Cutting sequences, regular polygons and Veech groups

Corinna Ulcigrai

University of BRISTOL

Linear trajectories and cutting sequences

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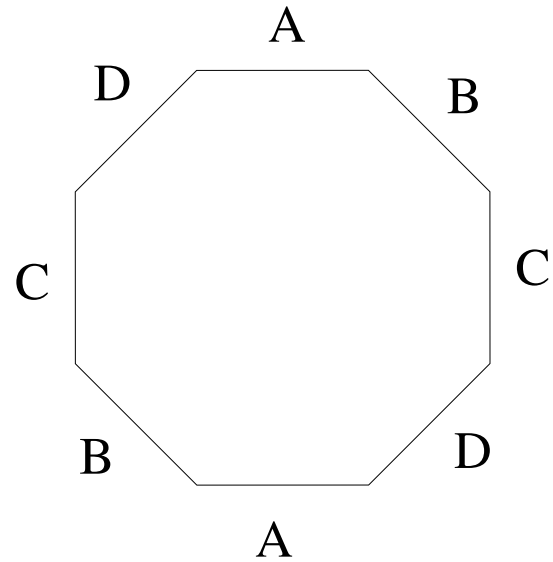
Label pairs of sides by $\{A, B, C, D\}$.



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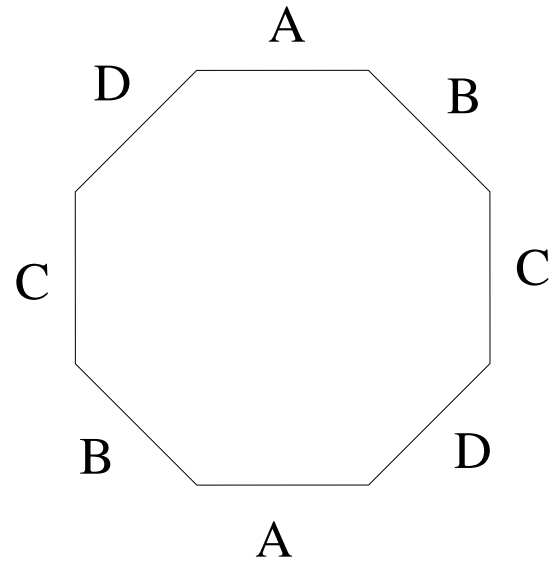

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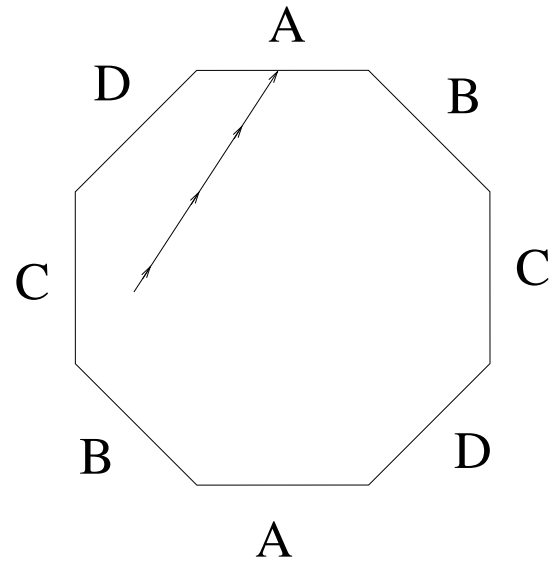

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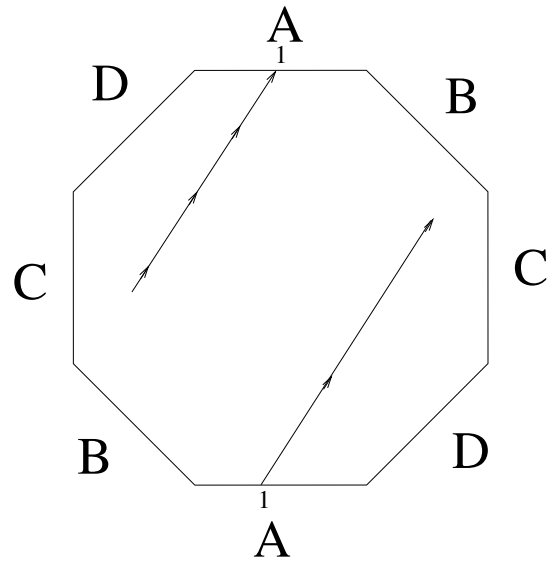

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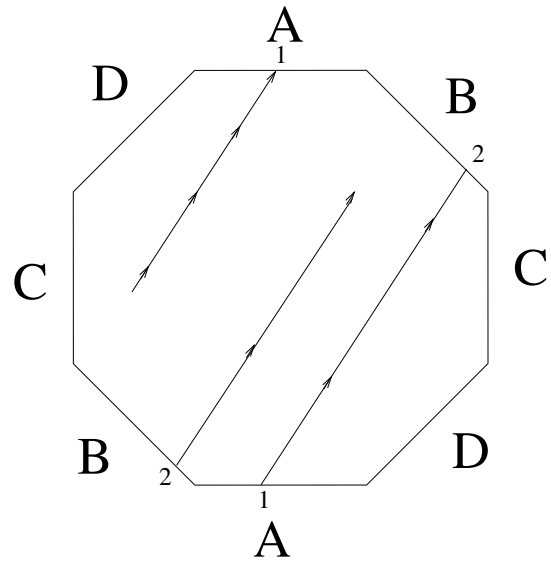

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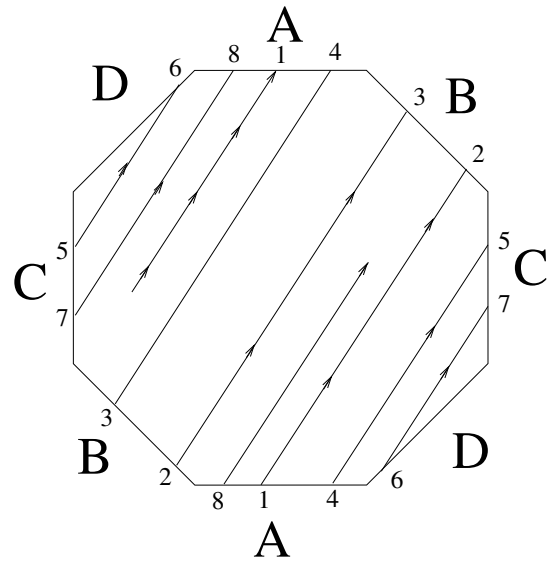

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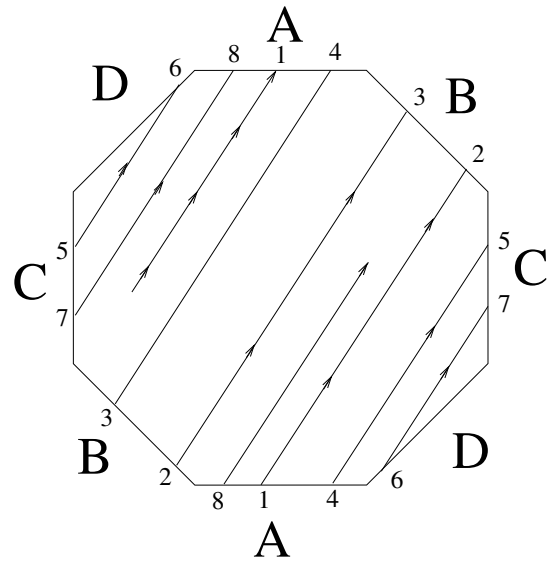

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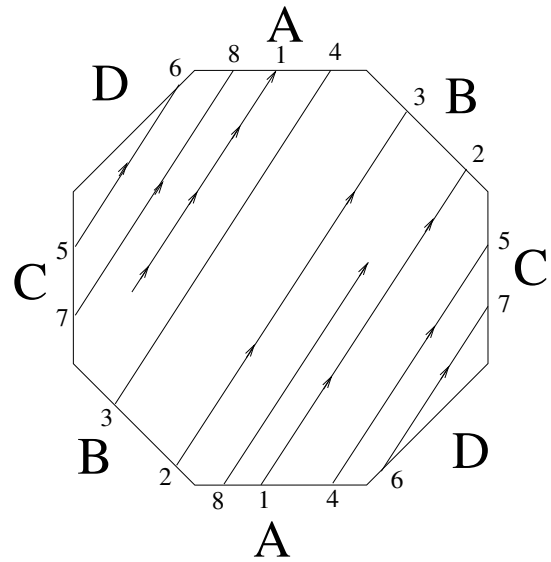

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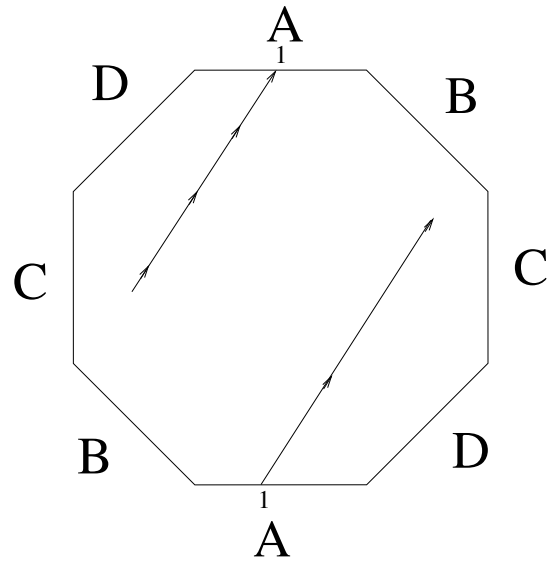

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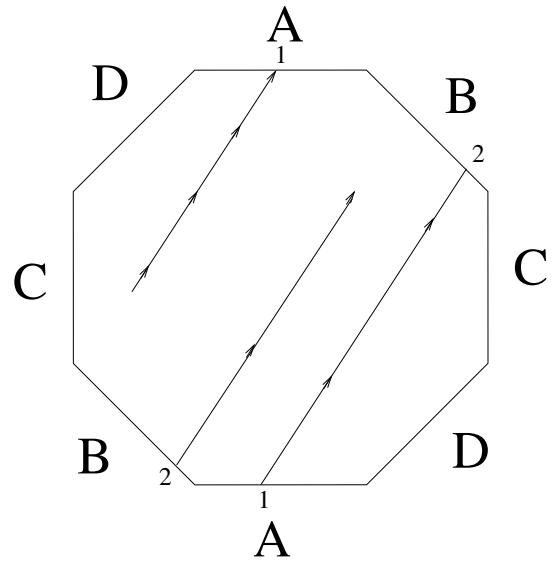

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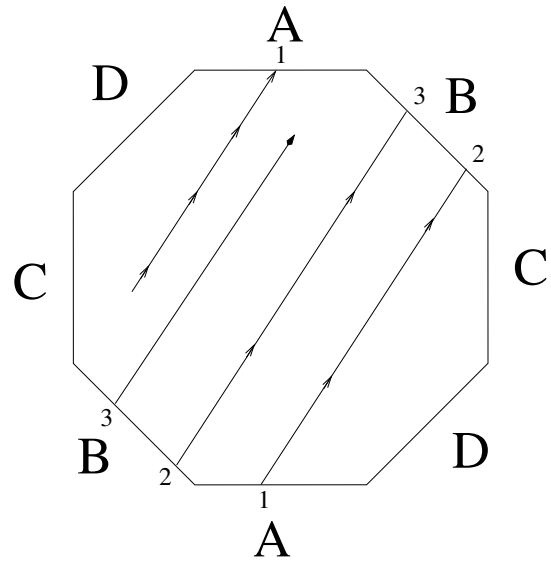

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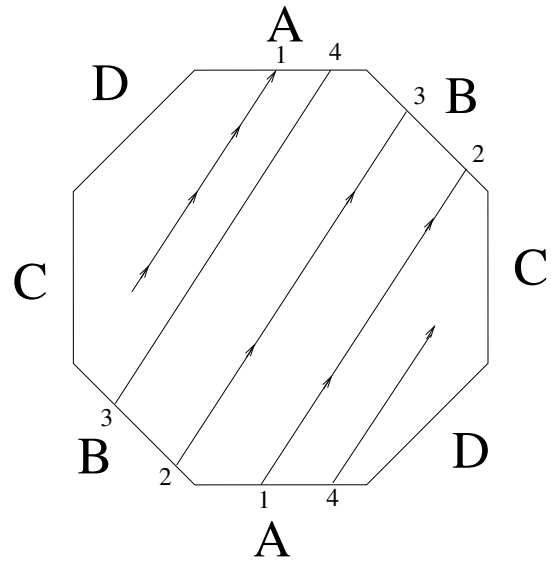

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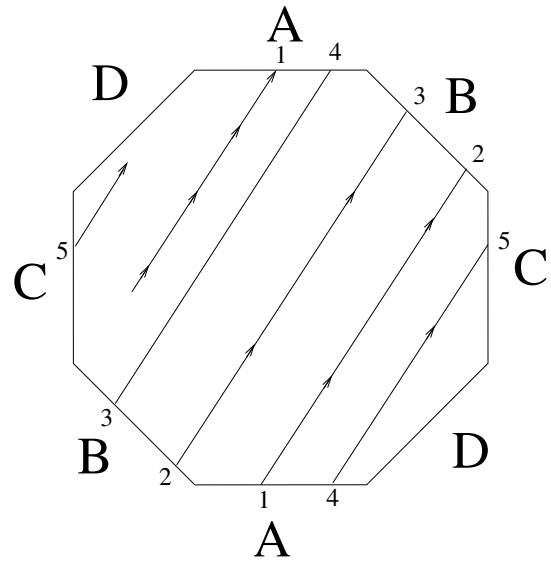

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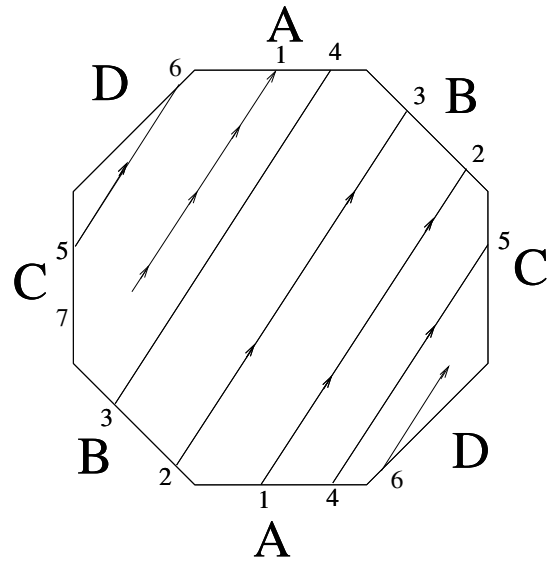

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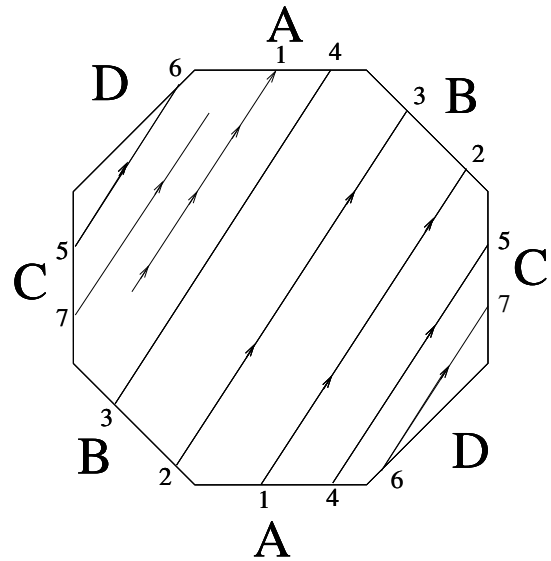

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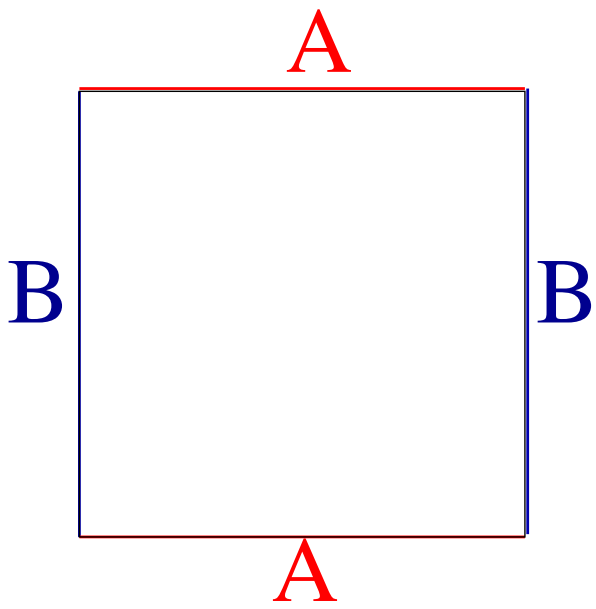
A classical case: Sturmian sequences

Consider the special case in which the polygon is a square.



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Sturmian sequences are characterized by having the smallest possible *complexity* among non-periodic sequences.

(Let $P_w(n)$ the number of words of lenght n which appear in the sequence w: $P_w(n) = n$ iff w is periodic. Sturmian sequences satisfy $P_w(n) = n + 1$.)

