_{\alpha/} & \longrightarrow & \mathcal{C}_{c/} \end{array}$$

that it is given by the discrete groupoid $\coprod_{\mathcal{C}_\alpha(c',c'')} [0]$, where $\mathcal{C}_\alpha(c',c'')$ is the set of morphisms $g : c' \to c''$ in $\mathcal{C}$ satisfying $g \circ \alpha = s$. Hence it follows that its coherent nerve is a discrete simplicial set, which is indeed a Kan complex. This shows that U is fibrant in the projective model structure. $\square$

5.6 Symmetric monoidal ∞-categories

As we have seen in Section 2, structured spectra can be defined using the Day convolution. Therefore, we want to apply the coherent nerve functor to the topological functor category $[\mathcal{C}, \mathbf{Top}^*_{cg}]$. We then get a corresponding ∞-category of functors, which should inherit a symmetric monoidal structure. But what is a monoidal structure on an ∞-category? To answer this question, we need to define a class of functors which incorporate all the information encoded in the definition of a symmetric monoidal structure. This section is devoted to introduce such functors, called Grothendieck opfibrations, and their corresponding analogues in the ∞-categorical setting. We use [Gro10] and [Lur09] as main references in this section.

Definition 5.25. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. Then specifying an object $d \in \mathcal{D}$ is equivalent to give a functor $d : [0] \to \mathcal{D}$. Now define the **fiber of F over d** to be the pullback $\mathcal{C}_d$ given by

$$\begin{array}{ccc} \mathcal{C}_d & \longrightarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow F \\ [0] & \xrightarrow{d} & \mathcal{D} \end{array}$$

Definition 5.26. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $f : c_1 \to c_2$ be a morphism in $\mathcal{C}$ over the morphism $\alpha : d_1 \to d_2$ in $\mathcal{D}$, i.e. we have that $F(f) = \alpha$. Then call f an **F-coCartesian morphism** if for all objects c_3 in $\mathcal{C}$ the hom set $\mathcal{C}(c_2, c_3)$ is the pullback given by the following diagram.

$$\begin{array}{ccc} \mathcal{C}(c_2, c_3) & \xrightarrow{(\cdot)\circ f} & \mathcal{C}(c_1, c_3) \\ F_{c_2,c_3}\downarrow & \lrcorner & \downarrow F_{c_1,c_3} \\ \mathcal{D}(F(c_2), F(c_3)) & \xrightarrow{(\cdot)\circ\alpha} & \mathcal{D}(F(c_1), F(c_3)) \end{array}$$

Remark 5.22. Notice that f is an F-coCartesian morphism if and only if for any morphism $h : c_1 \to c_3$ in $\mathcal{C}$ and $\gamma = F(h) : d_1 \to d_3$ it follows that for every $\beta : d_2 \to d_3$ such that $\gamma = \beta \circ \alpha$, there is a unique morphism $g : c_2 \to c_3$ such that $\beta = F(g)$ and $h = g \circ f$.

Definition 5.27. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is a **Grothendieck opfibration** if for all $c_1 \in \mathcal{C}$ and for all morphisms $\alpha : F(c_1) \to d_2$ in $\mathcal{D}$ there is an F-coCartesian morphism $f : c_1 \to c_2$ such that $F(f) = \alpha$.

The reason why we consider Grothendieck opfibrations is due to the following fact. Morphisms in $\mathcal{D}$ induce functors between the fibers of opfibrations over the source and target objects of the given morphism. More precisely, let $F : \mathcal{C} \to \mathcal{D}$ be a Grothendieck opfibration. Then let $\alpha : d_1 \to d_2$ be any morphism in $\mathcal{D}$ and define the functor

$$\alpha! : \mathcal{C}_{d_1} \to \mathcal{C}_{d_2}$$
$$c_1 \mapsto \text{codom}(f)$$

where c_1 is an object of $\mathcal{C}$ such that $F(c_1) = d_1$. Then since F is a Grothendieck opfibration, there exists a F-coCartesian morphism $f : c_1 \to c_2$ in $\mathcal{C}$ with $F(f) = \alpha$. Then $\text{codom}(f) = c_2$ which is clearly an object of $\mathcal{C}_{d_2}$.

Now let $g : c_1 \to \tilde{c_1}$ be an arrow in $\mathcal{C}_{d_1}$. Then by hypothesis there are two F-coCartesian morphisms $f : c_1 \to c_2$ and $\tilde{f} : \tilde{c_1} \to \tilde{c_2}$. Consider the morphism $h = \tilde{f} \circ g : c_1 \to \tilde{c_2}$ which has the property that $F(\tilde{f} \circ g) = F(\tilde{f}) \circ F(g) = F(\tilde{f}) = \alpha$. Now by Remark 5.22 it follows that for every $\beta : d_2 \to d_2$ such that $\gamma = F(h) = \beta \circ \alpha$ there is a unique morphism $\tilde{g} : c_2 \to \tilde{c_2}$ such that $\beta = F(\tilde{g})$. But as seen before $\gamma = F(h) = \alpha$ hence the only morphism is $\beta = \text{id}_{d_2}$. Hence there is a unique morphism $\tilde{g} : c_2 \to \tilde{c_2}$ such that $F(\tilde{g}) = \text{id}_{d_2}$, which shows that $\tilde{g}$ is a morphism in $\mathcal{C}_{d_2}$. Hence for morphisms, the functor is given by

$$\alpha! : \mathcal{C}_{d_1} \to \mathcal{C}_{d_2}$$
$$g : c_1 \to \tilde{c_2} \mapsto \tilde{g} : c_2 \to \tilde{c_2}$$

This property motivates the definition of Grothendieck opfibrations. An important observation is that by the above constructions one might assume that the association $d \mapsto \mathcal{C}_d$ is functorial, which is not true in general. This is due to the fact that for $\alpha : d_1 \to d_2$ and $\beta : d_2 \to d_3$ two composable morphisms in $\mathcal{D}$ one has $(\beta \circ \alpha)! \neq \beta! \circ \alpha!$. But one can show that there exists a unique natural isomorphism

$$(\beta \circ \alpha)! \cong \beta! \circ \alpha!$$

Now we want to use the theory of Grothendieck opfibrations to give an equivalent definition of a symmetric monoidal structure on a category $\mathcal{C}$.

Remark 5.23. Let $(\mathcal{C}, \otimes, 1, \sigma)$ be a symmetric monoidal category. Then define the category $\mathcal{C}^{\otimes}$ as follows.

(i) Objects are finite and possibly empty sequences of objects of $\mathcal{C}$ denoted by $[C_1, ..., C_m]$.

(ii) A morphism from $[C_1, ...C_n]$ to $[C'_1, ..., C'_m]$ is given by a pair $(\alpha, \{f_j\}_{1 \leq j \leq m})$ consisting of a morphism $\alpha : \langle n \rangle \to \langle m \rangle$ of finite pointed sets together with a family of morphisms

$$f_j : \bigotimes_{\alpha(i)=j} C_i \to C'_j$$

for all $1 \leq j \leq m$ in the category $\mathcal{C}$. Notice that if $\alpha^{-1}(j) = \emptyset$ for some j, then $f_j : 1_{\mathcal{C}} \to C'_j$.

(iii) Composition in $\mathcal{C}^{\otimes}$ is defined as follows. Given two morphisms $f : [C_1, ..., C_n] \to [C'_1, ..., C'_m]$ and $g : [C'_1, ..., C'_m] \to [C''_1, ..., C''_l]$, which are determined by the morphisms $\alpha : \langle n \rangle \to \langle m \rangle$ and $\beta : \langle m \rangle \to \langle l \rangle$, then the composition $g \circ f$ is the map determined by the morphism $\beta \circ \alpha : \langle n \rangle \to \langle l \rangle$ and the family of morphisms

$$(g \circ f)_k : \bigotimes_{(\beta \circ \alpha)(i)=k} C_i \xrightarrow{\cong} \bigotimes_{\beta(j)=k} \bigotimes_{\alpha(i)=j} C_i \xrightarrow{\bigotimes_{\beta(j)=k} f_j} \bigotimes_{\beta(j)=k} C'_j \xrightarrow{g_k} C''_k$$

for $1 \leq k \leq l$, i.e. $(g \circ f) = (\beta \circ \alpha, \{(g \circ f)_k\}_{1 \leq k \leq l})$.

Lemma 5.3. *For every monoidal category* $(\mathcal{C}, \otimes, 1)$*, the category* $\mathcal{C}^{\otimes}$ *is equipped with a forgetful functor*

$$\begin{aligned} p : \mathcal{C}^{\otimes} &\to \textbf{FinSet}^* \\ [C_1, ..., C_n] &\mapsto \langle n \rangle = \{1, ..., n\} \amalg \{*\} \\ (\alpha, \{f_j\}_{1 \leq j \leq m}) &\mapsto \alpha : \langle n \rangle \to \langle m \rangle \end{aligned}$$

to the category of finite pointed sets.

Proof. Since the functor is already defined on objects and morphisms, we only need to show that it preserves identity morphisms and compositions. By construction we have that p preserves composition and that the identity map $f : [C_1, ..., C_n] \to [C_1, ..., C_n]$ is given by the pair $(\mathrm{id}_{\langle n \rangle}, \{\mathrm{id}_{C_i}\}_{1 \leq i \leq n})$. Hence by applying the functor p we obtain the identity map on the finite pointed set $\langle n \rangle$. □

Using Definition 5.25 we denote the fiber of p over the set $\langle n \rangle$ by $\mathcal{C}^{\otimes}_{\langle n \rangle}$. Now we want to show that the functor p is in fact a Grothendieck opfibration, such that we have induced functors on the fibers.

Proposition 5.18. *If* $(\mathcal{C}, \otimes, 1, \sigma)$ *is a symmetric monoidal category, then the forgetful functor* $p : \mathcal{C}^{\otimes} \to \textbf{FinSet}^*$ *is a Grothendieck opfibration. Moreover, this functor satisfies the* ***Segal condition****. That is, the Segal maps*

$$\mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\times n}$$

are equivalences of categories for all $n \geq 0$*. Conversely, any Grothendieck opfibration* $p : \mathcal{D} \to \textbf{FinSet}^*$ *satisfying the Segal condition encodes a symmetric monoidal structure on* $\mathcal{C} = \mathcal{D}_{\langle 1 \rangle}$*.*

Proof. First we proof that a Grothendieck opfibration $p : \mathcal{D} \to \textbf{FinSet}^*$ satisfying the Segal condition induces a symmetric monoidal structure on $\mathcal{D}_{\langle 1 \rangle}$.
Consider the morphism in $\textbf{FinSet}^*$ given by

$$\begin{aligned} m : \langle 2 \rangle &\to \langle 1 \rangle \\ 1 &\mapsto 1 \\ 2 &\mapsto 1 \end{aligned}$$

Using that p is a Grothendieck opfibration we obtain an induced functor $m!$ on the fibers. Moreover, by the Segal condition the fibers can be identified with finite products of $\mathcal{D}_{\langle 1 \rangle} = \mathcal{C}$. Hence we obtain a functor

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\simeq} \mathcal{D}_{\langle 2 \rangle} \xrightarrow{m!} \mathcal{D}_{\langle 1 \rangle} = \mathcal{C}$$

This functor is denoted with $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and will be the monoidal product on $\mathcal{C}$. Notice that by the Segal condition $\mathcal{D}_{\langle 0 \rangle} \simeq [0]$. Moreover, the unique map $u : \langle 0 \rangle \to \langle 1 \rangle$ induces a functor

$$[0] \simeq \mathcal{D}_{\langle 0 \rangle} \xrightarrow{u!} \mathcal{D}_{\langle 1 \rangle} = \mathcal{C}$$

which classifies an object in $\mathcal{C}$ denoted by $1_{\mathcal{C}}$. This object will be the tensor unit. To construct the symmetric braiding σ, consider the morphism

$$\begin{aligned} s : \langle 2 \rangle &\to \langle 2 \rangle \\ 1 &\mapsto 2 \\ 2 &\mapsto 1 \end{aligned}$$

Notice that $m \circ s = m$ hence there is a natural isomorphism of functors from $\mathcal{D}_{\langle 2 \rangle}$ to $\mathcal{C}$.

$$\sigma : m! \cong m! \circ s!$$

Consider now the Segal map $(\rho^1!, \rho^2!) : \mathcal{D}_{\langle 2\rangle} \to \mathcal{C} \times \mathcal{C}$ and let $X, Y \in \mathcal{C}$ be two objects. Then there is an object $D \in \mathcal{D}_{\langle 2\rangle}$ such that $(\rho^1!, \rho^2!)(D) \cong (X, Y)$. Since σ is a natural isomorphism we have a family of isomorphisms in $\mathcal{C} \times \mathcal{C}$ given by

$$\sigma_D : m!(D) \cong (m! \circ s!)(D)$$

Now notice that under the hypothesis that $(\rho^1!, \rho^2!)(D) \cong (X, Y)$ it follows that $(\rho^1!, \rho^2!)(s!(D)) \cong (Y, X)$. Therefore, we have a family of isomorphisms

$$\sigma_{X,Y} : X \otimes Y \cong m!(D) \xrightarrow{\sigma_D} (m! \circ s!)(D) \cong Y \otimes X$$

which will be the braiding datum, satisfying the conditions of Definition 2.3. Now it is left to show that the constructed datum $(\mathcal{C}, \otimes, 1_{\mathcal{C}}, \sigma)$ defines a braided monoidal category, i.e. that there are natural isomorphisms

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

$$l_X : 1_{\mathcal{C}} \otimes X \to X \qquad\qquad r_X : X \otimes 1_{\mathcal{C}} \to X$$

and the axioms **A1 - A2** from Definition 2.1 are satisfied.
First consider the morphisms

$$\langle 3\rangle \xrightarrow{m\times \mathrm{id}} \langle 2\rangle \xrightarrow{m} \langle 1\rangle \qquad\qquad \langle 3\rangle \xrightarrow{\mathrm{id}\times m} \langle 2\rangle \xrightarrow{m} \langle 1\rangle$$

where we define

$$(m \times \mathrm{id})(i) = \begin{cases} 1 \text{ if } i = 1 \\ 1 \text{ if } i = 2 \\ 2 \text{ if } i = 3 \end{cases} \qquad\qquad (\mathrm{id} \times m)(i) = \begin{cases} 1 \text{ if } i = 1 \\ 2 \text{ if } i = 2 \\ 2 \text{ if } i = 3 \end{cases}$$

Then by composing with m we see that $(m \times \mathrm{id}) \circ m = (\mathrm{id} \times m) \circ m$, which induces a natural isomorphism

$$\alpha : (m \times \mathrm{id})! \circ m! \cong (\mathrm{id} \times m)! \circ m!$$

Using the Segal condition, we obtain a family of natural isomorphisms

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$$

which define precisely the associativity constraint.
Now consider the morphism $l : \langle 1\rangle \to \langle 2\rangle$ given by $l(1) = 2$ and the morphism $r : \langle 1\rangle \to \langle 2\rangle$ given by $r(1) = 1$. Then notice that $m \circ l = \mathrm{id}_{\langle 1\rangle} = m \circ r$. Hence there are natural isomorphisms

$$\mathfrak{l} : m! \circ l! \cong \mathrm{id}_{\mathcal{C}} \qquad \mathfrak{r} : m! \circ r! \cong \mathrm{id}_{\mathcal{C}}$$

Now using that $(\rho^1!, \rho^2!)$ defines an equivalence of categories, we obtain the natural isomorphisms

$$\mathfrak{l} : \otimes \circ (\rho^1!, \rho^2!) \circ l! \cong \mathrm{id}_{\mathcal{C}} \qquad \mathfrak{r} : \otimes \circ (\rho^1!, \rho^2!) \circ r! \cong \mathrm{id}_{\mathcal{C}}$$

By construction we have that

$$\begin{aligned} (\rho^1!, \rho^2!) \circ l! : \mathcal{C} &\to \mathcal{C} \times \mathcal{C} \\ X &\mapsto (1_{\mathcal{C}}, X) \end{aligned} \qquad\qquad \begin{aligned} (\rho^1!, \rho^2!) \circ r! : \mathcal{C} &\to \mathcal{C} \times \mathcal{C} \\ X &\mapsto (X, 1_{\mathcal{C}}) \end{aligned}$$

and therefore the above natural isomorphisms induce isomorphisms of the form

$$\mathfrak{l}_X : (\otimes \circ (\rho^1!, \rho^2!) \circ l!)(X) = 1_{\mathcal{C}} \otimes X \xrightarrow{\cong} X$$
$$\mathfrak{r}_X : (\otimes \circ (\rho^1!, \rho^2!) \circ r!)(X) = X \otimes 1_{\mathcal{C}} \xrightarrow{\cong} X$$

which show that $1_{\mathcal{C}}$ is indeed the tensor unit of $\otimes$. The strategy to prove the triangle, the pentagon and the hexagon axioms, is to find the right equalities of compositions of morphisms in the category $\mathbf{FinSet}^*$, such that they induce natural isomorphism of functors between the fibers. Then using the Segal condition we can transform those isomorphisms into the desired commutative diagrams in $\mathcal{C}$.

Conversely, we need to show that given a symmetric monoidal category $(\mathcal{C}, \otimes, 1, \sigma)$ the corresponding forgetful functor $p : \mathcal{C}^{\otimes} \to \mathbf{FinSet}^*$ is a Grothendieck opfibration. That is, for every object $[C_1, ..., C_n]$ and every morphism $\alpha : \langle n \rangle \to \langle m \rangle$ in $\mathbf{FinSet}^*$, there is a p-coCartesian morphism $f : [C_1, ..., C_n] \to [C'_1, ..., C'_m]$ such that $p(f) = \alpha$.

