Dynamical Systems Conference
in memory of Jean-Christophe Yoccoz

Corinna Ulcigrai

On Birkhoff sums and Roth type conditions for interval exchange maps

(based on joint work with
S. Marmi and J.-C. Yoccoz)

Collège de France, Paris, May 29 2017

University of BRISTOL

Interval exchange maps and Birkhoff sums

- Interval exchange map (i.e.m.)

  $T = (\pi, \lambda)$, where:
  - $T : I \to I$ where $I := [0, 1]$;
  - $\mathcal{A}$ alphabet with $|\mathcal{A}| = d$;
  - d subintervals I_α, $\alpha \in \mathcal{A}$;
  - π permutation on $\mathcal{A}$;
  - $\lambda = (\lambda_\alpha)_\alpha \in \mathcal{A}$;
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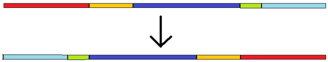

- Function $f\colon [0, 1] \to \mathbb{R}$;

  Consider its Birkhoff sums

$$S_n f(x) := \sum_{i=0}^{n} f(T^i x), \qquad x \in I.$$

- In this talk we will mostly focus on $f \in \Gamma(T)$, where $\Gamma(T) := \{f: \quad \text{piecewise constant}\}$
- f_α value on I_α, so $f \longleftrightarrow (f_\alpha, f_\beta, \ldots, f_\omega) \in \mathbb{R}^d$;
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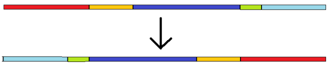

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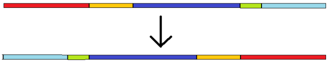

$\mathcal{A} := \{\alpha, \beta, \gamma, \delta, \varepsilon\}$

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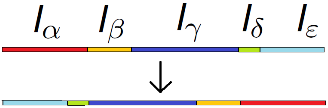

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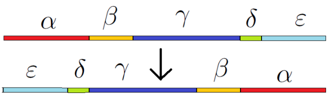

$$\pi = \begin{pmatrix} \alpha\,\beta\,\gamma\,\delta\,\varepsilon \\ \varepsilon\,\delta\,\gamma\,\beta\,\alpha \end{pmatrix}$$

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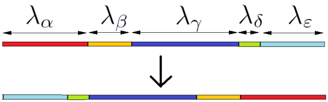

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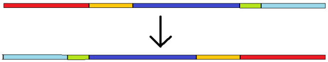

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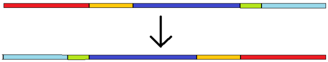

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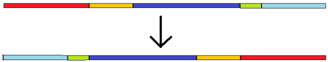

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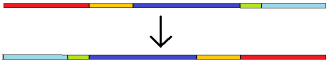

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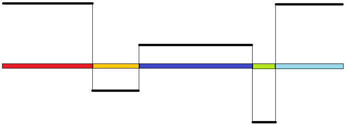

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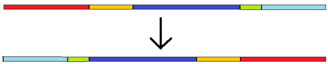

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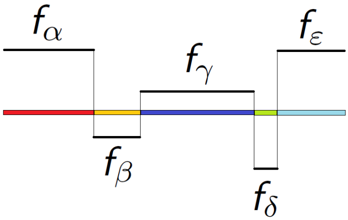

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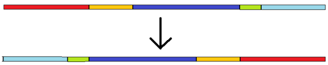

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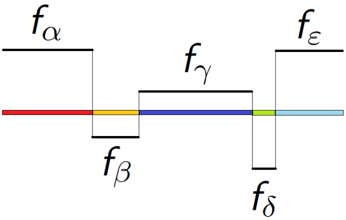

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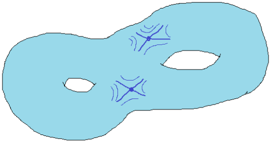
Translation surfaces and ergodic integrals

Geometric counterpart object:

- $M = M(\pi, \lambda, \tau)$ translation surface (flat metric, conical singularities)



- M can be represented as a zippered rectangle, i.e. union of rectangles R_α with glueings;
  λ_α width, q_α height of R_α
  τ "*zips heights*"; (π, τ) determine
  $q = (q_\alpha)_\alpha$ height vector ($q = -\Omega_\pi(\tau)$)

Remark: π "*knows*" about the genus g and number of conical singularities k;

$$d = 2g + k$$

- $\varphi_t : S \to S$ vertical linear flow;
  - *Poincaré section* is the i.e.m $T = (\pi, \lambda)$;
  - Given $f : S \to \mathbb{R}$, consider ergodic integrals

$$\int_0^T f(\varphi_t(x))dt, \qquad x \in M.$$

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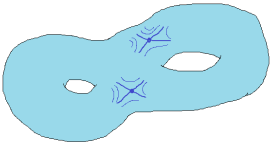

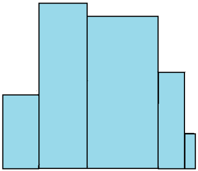

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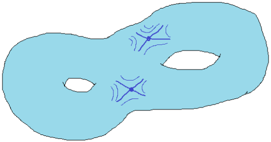

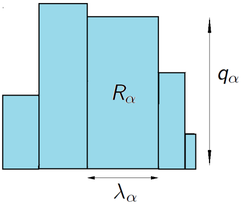

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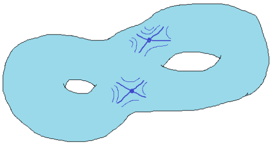

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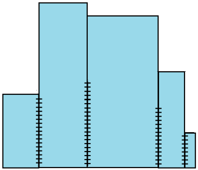

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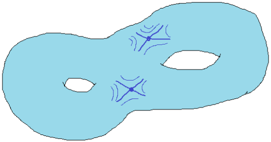

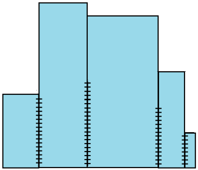

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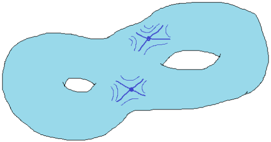

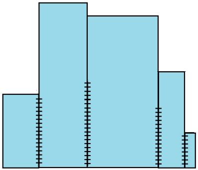

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Translation surfaces and ergodic integrals

Geometric counterpart object:

- $M = M(\pi, \lambda, \tau)$ translation surface (flat metric, conical singularities)
- M can be represented as a zippered rectangle, i.e. union of rectangles R_α with glueings;
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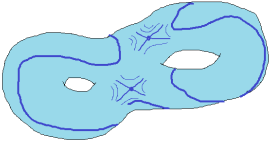

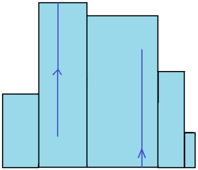

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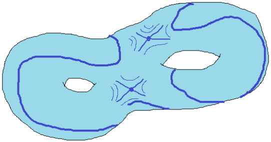