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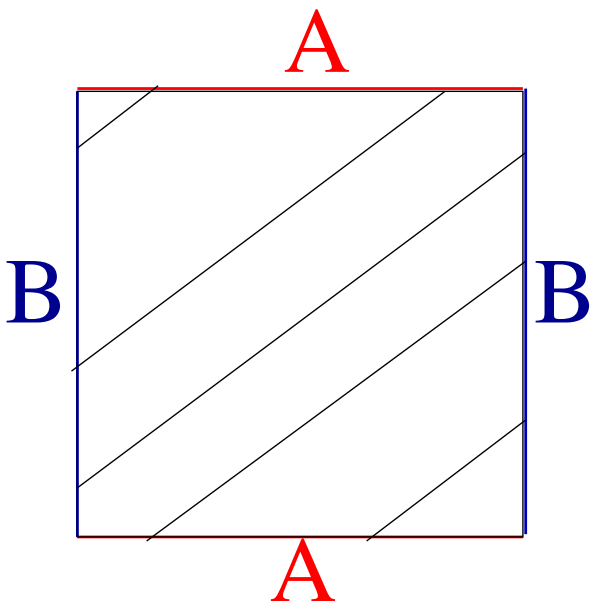

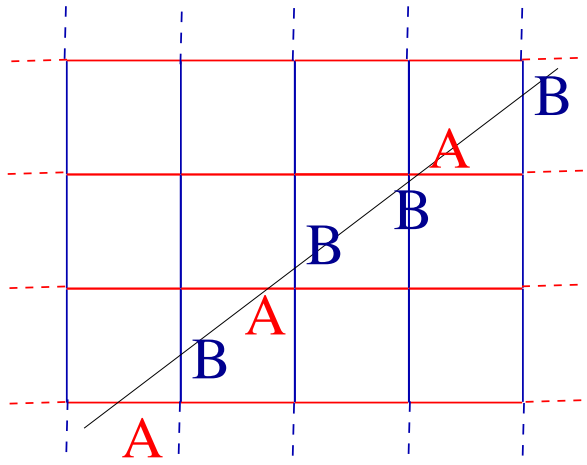

In this case the cutting sequence correspond to the sequence of horizontal (letter A) and vertical (letter B) sides crossed by a line in direction θ in a *square grid*.

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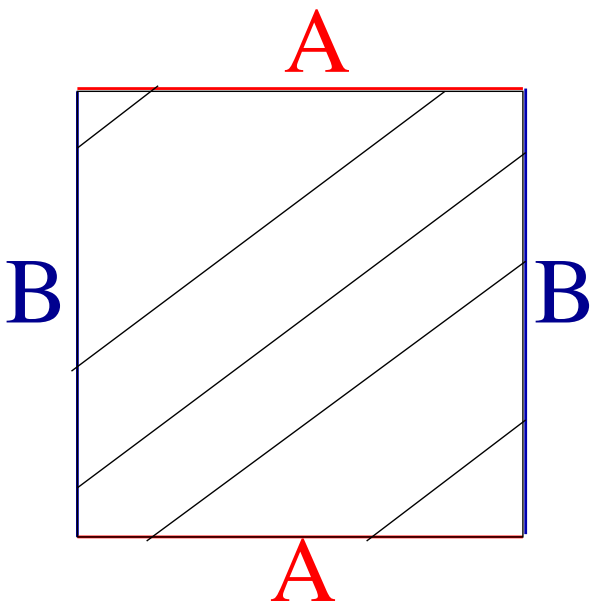

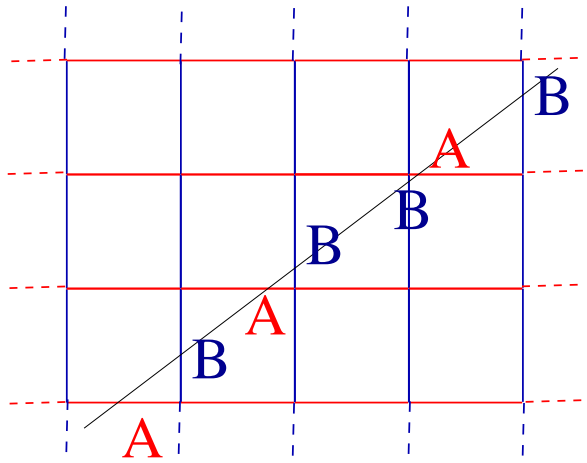

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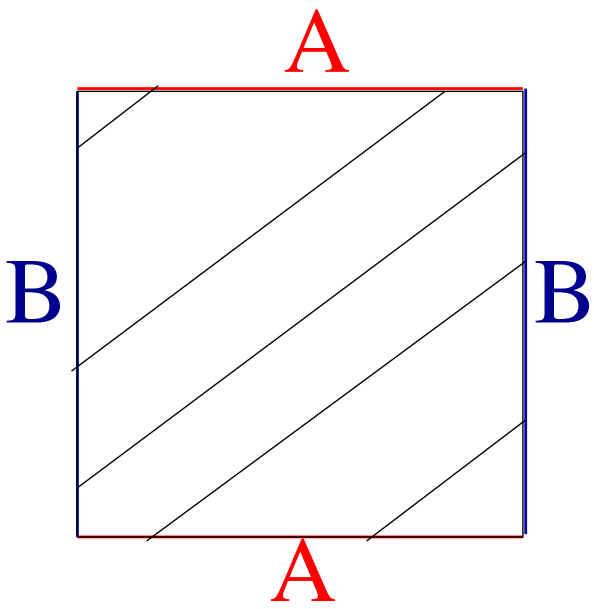

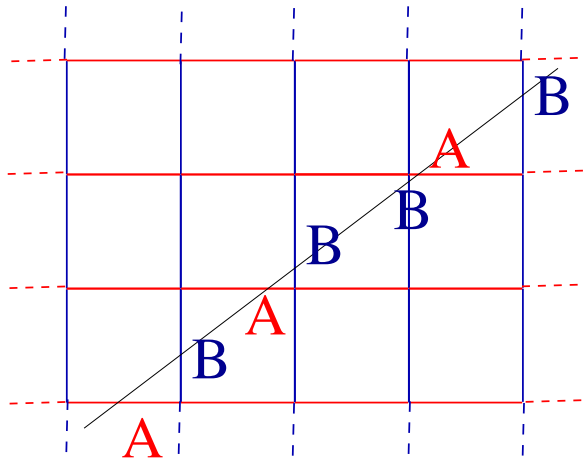

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Cutting sequences give a symbolic coding of the following systems:

- Translation surfaces
- Poligonal Billiards
- Interval exchange transformations

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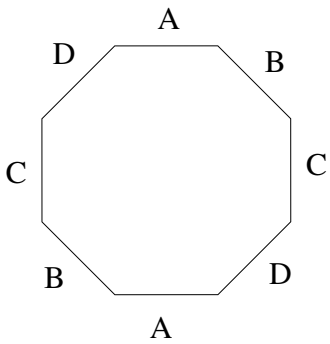

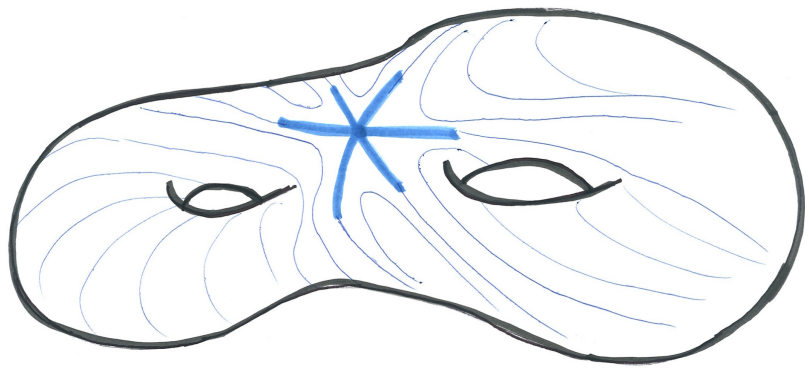

Glueing opposite sides one gets a surface of genus 2, with a flat metric with a singularity (it's a translation surface); φ_t^θ is the geodesic flow with respect to the flat metric;

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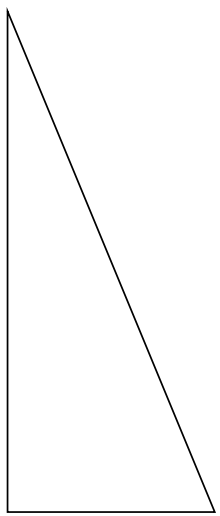

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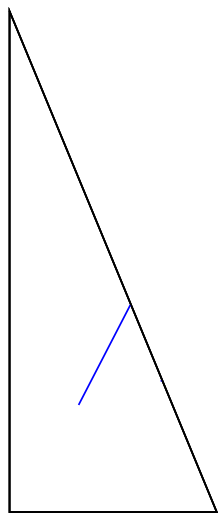

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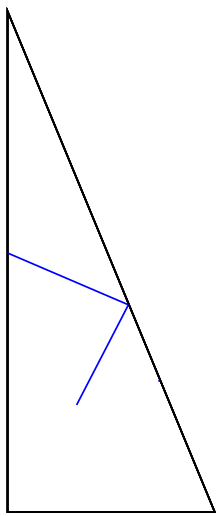

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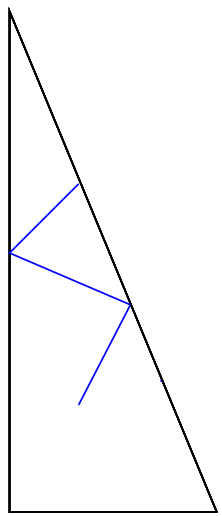

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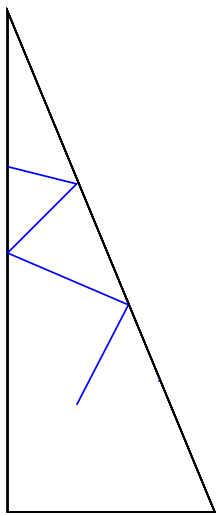

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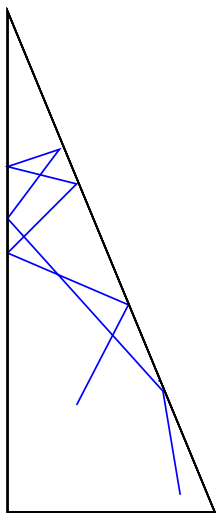

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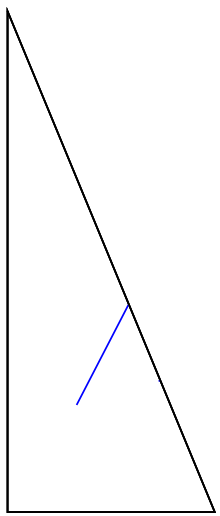

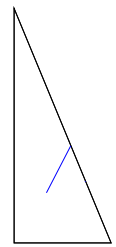

For example: A billiard with angles $\frac{\pi}{2}$, $\frac{\pi}{8}$ and $\frac{3\pi}{8}$ (motion of a particle with elastic reflections at sides) by a procedure called *unfolding* is equivalent to the flow φ_t^θ in the octagon.

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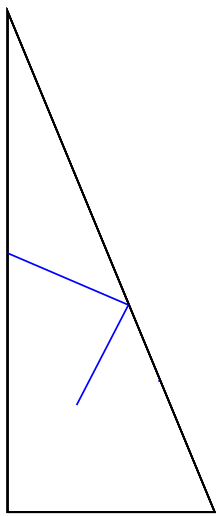

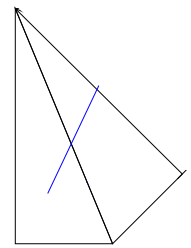

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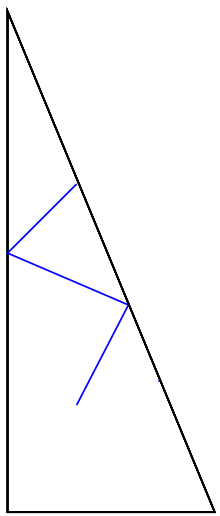

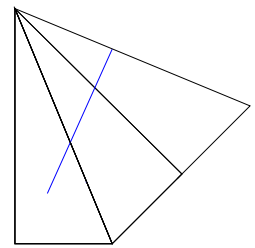

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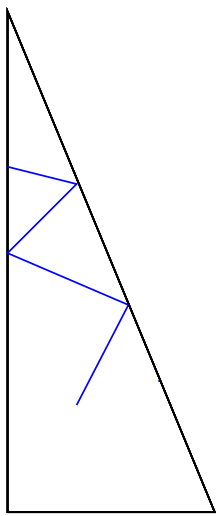

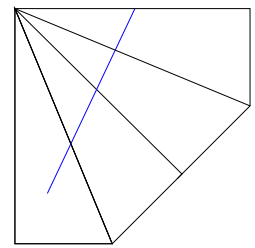

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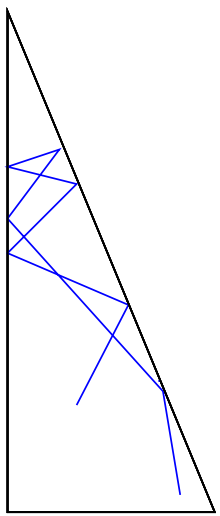

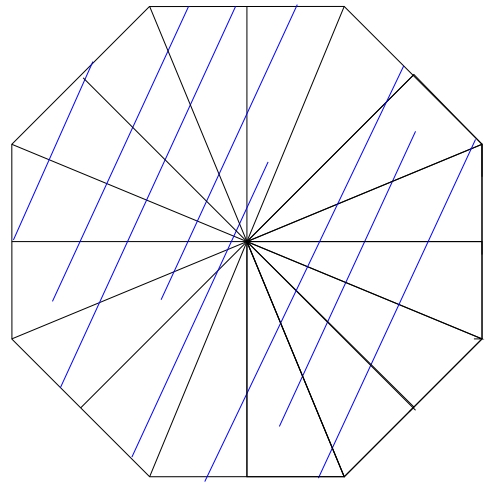

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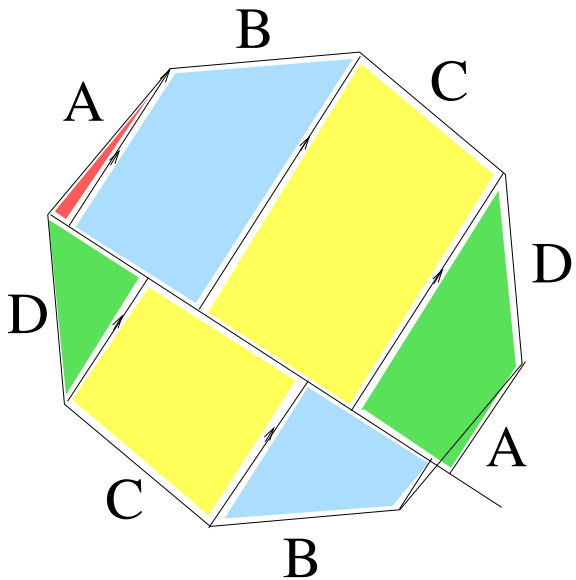

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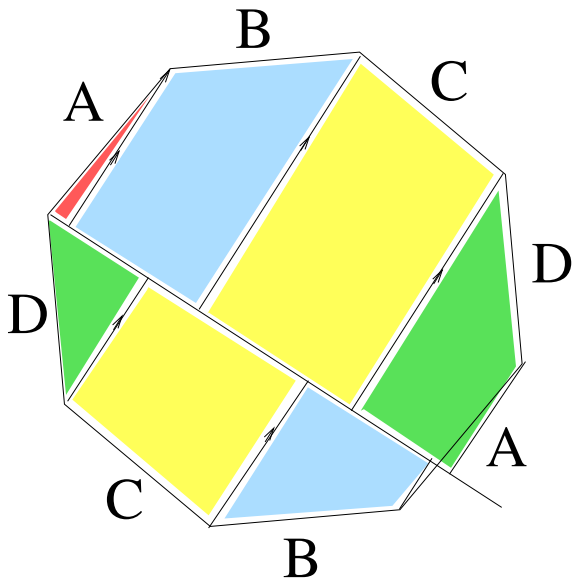

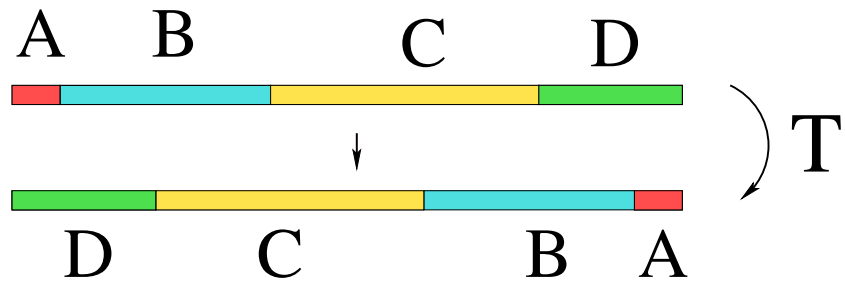

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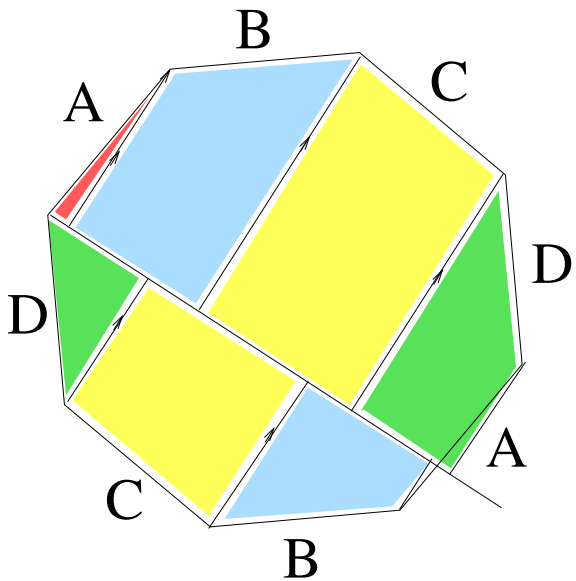

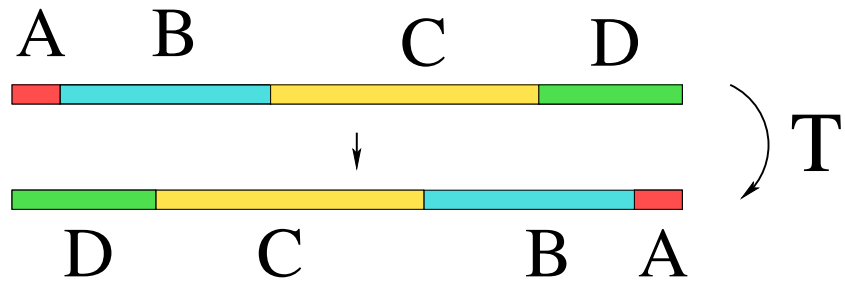

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Translation surfaces and IETs which come from regular polygons are not *generic*: techniques that are used for the generic setting do not apply here.