Let $[C_1, ..., C_n]$ be an object of $\mathcal{C}^{\otimes}$ and let $\alpha : \langle n \rangle \to \langle m \rangle$ be any morphism in $\mathbf{FinSet}^*$. Consider the morphism f given by the pair $f = (\alpha, \{\mathrm{id}_{\bigotimes_{\alpha(i)=j} C_i}\}_{1 \leq j \leq m})$. In other words f defines the following morphism.

$$f : [C_1, ..., C_n] \to \left[\bigotimes_{\alpha(i)=1} C_i, ..., \bigotimes_{\alpha(i)=m} C_i \right]$$

Then by definition $p(f) = \alpha$. Hence it is left to show that f is a p-coCartesian morphism. Therefore, let $h : [C_1, ..., C_n] \to [C''_1, ..., C''_k]$ be another morphism in $\mathcal{C}^{\otimes}$ given by the pair $h = (\gamma, \{h_t\}_{1 \leq t \leq k})$. By definition the morphisms h_t are of the form

$$h_t : \bigotimes_{\gamma(i)=t} C_i \to C''_t$$

and $\gamma : \langle n \rangle \to \langle k \rangle$. Then let $\beta : \langle m \rangle \to \langle k \rangle$ be a morphism such that $\gamma = \beta \circ \alpha$. We construct the morphism $g : [C'_1, ..., C'_m] \to [C''_1, ..., C''_k]$ as follows. For $1 \leq l \leq k$ define

$$g_l : \bigotimes_{\beta(j)=l} C'_j = \bigotimes_{\beta(j)=l} \bigotimes_{\alpha(i)=j} C_i = \bigotimes_{\gamma(i)=l} C_i \xrightarrow{h_l} C''_l$$

Then $g = (\beta, \{g_l\}_l)$ and therefore $p(g) = \beta$ and by construction $h = g \circ f$. This shows that f is indeed a p-coCartesian morphism and it is unique up to isomorphism. Hence p is a Grothendieck opfibration.

To show that p satisfies the Segal condition, let

$$\rho^j : \langle n \rangle \to \langle 1 \rangle$$

be the unique morphism such that $(\rho^j)^{-1}(1) = j$. Hence for any $n \geq 0$ there is a family of functors

$$\rho^j! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle} \simeq \mathcal{C}$$

Then by the universal property of the product, there is a unique morphism $(\rho^1!, ..., \rho^n!) : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\times n}$ such that the following diagram commutes for all $1 \leq j \leq n$.

$$\begin{array}{ccc} & & \mathcal{C}^{\times n} \\ & \overset{(\rho^1!, \cdots, \rho^n!)}{\nearrow} & \downarrow{\scriptstyle \pi_j} \\ \mathcal{C}^{\otimes}_{\langle n \rangle} & \xrightarrow[\rho^j!]{} & \mathcal{C} \end{array}$$

Now recall the definition of the induced functor $\rho^j!$ on the fibers. Let $[C_1, ..., C_n]$ be an object in the fiber and let $\tilde{\rho^j}$ denote the p-coCartesian lift of ρ^j. As we have seen above such a lift is given by the pair $\tilde{\rho^j} = (\rho^j, \mathrm{id}_{C_j}) : [C_1, ..., C_n] \to [C_j]$. Hence the induced functor is given by

$$[C_1, ..., C_n] \mapsto \mathrm{codom}(\tilde{\rho^j}) = [C_j]$$

which is simply the j-th projection. Hence the datum $\left(\mathcal{C}^{\otimes}_{\langle n \rangle}, \{\rho^j!\}_{1 \leq j \leq n}\right)$ satisfies the universal property of the product $\mathcal{C}^{\times n}$ in $\mathbf{Cat}$ up to equivalence of categories, which then implies the Segal condition. $\square$

Proposition 5.18 is a powerful tool as we can encode the structure of a symmetric monoidal category into a single morphism, satisfying a certain lifting property and satisfying the Segal condition. To define a monoidal structure on an ∞-category in the sense of Definition 2.1, all the higher morphisms would lead to an infinite amount of coherence diagrams. Using the above proposition the structure of the category $\mathbf{FinSet}^*$ will take care of those coherence conditions and allows a very concise definition of a symmetric monoidal ∞-category. To do so, we want to investigate how the nerve functor acts on Grothendieck opfibrations, i.e. for $F : \mathcal{C} \to \mathcal{D}$ an opfibration what can we say about the induced morphism on the nerve $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$. It will turn out that such morphisms are given by coCartesian fibrations of simplicial sets.

Definition 5.28. Let $p : X \to S$ be an inner fibration of simplicial sets. Let $f : \Delta^1 \to X$ specify an edge $x \to y$ in X. Then we say that f is a p-**cartesian edge** if the induced map

$$X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$$

is a trivial Kan fibration.

Remark 5.24. Notice that given an edge $f : \Delta^1 \to X$ in X the inner fibration p induces a morphism $X_{/f} \to S_{/p(f)}$ where $p(f) : \Delta^1 \xrightarrow{f} X \xrightarrow{p} S$ defines an edge in S. By precomposing with the degeneracy map we obtain $y : \Delta^0 \xhookrightarrow{s_1} \Delta^1 \xrightarrow{f} X$, which then similarly induces a morphism $X_{/f} \to X_{/y}$. Analogously one defines the induced map $S_{/p(f)} \to S_{/p(y)}$. Moreover, the following diagram commutes by construction.

$$\begin{array}{ccc} X_{/f} & \longrightarrow & S_{/p(f)} \\ \downarrow & & \downarrow \\ X_{/y} & \longrightarrow & S_{/p(y)} \end{array}$$

Hence by the universal property of the pullback, there is a unique map

$$X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$$

Definition 5.29. A morphism of simplicial sets $p : X \to S$ is a **cartesian fibration** if

(i) the morphism p is an inner fibration.

(ii) for every edge $f : x \to y$ in S and every vertex $\tilde{y}$ in X with $p(\tilde{y}) = y$ there exists a p-cartesian edge $\tilde{f} : \tilde{x} \to \tilde{y}$ with $p(\tilde{f}) = f$.

We say that p is a **coCartesian fibration**, if the opposite morphism $p^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is a cartesian fibration.

Remark 5.25. The motivation behind the definition of coCartesian fibrations is that they are the ∞-categorical analogues of Grothendieck opfibrations.

Proposition 5.19. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. Then $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$ is an inner fibration. Moreover, $N(F)$ is a coCartesian fibration if and only if F is a Grothendieck opfibration*

Remark 5.26. To prove this proposition it is appropriate to use an equivalent and more tangible definition of a coCartesian fibration than the one given in Definition 5.29. Notice that for $p : K \to S$ a morphism of simplicial sets, we constructed the simplicial set $S_{/p}$ being the ∞-categorical analogue to the notion of over categories. This construction can be dualized to define the simplicial set $S_{p/}$ being the ∞-categorical analogue to under categories. Using this construction, we are able to restate Definition 5.29 as follows.

Definition 5.30. A morphism of simplicial sets $p : X \to S$ is a **coCartesian fibration** if

(i) the morphism p is an inner fibration.

(ii) for every vertex $c_1 \in X$ and every edge $\alpha : p(c_1) = d_1 \to d_2$ in S there is a p-coCartesian edge $f : c_1 \to c_2$ such that $p(f) = \alpha$.

Notice that $f : c_1 \to c_2$ is said to be a **p-coCartesian edge**, if the following morphism is a trivial Kan fibration

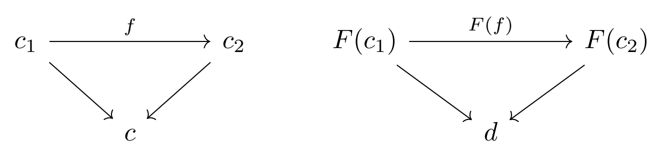
$$X_{f/} \to X_{c_1/} \times_{S_{p(c_1)/}} S_{p(f)/}$$

In order to prove Proposition 5.19 we need the following lemma.

Lemma 5.4. ([Gro10]) *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. Then a morphism $f : c_1 \to c_2$ is F-coCartesian if and only if the following functor is an isomorphism of categories*

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{D}_{F(c_1)/}} \mathcal{D}_{F(f)/}$$

Remark 5.27. Notice that $\mathcal{C}_{c_1/}$ and $\mathcal{D}_{F(c_1)/}$ denote the regular under categories under the objects c_1 and $F(c_1)$ respectively, whereas $\mathcal{C}_{f/}$ and $\mathcal{D}_{F(f)/}$ denote the categories with objects given by commutative triangles



and morphisms given by $g : c \to c'$ resp. $h : d \to d'$, such that the obvious diagrams commute.

proof of Proposition 5.19.
"⇒" Suppose we are given a Grothendieck opfibration $F : \mathcal{C} \to \mathcal{D}$ between ordinary categories. Then denote $C = N(\mathcal{C})$ and $D = N(\mathcal{D})$ and let $p = N(F)$ be the induced morphism of simplicial sets $p : C \to D$. Notice that by Example 5.2 we already have that C, D are ∞-categories and it follows that p is automatically an inner fibration of ∞-categories. Now let $c_1 \in C$ be a vertex. By construction, we can identify vertices of $N(\mathcal{C})$ with objects of $\mathcal{C}$, hence $c_1 \in \mathcal{C}$. Similarly edges in $N(\mathcal{D})$ can be identified with morphisms in $\mathcal{D}$, hence for any edge $\alpha : p(c_1) = d_1 \to d_2$ in D we can consider $\alpha : F(c_1) = d_1 \to d_2$ a morphism in $\mathcal{D}$. Since F is an opfibration there is an F-coCartesian lift $f : c_1 \to c_2$ of α. Then f can be identified with an edge $f : c_1 \to c_2$ in C. Now we apply Lemma 5.4 to the given F-coCartesian morphism $f : c_1 \to c_2$. By construction it follows that the nerve functor is compatible with the coslice construction, i.e. there are natural isomorphisms of simplicial sets

$$N(\mathcal{C}_{f/}) \cong N(\mathcal{C})_{f/} = C_{f/} \qquad N(\mathcal{C}_{c_1/}) \cong N(\mathcal{C})_{c_1/} = C_{c_1/}$$
$$N(\mathcal{D}_{F(f)/}) \cong N(\mathcal{D})_{N(F)(f)/} = D_{p(f)/} \qquad N(\mathcal{D}_{F(c_1)/}) \cong N(\mathcal{D})_{N(F)(c_1)/} = D_{p(c_1)}$$

Therefore, it follows that

$$C_{f/} \cong N(\mathcal{C}_{f/}) \to N(\mathcal{C}_{c_1/} \times_{\mathcal{D}_{F(c_1)/}} \mathcal{D}_{F(f)/}) \cong C_{c_1/} \times_{D_{p(c_1)/}} D_{p(f)/}$$

is an isomorphism of simplicial sets and therefore a trivial Kan fibration. This shows that $f : c_1 \to c_2$ considered as an edge in C is indeed p-coCartesian and hence p is a coCartesian fibration.

"⇐" Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $p : C \to D$ denote the morphism of simplicial sets induced by the nerve functor as before. Assume now that $N(F) = p$ is a coCartesian fibration. We need to show that F is a Grothendieck opfibration, i.e. for every $c_1 \in \mathcal{C}$ and every morphism $\alpha : F(c_1) = d_1 \to d_2$ in $\mathcal{D}$ there is a F-coCartesian lift $f : c_1 \to c_2$ of α. Therefore, let $c_1 \in \mathcal{C}$ and let $\alpha : d_1 \to d_2$ be any morphism in $\mathcal{D}$.

Similarly as before we can identify objects of $\mathcal{C}$ and $\mathcal{D}$ with vertices of C and D respectively, and morphisms with edges in the same way. Hence we can look at $c_1 : \Delta^0 \to C$ and $\alpha : \Delta^1 \to D$. Since p is a coCartesian fibration there is a p-coCartesian edge $f : c_1 \to c_2$ such that $p(f) = \alpha$. This shows that there is a morphism $f : c_1 \to c_2$ in $\mathcal{C}$ such that $F(f) = \alpha$. It is left to show that the morphism f is F-coCartesian. That is, the following morphism needs to be an isomorphism of categories.

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{D}_{F(c_1)/}} \mathcal{D}_{F(f)/}$$

Using the fact that $f : \Delta^1 \to C$, considered as an edge in C, is p-coCartesian implies that

$$C_{f/} \to C_{c_1/} \times_{D_{p(c_1)/}} D_{p(f)/}$$

is a trivial Kan fibration. Again using the fact that the nerve functor is compatible with the coslice construction and preserves limits, it follows that

$$N(\mathcal{C}_{f/}) \to N(\mathcal{C}_{c_1/} \times_{\mathcal{D}_{F(c_1)/}} \mathcal{D}_{F(f)/})$$

is a trivial Kan fibration. Using Theorem 5.6 it follows that the functor

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{D}_{F(c_1)/}} \mathcal{D}_{F(f)/}$$

is indeed an isomorphism of categories, which shows that f is F-coCartesian. It follows that F is a Grothendieck opfibration. □

Theorem 5.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. Then F is an isomorphism of categories if the induced map $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$ is a trivial Kan fibration.*

Proof. Assume $N(F)$ is a trivial Kan fibration. By Theorem 3.6.4 in [Hov99] the map $N(F)$ has the right lifting property with respect to the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for all $n \geq 0$.

For $n = 0$ we have that for any vertex $d : \Delta^0 \to N(\mathcal{D})$ there is a lift $c : \Delta \to N(\mathcal{C})$, making the following diagram commutative.

$$\begin{array}{ccc} \emptyset & \longrightarrow & N(\mathcal{C}) \\ \downarrow & \overset{c}{\nearrow} & \downarrow N(F) \\ \Delta^0 & \xrightarrow[d]{} & N(\mathcal{D}) \end{array}$$

The lift c defines a vertex in $N(\mathcal{C})$ and hence can be identified with an object $c \in \mathcal{C}$, having the property that $F(c) = d$. This shows that the functor F is surjective, i.e. for every object $d \in \mathcal{D}$ there exists an object $c \in \mathcal{C}$ with $F(c) = d$.

Let $n = 1$. Recall that a morphism in D is the same as an edge in $N(\mathcal{D})$. Moreover, edges in $N(\mathcal{D})$ can be written as morphisms $\Delta^1 \to N(\mathcal{D})$. Let now $f : d_1 \to d_2$ be such an edge. Then by the case $n = 0$ there are vertices c_1, c_2 such that $N(F)(c_i) = d_i$ for $i = 1, 2$. Let $\partial\Delta^1 \to N(\mathcal{C})$ be the morphism which associates the two boundary vertices to the vertices c_1 and c_2. Then the lifting property implies that there is a lift $\tilde{f} : \Delta^1 \to N(\mathcal{C})$ such that the following diagram commutes.

$$\begin{array}{ccc} \partial\Delta^1 & \longrightarrow & N(\mathcal{C}) \\ \big\downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow N(F) \\ \Delta^1 & \xrightarrow[f]{} & N(\mathcal{D}) \end{array}$$

This edge corresponds to a morphism $\tilde{f} : c_1 \to c_2$ in $\mathcal{C}$, such that $F(\tilde{f}) = f$. Hence for any $c_1, c_2 \in \mathcal{C}$ the map

$$F_{c_1,c_2} : \mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \to \mathrm{Hom}_{\mathcal{D}}(F(c_1), F(c_2))$$

is surjective, i.e. the functor F is full.

Let $n = 2$ and let $\partial\Delta^2 \to N(\mathcal{C})$ denote the triangle

$$\begin{array}{ccc} c_1 & & \\ {\scriptstyle \mathrm{id}_{c_1}}\uparrow & \searrow^{f} & \\ c_1 & \xrightarrow[g]{} & c_2 \end{array}$$

Then the lifting property implies that for any commutative triangle $\sigma : \Delta^2 \to N(\mathcal{D})$ given by

$$\begin{array}{ccc} F(c_1) & & \\ {\scriptstyle \mathrm{id}_{F(c_1)}}\uparrow & \searrow^{F(f)} & \\ F(c_1) & \xrightarrow[F(g)]{} & F(c_2) \end{array}$$

there is a lift $\tilde{\sigma} : \Delta^2 \to N(\mathcal{C})$, turning the triangle

$$\begin{array}{ccc} c_1 & & \\ {\scriptstyle \mathrm{id}_{c_1}}\uparrow & \searrow^{f} & \\ c_1 & \xrightarrow[g]{} & c_2 \end{array}$$

into a commutative diagram. This shows that for any $c_1, c_2 \in \mathcal{C}$ the map F_{c_1,c_2} is injective, i.e. the functor F is faithful.

Using the lifting properties for $n = 0, 1, 2$ we have showed that F is a surjective fully faithful functor, which defines an isomorphism of categories. □

We have seen that monoidal structures on ordinary categories can be encoded by Grothendieck opfibrations. Those opfibrations correspond to coCartesian fibrations via the nerve functor. Hence we are now able to define symmetric monoidal ∞-categories.

Definition 5.31. A **symmetric monoidal ∞-category** is a coCartesian fibration of simplicial sets

$$p : \mathcal{C} \to N(\mathbf{FinSet}^*)$$

satisfying the Segal condition. That is, the Segal maps are Joyal equivalences of simplicial sets for all $n \geq 0$

$$(\rho^1!, ..., \rho^n!) : \mathcal{C}_{\langle n \rangle} \to \left(\mathcal{C}_{\langle 1 \rangle}\right)^{\times n}$$

Remark 5.28. Similarly as a Grothendieck opfibration defines a family of categories fibered over a base category, the above coCartesian fibration p defines a family of ∞-categories fibered over $N(\mathbf{FinSet}^*)$. Indeed, one defines the fiber of p over a vertex $\langle n \rangle \in N(\mathbf{FinSet}^*)$ as the pullback

$$\begin{array}{ccc} \mathcal{C}_{\langle n \rangle} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^0 & \xrightarrow[\langle n \rangle]{} & N(\mathbf{FinSet}^*) \end{array}$$

Using that p is an inner fibration, it follows that the fiber $\mathcal{C}_{\langle n \rangle}$ is an ∞-category for all $n \geq 0$. Moreover, any edge $\alpha : \langle n \rangle \to \langle m \rangle$ in $N(\mathbf{FinSet}^*)$ defines a unique functor

$$\alpha! : \mathcal{C}_{\langle n \rangle} \to \mathcal{C}_{\langle m \rangle}$$

of ∞-categories.