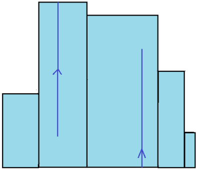

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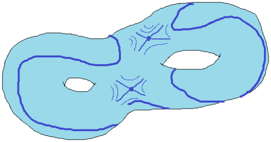

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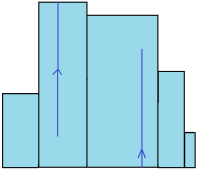

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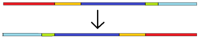
Ergodicity and deviations of ergodic averages



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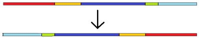

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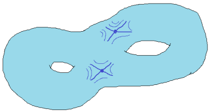

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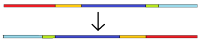

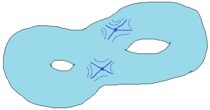

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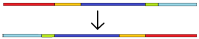

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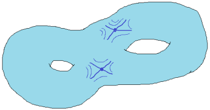

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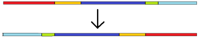

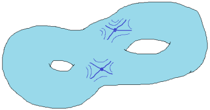

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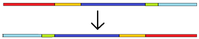

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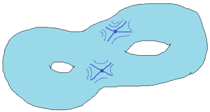

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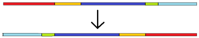

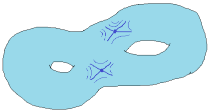

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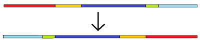

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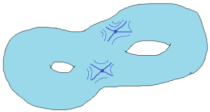

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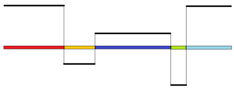

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Lower bounds on deviations

What about *lower bounds*?

Consider e.g. piecewise constant functions
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where $\quad dim(F+/F_0) = g, \; dim(F_0/F_-) = k-1, \; dim(F_-) = g$

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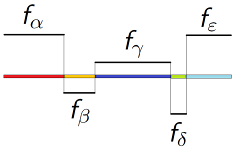

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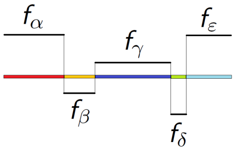

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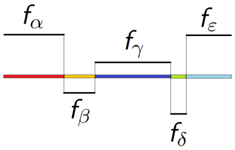

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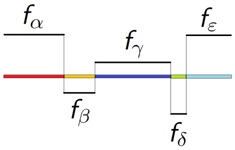

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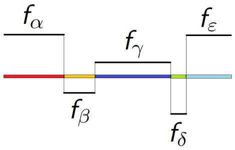

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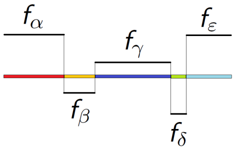

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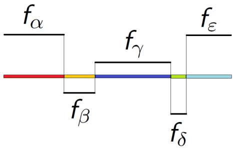

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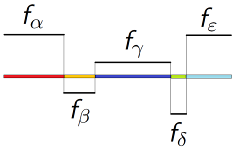

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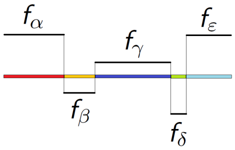

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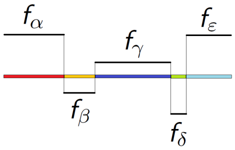

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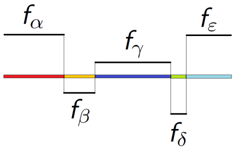

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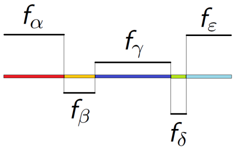

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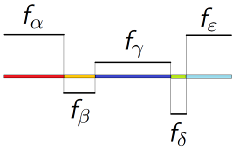

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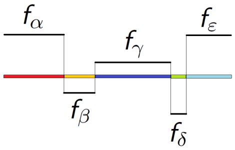

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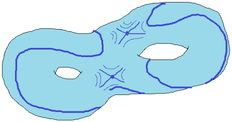

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Reference: Giovanni Forni, (Annals Math., 2002)

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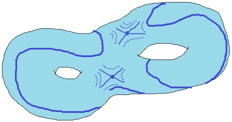

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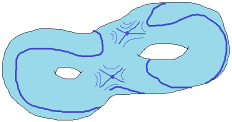

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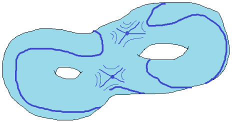

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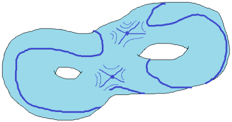

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Graphs of Birkhoff sums

Question: What can we say beyond the deviations asymptotic size?

- Take $f \in \Gamma(T)$, fix $x_0 \in I$.
- Plot the graph of the Birkhoff sums $S_k f(x_0)$, for $k = 0, 1, \ldots, n$.

Piecewise affine function with vertices $(k, S_k f(x))$, $k = 0, 1, \ldots, n$.

- Call $\Omega_n = \Omega_n(f, T, x_0)$ the rescaled plot, where the x-axis is rescaled to $[0, 1]$, by mapping $x \mapsto x/n$.

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- Call $\Omega_n = \Omega_n(f, T, x_0)$ the rescaled plot, where the x-axis is rescaled to $[0, 1]$, by mapping $x \mapsto x/n$.

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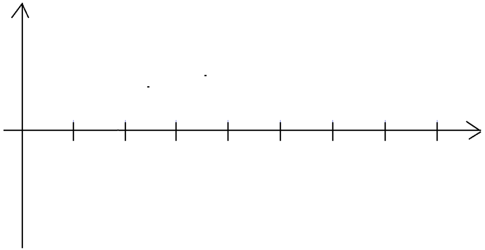

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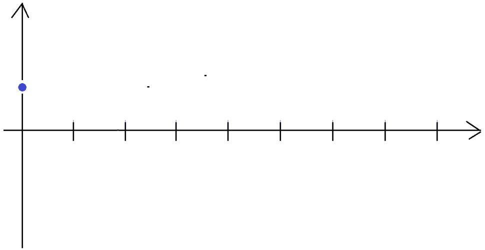

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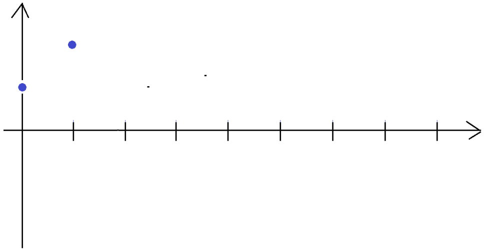

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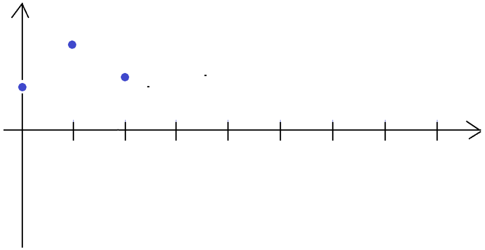