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Characterization of cutting sequences

Problem:

Describe explicitely the symbolic sequences which are cutting sequences of trajectories.

In particular, answer the following questions:

Q1) Which sequences in $\{A, B, C, D\}^{\mathbb{Z}}$ are cutting sequences?

Q2) Given a cutting sequence, can one recover the direction of the trajectory?
Given a *finite* piece of a cutting sequence, can one recover a sector of possible directions?

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  - Connection with Continued Fractions;
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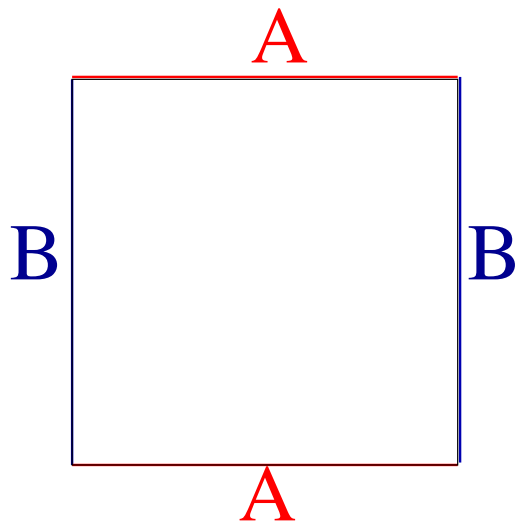
Let D_4 be the group of isometries of the square.

The letters $\{A, B\}$ are invariant under vertical symmetry and horizontal symmetry and are exachanged if we reflect diagonally



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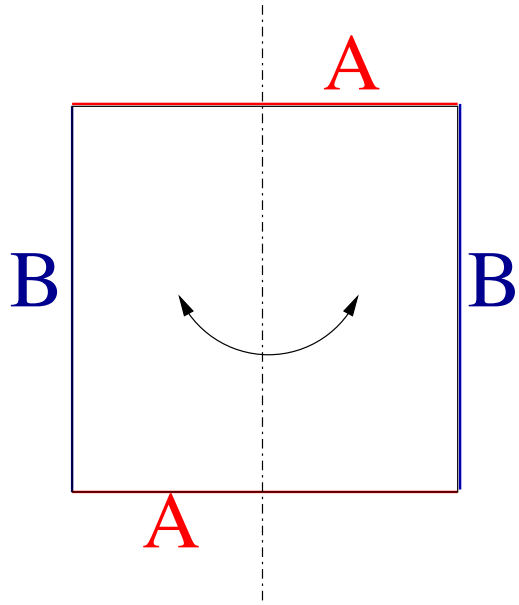

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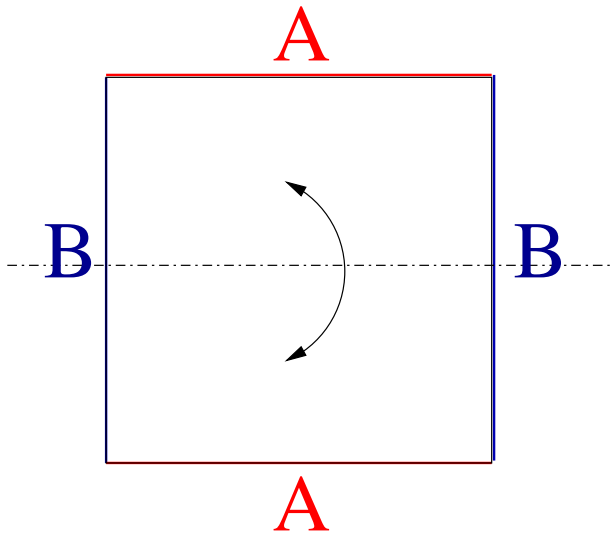

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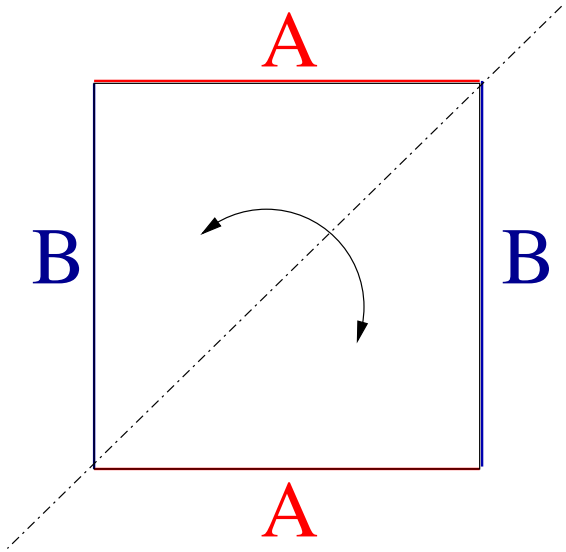

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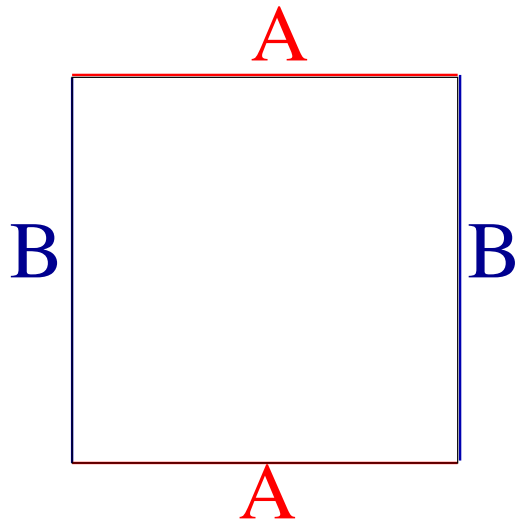

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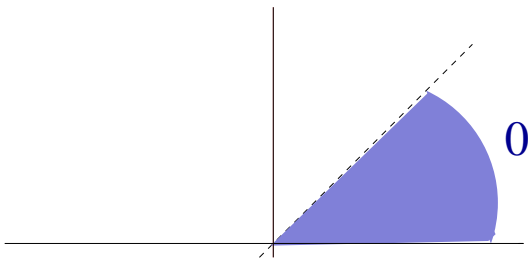

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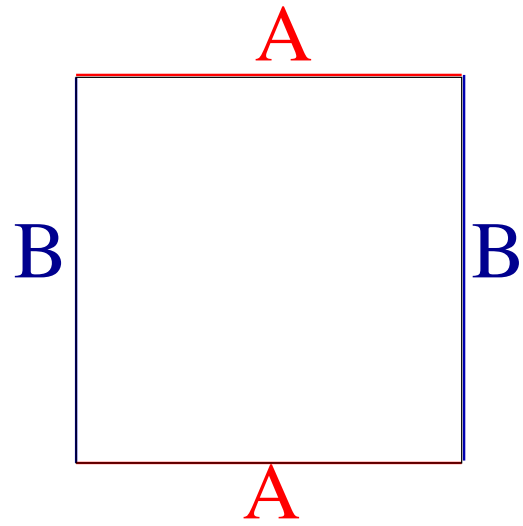

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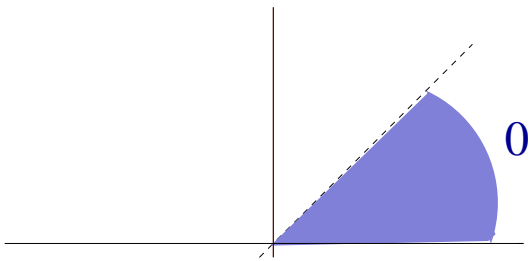
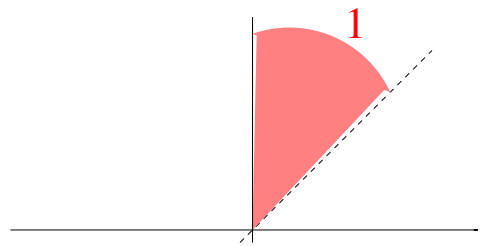
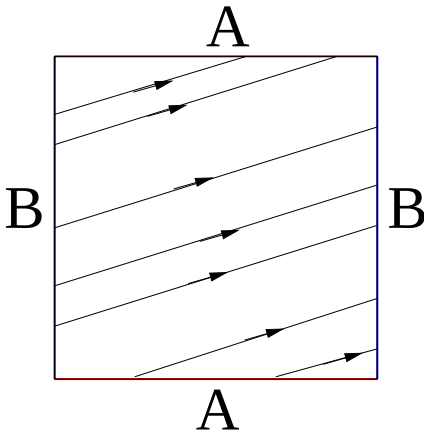




Let $\Sigma_1 := \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ be the other sector.

The square: possible transitions

- If $\theta \in \Sigma_0 := \left[0, \frac{\pi}{4}\right]$, AA does not appear:



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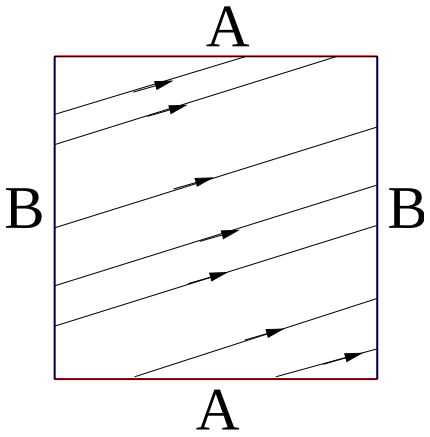

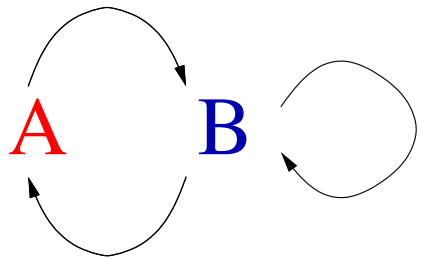

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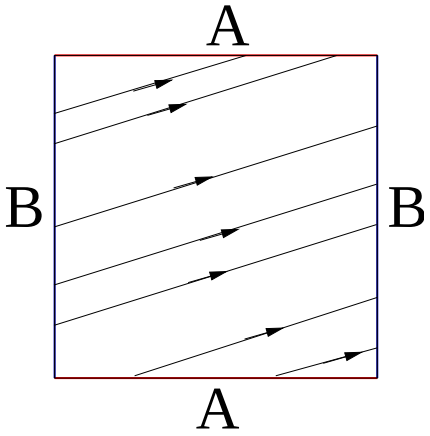

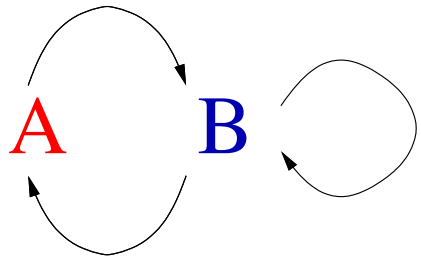

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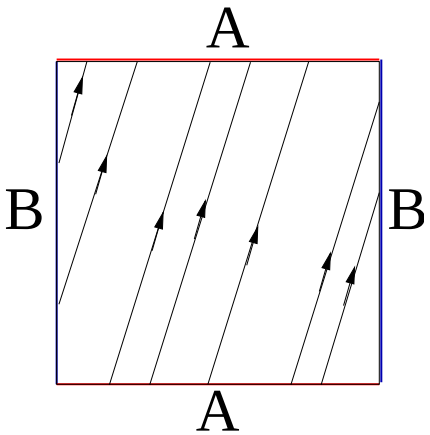
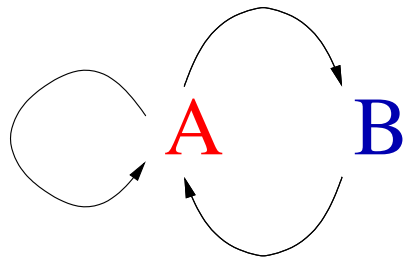
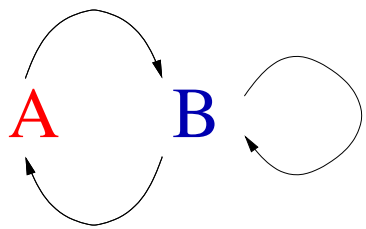




Admissible sequences

Definition

A sequence $w \in \{A, B\}^{\mathbb{Z}}$ is *admissibile* if it gives a infinite path on one of these two diagrams:



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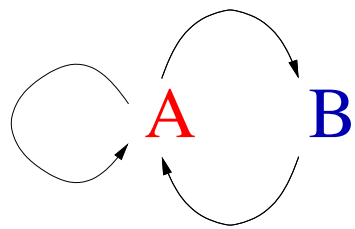

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In this case, we say that w is admissible in $\mathscr{D}_0$ or $\mathscr{D}_1$ respectively.

Lemma

A square cutting sequence is admissible.

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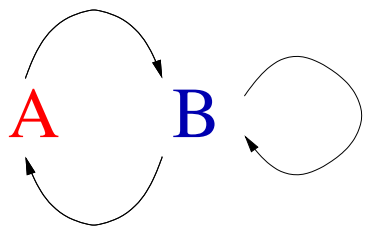

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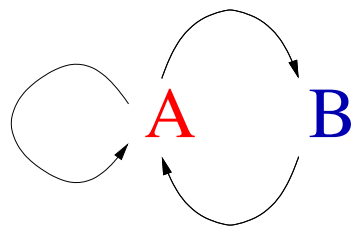

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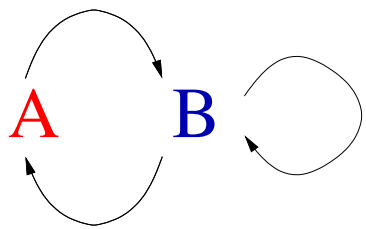

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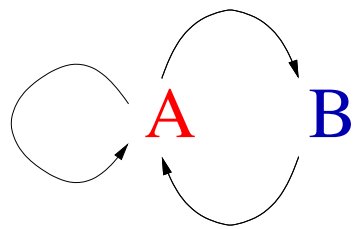

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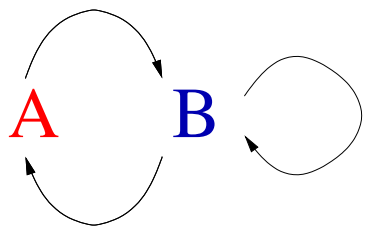

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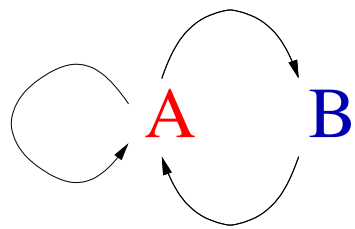

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Derivable sequences

Definition (Derived sequence)

Let w be the cuttings sequence of a trajectory with $\theta \in \Sigma_0$. The derived sequence w' is obtained erasing one B from each block of Bs. If $\theta \in \Sigma_1$, w' is obtained erasing one A from each block of As.

Example

$w = \ldots ABBBABBBABBBBABBBABBB \ldots,$
$w' = \ldots ABBB \quad ABB \quad ABBB \quad ABB \quad ABB \quad \ldots$

Definition (Derivable Sequences)

A sequence $w \in \{A, B\}^{\mathbb{Z}}$ is *derivable* if it is admissible and the derived sequence is admissible.

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Characterization of Sturmian sequences

Theorem

A square cutting sequence is infinitely derivable.

Corollary

If w is a square cutting sequence with $\theta \in \Sigma_0$, the blocks of Bs have length n o $n+1$.

Example

The sequence $w = \ldots ABBBBABBABBBBA \ldots$ is NOT a square cutting sequence. Indeed: $w' = \ldots ABBBABABBBA \ldots$ and $w'' = \ldots ABBAABBA \ldots$ which is not admissible.

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Let w be infinitely derivable. Then w belongs to the closure of square cutting sequences.

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Direction recognition and continued fractions

Let w a square cutting sequence, *for example:*
w $= \ldots$ BBBABBABBABBABBBABBAABBBABBABBABBA$\ldots$

Let us define $\{a_n\}_n \in \mathbb{N}^{\mathbb{N}}$ as follows:

- let a_0 such that the blocks of Bs in w have length a_0 or $a_0 + 1$ (*in the example* $a_0 = 2$: BB *o* BBB)
- let a_1 such that the blocks of As in $w^{(a_0)}$ (*in the example* w") have length a_1 or $a_1 + 1$ (*in the example* $a_1 = 3$: AAA *o* AAAA)
- $\ldots$
- let a_n such that the blocks of As (n odd) or Bs (n even) in $w^{(a_0+\cdots+a_{n-1})}$ have length a_n or $a_n + 1$;

Theorem (Direction Recognition)

The direction θ of the trajectory with cutting sequence w is given by:

$$\theta = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_n + \cdots}}}}$$

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w = . . . BBBABBABBABBABBBABBAABBBABBABBABBA. . .
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w'' = . . . BAAAABAAABAAA. . .