So far we have seen that in the world of categories, symmetric monoidal structures can be described by Grothendieck opfibrations, whereas in the world of ∞-categories, symmetric monoidal structures are described by coCartesian fibrations. The link between those worlds is given by the regular nerve functor.

Considering now simplicial or topological categories with enriched monoidal structures, we want to investigate how the coherent nerve functors N_Δ and N_T act on enriched opfibrations. Eventually we will apply this machinery to the Day convolution of enriched functors.

Recall from Definition 2.3 that a topological symmetric monoidal structure on a topological category $\mathcal{C}$ is the datum $(\mathcal{C}, \otimes, 1_\mathcal{C}, a, r, l, \sigma)$, where $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a topological functor and a, r, l, σ are topological natural isomorphisms satisfying the triangle, the pentagon and the hexagon axiom. Similarly we define simplicial symmetric monoidal categories.

As before, we need to embody all the structure given by a simplicial symmetric monoidal category $(\mathcal{C}, \otimes, 1, \sigma)$ into a single simplicial functor $p : \mathcal{C}^\otimes \to \mathbf{FinSet}^*$. Notice that in this case we endow the category $\mathbf{FinSet}^*$ with the trivial simplicial enrichment. The category $\mathcal{C}^\otimes$ is then defined in the same way as before and inherits a canonical simplicial enrichment.

Lemma 5.5. *Let $(\mathcal{C}, \otimes, 1, \sigma)$ be a simplicial symmetric monoidal category. Then the category $\mathcal{C}^\otimes$ defined in Remark 5.23 is simplicially enriched.*

Proof. Let $X = [X_1, ..., X_n]$ and $Y = [Y_1, ..., Y_m]$ be two objects in $\mathcal{C}^\otimes$. First we define for any $\alpha \in \mathbf{FinSet}^*(\langle n \rangle, \langle m \rangle)$ the simplicial set

$$\mathcal{C}^\otimes_\alpha(X, Y) = \prod_{j=1}^{m} \mathcal{C}\left(\bigotimes_{i \in \alpha^{-1}(j)} X_i, Y_j \right) \tag{3}$$

by using the fact that $\mathcal{C}$ is simplicially enriched. The hom space of the category $\mathcal{C}^\otimes$ is then given by the simplicial set

$$\mathcal{C}^\otimes(X, Y) = \coprod_{\alpha \in \mathbf{FinSet}^*(\langle n \rangle, \langle m \rangle)} \mathcal{C}^\otimes_\alpha(X, Y)$$

Moreover, we have that for all objects X and Y there is a projection map

$$\begin{aligned} p_{X,Y} : \mathcal{C}^\otimes(X, Y) &\longrightarrow \mathbf{FinSet}^*(\langle n \rangle, \langle m \rangle) \\ (\alpha, \{f_j\}) &\longmapsto \alpha \end{aligned}$$

It is left to show that the composition induces simplicial maps on the hom spaces, i.e.

$$\begin{aligned} \mathcal{C}^\otimes(X, Y) \times \mathcal{C}^\otimes(Y, Z) &\to \mathcal{C}^\otimes(X, Z) \\ ((\alpha, \{f_j\}), (\beta, \{g_k\})) &\mapsto (\beta \circ \alpha, \{h_k\}) \end{aligned}$$

is a simplicial map. Hence let $X = [X_1, ..., X_n]$, $Y = [Y_1, ..., Y_m]$ and $Z = [Z_1, ..., Z_l]$ be objects in $\mathcal{C}^\otimes$ and let $\alpha : \langle n \rangle \to \langle m \rangle$, $\beta : \langle m \rangle \to \langle l \rangle$ be morphisms in $\mathbf{FinSet}^*$. Then we have by (3)

$$\mathcal{C}^\otimes_\alpha(X, Y) \times \mathcal{C}^\otimes_\beta(Y, Z) \cong \prod_{k=1}^{l} \prod_{j=1}^{m} \left(\mathcal{C}\left(\bigotimes_{\alpha(i)=j} X_i, Y_j \right) \times \mathcal{C}\left(\bigotimes_{\beta(j')=k} Y_{j'}, Z_k \right) \right)$$

By fixing some $k = 1, ..., l$ we have that the map

$$\begin{aligned} \rho_k : \left(\prod_{j=1}^{m} \mathcal{C}(\bigotimes_{\alpha(i)=j} X_i, Y_j) \right) \times \mathcal{C}\left(\bigotimes_{\beta(j')=k} Y_{j'}, Z_k \right) &\longrightarrow \mathcal{C}\left(\bigotimes_{\beta(j)=k} \bigotimes_{\alpha(i)=j} X_i, Z_k \right) \\ (\{f_j\}, g_k) &\longmapsto g_k \circ \bigotimes_{\beta(j)=k} f_j \end{aligned}$$

is simplicial, since by hypothesis the monoidal product $\bigotimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a simplicial functor. Therefore, we have that the map $\rho = \bigoplus_{k=1}^{l} \rho_k$ defines a simplicial map

$$\rho : \mathcal{C}^\otimes_\alpha(X, Y) \times \mathcal{C}^\otimes_\beta(Y, Z) \to \mathcal{C}^\otimes_{\beta \circ \alpha}(X, Z)$$

Then, by using the description of the hom space as coproduct, it follows that the composition is given by

$$\coprod_{\alpha,\beta} \rho : \mathcal{C}^{\otimes}(X,Y) \times \mathcal{C}^{\otimes}(Y,Z) \to \mathcal{C}^{\otimes}(X,Z)$$

which is clearly a simplicial map. This shows that the category $\mathcal{C}^{\otimes}$ is indeed simplicially enriched. □

Remark 5.29. It follows that each simplicial symmetric monoidal category defines a simplicial functor $p : \mathcal{C}^{\otimes} \to \mathbf{FinSet}^*$. This functor satisfies similar lifting properties as Grothendieck opfibrations, which are compatible with the simplicial enrichment. Such functors will be called simplicial Grothendieck opfibrations.

Definition 5.32. Let $F : \mathcal{C} \to \mathcal{D}$ be a simplicial functor between simplicial categories, and let $f : c_1 \to c_2$ be a morphism in $\mathcal{C}$ over the morphism $\alpha : d_1 \to d_2$ in $\mathcal{D}$, i.e. we have that $F(f) = \alpha$. Then call f a **simplicial** F**-coCartesian morphism** if for all objects c_3 in $\mathcal{C}$ the hom set $\mathcal{C}(c_2, c_3)$ is the pullback (in **sSet**) of the following diagram.

$$\begin{array}{ccc} \mathcal{C}(c_2,c_3) & \xrightarrow{(\cdot)\circ f} & \mathcal{C}(c_1,c_3) \\ {\scriptstyle F_{c_2,c_3}}\downarrow & \lrcorner & \downarrow{\scriptstyle F_{c_1,c_3}} \\ \mathcal{D}(F(c_2),F(c_3)) & \xrightarrow[(\cdot)\circ\alpha]{} & \mathcal{D}(F(c_1),F(c_3)) \end{array}$$

Definition 5.33. Let $F : \mathcal{C} \to \mathcal{D}$ be a simplicial functor between simplicial categories. Then F is a **simplicial Grothendieck opfibration** if for all $c_1 \in \mathcal{C}$ and for all morphisms $\alpha : F(c_1) \to d_2$ in $\mathcal{D}$ there is a simplicial F-coCartesian morphism $f : c_1 \to c_2$ such that $F(f) = \alpha$.

In the simplicial case we have a similar result as in the **Set**-enriched case.

Proposition 5.20. *If $(\mathcal{C}, \otimes, 1, \sigma)$ is a simplicial symmetric monoidal category, then the forgetful simplicial functor $p : \mathcal{C}^{\otimes} \to \mathbf{FinSet}^*$ is a simplicial Grothendieck opfibration. Moreover, this functor satisfies the* ***Segal condition****. That is, the Segal maps*

$$\mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\times n}$$

are equivalences of simplicial categories for all $n \geq 0$.

Proof. Since the category $\mathcal{C}^{\otimes}$ is canonically enriched over **sSet** as showed in Lemma 5.5 , this proof is similar to the proof of Proposition 5.20. □

Remark 5.30. In fact the theory of Grothendieck opfibrations can be generalized to $\mathcal{V}$-enriched categories, where $\mathcal{V}$ is a monoidal category satisfying some additional properties. Also a generalization of the Grothendieck construction to $\mathcal{V}$-enriched categories is possible, as shown in [BW18b].

Consider now a topological symmetric monoidal category $(\mathcal{C}, \otimes, 1, \sigma)$. Then, similarly as in the simplicial case, there is a topological functor $p : \mathcal{C}^{\otimes} \to \mathbf{FinSet}^*$. By applying the topological nerve functor we get a simplicial map of ∞-categories $N_T(p) : N_T(\mathcal{C}^{\otimes}) \to N(\mathbf{FinSet}^*)$. We have seen earlier, that in the **Set**-enriched case, this simplicial map is a coCartesian fibration. Therefore, we want to investigate, if the coherent nerve also produces such a coCartesian fibration. To do so, we first need to characterize simplicial F-coCartesian morphisms of simplicial functors $F : \mathcal{C} \to \mathcal{D}$.

Lemma 5.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a simplicial functor between fibrant simplicial categories, such that for all pairs of objects $c, c' \in \mathcal{C}$ the map*

$$\mathcal{C}(c, c') \to \mathcal{D}(F(c), F(c'))$$

is a Kan fibration. Moreover, let $f : c_1 \to c_2$ be a morphism in $\mathcal{C}$. Then the following are equivalent

(i) *The simplicial functor*

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \underset{\mathcal{D}_{F(c_1)/}}{\times} \mathcal{D}_{F(f)/}$$

is a Dwyer-Kan equivalence of simplicial categories.

(ii) *For all $c_3 \in \mathcal{C}$, the following diagram is homotopy Cartesian.*

$$\begin{array}{ccc} \mathcal{C}(c_2, c_3) & \xrightarrow{(-)\circ f} & \mathcal{C}(c_1, c_3) \\ {\scriptstyle F_{c_2,c_3}}\downarrow & & \downarrow{\scriptstyle F_{c_1,c_3}} \\ \mathcal{D}(F(c_2), F(c_3)) & \xrightarrow[(-)\circ F(f)]{} & \mathcal{D}(F(c_1), F(c_3)) \end{array}$$

Remark 5.31. Notice that a diagram of fibrant simplicial sets

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

where $B \to D$ is a Kan fibration, is said to be **homotopy Cartesian**, if there exists a Quillen weak equivalence from A to the homotopy pullback of the given diagram.

$$A \xrightarrow{\sim} B \times^h_D C$$

Proof. This lemma basically follows from Proposition 5.28 and Proposition 2.4.1.10. in [Lur09]. However, it would be more convenient to give a direct proof of the statement. Unfortunately such a proof could not be established by now. □

In comparison with Definition 5.32 of a simplicial coCartesian morphism, we notice that Lemma 5.28 gives a motivation to weaken the assumptions on the commutativity of the diagrams. Indeed, consider a simplicial functor $p : \mathcal{C} \to \mathcal{D}$ satisfying the assumptions of Lemma 5.28. If we assume that for any morphism $f : c_1 \to c_2$ in $\mathcal{C}$ the induced simplicial functor

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \underset{\mathcal{D}_{F(c_1)/}}{\times} \mathcal{D}_{F(f)/}$$

is a Dwyer-Kan equivalence, it follows that there is a Joyal equivalence

$$N_\Delta(\mathcal{C}_{f/}) \to N_\Delta(\mathcal{C}_{c_1/}) \underset{N_\Delta(\mathcal{D}_{F(c_1)/})}{\times} N_\Delta(\mathcal{D}_{F(f)/})$$

of ∞-categories. Now it follows from Definition 5.29 that the map $N_\Delta(p) : N_\Delta(\mathcal{C}) \to N_\Delta(\mathcal{D})$ is a coCartesian fibration, if and only if for every vertex $c_1 \in N_\Delta(\mathcal{C})$ and every edge $\alpha : p(c_1) = d_1 \to d_2$ in $N_\Delta(\mathcal{D})$ there is a $N_\Delta(p)$-coCartesian lift $f : c_1 \to c_2$ of α, i.e. the simplicial map

$$N_\Delta(\mathcal{C})_{f/} \to N_\Delta(\mathcal{C})_{c_1/} \underset{N_\Delta(\mathcal{D})_{F(c_1)/}}{\times} N_\Delta(\mathcal{D})_{\alpha/}$$

is a trivial Kan fibration. Comparing the two simplicial maps above, we notice that the behavior of the coherent nerve acting on the under category plays a crucial role. More precisely, we want look at the following simplicial map $N_\Delta(\mathcal{C}_{f/}) \to N_\Delta(\mathcal{C})_{f/}$. In fact we show in the next section, that the coherent nerve behaves well, acting on certain under categories resp. over categories. Therefore, we give the following definition.

Definition 5.34. Let $F : \mathcal{C} \to \mathcal{D}$ be a simplicial functor between simplicial categories, such that for all pairs of objects $c, c' \in \mathcal{C}$ the map $F_{c,c'} : \mathcal{C}(c, c') \to \mathcal{D}(F(c), F(c'))$ is a Kan fibration. Then F is a **weak simplicial Grothendieck opfibration** if for all $c_1 \in \mathcal{C}$ and for all morphisms $\alpha : F(c_1) \to d_2$ in $\mathcal{D}$ there is a lift $f : c_1 \to c_2$ of α such that the induced functor

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \underset{\mathcal{D}_{F(c_1)/}}{\times} \mathcal{D}_{\alpha/}$$

is a Dwyer-Kan equivalence.

5.7 The coherent nerve on slice categories

In the last section, the question arose whether the coherent nerve of the under category defined by a simplicial functor $F : \mathcal{D} \to \mathcal{C}$ is equivalent to the simplicial set under the simplicial map $N_\Delta(F)$. That is, if there is a Joyal equivalence

$$N_\Delta(\mathcal{C}_{F/}) \xrightarrow{\sim} N_\Delta(\mathcal{C})_{N_\Delta(F)/}$$

For the special case where $\mathcal{D}$ is either the simplicial category with one object $[0]$ or the simplicial category $[1]$ with two objects and a single non-identity morphism, we can give an affirmative answer. This section is devoted to prove this statement. We will use the theory of derived hom spaces also called homotopy function complexes. Therefore, we will introduce all the necessary theory in Appendix B, following chapters 15 to 17 in [Hir03].

A derived hom space is usually characterized by a cosimplicial resolution of an object in the corresponding Reedy model structure. Therefore, we want to consider objects K in the Joyal model structure on **sSet** and look for cosimplicial resolutions of those objects. The aim will be to find presentations of left derived hom spaces in $\mathbf{sSet}_{\text{Joyal}}$ and then use the coherent nerve adjunction to give presentations of left derived hom spaces in **sSet-Cat** with respect to the Bergner model structure. Since the Joyal model structure is not simplicial, the study of left derived hom spaces is rather technical. Therefore, we follow the paper of Dugger and Spivak [DS11] to find cosimplicial resolutions of objects in $\mathbf{sSet}_{\text{Joyal}}$.

Definition 5.35. Let S be a set and denote with G_S the groupoid with objects $\text{ob}(G_S) = S$ and a single morphism $a \to b$ for every $a, b \in S$. Then define the functor

$$\begin{aligned} E : \mathbf{Set} &\to \mathbf{sSet} \\ S &\mapsto N(G_S) \end{aligned}$$

For $n \in \mathbb{N}$ we write $E^n = E(\{0, 1, \ldots, n\})$

Proposition 5.21. ([DS11]) *For every $K \in \mathbf{sSet}$ the map $K \times E^n \to K$ is a trivial fibration in the Joyal model structure for any $n \in \mathbb{N}$. Moreover, the cosimplicial object*

$$\begin{aligned} \widetilde{K} : \Delta &\longrightarrow \mathbf{sSet} \\ [n] &\longmapsto K \times E^n \end{aligned}$$

defines a cosimplicial resolution for K, with respect to the Joyal model structure.

What we want to show next is that cosimplicial resolutions behave well under the join operation of simplicial sets.

Proposition 5.22. *Let K, S be two simplicial sets. Then the cosimplicial object*

$$\begin{aligned} (\widetilde{K * S})' : \Delta &\longrightarrow \mathbf{sSet} \\ [n] &\longmapsto (K \times E^n) * S \end{aligned}$$

is a cosimplicial resolution of $K * S$, with respect to the Joyal model structure.

Proof. Recall that with Δ_{K*S} we denote the constant cosimplicial object. Then we need to show that there is a Reedy weak equivalence $(\widetilde{K*S})' \to \Delta_{K*S}$ and that $(\widetilde{K*S})'$ is Reedy cofibrant. First notice that for any $[n] \in \Delta$ the map

$$(\widetilde{K*S})'^{\,n} = (K \times E^n) * S \longrightarrow K * S = \Delta^n_{K*S}$$

is a Joyal equivalence by Proposition 5.21. Hence it follows by the definition of the Reedy model structure ([Hir03]), that $(\widetilde{K*S})' \to \Delta_{K*S}$ is indeed a Reedy weak equivalence.