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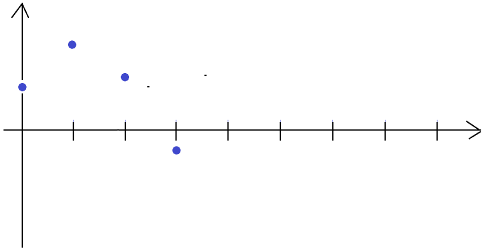

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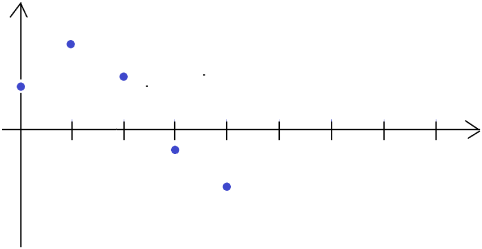

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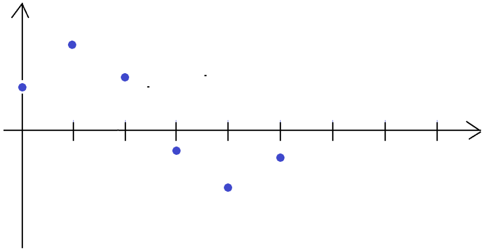

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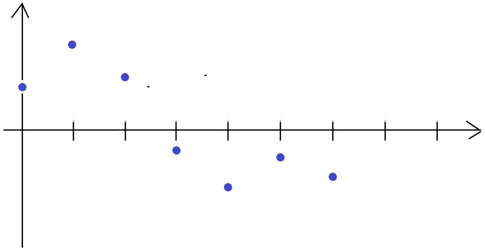

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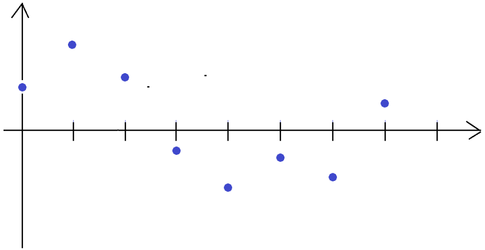

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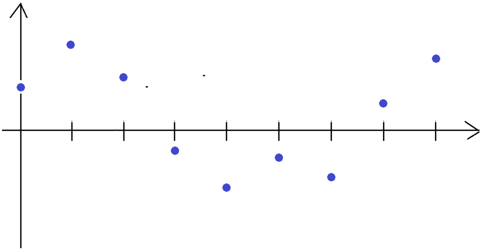

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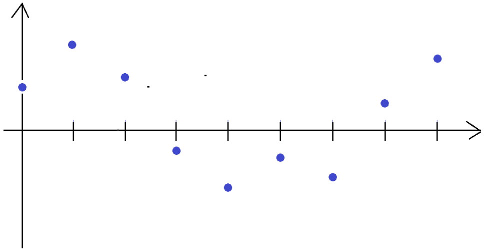

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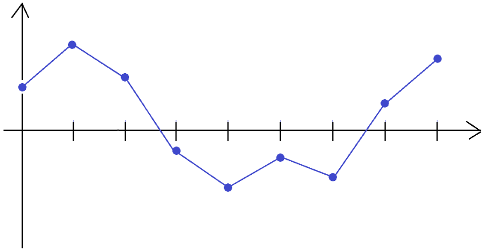

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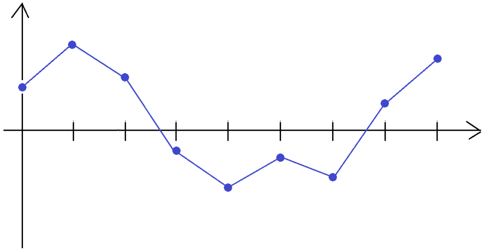

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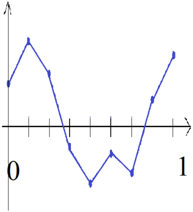

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Graphs of Birkhoff sums: examples of behaviour

The behaviour of the plot depends on whether:

- $f \in F_-(T)$,

- $f \in F_0(T) \backslash F_-(T)$

- $f \in F_+(T) \backslash F_0(T)$

(Credit for Figures: Stefano Marmi)

E.g. T i.e.m. with $d = 5$ (pseudo-Anosov), plot of $\Omega_n(f, T, 0)$, for $f \in \Gamma_0(T)$.

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The behaviour of the plot depends on whether:

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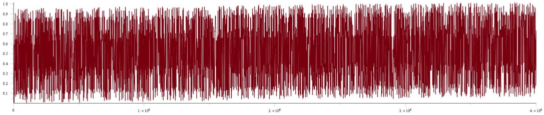

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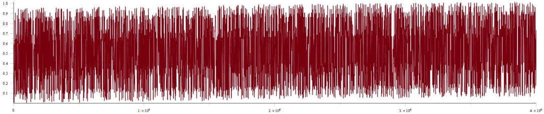

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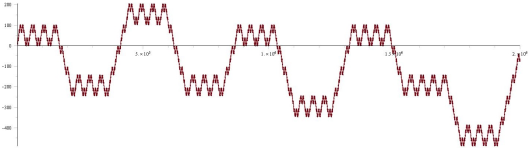

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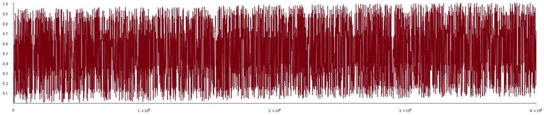

- $f \in F_0(T) \backslash F_{-}(T)$

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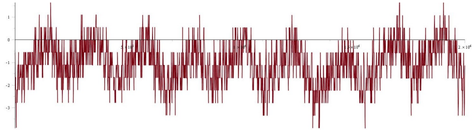

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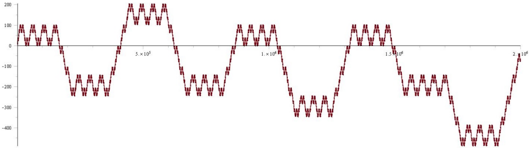

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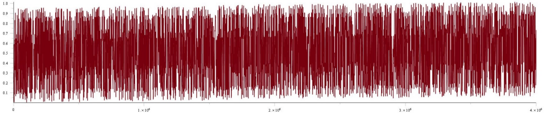

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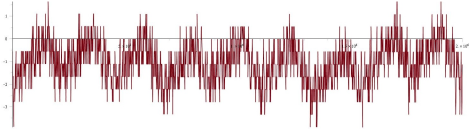

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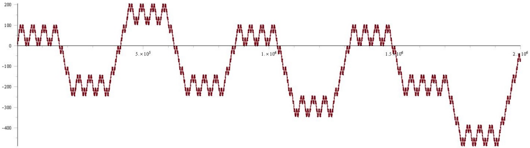

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Today: we will focus on describing the behaviour of $f \in F_0(T) \backslash F_-(T)$.