Let us define $\{a_n\}_n \in \mathbb{N}^{\mathbb{N}}$ as follows:

let a_0 such that the blocks of Bs in w have length a_0 or $a_0 + 1$ (*in the example* $a_0 = 2$: BB *o* BBB)

let a_1 such that the blocks of As in $w^{(a_0)}$ (*in the example* w'') have length a_1 or $a_1 + 1$ (*in the example* $a_1 = 3$: AAA *o* AAAA)

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let a_n such that the blocks of As (n odd) or Bs (n even) in $w^{(a_0+\cdots+a_{n-1})}$ have length a_n or $a_n + 1$;

Theorem (Direction Recognition)

The direction θ of the trajectory with cutting sequence w is given by:

$$\theta = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_n + \cdots}}}}$$

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Proof for Sturmian sequences

The key step is given by the following Lemma:

Lemma

If w is a square cutting sequence, also the derived sequence w' is a square cutting sequence.

Recalling that a square cutting sequence is clearly admissible, we have:

Corollary

Square cutting sequences are infinitely derivable.

Lemma

If w is a cutting sequence of a trajectory in direction θ, the derived sequence w' is a cutting sequence of a trajectory in direction θ', where $\theta' = F(\theta)$ and F is the Farey map in Figure.

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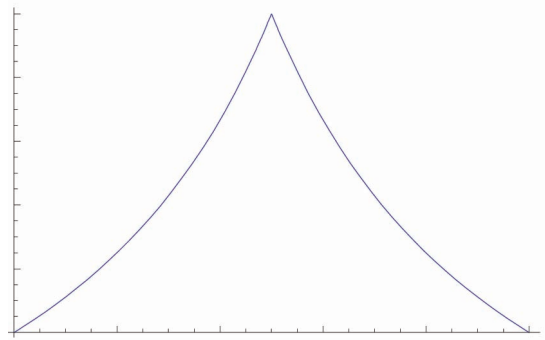

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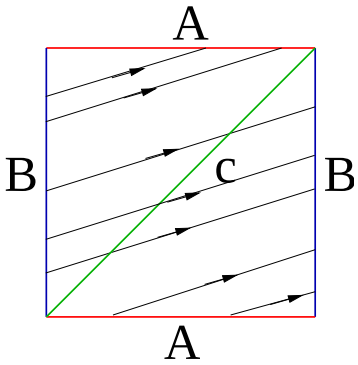

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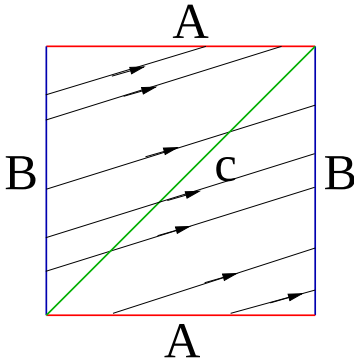

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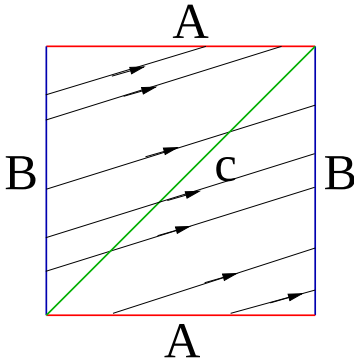

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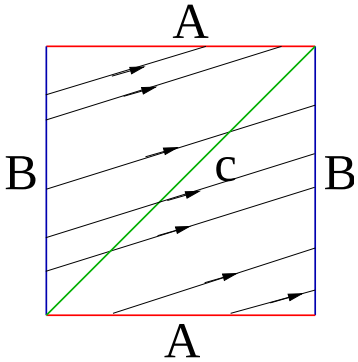

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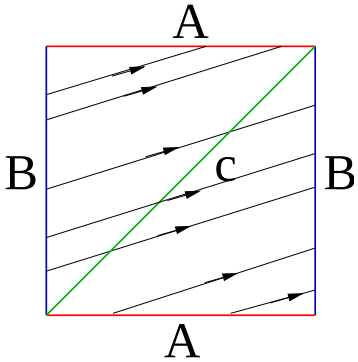

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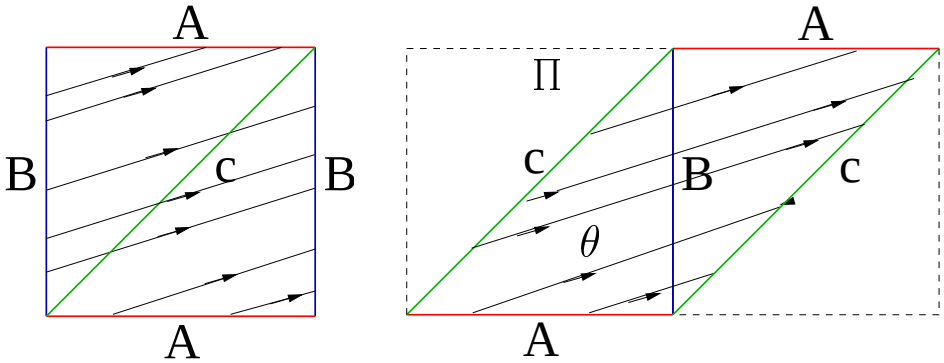

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Consider the cutting sequence u with respect to the parallelogram Π.

To obtain u from $\tilde{w}$ it is enough to drop the Bs.

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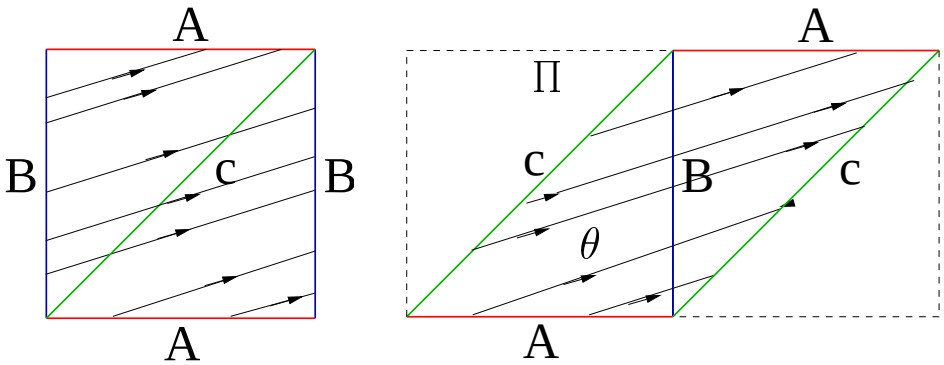

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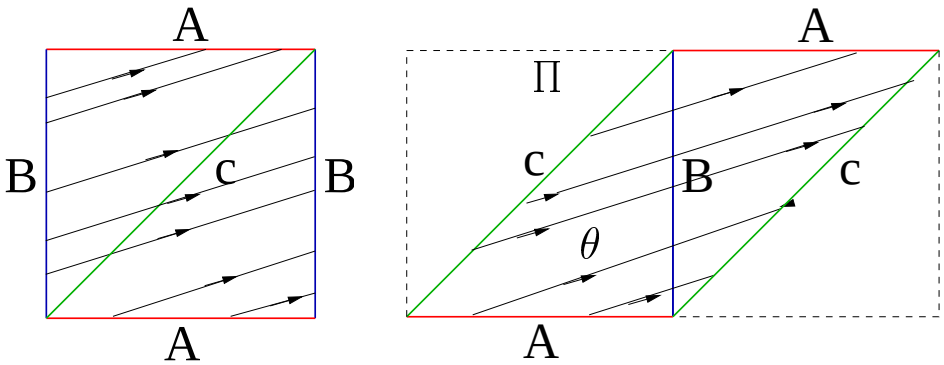

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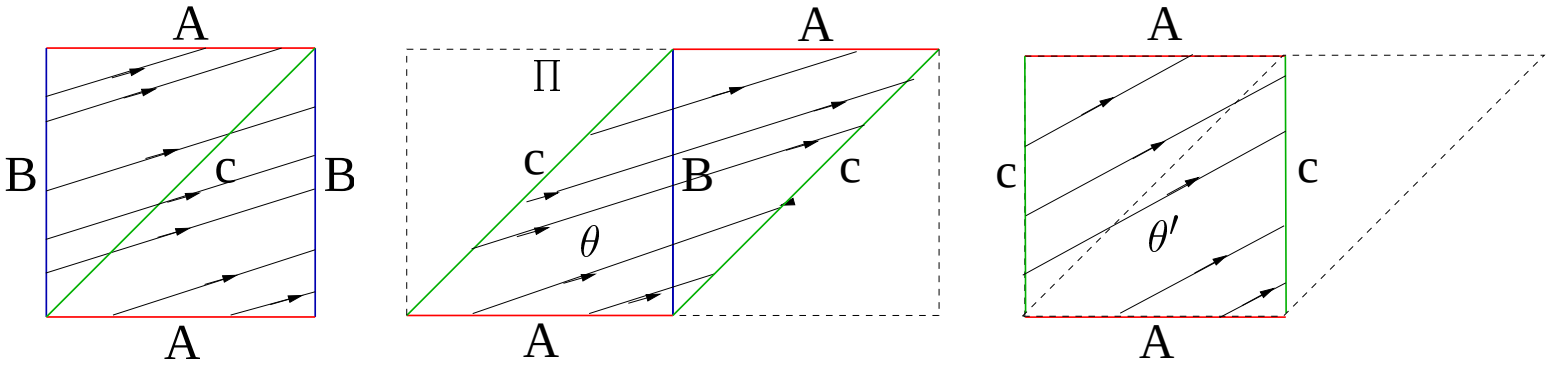

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Let us *renormalize*: we can transform Π in a square by the shear $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Let us transform back the Cs into Bs.

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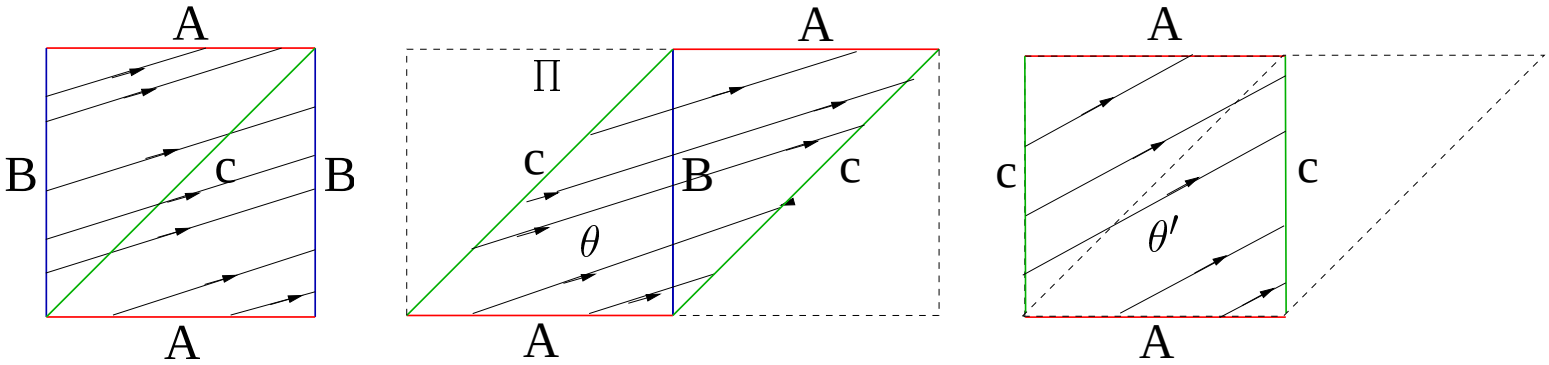

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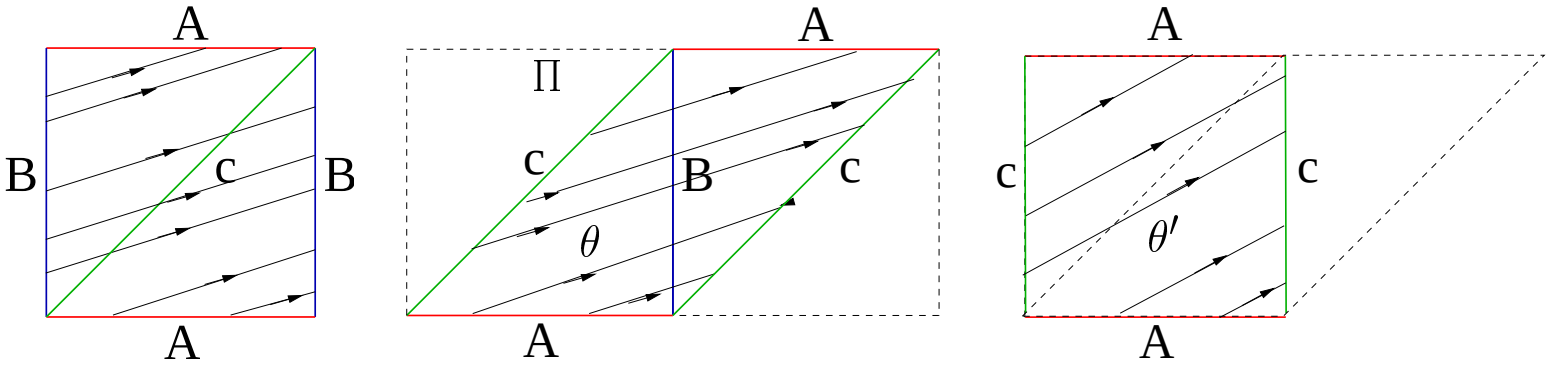

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The sequence thus obtained is the *derived sequence*.

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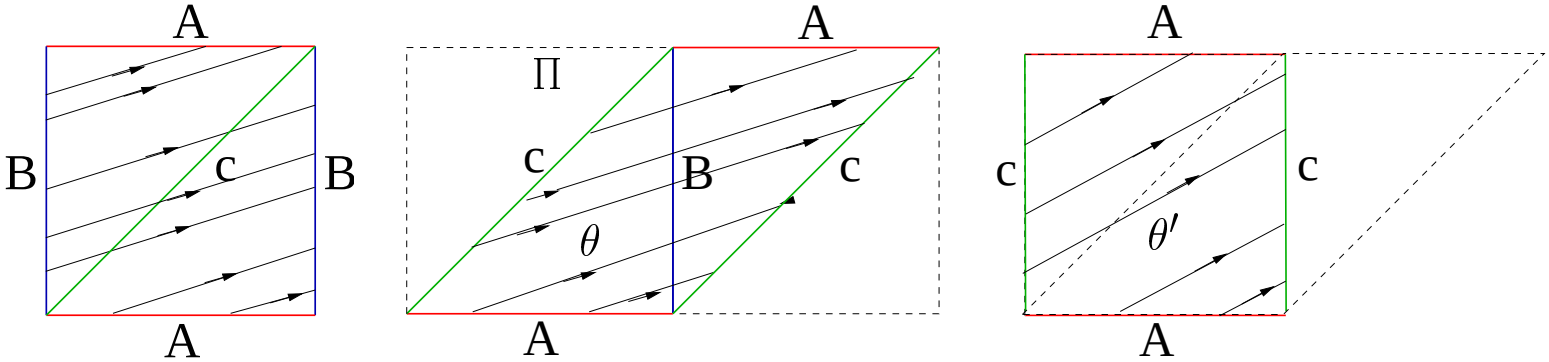

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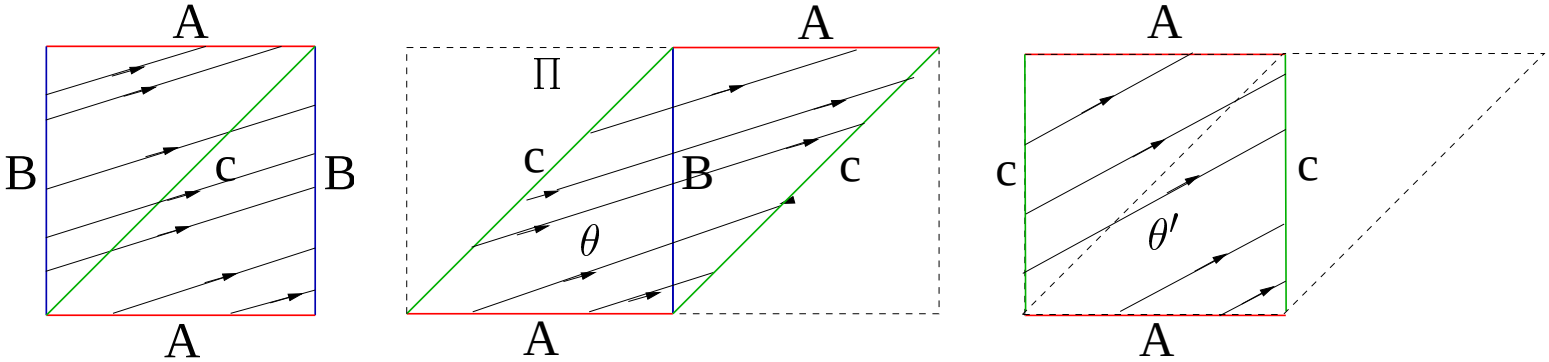