To show that $(\widetilde{K*S})'$ is cofibrant in the Reedy model structure, we need to show that for all $[m] \in \Delta$ the latching map

$$\Lambda'_m : L_m(\widetilde{K*S})' \to (\widetilde{K*S})'^{\,m}$$

is a cofibration in the Joyal model structure, i.e. is a monomorphism. By definition the latching object of $(\widetilde{K*S})'$ is the following colimit

$$L_m(\widetilde{K*S})' = \operatorname*{colim}_{[n]\hookrightarrow[m]} [(K \times E^n) * S]$$

taken over all injections $[n] \hookrightarrow [m]$ in Δ for fixed $[m]$. Notice that the join operation, considered as a functor,

$$- * S : \mathbf{sSet} \to \mathbf{sSet}_{S/}$$

preserves colimits (Remark 1.8.8.2 in [Lur09]). Therefore

$$\operatorname*{colim}_{[n]\hookrightarrow[m]} [(K \times E^n) * S] \cong \left[\operatorname*{colim}_{[n]\hookrightarrow[m]} (K \times E^n)\right] * S$$

Using Proposition 5.21 we have that the cosimplicial object $\widetilde{K}$ is a cosimplicial resolution of K. Hence in particular the latching maps

$$\Lambda_m : L_m\widetilde{K} \to K \times E^m$$

are monomorphisms for all m. It follows that

$$\Lambda'_m : L_m(\widetilde{K*S})' \cong (L_m\tilde{K}) * S \xrightarrow{\Lambda_m * S} (K \times E^n) * S$$

is also a monomorphism for all m, proving that $(\widetilde{K*S})'$ is indeed cofibrant. This shows that $\widetilde{K*S}$ is a cofibrant resolution of $K * S$. □

Remark 5.32. Notice that in the following, given a simplicial set Y or a simplicial category $\mathcal{D}$, we denote with RY and $R\mathcal{D}$ a fibrant approximation of Y and $\mathcal{D}$. That is, RY and $R\mathcal{D}$ are fibrant objects in the corresponding model categories, being weak equivalent to Y and $\mathcal{D}$ respectively.

Corollary 5.6. *For $X, Y \in \mathbf{sSet}_{\mathrm{Joyal}}$ simplicial sets, a left derived hom space from X to Y is given by the simplicial set*

$$\begin{aligned} \mathrm{map}_{\mathbf{sSet}}(X, Y) : \Delta^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ [n] &\longmapsto \mathbf{sSet}(X \times E^n, RY) \end{aligned}$$

Proof. This follows from Definition B.2 and Proposition 5.21. □

Corollary 5.7. *For $X, Y, S \in \mathbf{sSet}_{\mathrm{Joyal}}$ simplicial sets, a left derived hom space from $X * S$ to Y is given by the simplicial set*

$$\begin{aligned} \widetilde{\mathrm{map}}_{\mathbf{sSet}}(X * S, Y) : \Delta^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ [n] &\longmapsto \mathbf{sSet}\left((X \times E^n) * S, RY\right) \end{aligned}$$

Proof. This follows from Definition B.2 and Proposition 5.22. □

Having established some presentations of left derived hom spaces in $\mathbf{sSet}_{\text{Joyal}}$, we want to use the Quillen pair $\mathfrak{C} \vdash N_\Delta$ to give some presentations of left derived hom spaces in $\mathbf{sSet}\text{-}\mathbf{Cat}_{\text{Bergner}}$.

Proposition 5.23. *Let K, S be simplicial sets and let $\mathcal{D}$ be a simplicial category. Then left derived hom spaces in $\mathbf{sSet}\text{-}\mathbf{Cat}_{\text{Bergner}}$ are given by*

$$\begin{aligned} \operatorname{map}_{\mathbf{sSet}\text{-}\mathbf{Cat}}(\mathfrak{C}(K), \mathcal{D}) : \Delta^{\text{op}} &\longrightarrow \mathbf{sSet} \\ [n] &\longmapsto \mathbf{sSet}\text{-}\mathbf{Cat}\left(\mathfrak{C}(K \times E^n), R\mathcal{D}\right) \end{aligned}$$

from $\mathfrak{C}(K)$ to $\mathcal{D}$ and

$$\begin{aligned} \widetilde{\operatorname{map}}_{\mathbf{sSet}\text{-}\mathbf{Cat}}(\mathfrak{C}(K * S), \mathcal{D}) : \Delta^{\text{op}} &\longrightarrow \mathbf{sSet} \\ [n] &\longmapsto \mathbf{sSet}\text{-}\mathbf{Cat}\left(\mathfrak{C}\left((K \times E^n) * S\right), R\mathcal{D}\right) \end{aligned}$$

from $\mathfrak{C}(K * S)$ to $\mathcal{D}$.

Proof. This is an application of Proposition 16.2.1 in [Hir03] to the Quillen equivalence $N_\Delta \dashv \mathfrak{C}$. □

Proposition 5.24. *Let $\mathcal{G}$ be a simplicial category. Then the functor*

$$(-) * \mathcal{G} : \mathbf{sSet}\text{-}\mathbf{Cat}_{\text{Bergner}} \to \mathbf{sSet}\text{-}\mathbf{Cat}_{\text{Bergner}}$$

preserves cofibrations.

Proof. We need to show that for a given cofibration $f : \mathcal{C} \to \mathcal{D}$, the simplicial functor $f * \mathcal{G} : \mathcal{C} * \mathcal{G} \to \mathcal{D} * \mathcal{G}$ is again a cofibration, i.e. has the left lifting property with respect to all trivial fibrations. By assumption, let f be a cofibration. Moreover, notice that we have inclusion functors $i : \mathcal{C} \to \mathcal{C} * \mathcal{G}$ and $j : \mathcal{D} \to \mathcal{D} * \mathcal{G}$, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ i\ } & \mathcal{C} * \mathcal{G} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f * \mathcal{G}} \\ \mathcal{D} & \xrightarrow[\ j\ ]{} & \mathcal{D} * \mathcal{G} \end{array}$$

To show that $f * \mathcal{G}$ is a cofibration let $q : \mathcal{A} \to \mathcal{B}$ be a trivial fibration and consider functors $\varphi : \mathcal{C} * \mathcal{G} \to \mathcal{A}$ and $\psi : \mathcal{D} * \mathcal{G} \to \mathcal{B}$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C} * \mathcal{G} & \xrightarrow{\ \varphi\ } & \mathcal{A} \\ {\scriptstyle f * \mathcal{G}}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{D} * \mathcal{G} & \xrightarrow[\ \psi\ ]{} & \mathcal{B} \end{array}$$

Then the following diagram commutes.

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\ i\ } & \mathcal{C} * \mathcal{G} & \xrightarrow{\ \varphi\ } & \mathcal{A} \\ {\scriptstyle f}\downarrow & & \overset{h}{\nearrow} & & \downarrow{\scriptstyle q} \\ \mathcal{D} & \xrightarrow[\ j\ ]{} & \mathcal{D} * \mathcal{G} & \xrightarrow[\ \psi\ ]{} & \mathcal{B} \end{array}$$

By the fact that f is a cofibration there is a functor $h : \mathcal{D} \to \mathcal{A}$ such that the whole diagram commutes. Now we define the functor $\tilde{h} : \mathcal{D} * \mathcal{G} \to \mathcal{A}$ on objects as

$$
\begin{aligned}
\tilde{h} : \mathcal{D} * \mathcal{G} &\longrightarrow \mathcal{A} \\
d &\longmapsto h(d) \\
g &\longmapsto \varphi(g)
\end{aligned}
$$

and on morphisms as

$$
\begin{aligned}
\tilde{h} : \mathcal{D} * \mathcal{G} &\longrightarrow \mathcal{A} \\
\delta : d \to d' &\longmapsto h(\delta) \\
\gamma : g \to g' &\longmapsto \varphi(\gamma)
\end{aligned}
$$

The only missing morphisms are the ones of the form $d \to g$ for $d \in \mathcal{D}$ and $g \in \mathcal{G}$. Consider now the diagram

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & \mathcal{A}(h(d), \varphi(g)) \\
\downarrow & \nearrow & \downarrow q_{h(d),\varphi(g)} \\
\Delta^0 = (\mathcal{D} * \mathcal{G})(d, g) & \longrightarrow & \mathcal{B}(\psi(d), \psi(g))
\end{array}
$$

Using that q is a trivial fibration, it follows that the right vertical arrow $q_{h(d),\varphi(g)}$ is a trivial Kan fibration. Then by the lifting properties of trivial Kan fibrations, there is a unique dashed arrow making the diagram commute. This arrow precisely determines how $\tilde{h}$ acts on morphisms of the form $d \to g$. Hence the functor $\tilde{h}$ provides a lift and therefore $f * \mathcal{G}$ is cofibrant. □

Proposition 5.25. *Let $Y \in$ **sSet** be any simplicial set and let $\mathcal{D}$ be a simplicial category. Then the cosimplicial object given by*

$$[n] \longmapsto \mathfrak{C}(Y \times E^n) * \mathcal{D}$$

is a cosimplicial resolution of $\mathfrak{C}(Y) * \mathcal{D}$. Moreover, a left derived hom space is given by the simplicial set

$$
\begin{aligned}
\widehat{\operatorname{map}}_{\mathbf{sSet\text{-}Cat}} (\mathfrak{C}(Y) * \mathcal{D}, \mathcal{C}) : \Delta^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\
[n] &\longmapsto \mathbf{sSet\text{-}Cat} \left(\mathfrak{C}(Y \times E^n) * \mathcal{D}, \mathcal{C}\right)
\end{aligned}
$$

Proof. First we show that the cosimplicial object $[n] \mapsto \mathfrak{C}(Y \times E^n) * \mathcal{D}$ is Reedy weak equivalent to the constant cosimplicial object on $\mathfrak{C}(Y) * \mathcal{D}$. By Proposition 5.21 we know that for all $[n]$ there are weak equivalences in $\mathbf{sSet}_{\text{Joyal}}$

$$Y \times E^n \to Y$$

hence we have weak equivalences in $\mathbf{sSet\text{-}Cat}_{\text{Bergner}}$

$$\mathfrak{C}(Y \times E^n) \to \mathfrak{C}(Y)$$

then it follows that there are also weak equivalences

$$\mathfrak{C}(Y \times E^n) * \mathcal{D} \to \mathfrak{C}(Y) * \mathcal{D}$$

showing that the cosimplicial objects are indeed Reedy weak equivalent.

We are left to show that $[n] \mapsto \mathfrak{C}(Y \times E^n) * \mathcal{D}$ is Reedy cofibrant. First notice that the functor

$$(-) * \mathcal{D} : \mathbf{sSet\text{-}Cat} \to \mathbf{sSet\text{-}Cat}_{/\mathcal{D}}$$

has a right adjoint, hence preserves colimits. Then the latching object of the cosimplicial object $[n] \mapsto \mathfrak{C}(Y \times E^n) * \mathcal{D}$ at $[m]$ is given by the colimit

$$L_{[m]} \left(\widetilde{\mathfrak{C}(Y) * \mathcal{D}}\right) = \operatorname*{colim}_{[n] \hookrightarrow [m]} \left(\mathfrak{C}(Y \times E^n) * \mathcal{D}\right) \cong \left(\operatorname*{colim}_{[n] \to [m]} \mathfrak{C}(Y \times E^n)\right) * \mathcal{D} = \left(L_{[m]} \widetilde{\mathfrak{C}(Y)}\right) * \mathcal{D}$$

Hence using Proposition 5.23 we conclude that the latching map of $\widetilde{\mathfrak{C}(Y) * \mathcal{D}}$ is given by

$$\Lambda_m : L_{[m]}\left(\widetilde{\mathfrak{C}(Y) * \mathcal{D}}\right) \xrightarrow{\cong} \left(L_{[m]}\widetilde{\mathfrak{C}(Y)}\right) * \mathcal{D} \xrightarrow{\Lambda'_m * \mathcal{D}} \mathfrak{C}(Y \times E^m) * \mathcal{D}$$

where $\Lambda'_m : L_{[m]}\widetilde{\mathfrak{C}(Y)} \to \mathfrak{C}(Y \times E^m)$ is the latching map of the cosimplicial resolution $\widetilde{\mathfrak{C}(Y)}$. Therefore, we have that for all $[m]$ the map Λ'_m is a cofibration. Now using Proposition 5.24 it follows that for all $[m]$ the map $\Lambda'_m * \mathcal{D}$ is a cofibration, which shows that all latching maps Λ_m are indeed cofibrations. Hence the cosimplicial object $\widetilde{\mathfrak{C}(Y) * \mathcal{D}}$ is a cosimplicial resolution of $\mathfrak{C}(Y) * \mathcal{D}$. ☐

Now that we are given several presentations of left derived hom spaces in the Joyal and in the Bergner model structure, we can give a proof of the following proposition.

Proposition 5.26. *Let $F : \mathcal{D} \to \mathcal{C}$ be a simplicial functor where $\mathcal{D} \in \{[0], [1]\}$ and $\mathcal{C}$ is a fibrant simplicial category. Then there is a weak equivalence in the Joyal model structure on* **sSet**

$$N_\Delta(\mathcal{C}_{/F}) \xrightarrow{\sim} N_\Delta(\mathcal{C})_{/N_\Delta(F)}$$

Proof. The strategy to prove this proposition will be to use Theorem B.1. That is, we want to show that for any simplicial set Y the induced map of left derived hom spaces

$$g_* : \mathbb{L}\mathbf{sSet}(Y, N_\Delta(\mathcal{C}_{/F})) \to \mathbb{L}\mathbf{sSet}(Y, N_\Delta(\mathcal{C})_{/N_\Delta(F)})$$

is a Quillen weak equivalence. Then by Theorem B.1 it follows that

$$g : N_\Delta(\mathcal{C}_{/F}) \to N_\Delta(\mathcal{C})_{/N_\Delta(F)}$$

is a weak equivalence in the Joyal model structure.

First notice that $N_\Delta(\mathcal{C})$ is fibrant in the Joyal model structure, then by Proposition 1.2.9.3. in [Lur09] also $N_\Delta(\mathcal{C})_{/N_\Delta(F)}$ is fibrant.

First we look at the simplicial set given by

$$\begin{aligned} \mathrm{map}_{\mathbf{sSet}}\left(Y, N_\Delta(\mathcal{C})_{/N_\Delta(F)}\right)_n &= \mathbf{sSet}\left(Y \times E^n, N_\Delta(\mathcal{C})_{/N_\Delta(F)}\right) \\ &\cong \mathbf{sSet}_{N_\Delta(F)}\left((Y \times E^n) * S, N_\Delta(\mathcal{C})\right) \end{aligned}$$

where we take $S = N_\Delta(\mathcal{D})$. Notice that $\mathbf{sSet}_{N_\Delta(F)}\left((Y \times E^n) * S, N_\Delta(\mathcal{C})\right)$ is given by the pullback in **Set**

$$\begin{array}{ccc} \mathbf{sSet}_{N_\Delta(F)}\left((Y \times E^n) * S, N_\Delta(\mathcal{C})\right) & \longrightarrow & \mathbf{sSet}\left((Y \times E^n) * S, N_\Delta(\mathcal{C})\right) \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (-)_{|S}} \\ * & \xrightarrow[N_\Delta(F)]{} & \mathbf{sSet}\left(S, N_\Delta(\mathcal{C})\right) \end{array}$$

where the map $* \to \mathbf{sSet}\left(S, N_\Delta(\mathcal{C})\right)$ is characterizing the simplicial map $N_\Delta(F) : S \to N_\Delta(C)$ and the map

$$(-)_{|S} : \mathbf{sSet}\left((Y \times E^n) * S, N_\Delta(\mathcal{C})\right) \to \mathbf{sSet}\left(S, N_\Delta(\mathcal{C})\right)$$

is given by restriction of a simplicial map f to $f_{|S}$. Then using Corollary 5.7 we can write the pullback as follows.

$$\begin{array}{ccc} \mathbf{sSet}_{N_\Delta(F)}\left((Y \times E^n) * S, N_\Delta(\mathcal{C})\right) & \longrightarrow & \widetilde{\mathrm{map}}_{\mathbf{sSet}}\left(Y * S, N_\Delta(\mathcal{C})\right)_n \\ \big\downarrow & & \big\downarrow{\scriptstyle (-)_{|S}} \\ * & \xrightarrow[N_\Delta(F)]{} & \mathbf{sSet}\left(S, N_\Delta(\mathcal{C})\right) \end{array}$$

Then define the simplicial set $\widetilde{\mathrm{map}}_{N_\Delta(F)}(Y * S, N_\Delta(\mathcal{C}))$ as the homotopy pullback of the diagram

$$\begin{array}{ccc} \widetilde{\mathrm{map}}_{N_\Delta(F)}(Y * S, N_\Delta(\mathcal{C})) & \longrightarrow & \widetilde{\mathrm{map}}_{\mathbf{sSet}}(Y * S, N_\Delta(\mathcal{C})) \\ \downarrow & \lrcorner\mathrm{ho} & \downarrow \\ \Delta^0 & \longrightarrow & \Delta_{\mathbf{sSet}(S, N_\Delta(\mathcal{C}))} \end{array}$$

where $\Delta_{\mathbf{sSet}(S,N_\Delta(\mathcal{C}))}$ is the constant simplicial set on $\mathbf{sSet}\,(S, N_\Delta(\mathcal{C}))$. To show that the homotopy pullback agrees up to Quillen weak equivalence with the categorical pullback, we first notice that the Quillen model structure on **sSet** is proper and that the map

$$\widetilde{\mathrm{map}}_{\mathbf{sSet}}(Y * S, N_\Delta(\mathcal{C})) \to \Delta_{\mathbf{sSet}(S,N_\Delta(\mathcal{C}))}$$

is a Kan fibration. Then Corollary 13.3.8 in [Hir03] implies the desired result. It follows that there is a Quillen weak equivalence

$$\mathrm{map}_{\mathbf{sSet}}\left(Y, N_\Delta(\mathcal{C})_{/N_\Delta(F)}\right) \xrightarrow{\sim} \widetilde{\mathrm{map}}_{N_\Delta(F)}(Y * S, N_\Delta(\mathcal{C}))$$