Examples of Birkhoff sums with subpolynomial deviations

- Ex 1: $T = R_a$ rotation,

  $T(x) = x + a \mod 1$,

  $f = f_b = \mathbb{1}_{[0,b]} - b$.

  - If $b \in \{T^n(a), n \in \mathbb{N}\}$,

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- Ex 2: IETs with $k > 1$ (so that $dim F_0/F_- = k - 1 > 0$), $f \in F_0 \backslash F_-$ (e.g. $\pi = \begin{pmatrix} \alpha\beta\gamma\delta\varepsilon \\ \varepsilon\delta\gamma\beta\alpha \end{pmatrix}$ with d odd).
- Ex 3: Corrected characteristic functions $f_b - \chi$, given by:

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For a.e. $T = (\pi, \lambda, \tau)$, given $f_b = \mathbb{1}_{[0,b]}$, there exists $\chi \in \Gamma(T)$ such that $\tilde{f}_b := f_b - \chi \in F_0(T, b)$, i.e.

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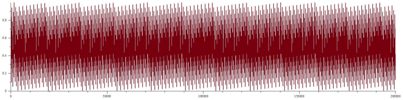

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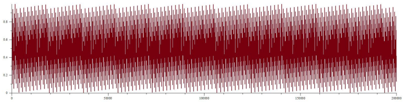

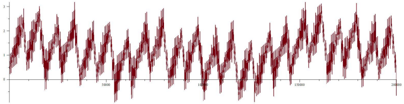

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  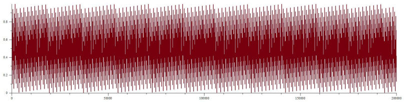

  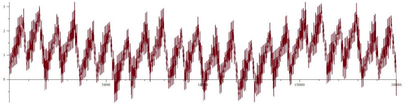

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Examples of Birkhoff sums with subpolynomial deviations

- Ex 1: $T = R_a$ rotation,

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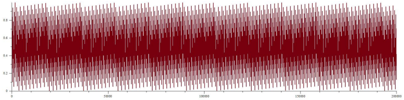

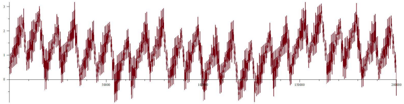

- Ex 2: IETs with $k > 1$ (so that $dim F_0/F_- = k-1 > 0$), $f \in F_0 \backslash F_-$ (e.g. $\pi = \begin{pmatrix} \alpha\beta\gamma\delta\varepsilon \\ \varepsilon\delta\gamma\beta\alpha \end{pmatrix}$ with d odd).

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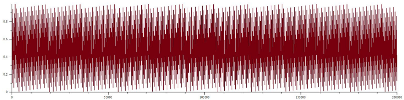

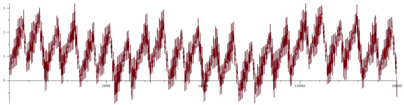

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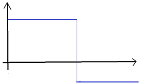

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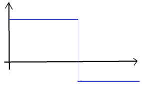

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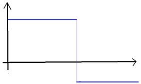

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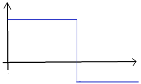

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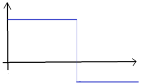

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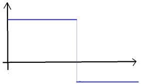

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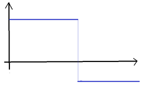

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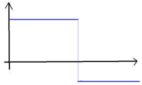

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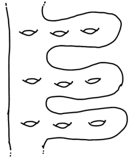

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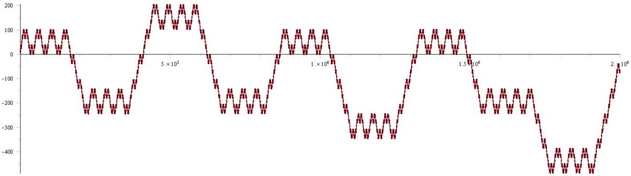

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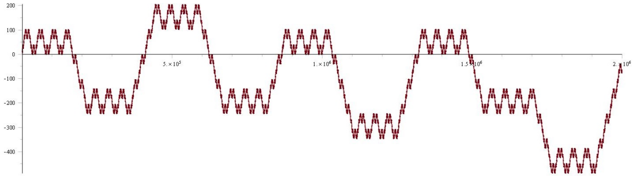

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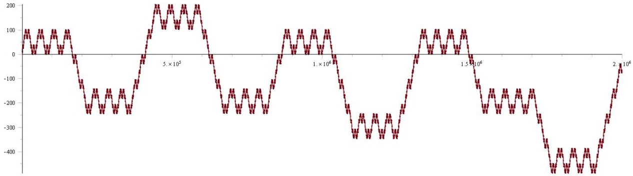

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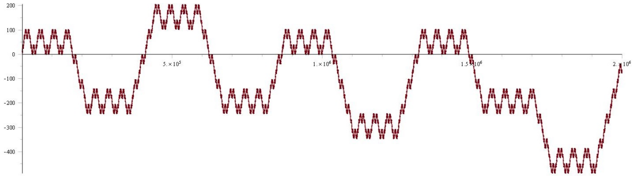

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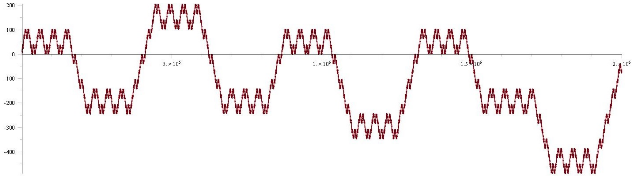

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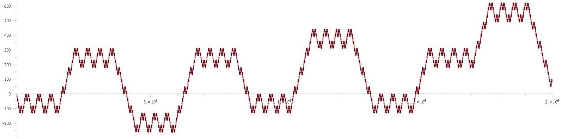
- The graphs Ω_n have a *fractal graph* structure; is there convergence?
  - NO: there are oscillations, since $S_n f(x) \sim n^\nu$ ($\nu > 0$)



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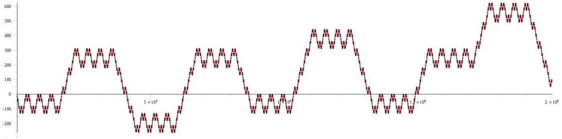

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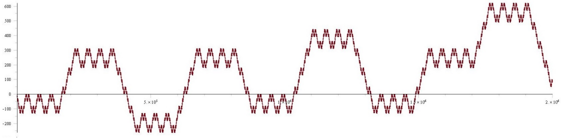

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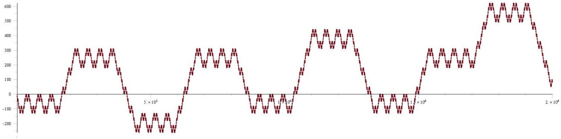