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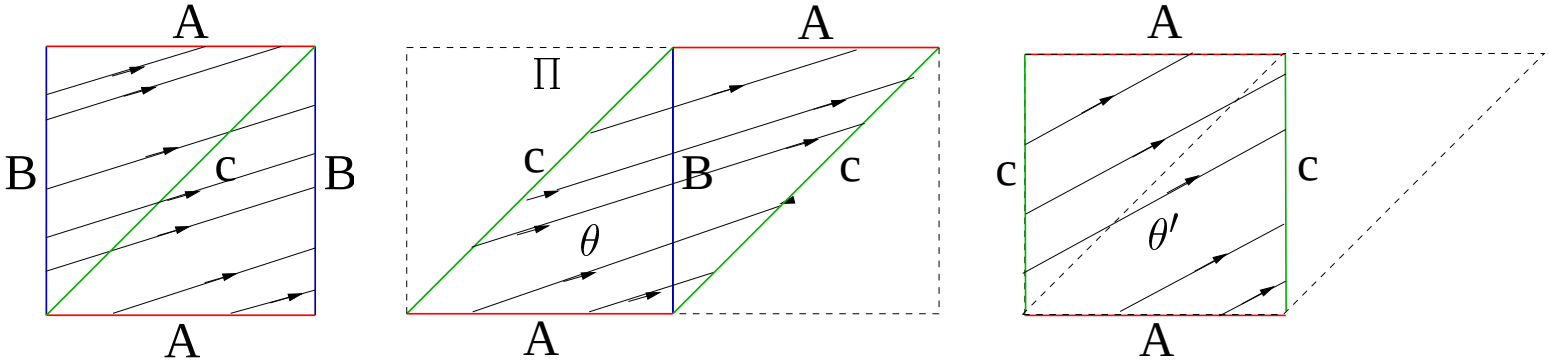

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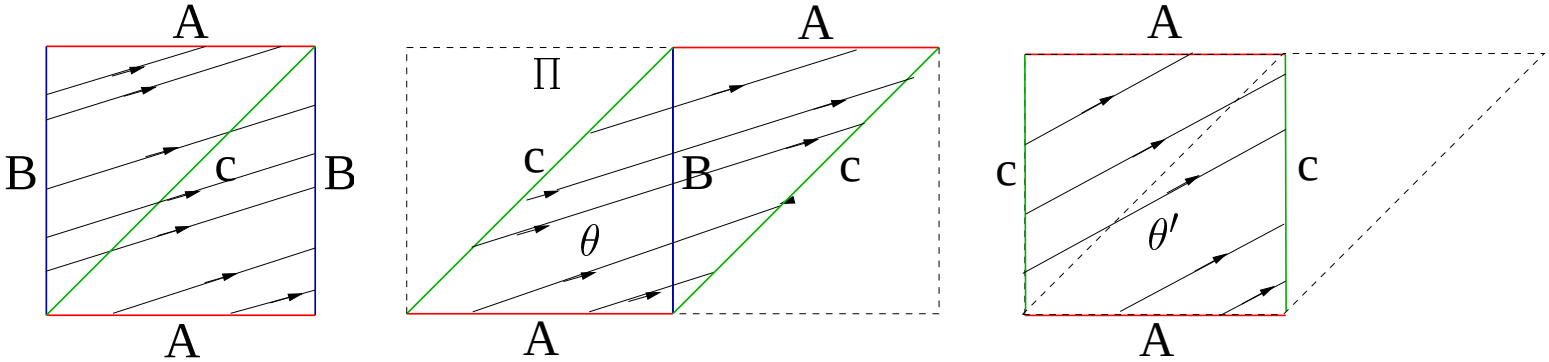

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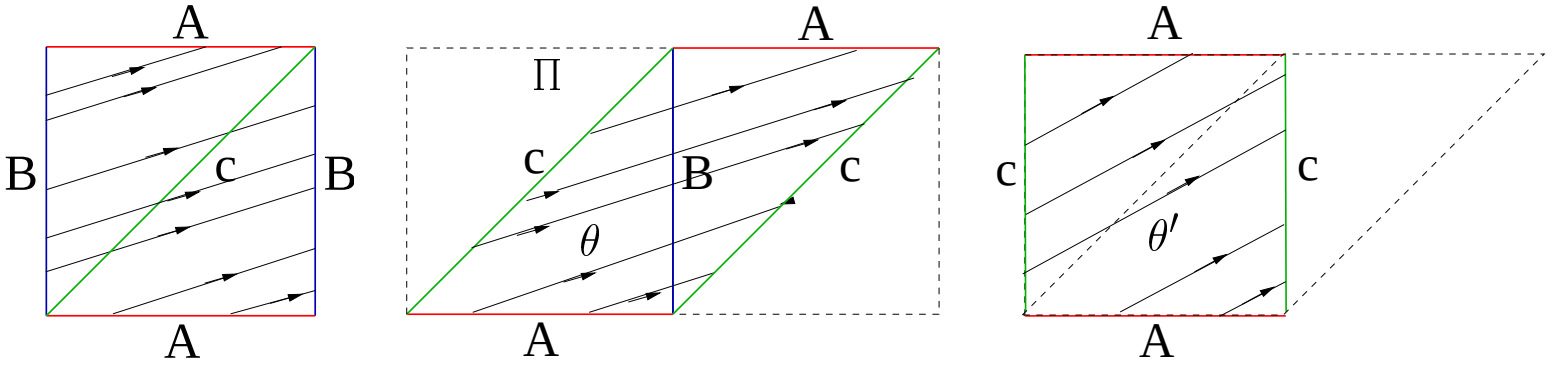

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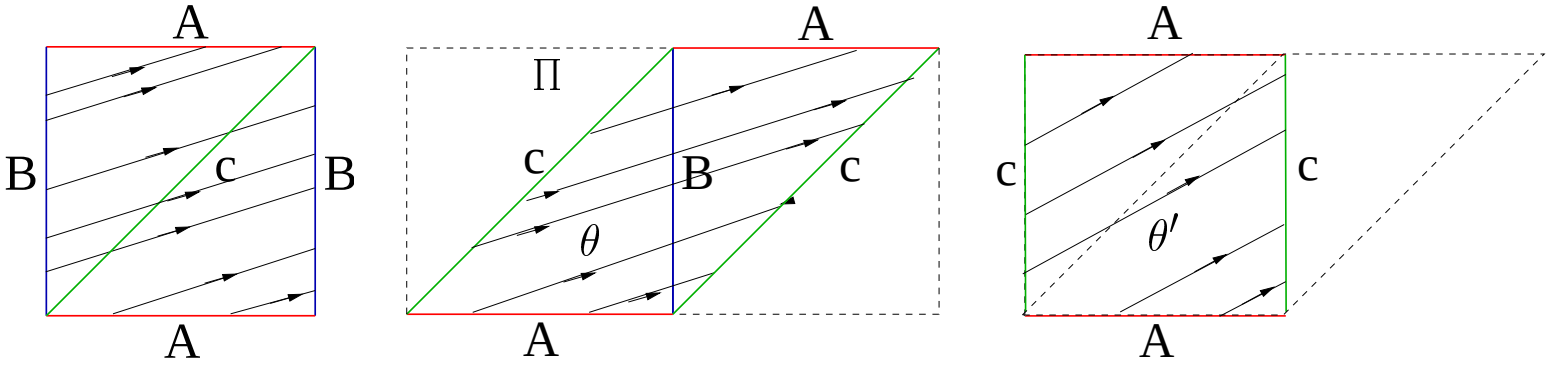

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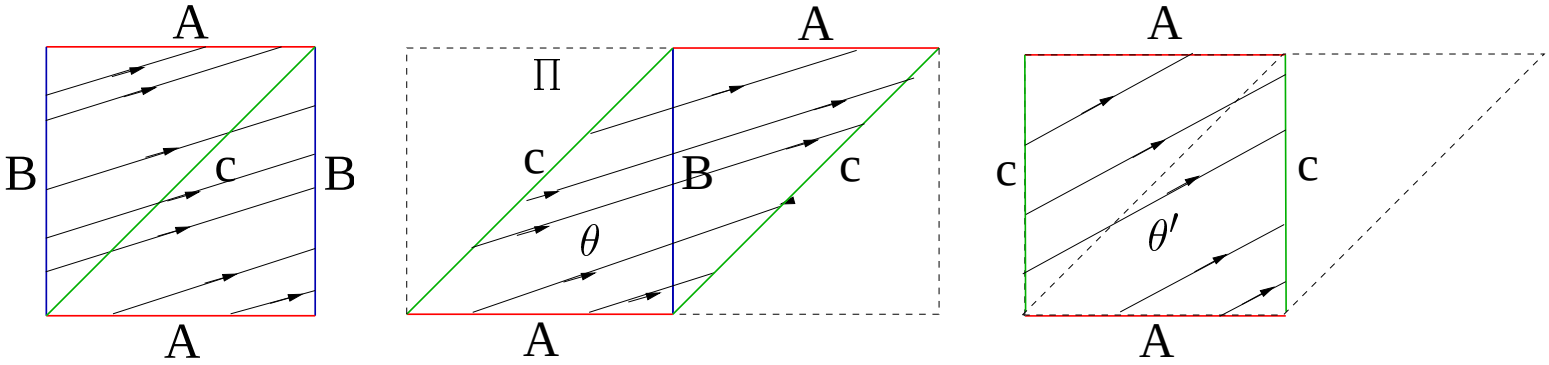

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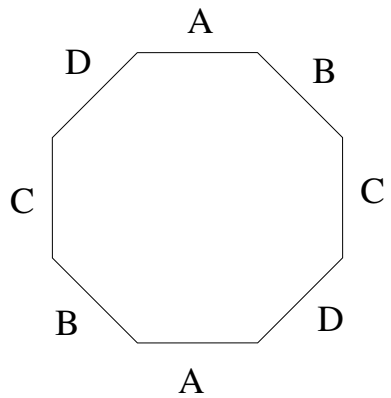
The octagon: isometries and sectors

Let D_8 be the isometries group of the octagon.



The letters $\{A, B, C, D\}$ are invariant with respect to a central symmetry. The other elements induce permutations of $\{A, B, C, D\}$ *for example:*

$A \mapsto C, C \mapsto A, B \mapsto B, D \mapsto D$.

Let us assume that the direction of the trajectory is $\theta \in \left[0, \frac{\pi}{8}\right]$. A fundamental domain for D_8 is $\Sigma_0 := \left[0, \frac{\pi}{8}\right]$. Thus, acting by an element of D_8, up to a permutation of the letters $\{A, B, C, D\}$, we can consider $\theta \in \Sigma_0 := \left[0, \frac{\pi}{8}\right]$.

The other sectors of angle $\pi/8$ are in order $\Sigma_1, \Sigma_2, \ldots, \Sigma_7$.

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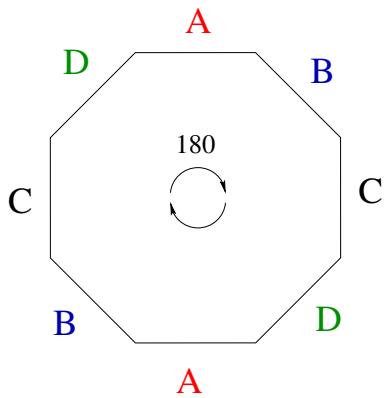

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The octagon: isometries and sectors

Let D_8 be the isometries group of the octagon.
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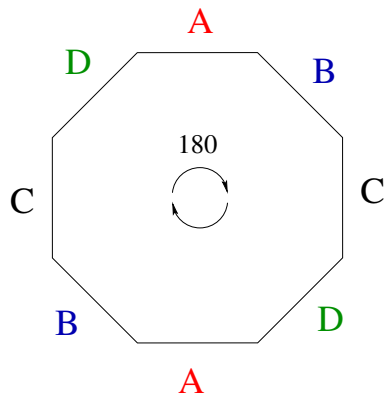

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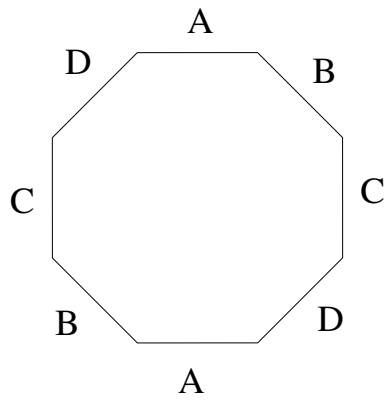

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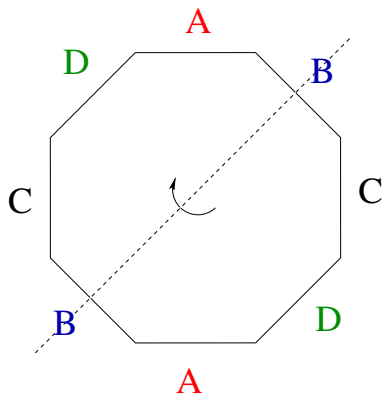

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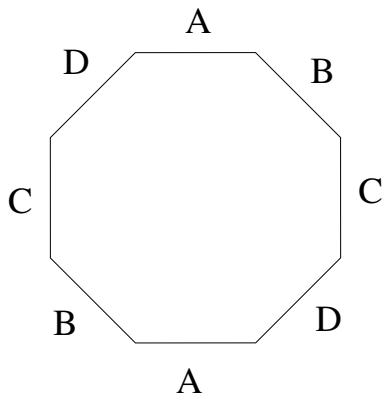

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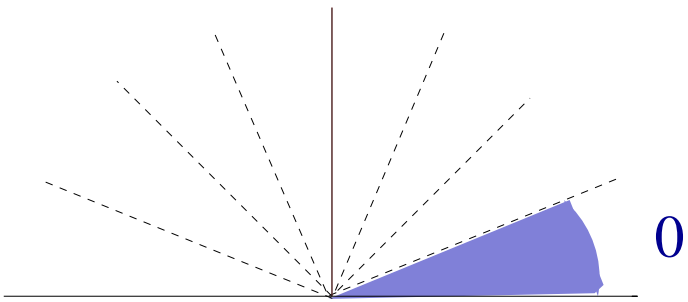

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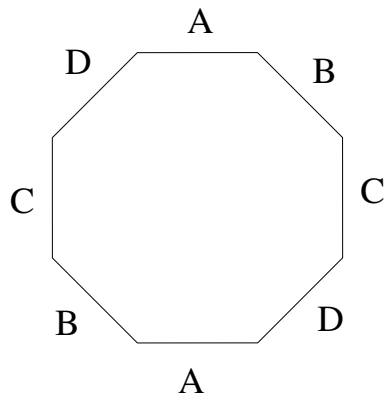

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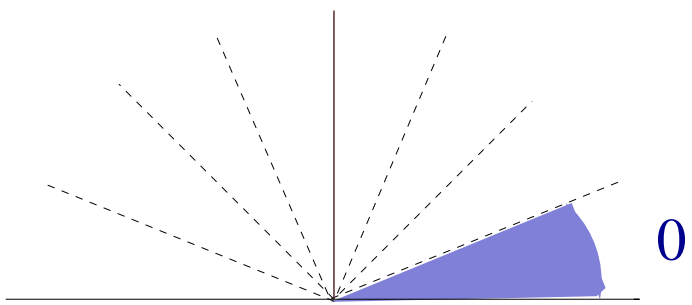

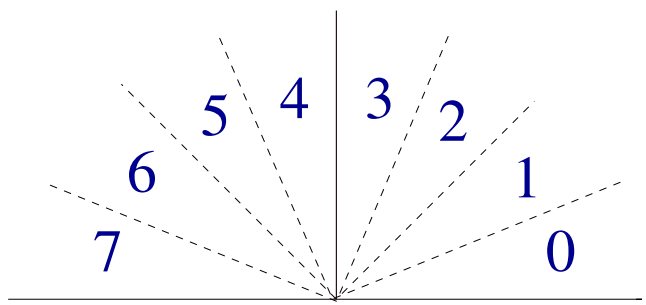

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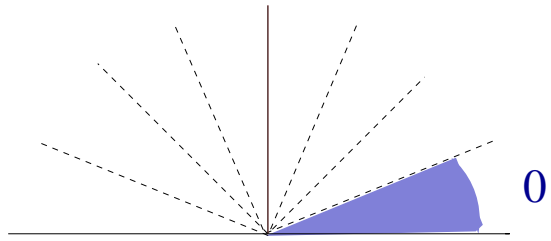
The octagon: allowed transitions in Σ_0

Let $\theta \in \Sigma_0 := \left[0, \frac{\pi}{8}\right]$.



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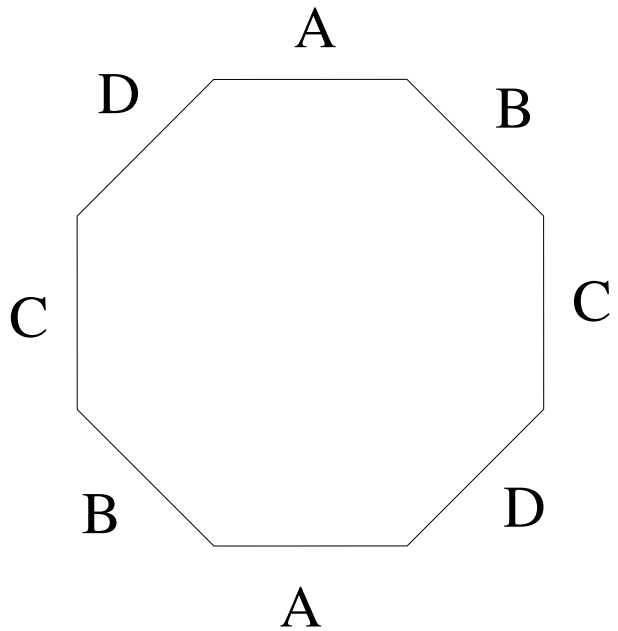

Each octagon cutting sequence of a trajectory in direction $\theta \in \Sigma_0$ determines a path in the diagram in Figure.