Moreover, using the Quillen pair $\mathfrak{C} \vdash N_\Delta$, it follows that we have the following natural isomorphisms

$$\begin{aligned} \mathbf{sSet}\left((Y \times E^n) * S, N_\Delta(\mathcal{C})\right) &\cong \mathbf{sSet\text{-}Cat}\left(\mathfrak{C}\left((Y \times E^n) * S\right), \mathcal{C}\right) \\ &= \widetilde{\mathrm{map}}_{\mathbf{sSet\text{-}Cat}}\left(\mathfrak{C}\left(Y * S\right), \mathcal{C}\right)_n \end{aligned}$$

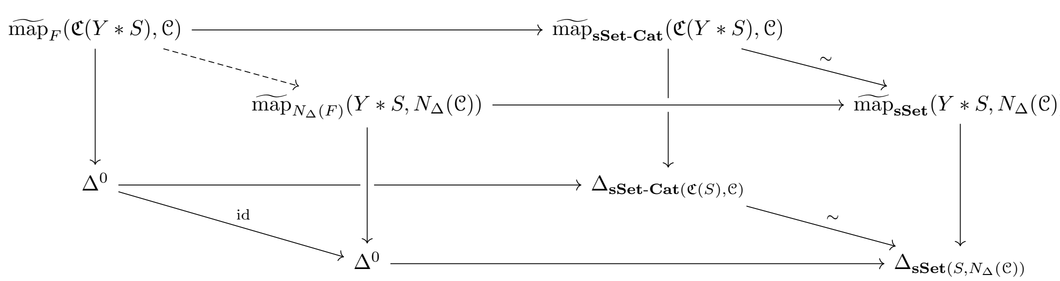
and

$$\mathbf{sSet}(S, N_\Delta(\mathcal{C})) \cong \mathbf{sSet\text{-}Cat}(\mathfrak{C}(S), \mathcal{C})$$

Hence we define $\widetilde{\mathrm{map}}_F(\mathfrak{C}(Y * S), \mathcal{C})$ as the homotopy pullback of the following diagram.

$$\begin{array}{ccc} \widetilde{\mathrm{map}}_F(\mathfrak{C}(Y * S), \mathcal{C}) & \longrightarrow & \widetilde{\mathrm{map}}_{\mathbf{sSet\text{-}Cat}}(\mathfrak{C}(Y * S), \mathcal{C}) \\ \downarrow & \lrcorner\mathrm{ho} & \downarrow \\ \Delta^0 & \longrightarrow & \Delta_{\mathbf{sSet\text{-}Cat}(\mathfrak{C}(S), \mathcal{C})} \end{array}$$

By definition it is clear that the following diagram commutes.



Then it follows by Proposition 13.3.14 in [Hir03] that the dashed arrow is a Quillen weak equivalence. Hence we have that there is a Quillen weak equivalence

$$\mathrm{map}_{\mathbf{sSet}}\left(Y, N_\Delta(\mathcal{C})_{/N_\Delta(F)}\right) \xrightarrow{\sim} \widetilde{\mathrm{map}}_F(\mathfrak{C}(Y * S), \mathcal{C})$$

On the other hand we have that

$$\mathrm{map}_{\mathbf{sSet}}(Y, N_\Delta(\mathcal{C}_{/F})) \cong \mathrm{map}_{\mathbf{sSet\text{-}Cat}}(\mathfrak{C}(Y), \mathcal{C}_{/F})$$

Then by looking at the components if follows

$$\mathrm{map}_{\mathbf{sSet\text{-}Cat}}(\mathfrak{C}(Y), \mathcal{C}_{/F})_n \cong \mathbf{sSet\text{-}Cat}_F\left(\mathfrak{C}(Y \times E^n) * \mathcal{D}, \mathcal{C}\right)$$

Similarly, we define the set $\mathbf{sSet\text{-}Cat}_F\left(\mathfrak{C}(Y \times E^n) * \mathcal{D}, \mathcal{C}\right)$ as the pullback in the following diagram.

$$\begin{array}{ccc} \mathbf{sSet\text{-}Cat}_F\left(\mathfrak{C}(Y \times E^n) * \mathcal{D}, \mathcal{C}\right) & \longrightarrow & \mathbf{sSet\text{-}Cat}\left(\mathfrak{C}(Y \times E^n) * \mathcal{D}, \mathcal{C}\right) \\ \big\downarrow & \lrcorner & \big\downarrow \\ * & \longrightarrow & \mathbf{sSet\text{-}Cat}(\mathcal{D}, \mathcal{C}) \end{array}$$

We notice that

$$\mathbf{sSet\text{-}Cat}\left(\mathfrak{C}(Y \times E^n) * \mathcal{D}, \mathcal{C}\right) = \widehat{\mathrm{map}}_{\mathbf{sSet\text{-}Cat}}(\mathfrak{C}(Y) * \mathcal{D}, \mathcal{C})_n$$

Therefore, we define $\widehat{\mathrm{map}}_F(\mathfrak{C}(Y) * \mathcal{D}, \mathcal{C})$ as the homotopy pullback of the following diagram.

$$\begin{array}{ccc} \widehat{\mathrm{map}}_F(\mathfrak{C}(Y) * \mathcal{D}, \mathcal{C}) & \longrightarrow & \widehat{\mathrm{map}}_{\mathbf{sSet\text{-}Cat}}(\mathfrak{C}(Y) * \mathcal{D}, \mathcal{C}) \\ \big\downarrow & \lrcorner^{\mathrm{ho}} & \big\downarrow \\ \Delta^0 & \longrightarrow & \Delta_{\mathbf{sSet\text{-}Cat}(\mathcal{D}, \mathcal{C})} \end{array}$$

Similarly as before we show, using properness of the Quillen model structure and the fact that the map

$$\widehat{\mathrm{map}}_{\mathbf{sSet\text{-}Cat}}(\mathfrak{C}(Y) * \mathcal{D}, \mathcal{C}) \to \Delta_{\mathbf{sSet\text{-}Cat}(\mathcal{D}, \mathcal{C})}$$

is a Kan fibration, that the categorical pullback is Quillen weak equivalent to the homotopy pullback. This shows that there is a Quillen weak equivalence

$$\mathrm{map}_{\mathbf{sSet}}(Y, N_\Delta(\mathcal{C}_{/F})) \xrightarrow{\sim} \widehat{\mathrm{map}}_F(\mathfrak{C}(Y) * \mathcal{D}, \mathcal{C})$$

Now notice that $\mathfrak{C}\left(N_\Delta\left([0]\right)\right) \cong [0]$ and $\mathfrak{C}\left(N_\Delta\left([1]\right)\right) \cong [1]$ hence it follows that there is an isomorphisms

$$\mathbf{sSet\text{-}Cat}\left(\mathfrak{C}(N_\Delta(\mathcal{D}), \mathcal{C}\right) \cong \mathbf{sSet\text{-}Cat}\left(\mathcal{D}, \mathcal{C}\right)$$

since we just consider $\mathcal{D} = [0]$ or $\mathcal{D} = [1]$. It follows that the constant simplicial sets are isomorphic, i.e.

$$\Delta_{\mathbf{sSet\text{-}Cat}(\mathfrak{C}(N_\Delta(\mathcal{D}), \mathcal{C})} \cong \Delta_{\mathbf{sSet\text{-}Cat}(\mathcal{D}, \mathcal{C})}$$

Consider the two simplicial categories $\mathfrak{C}(Y) * \mathcal{D}$ and $\mathfrak{C}(Y * N_\Delta(D))$. By Corollary 4.2.1.4. in [Lur09] there is a Dwyer-Kan equivalence of simplicial categories

$$\mathfrak{C}(Y * N_\Delta(D)) \xrightarrow{\sim} \mathfrak{C}(Y) * \mathfrak{C}(N_\Delta(D))$$

Using the fact that there is a Dwyer-Kan equivalence of simplicial categories

$$\mathfrak{C}(N_\Delta(\mathcal{D})) \xrightarrow{\sim} \mathcal{D}$$

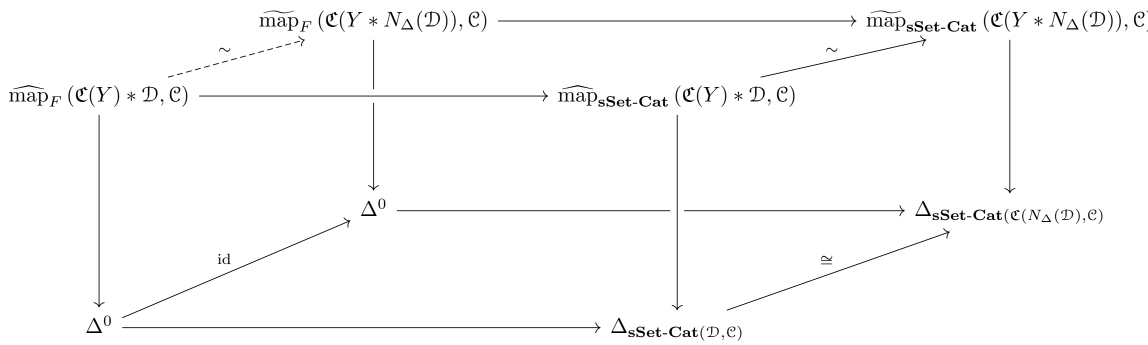
it follows that there is a Dwyer-Kan equivalence

$$\mathfrak{C}(Y * N_\Delta(\mathcal{D})) \xrightarrow{\sim} \tilde{\mathfrak{C}}(Y) * \mathcal{D}$$

Then by Theorem B.1 there is a Quillen weak equivalence of left derived hom spaces

$$\widehat{\mathrm{map}}_{\mathbf{sSet\text{-}Cat}}(\mathfrak{C}(Y) * \mathcal{D}, \mathcal{C}) \xrightarrow{\sim} \widetilde{\mathrm{map}}_{\mathbf{sSet\text{-}Cat}}\left(\mathfrak{C}\left(Y * N_\Delta(\mathcal{D})\right), \mathcal{C}\right)$$

By looking at the homotopy pullback diagrams it follows



that there is a Quillen weak equivalence

$$\widehat{\mathrm{map}}_F\left(\mathfrak{C}(Y) * \mathcal{D}, \mathcal{C}\right) \xrightarrow{\sim} \widetilde{\mathrm{map}}_F\left(\mathfrak{C}(Y * N_\Delta(\mathcal{D})), \mathcal{C}\right)$$

Connecting all the pieces, we end up with a diagram of Quillen weak equivalences

$$\begin{array}{ccc} \widehat{\mathrm{map}}_F\left(\mathfrak{C}(Y) * \mathcal{D}, \mathcal{C}\right) & \xrightarrow{\sim} & \widetilde{\mathrm{map}}_F\left(\mathfrak{C}(Y * N_\Delta(\mathcal{D})), \mathcal{C}\right) \\ \sim\uparrow & & \uparrow\sim \\ \mathrm{map}_{\mathbf{sSet}}(Y, N_\Delta(\mathcal{C}_{/F})) & \dashrightarrow & \mathrm{map}_{\mathbf{sSet}}\left(Y, N_\Delta(\mathcal{C})_{/N_\Delta(F)}\right) \end{array}$$

Then by the 2-out-of-3 property it follows that the map

$$\mathrm{map}_{\mathbf{sSet}}(Y, N_\Delta(\mathcal{C}_{/F})) \to \mathrm{map}_{\mathbf{sSet}}\left(Y, N_\Delta(\mathcal{C})_{/N_\Delta(F)}\right)$$

is a Quillen weak equivalence of left derived hom spaces. Since we choose Y arbitrarily, this is true for all $Y \in \mathbf{sSet}$. Hence Theorem B.1 implies that the map

$$N_\Delta(\mathcal{C}_{/F}) \to N_\Delta(\mathcal{C})_{/N_\Delta(F)}$$

is a Joyal weak equivalence. □

Now using that the Quillen model structure on **sSet** is a left Bousfield localization of the Joyal model structure ([JT06]), it follows that Joyal weak equivalences are also Quillen weak equivalences.

Proposition 5.27. ([JT06]) *A Joyal equivalence between simplicial sets is a Quillen weak equivalence.*

Corollary 5.8. *In the setting of Proposition 5.26 there is a Quillen weak equivalence*

$$N_\Delta(\mathcal{C}_{/F}) \to N_\Delta(\mathcal{C})_{/N_\Delta(F)}$$

Remark 5.33. From the proof of Proposition 5.26, it follows that the same result also holds for the dual statement, i.e. for $F : \mathcal{D} \to \mathcal{C}$ a simplicial functor, where $\mathcal{D} = [0], [1]$, there is a Joyal weak equivalence

$$N_\Delta(\mathcal{C}_{F/}) \to N_\Delta(\mathcal{C})_{N_\Delta(F)/}$$

which is in particular also a Quillen weak equivalence.

We notice that given a simplicial category $\mathcal{C}$ and a morphism f in $\mathcal{C}$, the under category $\mathcal{C}_{f/}$ is then defined by identifying the morphisms f with the simplicial functor $f : [1] \to \mathcal{C}$. Then Remark 5.33 implies that there is a Quillen weak equivalence

$$N_\Delta(\mathcal{C}_{f/}) \xrightarrow{\sim} N_\Delta(\mathcal{C})_{f/}$$

Having established this important property of the coherent nerve functor, we can state the following proposition.

Proposition 5.28. *Let $p : \mathcal{C} \to \mathcal{D}$ be a simplicial functor between fibrant simplicial categories, such that for all pairs of objects $c, c' \in \mathcal{C}$ the map*

$$\mathcal{C}(c, c') \to \mathcal{D}(F(c), F(c'))$$

is a Kan fibration. Then p is a weak simplicial Grothendieck opfibration if and only if

$$N_\Delta(p) : N_\Delta(\mathcal{C}) \to N_\Delta(\mathcal{D})$$

is a coCartesian fibration.

Proof. "⇒":
Suppose that p is a weak simplicial Grothendieck opfibration. By Proposition 5.12 it is clear that $N_\Delta(p)$ is an inner fibration. Hence we only need to show that for any vertex c_1 in $N_\Delta(\mathcal{C})$ and every edge $\alpha : N_\Delta(p)(c_1) =: d_1 \to d_2$ in $N_\Delta(\mathcal{D})$, there is a $N_\Delta(p)$-coCartesian lift $f : c_1 \to c_2$ of α. Indeed, let c_1 be such a vertex and $\alpha : d_1 \to d_2$ such an edge. Then c_1 can be regarded as an object of $\mathcal{C}$ and the edge $\alpha : d_1 \to d_2$ can be identified with a morphism $\alpha : d_1 \to d_2$ in $\mathcal{D}$. By hypothesis, p is a weak simplicial Grothendieck opfibration. Hence there exists a lift $f : c_1 \to c_2$ of α in $\mathcal{C}$, such that there is a Dwyer-Kan equivalence of simplicial categories

$$\mathcal{C}_{f/} \xrightarrow{\sim} \mathcal{C}_{c_1/} \underset{\mathcal{D}_{p(c_1)/}}{\times} \mathcal{D}_{p(f)/}$$

By applying the coherent nerve we obtain the following Joyal equivalence

$$N_\Delta(\mathcal{C}_{f/}) \xrightarrow{\sim} N_\Delta(\mathcal{C}_{c_1/}) \underset{N_\Delta(\mathcal{D}_{p(c_1)/})}{\times} N_\Delta(\mathcal{D}_{p(f)/})$$

Now we apply Proposition 5.26 an get Joyal equivalences of the form

$$N_\Delta(\mathcal{C}_{f/}) \simeq N_\Delta(\mathcal{C})_{f/} \qquad N_\Delta(\mathcal{C}_{c_1/}) \simeq N_\Delta(\mathcal{C})_{c_1/} \tag{4}$$

$$N_\Delta(\mathcal{D}_{p(c_1)/}) \simeq N_\Delta(\mathcal{D})_{d_1/} \qquad N_\Delta(\mathcal{D}_{p(f)/}) \simeq N_\Delta(\mathcal{D})_{\alpha/} \tag{5}$$

Now we want to show that the pullbacks

$$N_\Delta(\mathcal{C}_{c_1/}) \underset{N_\Delta(\mathcal{D}_{d_1/})}{\times} N_\Delta(\mathcal{D}_{\alpha/}) \quad \text{and} \quad N_\Delta(\mathcal{C})_{c_1/} \underset{N_\Delta(\mathcal{D})_{d_1/}}{\times} N_\Delta(\mathcal{D})_{\alpha/}$$

are both Joyal equivalent to the corresponding homotopy pullbacks in the Joyal model structure. Since all the objects in the pullback diagrams are fibrant, we only need to verify that at least one morphism in each of the pullbacks is a Joyal fibration. For the right hand side, we notice that since p is a weak simplicial Grothendieck opfibration, it follows that by Proposition 5.16 the map $N_\Delta(p) : N_\Delta(\mathcal{C}) \to N_\Delta(\mathcal{D})$ is a Joyal fibration. Then by Proposition 5.13 we can decompose the map $N_\Delta(\mathcal{C})_{c_1/} \to N_\Delta(\mathcal{D})_{d_1/}$ as follows.

$$\begin{array}{ccccc} N_\Delta(\mathcal{C})_{c_1/} & \longrightarrow & N_\Delta(\mathcal{C}) \underset{N_\Delta(\mathcal{D})}{\times} N_\Delta(\mathcal{D})_{d_1/} & \longrightarrow & N_\Delta(\mathcal{D})_{d_1/} \\ & & \downarrow & \lrcorner & \downarrow \\ & & N_\Delta(\mathcal{C}) & \longrightarrow & N_\Delta(\mathcal{D}) \end{array}$$

Notice that the left horizontal morphism is a left fibration by Proposition 5.13. The right horizontal morphism is a Joyal fibration, since fibrations are preserved under pullbacks. Then the composition

$$N_\Delta(\mathcal{C})_{c_1/} \to N_\Delta(\mathcal{D})_{d_1/}$$

is indeed a Joyal fibration. Therefore, the pullback $N_\Delta(\mathcal{C})_{c_1/} \underset{N_\Delta(\mathcal{D})_{d_1/}}{\times} N_\Delta(\mathcal{D})_{\alpha/}$ is Joyal equivalent to the homotopy pullback in $\mathbf{sSet}_{\text{Joyal}}$.