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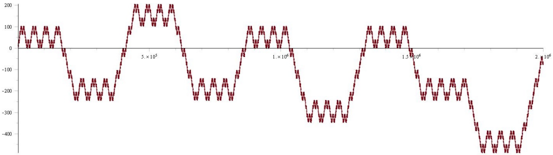

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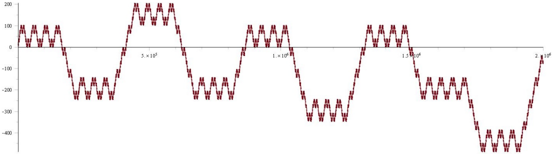

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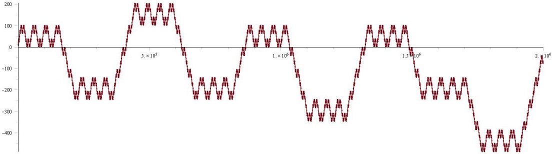

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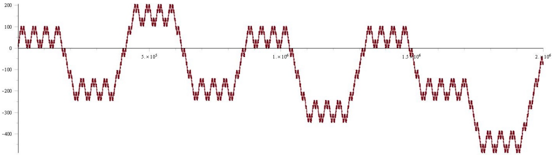

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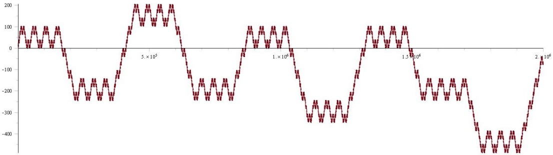

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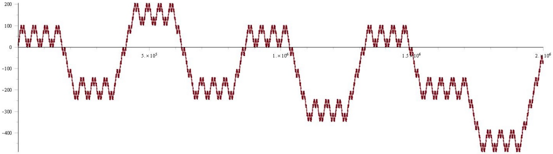

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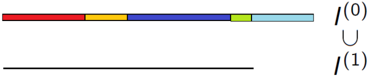

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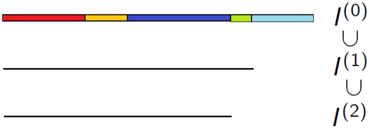

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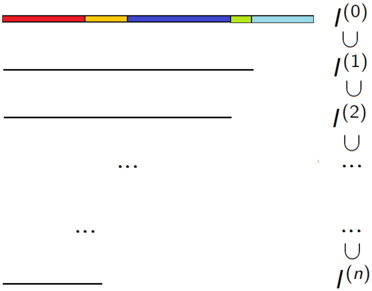

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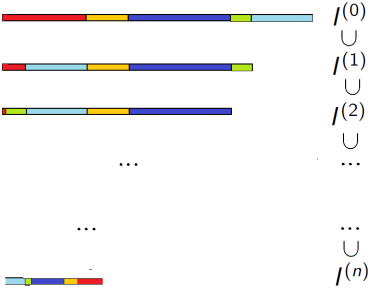

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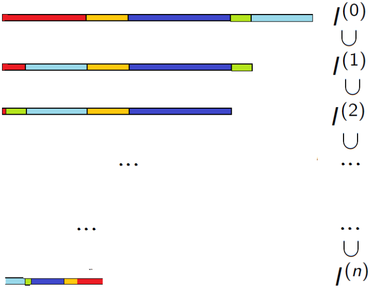

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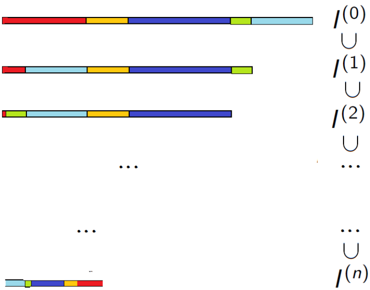

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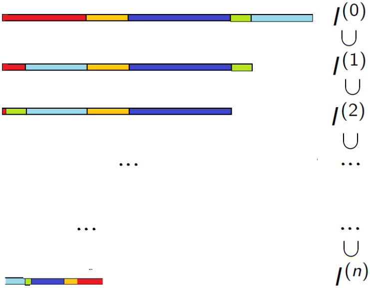

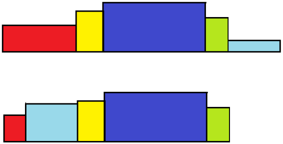

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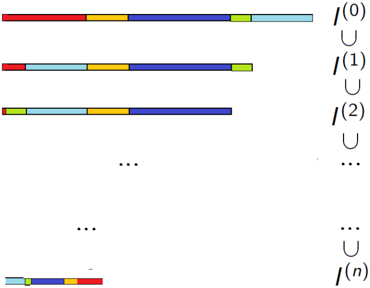

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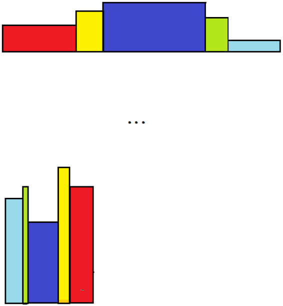

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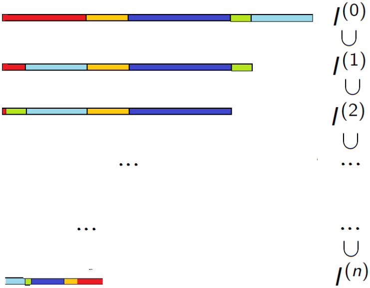

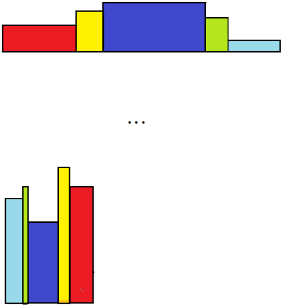

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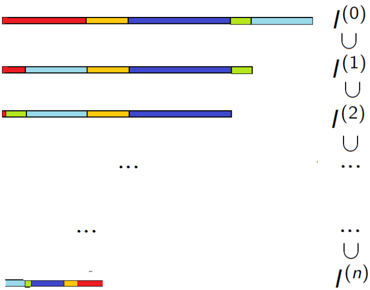

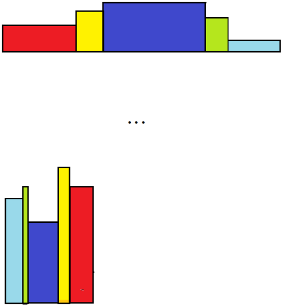

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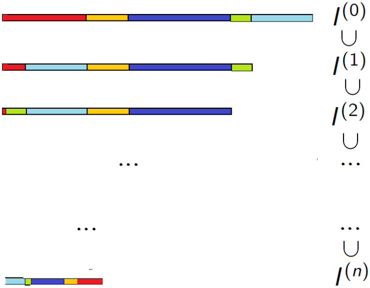