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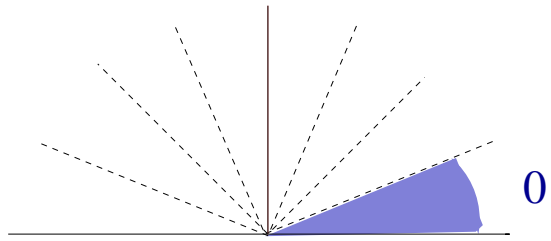

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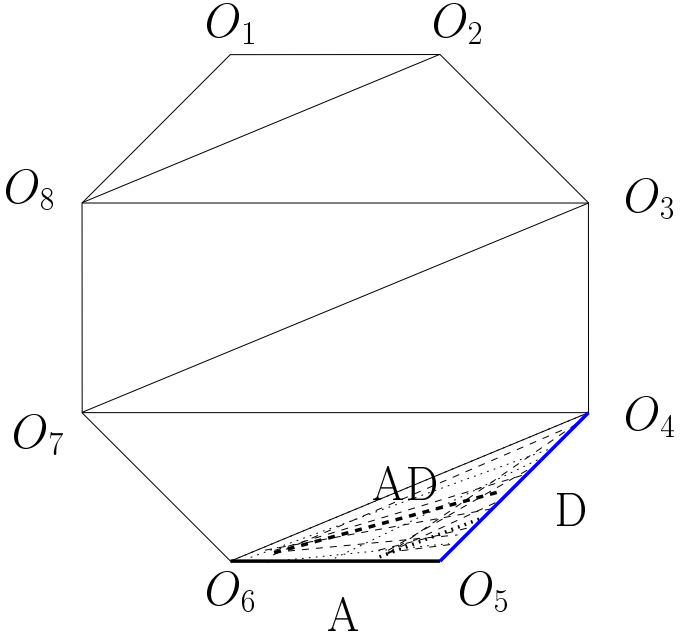

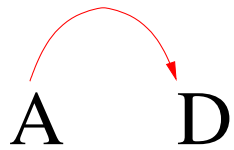

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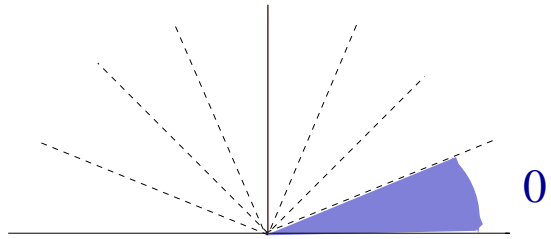

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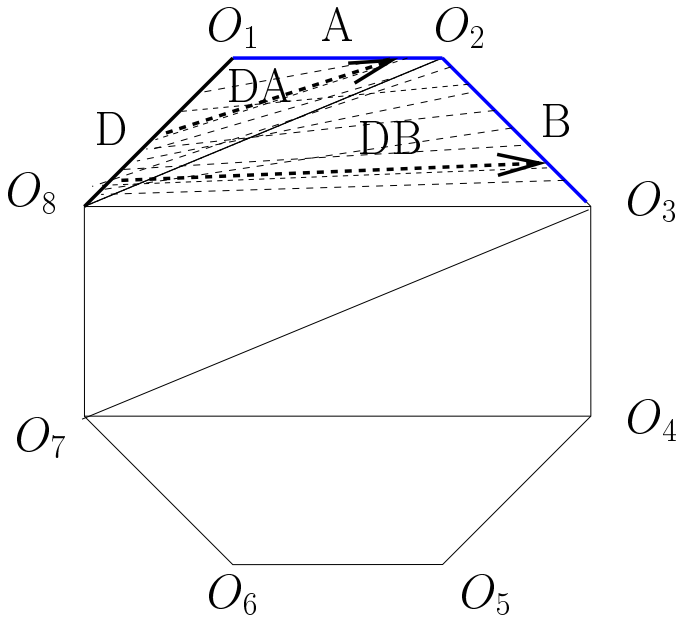

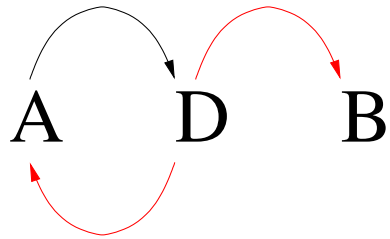

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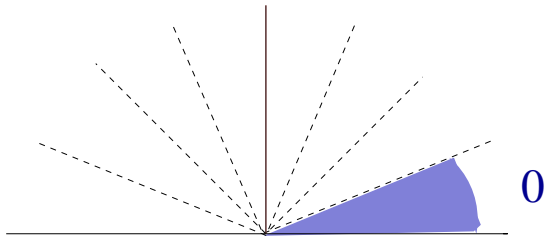

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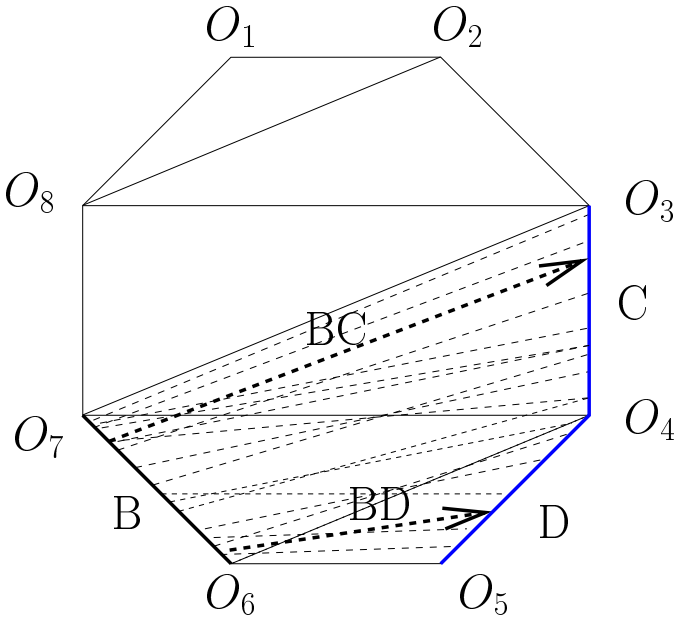

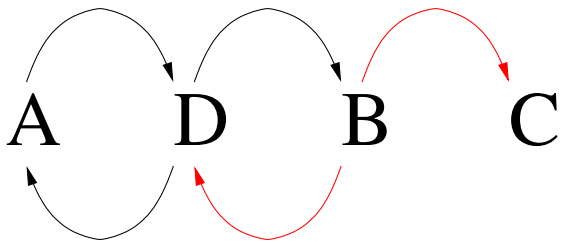

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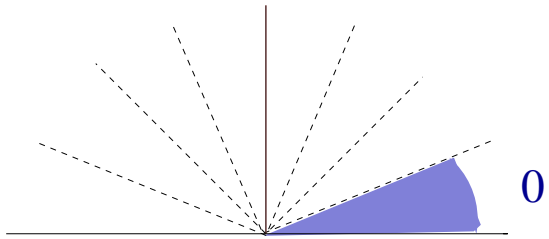

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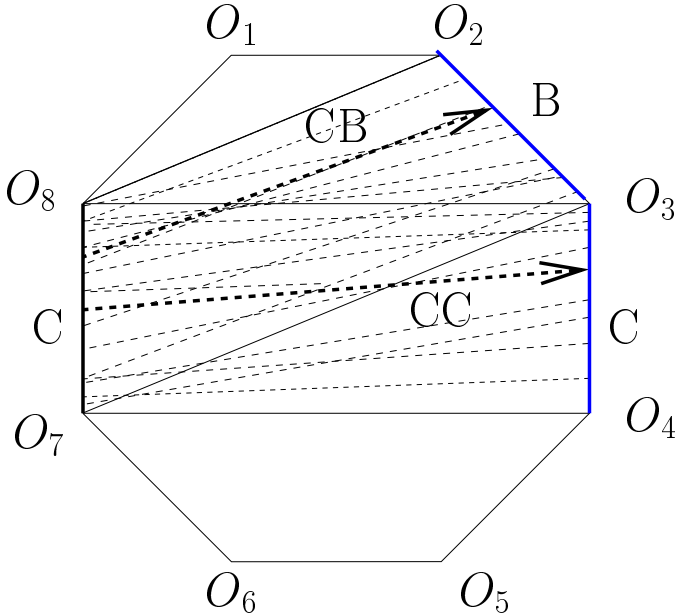

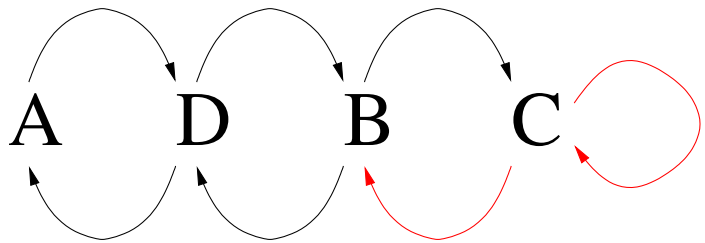

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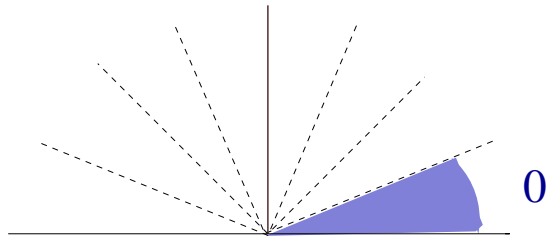

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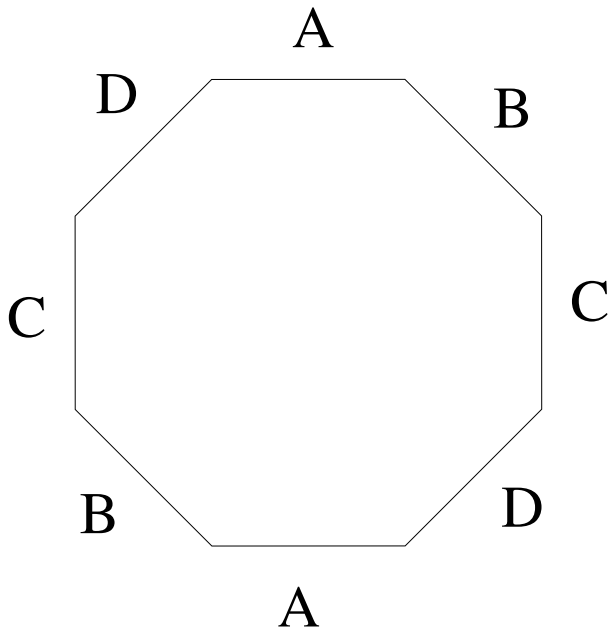

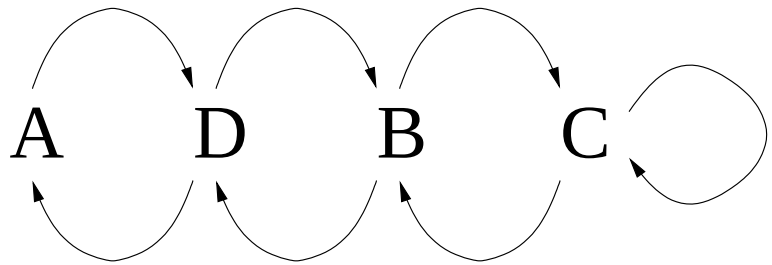

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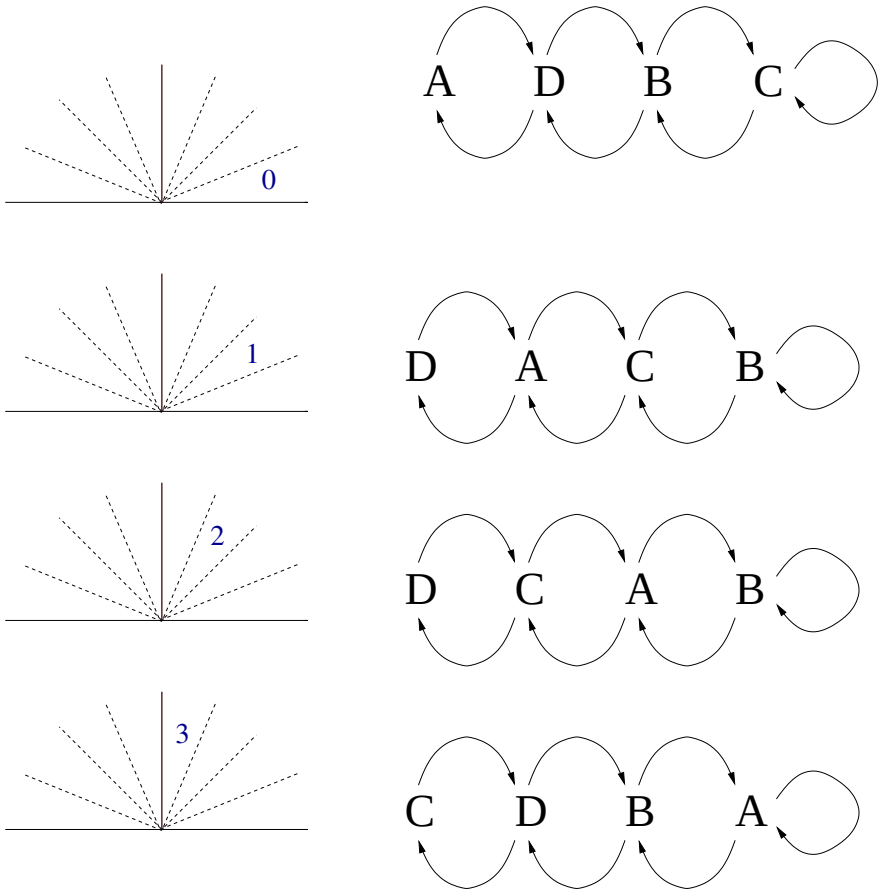
The octagon: possible transitions

Permuting the letters we obtain the diagrams corresponding to the other sectors:



Admissible sequences

Definition

A sequence $w \in \{A, B\}^{\mathbb{Z}}$ is *admissible* if it gives an infinite path on one of the following diagrams:

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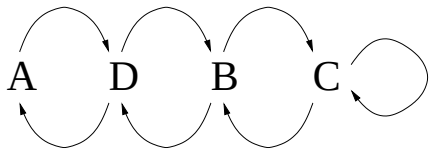
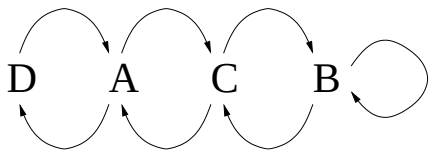
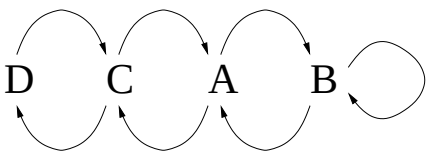


$\mathscr{D}_1$



$\mathscr{D}_2$



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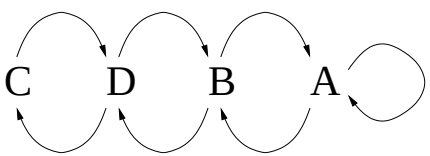

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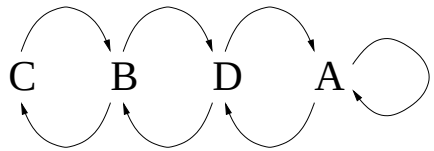
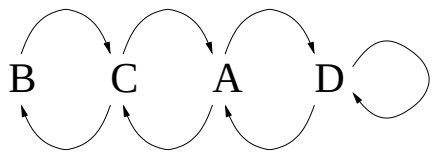
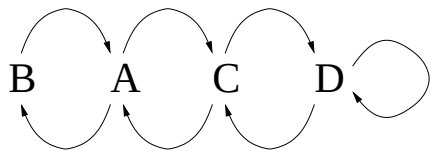


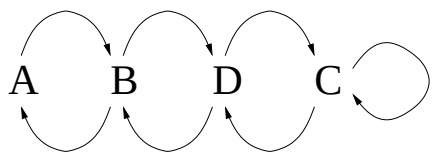
$\mathscr{D}_5$



$\mathscr{D}_6$



$\mathscr{D}_7$



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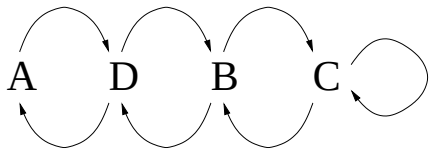

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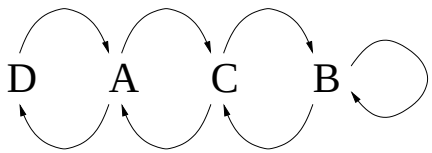

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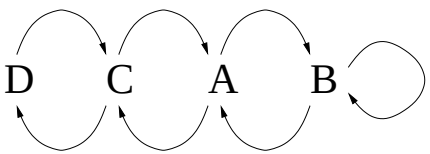

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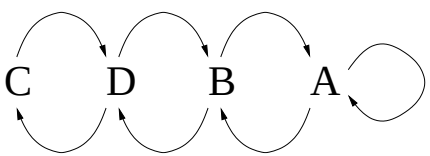

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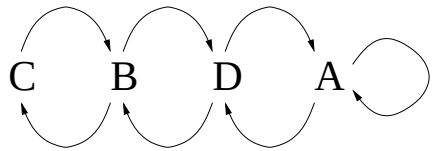

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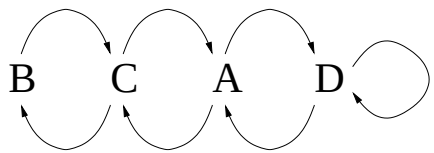

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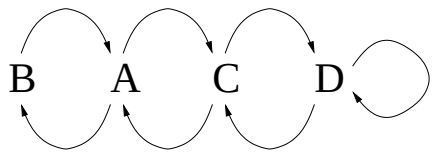

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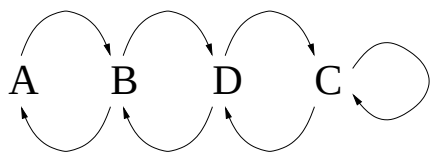

Lemma

An octagon cutting sequence is admissible.

Derived sequences

Definition

A letter in $\{A, B, C, D\}$ is *sandwiched* if it is preceeded and followed by the same letter.