For the left hand side, we use Proposition 5.17 to show that the morphism

$$N_\Delta(\mathcal{D}_{\alpha/}) \to N_\Delta(\mathcal{D}_{d_1/})$$

is a left fibration, hence it is in particular also a Joyal fibration. This shows that also the pullback $N_\Delta(\mathcal{C}_{c_1/}) \underset{N_\Delta(\mathcal{D}_{d_1/})}{\times} N_\Delta(\mathcal{D}_{\alpha/})$ is indeed Joyal equivalent to the homotopy pullback in the Joyal model structure. Since we have Joyal equivalences on the objects given by (4) and (5) it follows, using the fact that the Joyal model structure is framed (Remark 5.34), that there is an induced weak equivalence on the pullbacks.

$$N_\Delta(\mathcal{C}_{c_1/}) \underset{N_\Delta(\mathcal{D}_{d_1/})}{\times} N_\Delta(\mathcal{D}_{\alpha/}) \xrightarrow{\sim} N_\Delta(\mathcal{C})_{c_1/} \underset{N_\Delta(\mathcal{D})_{d_1/}}{\times} N_\Delta(\mathcal{D})_{\alpha/}$$

This provides the following commutative diagram.

$$\begin{array}{ccc} N_\Delta(\mathcal{C}_{f/}) & \longrightarrow & N_\Delta(\mathcal{C}_{c_1/}) \underset{N_\Delta(\mathcal{D}_{p(c_1)/})}{\times} N_\Delta(\mathcal{D}_{p(f)/}) \\ \downarrow & & \downarrow \\ N_\Delta(\mathcal{C})_{f/} & \dashrightarrow & N_\Delta(\mathcal{C})_{c_1/} \underset{N_\Delta(\mathcal{D})_{d_1/}}{\times} N_\Delta(\mathcal{D})_{\alpha/} \end{array}$$

Then by the 2-out-of-3 property, it follows that the dashed map is also a Joyal equivalence.

Consider now the following diagram

$$* \subseteq \Delta^1 \xrightarrow{f} N_\Delta(\mathcal{C}) \xrightarrow{N_\Delta(p)} N_\Delta(D)$$

Using that $N_\Delta(p)$ is an inner fibration, it follows by Proposition 5.13 that the dashed map above is a left fibration. Since it is also a Joyal equivalence it follows by Proposition 5.14 that for all vertices s in $N_\Delta(\mathcal{C})_{c_1/} \underset{N_\Delta(\mathcal{D})_{d_1/}}{\times} N_\Delta(\mathcal{D})_{\alpha/}$ the induced map

$$\left(N_\Delta(\mathcal{C})_{f/}\right)_s \to \left(N_\Delta(\mathcal{C})_{c_1/} \underset{N_\Delta(\mathcal{D})_{d_1/}}{\times} N_\Delta(\mathcal{D})_{\alpha/}\right)_s \cong *$$

is a homotopy equivalence. This shows that all fibers $\left(N_\Delta(\mathcal{C})_{f/}\right)_s$ are contractible, which then implies using Proposition 5.15 that the map

$$N_\Delta(\mathcal{C})_{f/} \to N_\Delta(\mathcal{C})_{c_1/} \underset{N_\Delta(\mathcal{D})_{d_1/}}{\times} N_\Delta(\mathcal{D})_{\alpha/}$$

is a trivial Kan fibration. This shows that f is indeed a coCartesian lift of α, which shows that $N_\Delta(p)$ is a coCartesian fibration.

"$\Leftarrow$":
Suppose now that $q := N_\Delta(p) : N_\Delta(\mathcal{C}) \to N_\Delta(\mathcal{D})$ is a coCartesian fibration. Hence for every vertex c_1 in $N_\Delta(\mathcal{C})$ and every edge $\alpha : q(c_1) = d_1 \to d_2$ in $N_\Delta(\mathcal{D})$ there is a coCartesian lift f, i.e.

$$N_\Delta(\mathcal{C})_{f/} \to N_\Delta(\mathcal{C})_{c_1/} \underset{N_\Delta(\mathcal{D})_{d_1/}}{\times} N_\Delta(\mathcal{D})_{\alpha/}$$

is a trivial Kan fibration. Since the Quillen model structure is a left Bousfield localization of the Joyal model structure, it follows that the class of Kan fibrations equals the class of trivial Joyal fibrations. Consider for example Proposition 3.3.3. in [Hir03] as a reference. Hence the simplicial map

$$N_\Delta(\mathcal{C})_{f/} \to N_\Delta(\mathcal{C})_{c_1/} \underset{N_\Delta(\mathcal{D})_{d_1/}}{\times} N_\Delta(\mathcal{D})_{\alpha/}$$

is a Joyal equivalence. Then using Proposition 5.26 it follows that the functor

$$\mathcal{C}_{f/} \xrightarrow{\sim} \mathcal{C}_{c_1/} \underset{\mathcal{D}_{p(c_1)/}}{\times} \mathcal{D}_{p(f)/}$$

is indeed a Dwyer-Kan equivalence of simplicial categories. This shows now that $p : \mathcal{C} \to \mathcal{D}$ is a weak simplicial Grothendieck opfibration. $\square$

Remark 5.34. Consider a model category $\mathcal{M}$ and $X, Y : \mathcal{C} \to \mathcal{M}$ small diagrams in $\mathcal{M}$ such that for all $\alpha \in \mathcal{C}$ the objects $X(\alpha)$ and $Y(\alpha)$ are fibrant. Moreover, let $f : X \to Y$ be a map of $\mathcal{C}$-diagrams in $\mathcal{M}$. In general it is not true, that if we are given object wise weak equivalences

$$f_\alpha : X(\alpha) \xrightarrow{\sim} Y(\alpha)$$

that the induced map of homotopy limits $\mathrm{holim}(X) \to \mathrm{holim}(Y)$ is also a weak equivalence. However, it holds if the model category $\mathcal{M}$ is assumed to be framed, or if in particular the model category is right proper. Since the Joyal model structure on **sSet** is not right proper, we need the fact that it is framed. Indeed, by Proposition 5.21 it follows that to any simplicial set K, we can associate a cosimplicial resolution $\widetilde{K}$ of K, having the property that $\widetilde{K}^0 \cong K$. Now notice that **sSet** endowed with the cartesian product and the Joyal model structure is a monoidal model category. Therefore, it follows that to any simplicial set K we can associate a simplicial resolution $\widehat{K}$, having the property that $K \cong \widehat{K}_0$. This eventually shows that $\mathbf{sSet}_{\mathrm{Joyal}}$ is a framed model category according to Definition 16.6.21. in [Hir03].

The following theorem now shows that any symmetric monoidal topological category $(\mathcal{C}, \otimes, 1, \sigma)$ gives rise to a symmetric monoidal ∞-category $N_\Delta(\mathcal{C})$. This fact might seem surprising, but actually an even more general result has been showed by Nikolaus and Sagave in [NS15]. They showed that any presentably symmetric monoidal ∞-category is represented by a symmetric monoidal model category. Now as we have seen in the introduction of this chapter, any simplicial category $\mathcal{C}$ gives rise to a model category $\mathcal{M}$, having the property that its simplicial localization $L\mathcal{M}$ is weakly equivalent in the Bergner model structure to $\mathcal{C}$. Hence if we assume $\mathcal{C}$ to be symmetric monoidal, if follows that there is a symmetric monoidal model category $\mathcal{M}$ such that $L\mathcal{M} \simeq \mathcal{C}$.

Theorem 5.7. *Let $(\mathcal{C}, \otimes, 1, \sigma)$ be a symmetric monoidal topological category. Then the associated ∞-category $N_T(\mathcal{C})$ is symmetric monoidal.*

Proof. Let $\mathcal{C}$ be a symmetric monoidal topological category. Then the associated fibrant simplicial category Sing $\mathcal{C}$ can be endowed with a symmetric monoidal structure. Hence it follows by Proposition 5.20 that there is a simplicial Grothendieck opfibration

$$p : (\mathrm{Sing}\ \mathcal{C})^{\otimes} \to \mathbf{FinSet}^*$$

which satisfies the Segal condition. We need to show now that the induced map

$$N_\Delta(p) : N_\Delta\left((\mathrm{Sing}\ \mathcal{C})^{\otimes}\right) \to N_\Delta(\mathbf{FinSet}^*)$$

is a coCartesian fibration. First notice, by definition of the category $(\mathrm{Sing}\ \mathcal{C})^{\otimes}$ and by the fact that Sing $\mathcal{C}$ is locally fibrant, that the category $(\mathrm{Sing}\ \mathcal{C})^{\otimes}$ is also locally fibrant. Since $\mathbf{FinSet}^*$ is trivially enriched, it is also fibrant with hom spaces given by discrete simplicial sets. Moreover, it follows by Proposition 5.11 that $N_\Delta(\mathbf{FinSet}^*) = N(\mathbf{FinSet}^*)$. We want to apply Proposition 5.28 to show that the induced map is a coCartesian fibration. First notice that given objects $X = [X_1, ..., X_m]$ and $Z = [Z_1, ..., Z_k]$ the hom space of $(\mathrm{Sing}\ \mathcal{C})^{\otimes}$ is defined as follows.

$$(\mathrm{Sing}\ \mathcal{C})^{\otimes}(X, Z) = \coprod_{\alpha \in \mathbf{FinSet}^*(\langle m \rangle, \langle k \rangle)} (\mathrm{Sing}\ \mathcal{C})^{\otimes}_{\alpha}(X, Y)$$

Therefore, the simplicial map induced by the functor p on the hom spaces is given by

$$\coprod_{\alpha \in \mathbf{FinSet}^*(\langle m \rangle, \langle k \rangle)} (\mathrm{Sing}\ \mathcal{C})^{\otimes}_{\alpha}(X, Y) \longrightarrow \mathbf{FinSet}^*(p(X), p(Z)) = \mathbf{FinSet}^*(\langle m \rangle, \langle k \rangle)$$

which is just the natural projection and therefore a Kan fibration. This shows that the functor p satisfies all the necessary properties, such that we can apply Proposition 5.28. Since by hypothesis

$$p : (\mathrm{Sing}\ \mathcal{C})^{\otimes} \to \mathbf{FinSet}^*$$

is a simplicial Grothendieck opfibration it is in particular also a weak simplicial Grothendieck opfibration. Then Proposition 5.28 implies that the map

$$N_\Delta(p) : N_\Delta\left((\text{Sing }\mathcal{C})^{\otimes}\right) \to N_\Delta(\mathbf{FinSet}^*) = N(\mathbf{FinSet}^*)$$

is a coCartesian fibration. Moreover, we notice by the definition of the topological nerve N_T and the fact that the functor Sing : $\mathbf{Top}_{cg}$-$\mathbf{Cat} \to \mathbf{sSet}$-$\mathbf{Cat}$ is a right adjoint, that

$$N_T\left(\mathcal{C}^{\otimes}\right) \cong N_\Delta\left((\text{Sing }\mathcal{C})^{\otimes}\right)$$

Therefore, we have that

$$q : N_T\left(\mathcal{C}^{\otimes}\right) \xrightarrow{\cong} N_\Delta\left((\text{Sing }\mathcal{C})^{\otimes}\right) \xrightarrow{N_\Delta(p)} N(\mathbf{FinSet}^*)$$

is a coCartesian fibration. Since p satisfies the Segal condition, we have that for all $n \geq 0$ the functors

$$(\text{Sing }\mathcal{C})^{\otimes}_{\langle n\rangle} \to (\text{Sing }\mathcal{C})^{\times n}$$

are equivalences of simplicial categories. In particular these are Dwyer-Kan equivalences and therefore the induced map

$$N_\Delta\left((\text{Sing }\mathcal{C})^{\otimes}_{\langle n\rangle}\right) \longrightarrow N_\Delta\left((\text{Sing }\mathcal{C})^{\times n}\right)$$

is a Joyal equivalence. Then the following composition is still a Joyal equivalence.

$$N_\Delta\left((\text{Sing }\mathcal{C})^{\otimes}\right)_{\langle n\rangle} \cong N_\Delta\left((\text{Sing }\mathcal{C})^{\otimes}_{\langle n\rangle}\right) \longrightarrow N_\Delta\left((\text{Sing }\mathcal{C})^{\times n}\right) \cong N_\Delta\left(\text{Sing }\mathcal{C}\right)^{\times n}$$

This shows that for all $n \geq 0$ the induced Segal maps

$$N_T(\mathcal{C}^{\otimes})_{\langle n\rangle} \xrightarrow{\sim} N_T(\mathcal{C}^{\otimes})^{\times n}$$

are Joyal equivalences, which implies that q indeed defines a symmetric monoidal structure on $N_T(\mathcal{C})$.

□

5.8 The Day convolution on the ∞-category of enriched functors

To define an equivalent notion of topological Day convolution in the ∞-categorical setting, we first need to specify the meaning of "topological" in this context. To do so, we need the notion of the ∞-category of spaces.

Definition 5.36. Let **Kan** denote the full subcategory of **sSet** spanned by the Kan complexes. Notice that the simplicial mapping space makes **Kan** into a simplicial category. Then we define $\mathcal{S} := N_\Delta(\mathbf{Kan})$ to be the **∞-category of spaces**.

Remark 5.35. First notice that even if we call $\mathcal{S}$ the ∞-category of spaces, we do not know yet that $\mathcal{S}$ is indeed an ∞-category. Moreover, by considering the Quillen model structure on **sSet** we notice that $\mathbf{Kan} = \mathbf{sSet}_{\mathrm{cf}}$ can be regarded as the subcategory of fibrant and cofibrant objects. Indeed, since the fibrant objects are precisely Kan complexes and any simplicial set is cofibrant. To see that $\mathcal{S}$ is indeed an ∞-category, we need the following result.

Proposition 5.29. *The full subcategory* **Kan** *is a locally fibrant simplicial category. Consequently, the simplicial set* $\mathcal{S}$ *is an* ∞*-category.*

Proof. We need to show that for all $X, Y \in \mathbf{Kan}$ the corresponding mapping space $\mathrm{Map}_{\mathbf{sSet}}(X, Y)$ is a Kan complex. Recall that in the Quillen model structure on **sSet** every object is cofibrant and the fibrant objects are given by Kan complexes. Using the fact that $(\mathbf{sSet}, \times, *)$ endowed with the Quillen model structure is a symmetric monoidal model category (Proposition 4.2.8 in [Hov99]), it follows that

$$\times : \mathbf{sSet} \times \mathbf{sSet} \to \mathbf{sSet}$$

is a Quillen bifunctor. By Remark 4.2.3 in [Hov99] we have that for any fibrant object Y in **sSet**, the functor

$$[-, Y] : \mathbf{sSet} \to \mathbf{sSet}^{\mathrm{op}}$$

is a left Quillen functor and in particular preserves cofibrations. Given any simplicial set X we know that it is cofibrant, i.e. there is a cofibration $* \to X$ which is preserved under the above functor. That is, the map

$$[*, Y] \to [X, Y]$$

in $\mathbf{sSet}^{\mathbf{op}}$ is a cofibration, which implies that

$$[X, Y] \to [*, Y] \cong Y$$

is a fibration in **sSet**. Since we choose Y to be fibrant, it follows now that also $[X, Y]$ is fibrant, i.e. the mapping space is a Kan complex. Since this is true for any X and any fibrant Y, it follows that **Kan** is locally fibrant.
Now we apply Remark 5.14 which implies that $\mathcal{S} = N_\Delta(\mathbf{Kan})$ is an ∞-category. □

Given a simplicial category $\mathcal{C}$, we showed that we are able to endow the enriched functor category $[\mathcal{C}, \mathbf{Kan}]$ with a Day convolution tensor product. The corresponding ∞-category of enriched functors should be given by the simplicial set $[N_\Delta(\mathcal{C}), \mathcal{S}]$. Indeed, the following proposition shows that under certain assumptions, the inner hom space $[-, -]$ in **sSet** is an ∞-category.

Proposition 5.30. ([Lur09]) *Let K be any simplicial set. Then the following statements hold.*

(i) *For every ∞-category $\mathcal{C}$, the simplicial set $[K, \mathcal{C}]$ is an ∞-category.*

(ii) *Let $\mathcal{C} \to \mathcal{D}$ be a Joyal equivalence of ∞-categories, then the induced map $[K, \mathcal{C}] \to [K, \mathcal{D}]$ is also a Joyal equivalence.*

(iii) *Let $\mathcal{C}$ be an ∞-category and $K \to K'$ a Joyal equivalence, then the induced map $[K', \mathcal{C}] \to [K, \mathcal{C}]$ is also a Joyal equivalence.*

Corollary 5.9. *For any simplicial category $\mathcal{C}$, the simplicial set $[N_\Delta(\mathcal{C}), \mathcal{S}]$ of functors into the ∞-category of spaces, is again an ∞-category.*

Proof. This follows directly from Proposition 5.29 and Proposition 5.30. □

Similarly as we constructed the Day convolution on the category of pointed enriched functors $[\mathcal{C}, \mathbf{Top}^*_{cg}]$, we can define the Day convolution in the same way on the topological category of functors $[\mathcal{C}, \mathbf{Top}_{cg}]$. Notice that here we consider enriched functors between unpointed categories. In the unpointed case, we just consider the cartesian product on $\mathbf{Top}_{cg}$ instead of the smash product on $\mathbf{Top}^*_{cg}$, which allows us to define the Day convolution in the same way. The reason why we used pointed spaces in the construction of the Day convolution is a geometrical one. Namely, we wanted that the sphere spectrum, playing the role of the tensor unit, induces structure maps of the form $S^1 \wedge X_n \to X_{n+1}$ to the $\mathcal{C}$-diagrams in $\mathbf{Top}^*_{cg}$. Nevertheless, we are going to restrict ourselves to unpointed enriched functors. This is mostly due to notational reasons and also because the pointed case can be handled easily by considering only slight modifications.