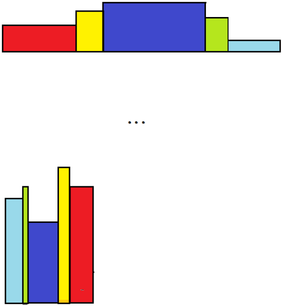

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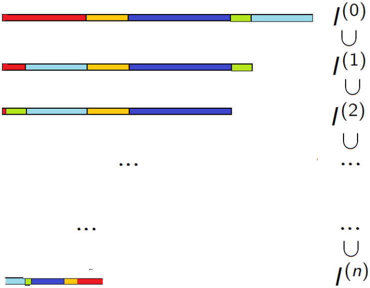

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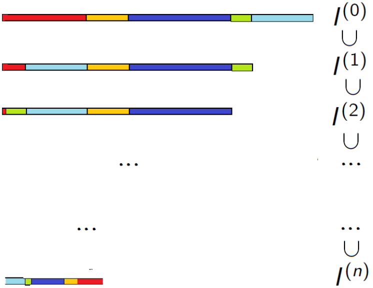

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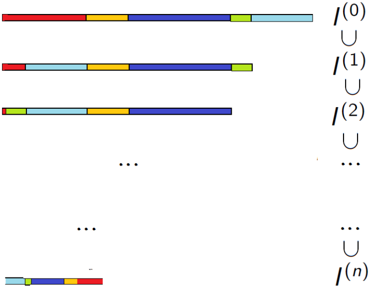

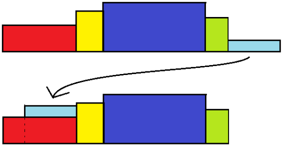

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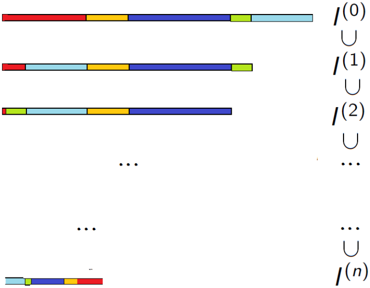

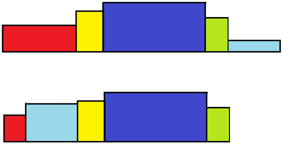

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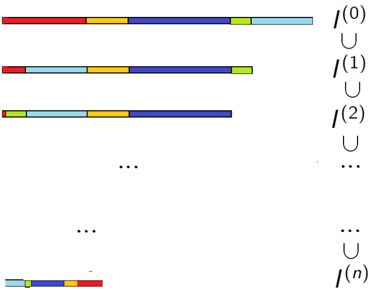

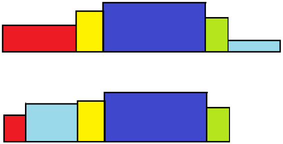

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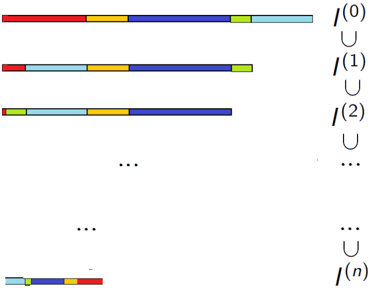

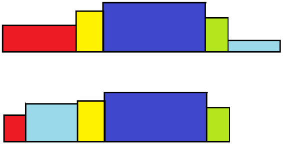

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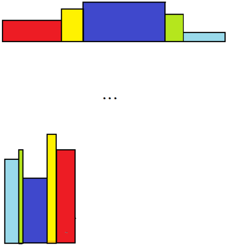

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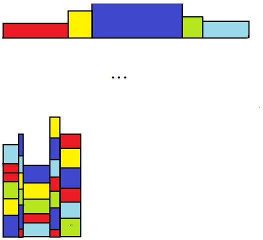

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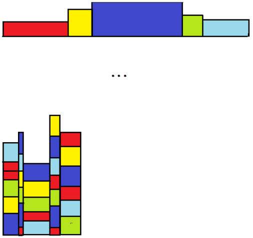

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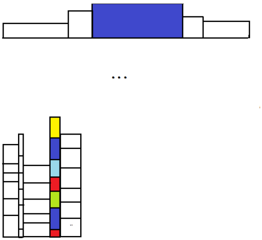

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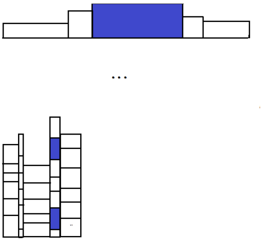

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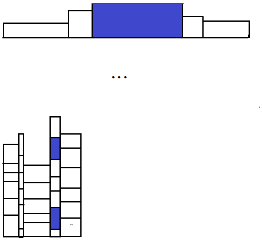

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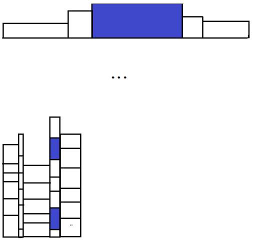

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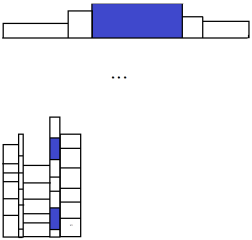

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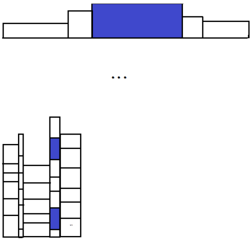

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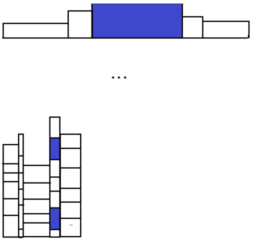

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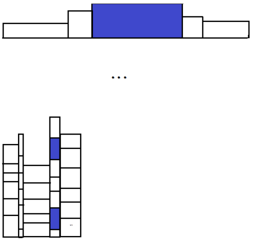

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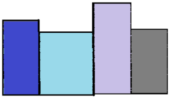
Constructing Limit Shapes

- Let (π, λ, τ) Oseledets generic.
- Fix $f \in E_i$, $i > 0$, so $\lim_n \frac{\log B(n) f}{n} = \nu_i > 0$.
- Renormalize *backwards*: consider $T^{(-n)}$, $q^{(-n)}$, $f^{(-n)}$, for $-n < 0$.