Example

If w $= \ldots$ D A D B C C B C C B D A D B C B D B D B C B D $\ldots$,
w' $= \ldots$ A

Definition (Derivable sequences)

A sequence $w \in \{A, B, C, D\}^{\mathbb{Z}}$ is *derivable* if it is admissible and its derivative is still admissible. The sequence w is *infinitely derivable* if each of its derivatives is derivable.

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Definition (Derivable sequences)

A sequence $w \in \{A, B, C, D\}^{\mathbb{Z}}$ is *derivable* if it is admissible and its derivative is still admissible. The sequence w is *infinitely derivable* if each of its derivatives is derivable.

Derived sequences

Definition

A letter in $\{A, B, C, D\}$ is *sandwiched* if it is preceeded and followed by the same letter.

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Necessary condition and sequence of sectors

Theorem

An octagon cutting sequence is infinitly derivable.

The converse is not true, but we can describe exactly the condition which one needs to add.

Definition

Let w be infinitly derivable and let $w^{(n)}$ be the n^{th} derived sequence. The sequence $\{s_k\}_{k\in\mathbb{N}} \in \{0,1,\ldots,7\}^{\mathbb{N}}$ is a *sequence of sectors* for w if for each k, $w^{(k)}$ gives a path on the diagram $\mathscr{D}_{s_k}$.

Example

w= C C C B C C B D B

$s_0 = 0$

w'= C B D B D

$s_1 = 4$

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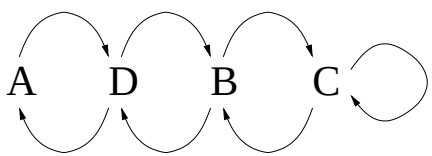

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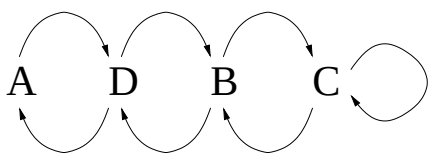

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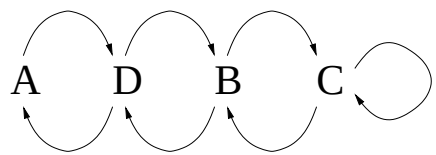

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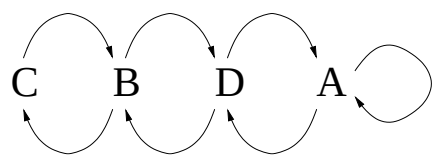
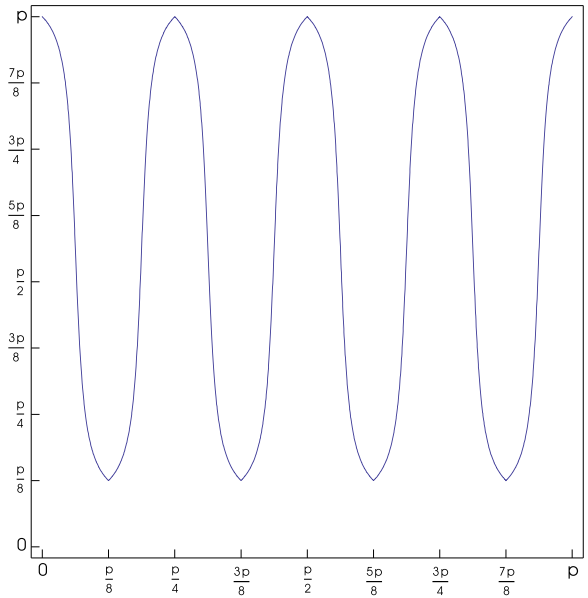


Octagon Continued Fractions

Let $F : [0, \pi] \to [\pi/8, \pi]$ the following map, that we call *Octagon Farey map*:



Definition

The octagon continued fraction expansion of θ is

$$\theta = [s_0, s_1, s_2, \ldots, s_k, \ldots]_O \quad \text{iff} \quad \{\theta\} = \cap_k F_{s_0}^{-1} F_{s_1}^{-1} \ldots F_{s_k}^{-1}[0, \pi].$$

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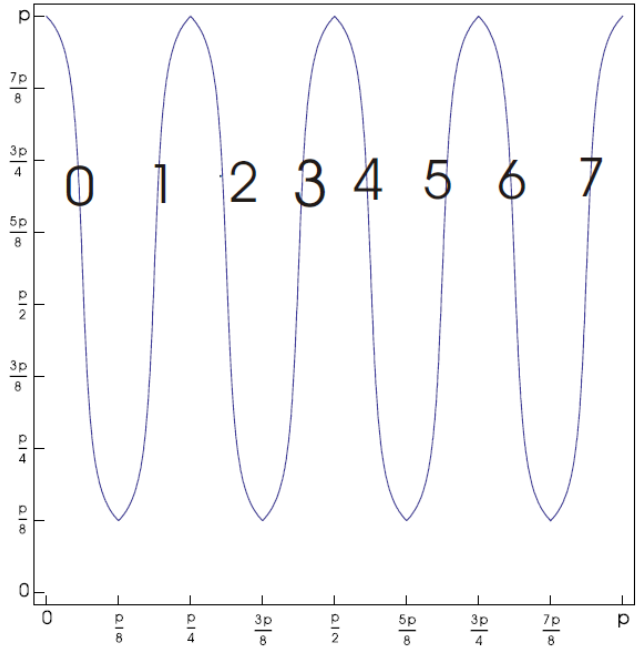

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Direction Recognitionn

Let w be an octagon cutting sequence.

Lemma

If w is not a periodic sequence, the sequence of sectors $\{s_k\}_{k\in\mathbb{N}}$ is uniquely determined. In particular, each derivative $w^{(k)}$ is admissible in an unique $\mathscr{D}_{s_k}$.

Theorem

If w is not periodic, there is a unique sequence of sectors $\{s_k\}_{k\in\mathbb{N}}$ for w and the direction of the trajectories with cutting sequence w is given by

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Ideas from proofs

As in the case of the square, the theorems follow if we prove that:

Theorem

If w is an octagon cutting sequence, also the derived sequence w' is an octagon cutting sequence.

Furthermore, if w is the cutting sequence of a trajectory in direction θ, the derived sequence w' is a cutting sequence of a trajectory in direction $\theta' = F(\theta)$, where F is the octagon Farey map.

To prove it, one uses an argument in the same spirit of:

i.e. one uses renormalization in the space of affine deformations of the octagon.

Regular polygons are rich of affine automorphisms.

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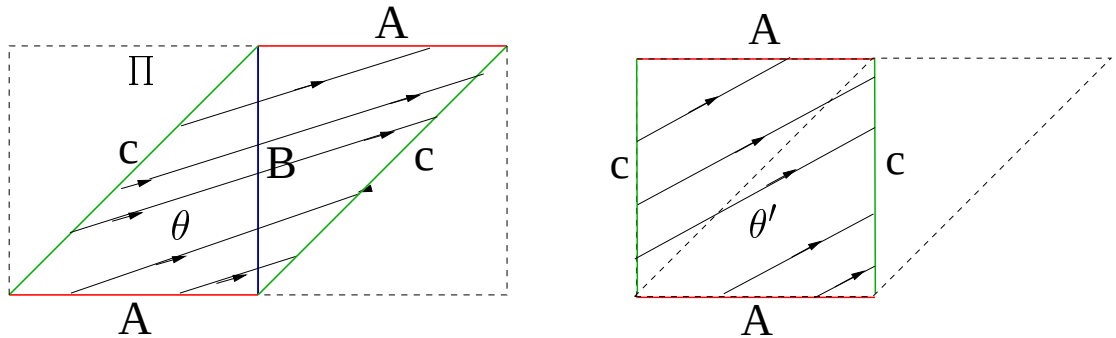

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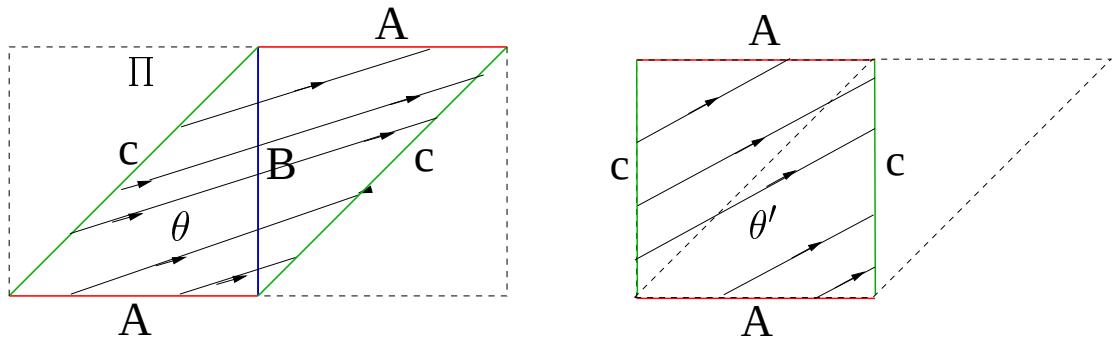

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Affine automorphism and Veech group

Let S_O be the surface obtained glueing opposite sides of the octagon by translations: S_O is an example of a *translation surface*, i.e. it has an atlas whose changes of coordinates are of the form $z \mapsto z + c$.

Definition

An automorphism $\Psi : S \mapsto S$ of a translation surface S is an affine automorphism if it is affine in each chart and $D\Psi(z)$ is independent on $z \in S$. Let

$$Aff(S) = \{\Psi, \quad \Psi \text{ affine automorphism}\}.$$

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Se $\Psi \in D_8$
is an isometry of O,
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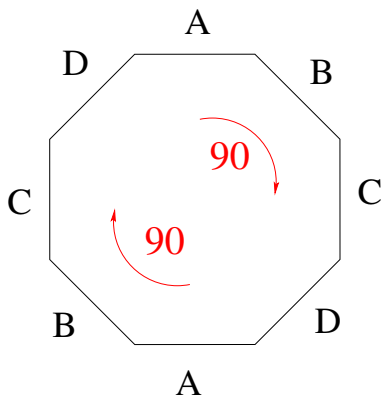

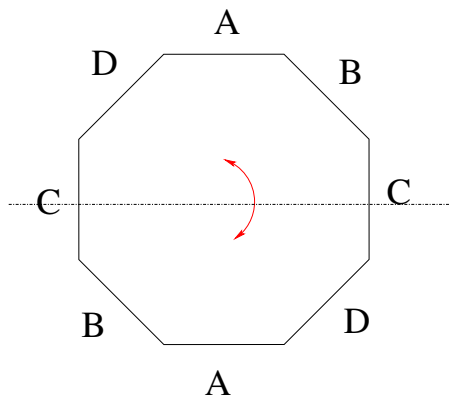

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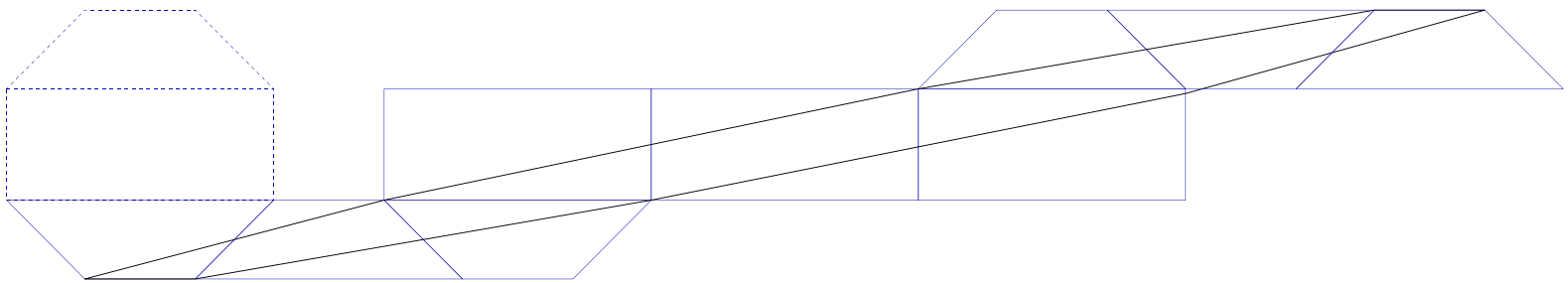
Example (2)

The matrix $\begin{pmatrix} 1 & 2(1+\sqrt{2}) \\ 0 & 1 \end{pmatrix} \in V(O)$.



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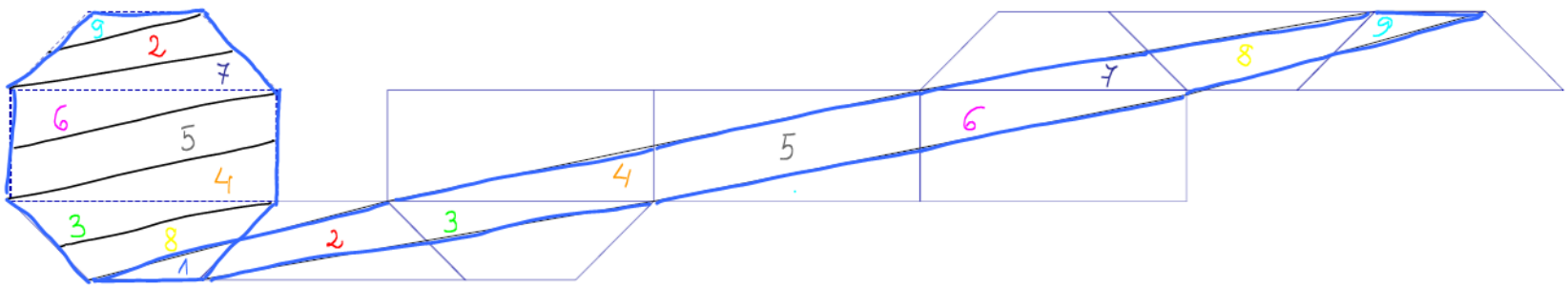

The Veech group of the octagon

The Veech group $V(S)$ is the group of linear parts of $Aff(S)$:

$$V(S) = \{D\Psi, \quad \Psi \in Aff(S)\} \subset SL(2,\mathbb{R})^{\pm}.$$

Ex 1

$$V(\mathbb{T}^2)^+ = \quad < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} > \quad = SL(2,\mathbb{Z}).$$

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If S is a translation surface glued from a regular polygon, $V(S)$ is a lattice in $SL(2,\mathbb{R})^{\pm}$ (Veech)

The surfaces S for which $V(S)$ is a lattice are actively researched in Teichmüller dynamics.

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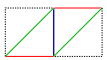

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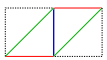

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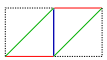

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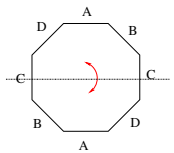

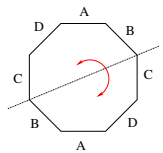

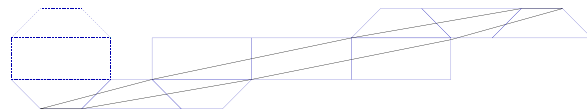

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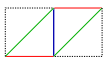

$$V(\mathbb{T}^2)^+ = \quad < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} > \quad = SL(2,\mathbb{Z}).$$

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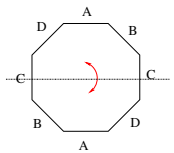

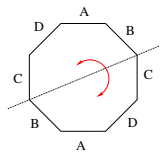

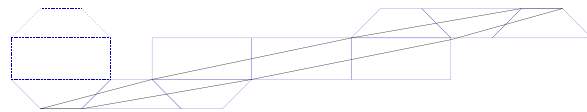

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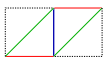

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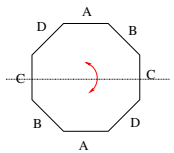

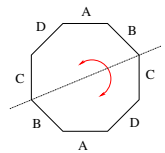

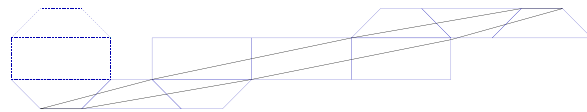

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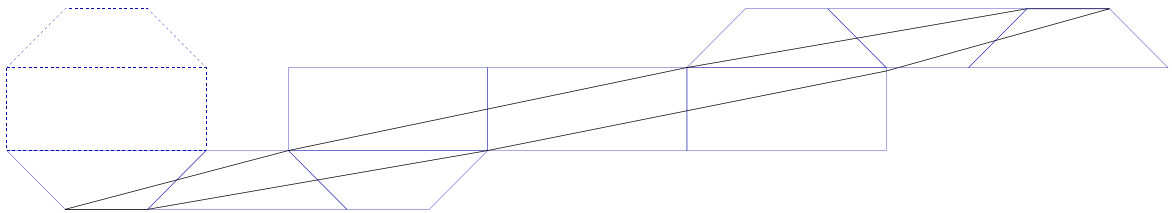
Derivation and renormalization in the octagon

Let w be the cutting sequence of a trajectory in direction $\theta \in \Sigma_0$. Let $O' = \begin{pmatrix} -1 & 2(1+\sqrt{2}) \\ 0 & 1 \end{pmatrix} O$.

Lemma

The derived sequence w' coincides with the cutting sequence of the same trajectory in direction θ with respect to the sides of O'.