Corollary 5.10. *Let $\mathcal{C}$ be a topological category and consider the symmetric monoidal topological category $\left([\mathcal{C}, \mathbf{Top}_{cg}], \underset{\mathrm{Day}}{\otimes}, 1, \sigma\right)$. Then the ∞-category of enriched functors $N_T([\mathcal{C}, \mathbf{Top}_{cg}])$ carries a symmetric monoidal structure induced by the Day-Convolution.*

Proof. This is the application of Theorem 5.7 to the topological symmetric monoidal category $\left([\mathcal{C}, \mathbf{Top}_{cg}], \underset{\mathrm{Day}}{\otimes}, 1, \sigma\right)$. □

5.9 Outlook

Using Theorem 5.7 we were able to show, under the condition that $\mathcal{C}$ is a symmetric monoidal topological category, that the topological Day convolution passes down the coherent nerve functor to a symmetric monoidal structure on the ∞-category $N_T\left([\mathcal{C}, \mathbf{Top}_{cg}]\right)$ of enriched functors. That is, we are given a coCartesian fibration

$$p : N_T\left([\mathcal{C}, \mathbf{Top}_{cg}]^{\otimes}_{\mathrm{Day}}\right) \to N(\mathbf{FinSet}^*)$$

satisfying the Segal condition. However, this result is not yet in the desired form. We would like to give a Day convolution on the ∞-category $[K, \mathcal{S}]$ of functors into the ∞-category of spaces. Such a Day convolution exists, if we assume K to be a symmetric monoidal ∞-category, as for example showed in [Gla13]. By definition we then have a coCartesian fibration

$$[K, \mathcal{S}]^{\otimes} \longrightarrow N(\mathbf{FinSet}^*)$$

satisfying the Segal condition. Choosing $\mathcal{C}$ to be a symmetric monoidal topological category, we have again by Theorem 5.7 that the corresponding ∞-category $N_T(\mathcal{C})$ is symmetric monoidal. Hence for $K = N_T(\mathcal{C})$ we obtain a Day convolution on $[N_T(\mathcal{C}), \mathcal{S}]$, i.e. a coCartesian fibration

$$q : [N_T(\mathcal{C}), \mathcal{S}]^{\otimes} \longrightarrow N(\mathbf{FinSet}^*)$$

We notice that especially the relation between the simplicial sets $N_T([\mathcal{C}, \mathbf{Top}_{cg}])$ and $[N_T(\mathcal{C}), N_T(\mathbf{Top}_{cg})]$ plays an important role in the comparison between the two monoidal structures p and q.

Suppose we are given two topological categories $\mathcal{C}$ and $\mathcal{D}$. Then it follows from Proposition 5.2 that

$$\begin{aligned} \mathbf{Top\text{-}Cat}([\mathcal{C}, \mathcal{D}], [\mathcal{C}, \mathcal{D}]) &\cong \mathbf{Top\text{-}Cat}([\mathcal{C}, \mathcal{D}] \times \mathcal{C}, \mathcal{D}) \\ \mathrm{id} &\mapsto \varphi : [\mathcal{C}, \mathcal{D}] \times \mathcal{C} \to \mathcal{D} \end{aligned}$$

Therefore, by applying the coherent nerve we obtain

$$\begin{aligned} \mathbf{sSet}\left(N_T([\mathcal{C}, \mathcal{D}]) \times N_T(\mathcal{C}), N_T(\mathcal{D})\right) &\cong \mathbf{sSet}\left(N_T([\mathcal{C}, \mathcal{D}]), [N_T(\mathcal{C}), N_T(\mathcal{D})]\right) \\ N_T(\varphi) &\mapsto \psi : N_T([\mathcal{C}, \mathcal{D}]) \to [N_T(\mathcal{C}), N_T(\mathcal{D})] \end{aligned}$$

This shows that there is a canonical map of ∞-categories

$$\psi : N_T([\mathcal{C}, \mathcal{D}]) \to [N_T(\mathcal{C}), N_T(\mathcal{D})]$$

inducing the following diagram.

$$\begin{array}{ccc} N_T\left([\mathcal{C}, \mathbf{Top}_{cg}]^{\otimes}_{\mathrm{Day}}\right) & \xrightarrow{\quad \psi^{\otimes} \quad} & [N_T(\mathcal{C}), \mathcal{S}]^{\otimes} \\ & {}_{p}\searrow \quad \swarrow_{q} & \\ & N(\mathbf{FinSet}^*) & \end{array}$$

Assume now that the map ψ is a Joyal equivalence and that the diagram commutes. Then if $\psi^{\otimes}$ carries p-coCartesian edges to q-coCartesian edges, it follows that these two different notions of Day convolution are equivalent in the coCartesian model structure on the category of marked simplicial sets over $N(\mathbf{FinSet}^*)$, denoted by $\mathbf{sSet}^{+}{}_{/N(\mathbf{FinSet}^*)}$. However, neither the commutativity nor the fact that ψ is a weak equivalence follows immediately from the established theory, motivating further investigation.

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A Model categories

As model categories and model structures are used widely throughout this paper, this section will give a short introduction following Chapter 1 in [Hov99].

A.1 Model categories and their homotopy categories

Let $\mathcal{C}$ be a category, then we denote with $\mathrm{Map}\mathcal{C}$ the category whose objects are the morphisms of $\mathcal{C}$ and whose morphisms $f \to g$ are given by commutative squares in $\mathcal{C}$.

$$\begin{array}{ccc} x & \longrightarrow & x' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ y & \longrightarrow & y' \end{array}$$

Definition A.1.

(i) A map f in $\mathcal{C}$ is a **retract** of a map g in $\mathcal{C}$, if there is a commutative diagram of the form

$$\begin{array}{ccccc} x & \longrightarrow & x' & \longrightarrow & x \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ y & \longrightarrow & y' & \longrightarrow & y \end{array}$$

where the horizontal composites are identities.

(ii) A **functorial factorization** is an ordered pair (α, β) of functors $\mathrm{Map}\mathcal{C} \to \mathrm{Map}\mathcal{C}$, such that for all $f \in \mathrm{Map}\mathcal{C}$ it holds $\beta(f) \circ \alpha(f) = f$. This means in particular that the domain of $\alpha(f)$ is the domain of f, the codomain of $\alpha(f)$ is the domain of $\beta(f)$ and the codomain of $\beta(f)$ is the codomain of f.

Remark A.1. Recall that in a general category an object x is a retract of an object y if there are morphisms $i : x \to y$ and $r : y \to x$ such that the composition $r \circ i = \mathrm{id}_x$. Hence it is easy to see that the above definition of a retract coincides with the notion of a retract in the category $\mathrm{Map}\mathcal{C}$.

Definition A.2. Suppose $i : a \to b$ and $p : x \to y$ are maps in the category $\mathcal{C}$. Then i has the **left lifting property with respect to** p and p has the **right lifting property with respect to** i, if for every commutative diagram of the form

$$\begin{array}{ccc} a & \xrightarrow{f} & x \\ {\scriptstyle i}\downarrow & {\scriptstyle h}\nearrow & \downarrow{\scriptstyle p} \\ b & \xrightarrow[g]{} & y \end{array}$$

there exists a lift $h : b \to x$ such that $h \circ i = f$ and $p \circ h = g$.

Definition A.3. A **model structure** on a category $\mathcal{C}$ is the data of three subcategories of $\mathrm{Map}\mathcal{C}$ called weak equivalences, cofibrations and fibrations, and two functorial factorizations (α, β) and (γ, δ) satisfying the following axioms.

M1 (2-out-of-3) : If f and g are two composable morphisms of $\mathcal{C}$ and two of f, g and $g \circ f$ are weak equivalences, then so is the third.

M2 (Retracts) : If f and g are morphisms of $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence, cofibration, or fibration, then so is f.

M3 (Lifting property) : A morphism is said to be a **trivial cofibration** resp. **trivial fibration**, if it is both a weak equivalence and a cofibration resp. fibration. Then trivial cofibrations have the left lifting property with respect to fibrations and trivial fibrations have the right lifting property with respect to cofibrations.

M4 (Factorization) : For any morphism f in $\mathcal{C}$, $\alpha(f)$ is a cofibration and $\beta(f)$ is a trivial fibration and $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration. Hence every morphism can be factored as a composition of trivial cofibrations/fibrations and fibrations/cofibrations.

Definition A.4. A **model category** is a category $\mathcal{C}$ with all small limits and small colimits together with a model structure.

Remark A.2. A category with all small (co-)limits is called (co-)complete. This means that every model category has an initial object, the colimit of the empty diagram, and a terminal object, the limit of the empty diagram. If the morphism from the initial object to the terminal object is an isomorphism, we say that the category is **pointed**.

Definition A.5. An object x of a model category $\mathcal{C}$ is called **cofibrant**, if the morphism $0 \to x$ from the initial object 0 to x is a cofibration. Similarly an object x is called **fibrant** if the morphism $x \to *$ from x to the terminal object $*$ is a fibration.

Let $\mathcal{C}$ be a pointed model category and $*$ its terminal object. Then we denote with $\mathcal{C}^*$ the category under the object $*$. Objects of $\mathcal{C}^*$ are pairs (X, v) where $X \in \mathcal{C}$ and $v : * \to X$. Consider now the forgetful functor $U : \mathcal{C}^* \to \mathcal{C}$, which has a left adjoint functor $(-)_+ : \mathcal{C} \to \mathcal{C}^*$ given by the assignment $X \mapsto X_+ = X \amalg *$. These two functors define an equivalence of the categories $\mathcal{C}$ and $\mathcal{C}^*$. Since $\mathcal{C}$ is a model category the question arises, if the model structure transfers through the mentioned adjunction to the category $\mathcal{C}^*$.

Proposition A.1. ([Hov99]) *Suppose $\mathcal{C}$ is a model category. Then we define a map f in $\mathcal{C}^*$ to be a cofibration, fibration, or weak equivalence if and only if $U(f)$ is a cofibration, fibration, or weak equivalence in the model structure on $\mathcal{C}$. Then $\mathcal{C}^*$ is also a model category.*

Given a model category $\mathcal{C}$, apply the functorial factorization (α, β) to the unique morphism $i_X : 0 \to X$ for any object $X \in \mathcal{C}$. It follows that the morphism $\alpha(i_X) : 0 \to QX$ is a cofibration and $\beta(i_X) : QX \to X$ a trivial fibration. Hence there is a unique way to assign to each object $X \in \mathcal{C}$ its **cofibrant replacement**, denoted by QX. Similarly the functorial factorization (γ, δ) applied to the unique morphism $r_X : X \to *$ gives a trivial cofibration $\gamma(r_X) : X \to RX$ and a fibration $\delta(r_X) : RX \to *$. Hence there is also a unique way to assign to each object $X \in \mathcal{C}$ its **fibrant replacement**, denoted by RX. These assignments define two functors

$$Q : \mathcal{C} \to \mathcal{C} \qquad\qquad R : \mathcal{C} \to \mathcal{C}$$
$$X \mapsto QX \qquad\qquad X \mapsto RX$$

called cofibrant/fibrant replacement functors.

Lemma A.1. ([Hov99]) *Let $\mathcal{C}$ be a model category. Then a morphism is a cofibration resp. a trivial cofibration if and only if it has the left lifting property with respect to all trivial fibrations resp. all fibrations. Dually a morphism is a fibration resp. a trivial fibration if and only if it has the right lifting property with respect to all trivial cofibrations resp. all cofibrations.*

Remark A.3. Lemma A.1 shows that the axioms for a model category are overdetermined. That is given the two subcategories of weak equivalences and cofibrations, the subcategory of fibrations is determined by the right lifting property and similarly given weak equivalences and fibrations, the cofibrations are determined by the left lifting property.

The following corollary is very useful when working in concrete model categories such as **Top** or **sSet**.

Corollary A.1. *In a model category $\mathcal{C}$ cofibrations resp. trivial cofibrations are closed under pushouts. That is, given a pushout square where f is a cofibration resp. a trivial cofibration*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & Y \end{array}$$

then g is also a cofibration resp. a trivial cofibration. Dually fibrations resp. trivial fibrations are closed under pullbacks.

Given a model category $\mathcal{C}$ with classes of weak equivalences, fibrations and cofibrations one might want to look at objects of $\mathcal{C}$ up to weak equivalence. Since weak equivalences do not need to be isomorphisms in the category $\mathcal{C}$ we construct the homotopy category of $\mathcal{C}$ by formally inverting the weak equivalences.

Definition A.6. Let $\mathcal{C}$ be a category and $\mathcal{W}$ be a subcategory of weak equivalences. Then we define the quiver $Q(\mathcal{C}, \mathcal{W}^{-1})$ as follows

$$\begin{aligned} Q(\mathcal{C}, \mathcal{W}^{-1}) : \mathcal{X} &\to \textbf{Set} \\ X_0 &\mapsto \mathrm{Ob}(\mathcal{C}) \\ X_1 &\mapsto \mathrm{Mor}(\mathcal{C}) \cup \{w^{-1} \mid w \in \mathcal{W}\} \end{aligned}$$

and where

$$\begin{aligned} Q(s) : \mathrm{Mor}(\mathcal{C}) \cup \{w^{-1} \mid w \in \mathcal{W}\} &\to \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Mor}(\mathcal{C}) \ni f &\mapsto \mathrm{dom}(f) \\ \{w^{-1} \mid w \in \mathcal{W}\} \ni w^{-1} &\mapsto \mathrm{cod}(w) \end{aligned} \qquad \begin{aligned} Q(t) : \mathrm{Mor}(\mathcal{C}) \cup \{w^{-1} \mid w \in \mathcal{W}\} &\to \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Mor}(\mathcal{C}) \ni f &\mapsto \mathrm{cod}(f) \\ \{w^{-1} \mid w \in \mathcal{W}\} \ni w^{-1} &\mapsto \mathrm{dom}(w) \end{aligned}$$

Then we define the free category $F(\mathcal{C}, \mathcal{W}^{-1})$ as the path category of the quiver $Q(\mathcal{C}, \mathcal{W}^{-1})$, i.e.

$$F(\mathcal{C}, \mathcal{W}^{-1}) := PQ(\mathcal{C}, \mathcal{W}^{-1})$$

Remark A.4. Notice that objects of $F(\mathcal{C}, \mathcal{W}^{-1})$ are objects of $\mathcal{C}$ and morphisms are finite strings of composable morphisms $(f_1, ..., f_n)$ where f_i is either a morphism of $\mathcal{C}$ or the reversal w_i^{-1} of a morphism $w_i \in \mathcal{W}$.

Definition A.7. Let $\mathcal{C}$ and $\mathcal{W}$ be as above. Then we define the category $\mathcal{C}\left[\mathcal{W}^{-1}\right]$ as the quotient category of $F(\mathcal{C}, \mathcal{W}^{-1})$ by the relations

$$\begin{aligned} &1_X \sim (1_X) \text{ for all objects } X \in \mathcal{C} \\ &(g \circ f) \sim (f, g) \text{ for all composable arrows } f, g \in \mathcal{C} \\ &\mathrm{id}_{\mathrm{dom}(w)} \sim (w, w^{-1}) \text{ and } \mathrm{id}_{\mathrm{cod}(w)} \sim (w^{-1}, w) \text{ for all } w \in \mathcal{W} \end{aligned}$$

The category $\mathcal{C}\left[\mathcal{W}^{-1}\right]$ is called the **localization of $\mathcal{C}$ by $\mathcal{W}$**.

Remark A.5. In the case where $\mathcal{C}$ is a model category and $\mathcal{W}$ the subcategory of weak equivalences, we denote the localization by $\mathrm{Ho}\,\mathcal{C}$, which is also called the **homotopy category** of $\mathcal{C}$. Localizations in general may not behave very well. That is, given a locally small category its localization by some subcategory may not be locally small anymore. By treating categories in full generality this does not generate any issue, but as we restricted ourselves to treat only locally small categories one should always be aware. Thankfully it can be shown that the localization of a model category by its weak equivalences always defines a locally small category.

The localization of a category can be characterized by a universal property. Note that there is a functor $\gamma : \mathcal{C} \to \mathcal{C}\left[\mathcal{W}^{-1}\right]$ being the identity on objects and taking morphisms of $\mathcal{W}$ to isomorphisms in $\mathcal{C}\left[\mathcal{W}^{-1}\right]$.

Lemma A.2. ([Hov99]) *Let $\mathcal{C}$ be a category and $\mathcal{W}$ a subcategory.*

(i) *If $F : \mathcal{C} \to \mathcal{D}$ is a functor that sends morphisms of $\mathcal{W}$ to isomorphisms in $\mathcal{D}$, then there exists a unique functor $F\left[\mathcal{W}^{-1}\right] : \mathcal{C}\left[\mathcal{W}^{-1}\right] \to \mathcal{D}$ such that $\left(F\left[\mathcal{W}^{-1}\right]\right) \circ \gamma = F$.*

(ii) *Suppose $G : \mathcal{C} \to \mathcal{D}$ is a functor taking morphisms of $\mathcal{W}$ to isomorphisms and being equipped with the universal property of (i), then there is a unique isomorphism $\mathcal{C}\left[\mathcal{W}^{-1}\right] \xrightarrow{F} \mathcal{D}$ such that $F \circ \gamma = \delta$.*

As we will see later, it is important to consider the following full subcategories of a model category $\mathcal{C}$.