- Fix $\alpha \in \mathcal{A}$; q_α is subdivided into $q_\beta^{(-n)}$ time increments.
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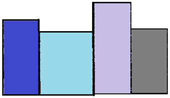

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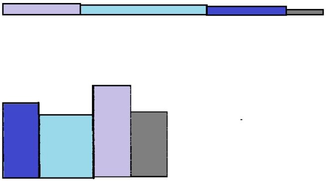

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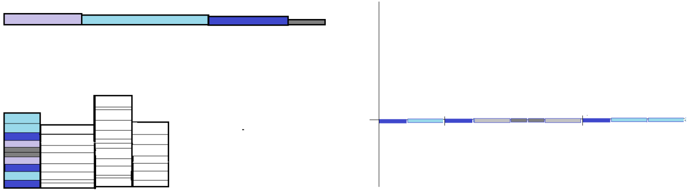

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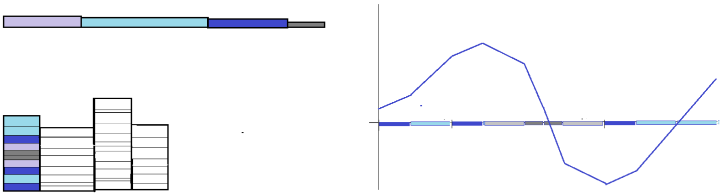

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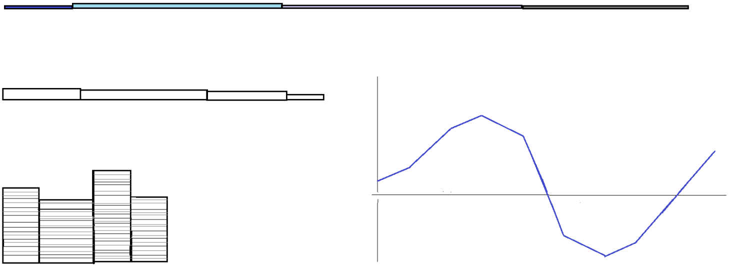

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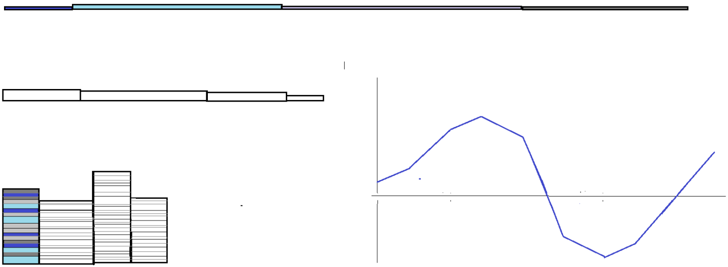

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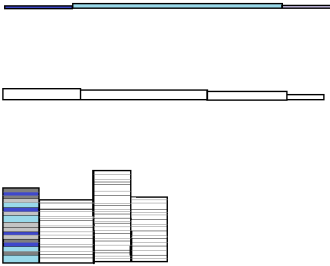

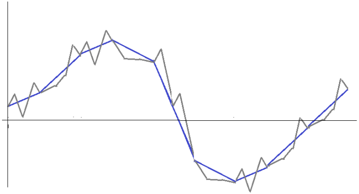

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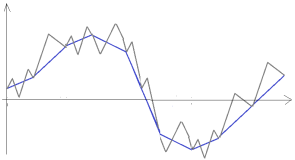

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(exponentially fast in $-n$).

Rk: the limit shape Ω_α^i is a ν-Holder function on $[0, q_\alpha^{(0)}]$ $\forall \nu < \frac{\nu_i}{\nu_1}$.

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The graph of $S_k f_i$ Birkoff sums over $T = T^{(0)}$ for $x_0 = 0$, for $k = 0, \dots, q_{\alpha_0}^{(n)}$, $n > 0$, *rescaled*, approches as $n \to +\infty$ the (moving) limit shape

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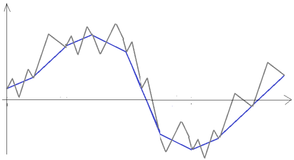

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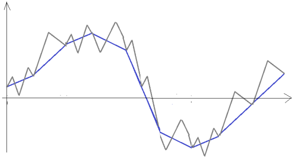

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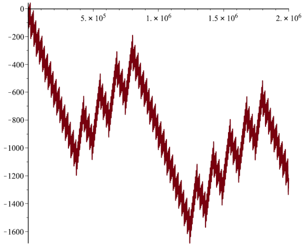

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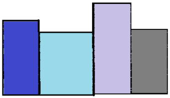

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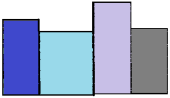

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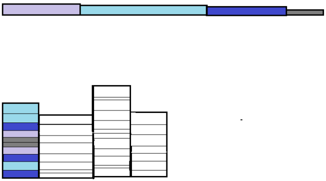

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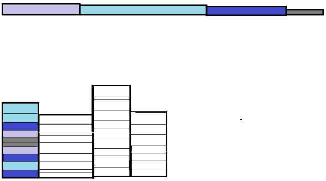

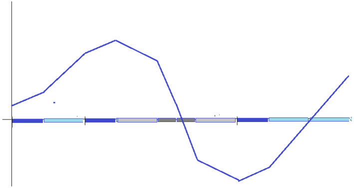

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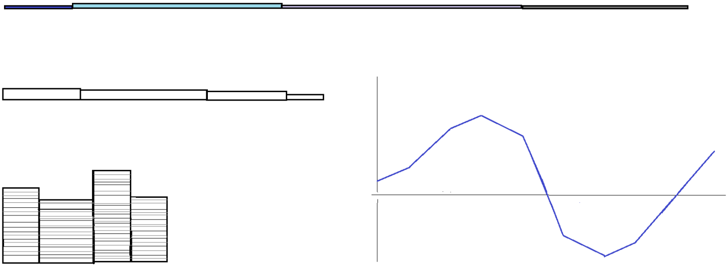

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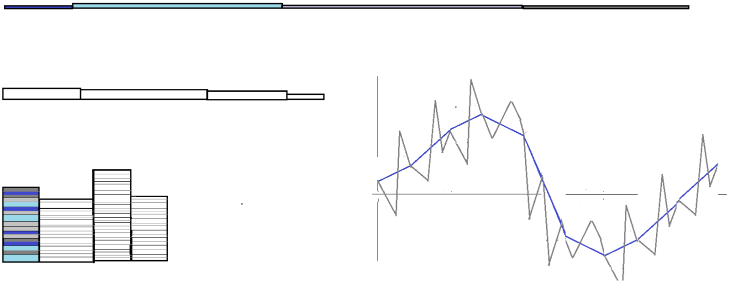

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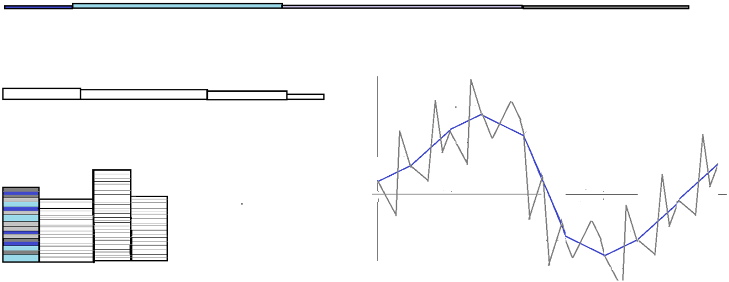