- $\mathcal{C}_c$ the full subcategory of cofibrant objects
- $\mathcal{C}_f$ the full subcategory of fibrant objects
- $\mathcal{C}_{cf}$ the full subcategory of cofibrant and fibrant objects

which all carry the "same" model structure up to weak equivalence, which is shown in the next Proposition.

Proposition A.2. ([Hov99]) *Let $\mathcal{C}$ be a model category and let $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ denote the full subcategories defined above. Then the corresponding inclusion functors induce equivalences of categories*

$$\operatorname{Ho}\mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}_c \to \operatorname{Ho}\mathcal{C}$$
$$\operatorname{Ho}\mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}_f \to \operatorname{Ho}\mathcal{C}$$

The above definition of the homotopy category of a model category is very abstract and does not really motivate its name. Hence by a second approach we will define an abstract notion of homotopy equivalence between arrows in a model category and use this notion to give a more tangible construction of the homotopy category.

Definition A.8. Let $\mathcal{C}$ be a model category, and let $f, g : B \to X$ be two morphisms in $\mathcal{C}$.

(i) A **cylinder object** for B is a factorization of the fold map $\nabla : B \amalg B \to B$ into a cofibration $B \amalg B \xrightarrow{i_0 + i_1} \operatorname{Cyl}(B)$ followed by a weak equivalence $\operatorname{Cyl}(B) \xrightarrow{s} B$.

(ii) A **path object** for X is a factorization of the diagonal map $\Delta : X \to X \times X$ into a weak equivalence $X \xrightarrow{r} \operatorname{Path}(X)$ followed by a fibration $\operatorname{Path}(X) \xrightarrow{(p_0,p_1)} X \times X$.

(iii) A **left homotopy** from f to g is a morphism $H : \operatorname{Cyl}(B) \to X$ for some cylinder object for B, such that $H \circ i_0 = f$ and $H \circ i_1 = g$. If such a left homotopy exists, we say that f and g are left homotopic, written $f \overset{l}{\sim} g$.

(iv) A **right homotopy** from f to g is a morphism $h : B \to \operatorname{Path}(X)$ for some path object for X, such that $p_0 \circ h = f$ and $p_1 \circ h = g$. If such a right homotopy exists, we say that f and g are right homotopic, written $f \overset{r}{\sim} g$.

(v) We say that f and g are **homotopic**, written $f \sim g$, if they are both left and right homotopic. Moreover a morphism $f : B \to X$ is said to be a homotopy equivalence, if there is a morphism $k : X \to B$ such that $f \circ k \sim \mathrm{id}_X$ and $k \circ f \sim \mathrm{id}_B$.

Remark A.6. Notice that in the above definition the morphisms i_0, i_1 and p_0, p_1 are the corresponding inclusion and projection morphisms, such that the following diagrams commute for a left homotopy H and a right homotopy h between f and g.

$$
\begin{array}{ccc}
\mathrm{Cyl}(B) & \xrightarrow{H} & X \\
{\scriptstyle i_0+i_1}\uparrow & & \uparrow{\scriptstyle f} \\
B \amalg B & \xleftarrow[i_0]{} & B
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Cyl}(B) & \xrightarrow{H} & X \\
{\scriptstyle i_0+i_1}\uparrow & & \uparrow{\scriptstyle g} \\
B \amalg B & \xleftarrow[i_1]{} & B
\end{array}
\qquad
\begin{array}{ccc}
B & \xrightarrow{h} & \mathrm{Path}(X) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (p_0,p_1)} \\
X & \xleftarrow[p_0]{} & X \times X
\end{array}
\qquad
\begin{array}{ccc}
B & \xrightarrow{h} & \mathrm{Path}(X) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (p_0,p_1)} \\
X & \xleftarrow[p_1]{} & X \times X
\end{array}
$$

Moreover it follows that the natural choices for cylinder and path objects are the objects $B \times I$ and X^I, which are defined to be the objects occurring in the functorial factorization of the fold map and the diagonal map. Those objects are thus part of the datum of a model category.

We will see in the following proposition that left and right homotopies define an equivalence relation on certain morphisms of the model category $\mathcal{C}$, which allows us to talk about morphisms up to homotopy.

Proposition A.3. ([Hov99]) *Let $\mathcal{C}$ be a model category and let B be a cofibrant object and X a fibrant object of $\mathcal{C}$. Then the left homotopy and right homotopy relations coincide and define an equivalence relation on $\mathcal{C}(B, X)$.*

Corollary A.2. *The homotopy relation on the morphisms of $\mathcal{C}_{cf}$ is an equivalence relation and is compatible with composition. Hence the quotient category $\mathcal{C}_{cf}/_{\sim}$ exists.*

Now we notice that the quotient category $\mathcal{C}_{cf}/_{\sim}$ implements our intuition of a homotopy category of a model category. Hence it is left to show that $\mathcal{C}_{cf}/_{\sim}$ and $\mathrm{Ho}\,\mathcal{C}$ define indeed the same category up to equivalence. To do so, we fist notice that the notion of homotopy equivalence and weak equivalence coincide in the cofibrant and fibrant setting.

Proposition A.4. ([Hov99]) *Let $\mathcal{C}$ be a model category. Then a morphism in $\mathcal{C}_{cf}$ is a weak equivalence if and only if it is a homotopy equivalence.*

Corollary A.3. *Let $\mathcal{C}$ be a model category and let $\gamma : \mathcal{C}_{cf} \to \mathrm{Ho}\,\mathcal{C}_{cf}$ and $\delta : \mathcal{C}_{cf} \to \mathcal{C}_{cf}/_{\sim}$ be the canonical functors. Then there is a unique isomorphism of categories $\mathcal{C}_{cf}/_{\sim} \xrightarrow{j} \mathrm{Ho}\,\mathcal{C}_{cf}$ such that $j \circ \delta = \gamma$. Moreover j is the identity on objects.*

Proof. One needs to show that the pair $(\delta, \mathcal{C}_{cf}/_{\sim})$ has the universal property of the localization of $\mathcal{C}_{cf}$ with respect to weak equivalences. For details see Corollary 1.2.9. in [Hov99]. □

The following theorem, providing the desired categorical equivalence between these two notions of a homotopy category can be considered the fundamental theorem of model categories.

Theorem A.1. *Let $\mathcal{C}$ be a model category. Let $\gamma : \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ denote the canonical functor, and let Q denote the cofibrant replacement functor of $\mathcal{C}$ and let R denote the fibrant replacement functor. Then*

(i) *the inclusion $\mathcal{C}_{cf} \to \mathcal{C}$ induces an equivalence of categories*

$$\mathcal{C}_{cf}/_{\sim} \longrightarrow \mathrm{Ho}\,\mathcal{C}$$

(ii) *there are natural isomorphisms*

$$\mathcal{C}(QRX, QRY)/_{\sim} \cong \mathrm{Ho}\,\mathcal{C}(\gamma(X), \gamma(Y)) \cong \mathcal{C}(RQX, RQY)/_{\sim}$$

(iii) *if $f : A \to B$ is a morphism in $\mathcal{C}$ such that $\gamma(f)$ is an isomorphism in $\mathrm{Ho}\,\mathcal{C}$, then f is a weak equivalence.*

A.2 Quillen functors and derived functors

The following will be a short introduction to Quillen functors and derived functors with the aim to give a profound definition of Quillen equivalences and their properties.
In the following let $\mathcal{C}$ and $\mathcal{D}$ be two model categories, unless specified otherwise.

Definition A.9.

(i) A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a **left Quillen functor** if F is a left adjoint and preserves cofibrations and trivial cofibrations.

(ii) A functor $U : \mathcal{D} \to \mathcal{C}$ is said to be a **right Quillen functor** if G is a right adjoint and preserves fibrations and trivial fibrations.

(iii) A pair of adjoint functors $F \dashv G$ is said to be a **Quillen adjunction** or **Quillen pair** if F is a left Quillen functor.

Remark A.7. It can be shown using Ken Brown's Lemma that every left Quillen functor preserves weak equivalences between cofibrant objects and that every right Quillen functor preserves weak equivalences between fibrant objects. Moreover it is easy to see that a pair of adjoint functors $F \dashv G$ between to model categories is a Quillen adjunction if and only if G is a right Quillen functor. For details see Lemma 1.3.4. in [Hov99].

Definition A.10.

(i) If $F : \mathcal{C} \to \mathcal{D}$ is a left Quillen functor, define the **total left derived functor** $LF : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ to be the composite

$$\operatorname{Ho}\mathcal{C} \xrightarrow{\operatorname{Ho}Q} \operatorname{Ho}\mathcal{C}_c \xrightarrow{\operatorname{Ho}F} \operatorname{Ho}\mathcal{D}$$

Given a natural transformation $\tau : F \to F'$ of left Quillen functors, define the **total derived natural transformation** $L\tau : LF \to LF'$ to be the composite $\operatorname{Ho}\tau \circ \operatorname{Ho}Q$, so that one has $(L\tau)_X = \tau_{QX}$ for every $X \in \operatorname{Ho}\mathcal{C}$.

(ii) If $G : \mathcal{D} \to \mathcal{C}$ is a right Quillen functor, define the **total right derived functor** $RG : \operatorname{Ho}\mathcal{D} \to \operatorname{Ho}\mathcal{C}$ to be the composite

$$\operatorname{Ho}\mathcal{D} \xrightarrow{\operatorname{Ho}R} \operatorname{Ho}\mathcal{D}_f \xrightarrow{\operatorname{Ho}G} \operatorname{Ho}\mathcal{C}$$

Given a natural transformation $\eta : G \to G'$ of right Quillen functors define the **total derived natural transformation** $R\eta : RG \to RG'$ to be the composite $\operatorname{Ho}\eta \circ \operatorname{Ho}R$, so that one has $(R\eta)_Y = \eta_{RY}$ for every $Y \in \operatorname{Ho}\mathcal{D}$.

Given a Quillen adjunction $F \dashv G$ between two categories, the corresponding total derived functors will create a pair of adjoint functors between the corresponding homotopy categories. The main goal of this section will be to investigate under what conditions this derived adjunction is in fact an equivalence of categories. Such an adjunction will be called a Quillen equivalence.

Proposition A.5. ([Hov99]) *Let $F \dashv G : \mathcal{C} \to \mathcal{D}$ be a Quillen adjunction. Then the total derived functors LF and RG form an adjoint pair of functors between the homotopy categories, i.e.*

$$LF \dashv GR : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$$

This adjunction is called the derived adjunction.

Definition A.11. A Quillen adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ is called a **Quillen equivalence**, if and only if for all cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, a morphism $f : FX \to Y$ is a weak equivalence in $\mathcal{D}$, if and only if $\varphi_{X,Y}(f) : X \to GY$ is a weak equivalence in $\mathcal{C}$.

Remark A.8. Notice that φ denotes the family of natural isomorphisms of hom sets

$$\varphi_{X,Y} : \mathcal{D}(FX, Y) \xrightarrow{\cong} \mathcal{C}(X, GY)$$

given by the adjoint pair $F \dashv G$.

Proposition A.6. ([Hov99]) *Let $F \dashv G : \mathcal{C} \to \mathcal{D}$ be a Quillen adjunction. Then the following are equivalent.*

(i) *$F \dashv G$ is a Quillen equivalence.*

(ii) *The derived adjunction $LF \dashv RG$ is an adjoint equivalence of categories.*

The next proposition will be a useful tool to check whether a Quillen adjunction is a Quillen equivalence.

Proposition A.7. ([Hov99]) *Let $F \dashv G : \mathcal{C} \to \mathcal{D}$ be a Quillen adjunction. Then the following are equivalent.*

(i) *$F \dashv G$ is a Quillen equivalence.*

(ii) *F reflects weak equivalences between cofibrant objects, i.e. for every morphism $f : X \to X'$ in $\mathcal{C}_c$ such that $F(f)$ is a weak equivalence in $\mathcal{D}$ then f is a weak equivalence in $\mathcal{C}$. Moreover for every fibrant Y in $\mathcal{D}$, the morphism $(FQG)(Y) \to Y$ is a weak equivalence.*

(iii) *G reflects weak equivalences between fibrant objects and, for every cofibrant X in $\mathcal{C}$ the morphism $X \to (GRF)(X)$ is a weak equivalence.*

B Derived hom spaces

In this section we will introduce a very powerful tool to detect weak equivalences in model categories. To do so, we associate to every pair of objects x, y in a model category $\mathcal{M}$ a simplicial set $\mathbb{L}\mathcal{M}(x, y)$ which we call the left derived hom space of x and y. In contrast to the hom sets of the homotopy category $\operatorname{Ho}\mathcal{M}$, the derived hom spaces carry higher homotopical coherence data. To introduce these spaces we will follow Chapter 15, 16 and 17 in [Hir03].

B.1 (Co)Simplicial resolutions

Definition B.1. Let $\mathcal{C}$ be a model category and let x be an object of $\mathcal{C}$.

(i) A **cosimplicial resolution** of x is a cofibrant approximation $Q(\Delta_x) \to \Delta_x$ in the Reedy model structure on $\mathcal{C}^{\Delta}$. Here Δ_x denotes the constant cosimplicial object given by $(\Delta_x)^n = x$ for all $[n] \in \Delta$.

(ii) A **simplicial resolution** of x is a fibrant approximation $\Delta_x^{\mathrm{op}} \to R(\Delta_x^{\mathrm{op}})$ in the Reedy model structure on $\mathcal{C}^{\Delta^{\mathrm{op}}}$. Here Δ_x^{op} denotes the constant simplicial object given by $(\Delta_x^{\mathrm{op}})_n = x$ for all $[n] \in \Delta^{\mathrm{op}}$.

Proposition B.1. ([Hir03]) *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen pair. Then*

(i) *if x is a cofibrant object of $\mathcal{C}$ and $Q(\Delta_x) \to \Delta_x$ is a cofibrant resolution of x, then $FQ(\Delta_x) \to \Delta_{F(x)}$ is a cofibrant resolution of $F(x)$.*

(ii) *if y is a fibrant object of $\mathcal{D}$ and $\Delta_y^{\mathrm{op}} \to R(\Delta_y^{\mathrm{op}})$ is a simplicial resolution of y, then $\Delta_{G(y)}^{\mathrm{op}} \to GR(\Delta_y^{\mathrm{op}})$ is a simplicial resolution of $G(y)$.*

B.2 Left derived hom spaces

Definition B.2. If $\mathcal{C}$ is a model category and x, y are objects of $\mathcal{C}$, then a **left derived hom space** from x to y is a triple

$$(Q(\Delta_x), Ry, \mathbb{L}\mathcal{C}(x,y))$$

where $Q(\Delta_x)$ is a cosimplicial resolution of x, Ry is a fibrant approximation of y and $\mathbb{L}\mathcal{C}(x,y)$ is the simplicial set defined by

$$\mathbb{L}\mathcal{C}(x,y)_n = \mathbf{sSet}(Q(\Delta_x)^n, Ry)$$

Proposition B.2. ([Hir03]) *If $\mathcal{C}$ is a model category and x, y are objects of $\mathcal{C}$, then a left derived hom space from x to y is a fibrant simplicial set in the Quillen model structure on* **sSet**.

Definition B.3. A **change of left derived hom spaces** is a triple (f, g, h), where

(i) $f : Q'(\Delta_x) \to Q(\Delta_x)$ is a map of cosimplicial resolutions of x.

(ii) $g : Ry \to R'y$ is a map of fibrant approximations of y.

(iii) $h : \mathbb{L}\mathcal{C}(x,y) \to \mathbb{L}'\mathcal{C}(x,y)$ is the map of simplicial sets induced by f and g.

Remark B.1. Notice that in the above definition, the left derived hom space $\mathbb{L}'\mathcal{C}(x,y)$ is the simplicial set defined by

$$\mathbb{L}'\mathcal{C}(x,y)_n = \mathbf{sSet}(Q'(\Delta_x)^n, R'y)$$

Proposition B.3. ([Hir03]) *If $\mathcal{C}$ is a model category and x and y are objects in $\mathcal{C}$, then a change of left derived hom spaces is a weak equivalence of fibrant simplicial sets in the Quillen model structure on* **sSet**.

Theorem B.1. ([Hir03]) *If $\mathcal{C}$ is a model category and $g : x \to y$ is a morphism in $\mathcal{C}$, then the following are equivalent.*

(i) *The morphism g is a weak equivalence in $\mathcal{C}$.*

(ii) *For every object z in $\mathcal{C}$ the morphism g induces a weak equivalence of left derived hom spaces*

$$g_* : \mathbb{L}\mathcal{C}(z,x) \to \mathbb{L}\mathcal{C}(z,y)$$

in the Quillen model structure on **sSet**.

(iii) *For every cofibrant object z in $\mathcal{C}$ the morphism g induces a weak equivalence of left derived hom spaces*

$$g_* : \mathbb{L}\mathcal{C}(z,x) \to \mathbb{L}\mathcal{C}(z,y)$$

in the Quillen model structure on **sSet**.

(iv) *For every object z in $\mathcal{C}$ the morphism g induces a weak equivalence of left derived hom spaces*

$$g^* : \mathbb{L}\mathcal{C}(z,x) \to \mathbb{L}\mathcal{C}(z,y)$$

in the Quillen model structure on **sSet**.

(v) *For every fibrant object z in $\mathcal{C}$ the morphism g induces a weak equivalence of left derived hom spaces*

$$g^* : \mathbb{L}\mathcal{C}(z,x) \to \mathbb{L}\mathcal{C}(z,y)$$

in the Quillen model structure on **sSet**.

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