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There is a full measure condition on (π, λ, τ) (dual Roth type) s.t. for any $f \in E_0 \backslash E_-$, $\forall \alpha \in \mathcal{A}$

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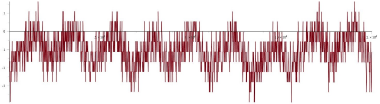

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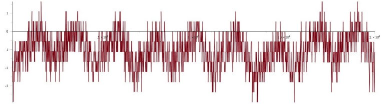

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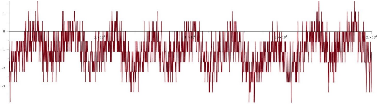

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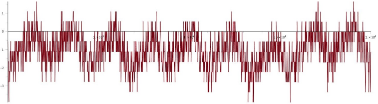

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Tools introduced and used for the proof of the Theorem:

- *Key duality* between *space* vs *time*
  (equivalently, *horizontal* vs *vertical* flows on M,
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- a new Diophantine Condition for i.e.m.s,
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  - the dual Rauzy-Veech cocycle;
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Classical Roth type conditions

- *Def:* a rotation number $a \in \mathbb{R}$ with continued fraction expansion $a = [a_1, a_2, \ldots, a_n, \ldots]$ and convergences p_n/q_n is of Roth type if $\exists C > 0$ such that

$$a_n = q_n^{\epsilon}, \qquad \forall \epsilon > 0.$$

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Absolute Roth type

- $B(0, n)$ act on $\mathbb{R}^d$ which can be indentified with $H_1(M, Sing, \mathbb{R})$ *relative homology*.
- Focus on the absolute homology $H_1(M, \mathbb{R}) \subset H_1(M, Sing, \mathbb{R})$;
  - define *positive acceleration* only with respect to absolute homology;
  - *Rk*: more natural geometrically, less *visible* from Ruazy-Veech induction, but it can be defined considering $H_\pi = Im\Omega_\pi$;
  - Modify condition (a) in (classical=relative) Roth type, to define *asbolute Roth type*;
- Marmi-U'-Yoccoz: the results by Marmi-Moussa-Yoccoz can be reproved assuming only the absolute Roth type condition;
- As a consequence, using
  - Eskin-Chaika, any M is Oseledetes generic in a.e. direction θ;

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Dual Roth type

Assume (M, π, τ) has no *horizontal saddle connections*. Iterate Rauzy Veech backward; (π, τ) determines a *backward rotation number*.

Lemma (Marmi-U-Yoccoz)

The backward rotation number is infinitely complete, i.e. for every $-n < 0$ there exists $-m < -n$ such that $B^\circ(-m, -n) > 0$.

Remark: very involved combinatorial proof (by Yoccoz)!

Corollary The backward positive acceleration is well defined.
Let $B^\circ(-n)$ denote the positive acceleration.

Definition

(π, τ) satisfy the *dual Roth type condition* if it satisfies

(a) Matrices growth for the dual cocycle: $\exists C > 0$ s.t.

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Proposition: [Marmi-U'-Yoccoz] dual Roth type has full measure.

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The backward rotation number is infinitely complete, i.e. for every $-n < 0$ *there exists* $-m < -n$ *such that* $B^{\circ}(-m, -n) > 0$.

Remark: very involved combinatorial proof (by Yoccoz)!

Corollary The backward positive acceleration is well defined.
Let $B^{\circ}(-n)$ denote the positive acceleration.

Definition

(π, τ) satisfy the *dual Roth type condition* if it satisfies

(a) Matrices growth for the dual cocycle: $\exists C > 0$ s.t.

$$\|B^{\circ}(n, n+1)\| \leq C\|B^{\circ}(0, n)\| \qquad \forall \epsilon > 0$$

(b) Spectral gap for dual cocycle/dual Birkhoff sums

(c) Coherence for dual cocycle/dual Birkhoff sums

Proposition: [Marmi-U'-Yoccoz] dual Roth type has full measure.

Dual Roth type

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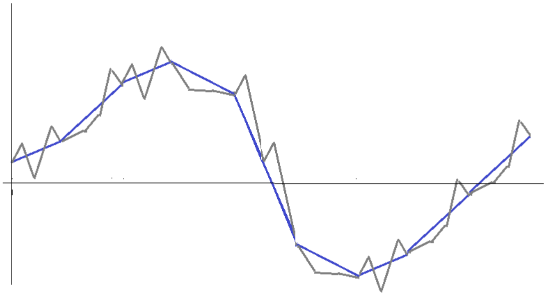
Dual Birkhoff sums in distributional convergence

- For $-m < -n$, compare $\Omega_{0,\alpha}^{(-m)} - \Omega_{0,\alpha}^{(-n)}$:



- Difference is a sum of rescaled versions of copies of $\Omega_{0,\beta}^{(-m-n)}(T^{(-n)})$, $\beta \in \mathcal{A}$.
- Sum over occurrences of a fixed subgraph $\Omega_{0,\beta}^{(-m-n)}(T^{(-n)})$ is a *dual Birkhoff sum*;
- to estimate $\int \left(\Omega_{0,\alpha}^{(-m)} - \Omega_{0,\alpha}^{(-n)}\right) \psi$ exploit estimates on dual special Birkhoff sums of Holder functions, which hold under the dual Roth type condition.

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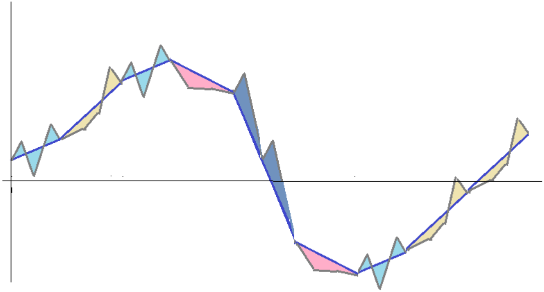

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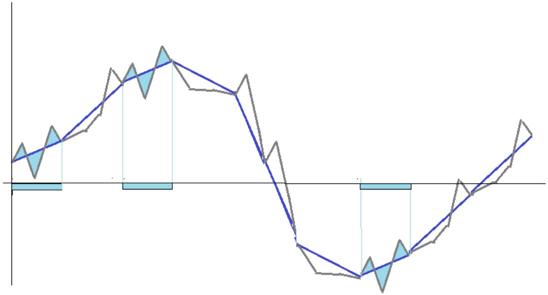

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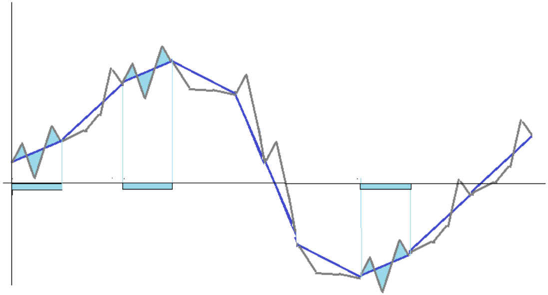

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