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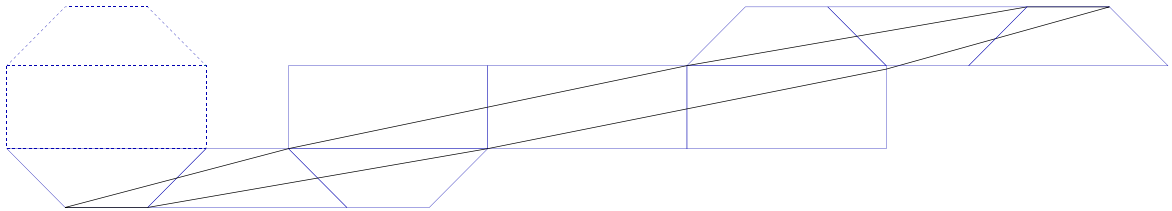

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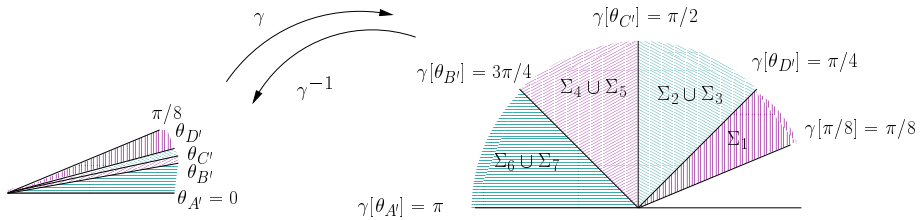

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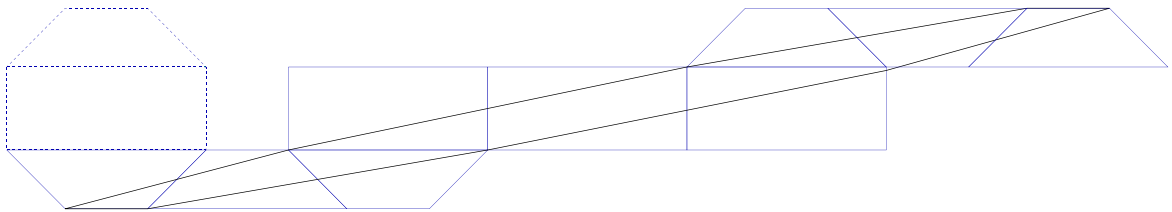

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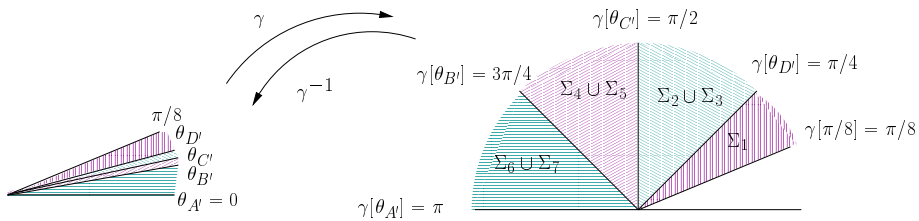

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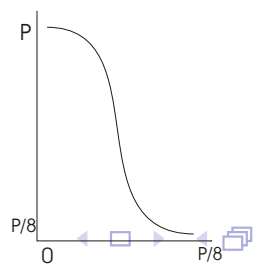

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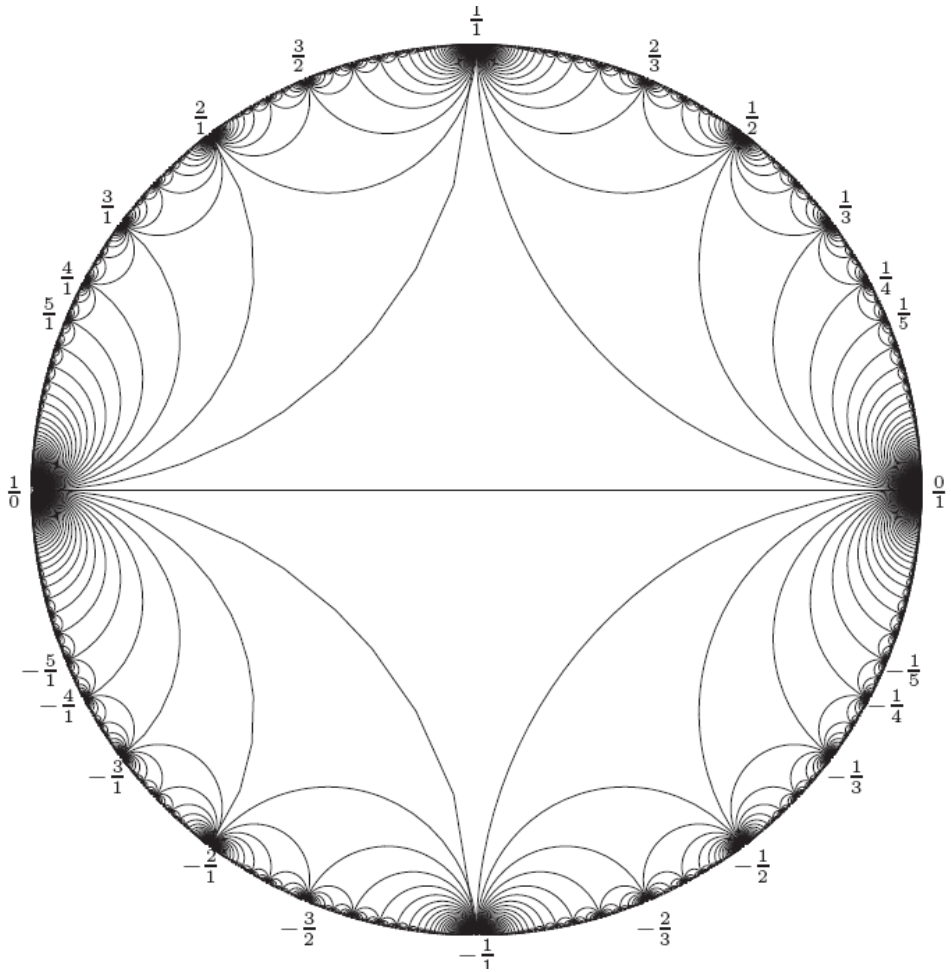

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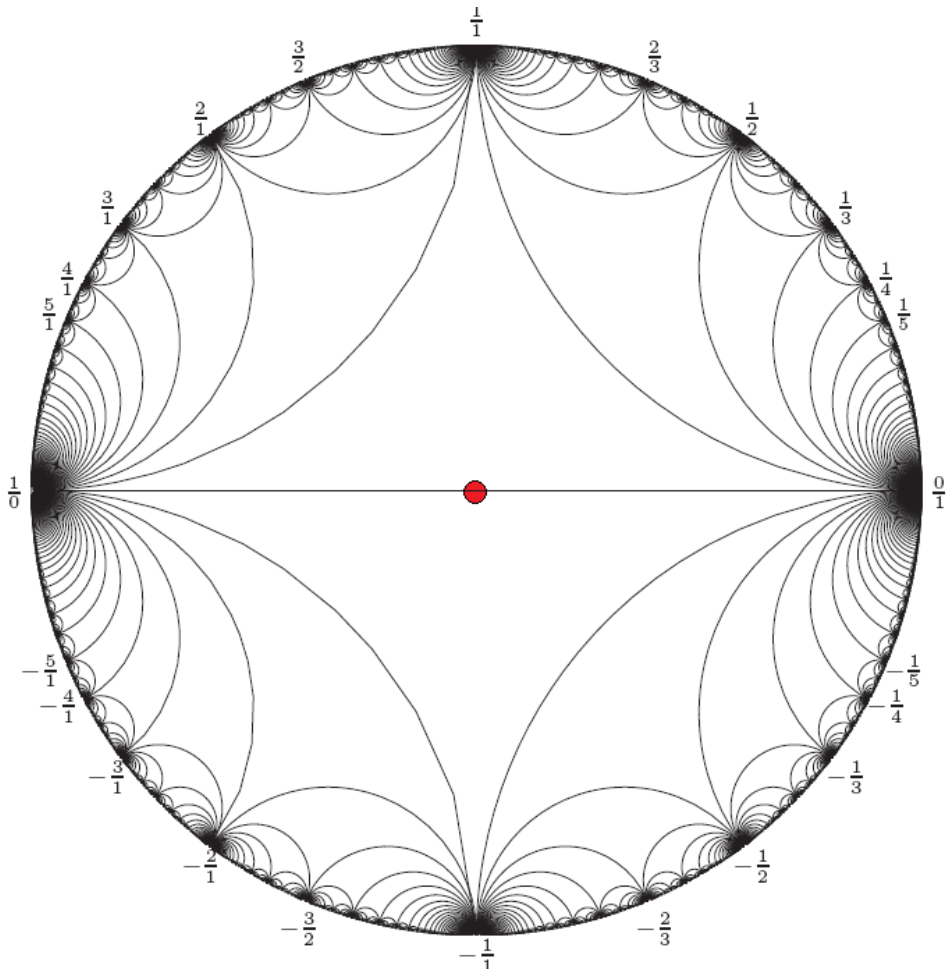

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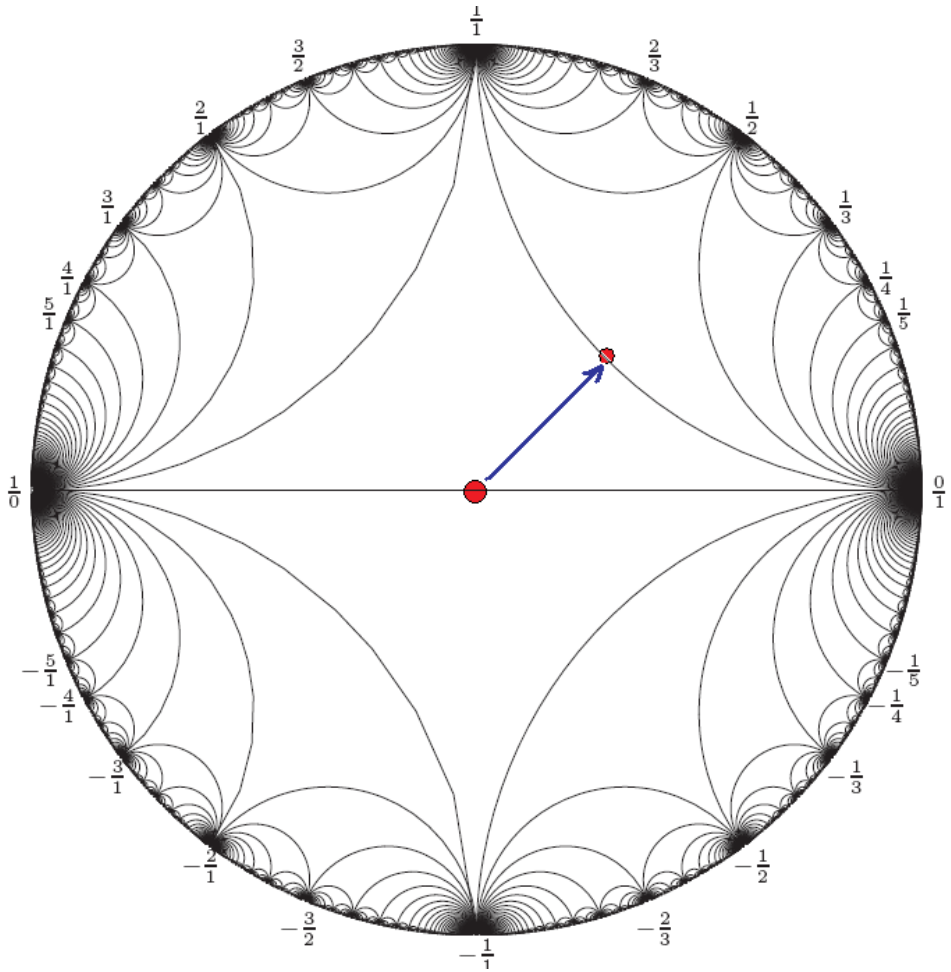

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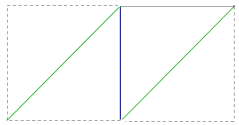

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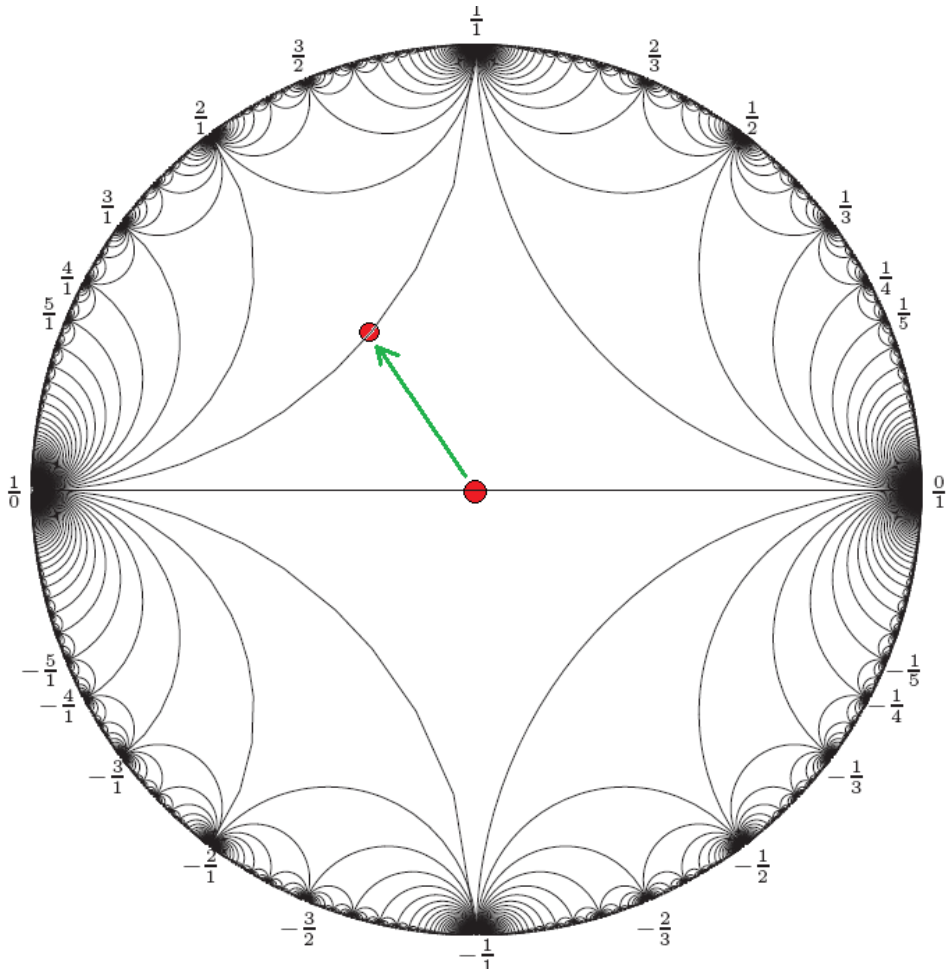

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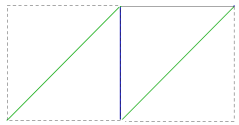

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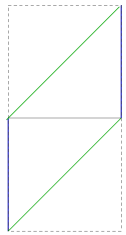

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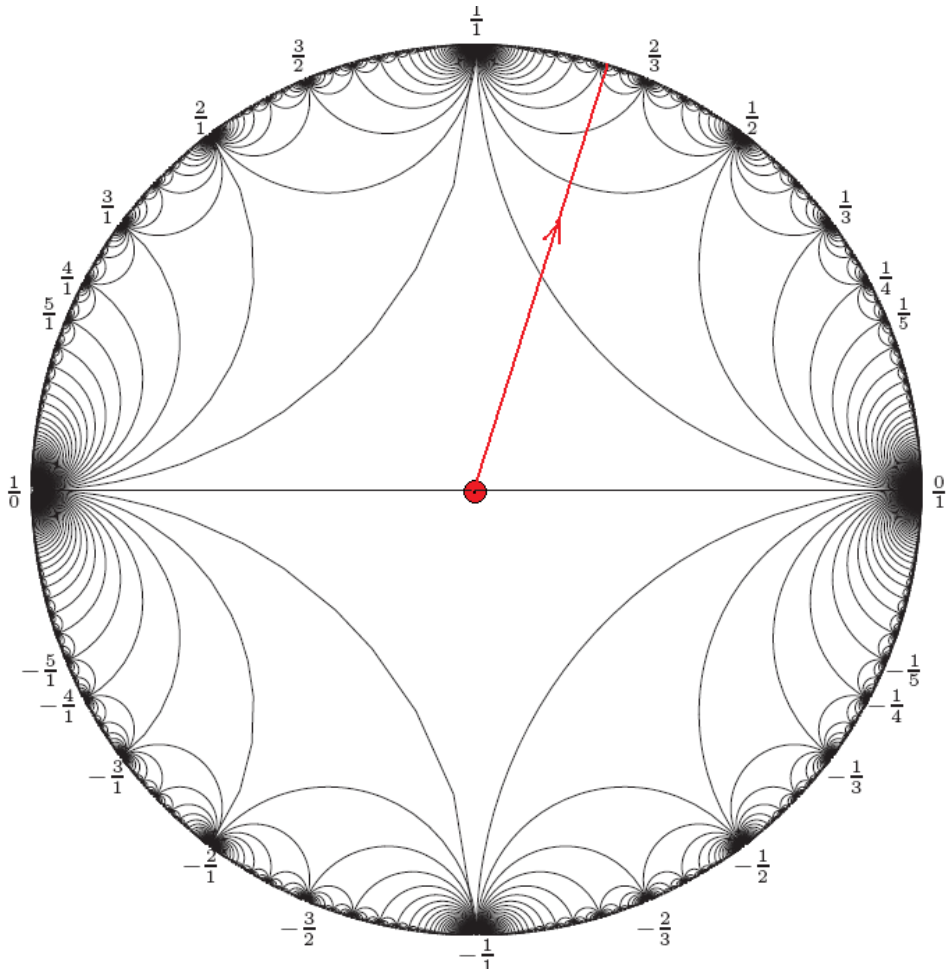

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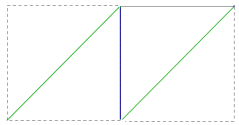

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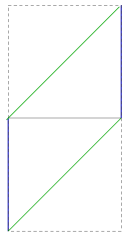

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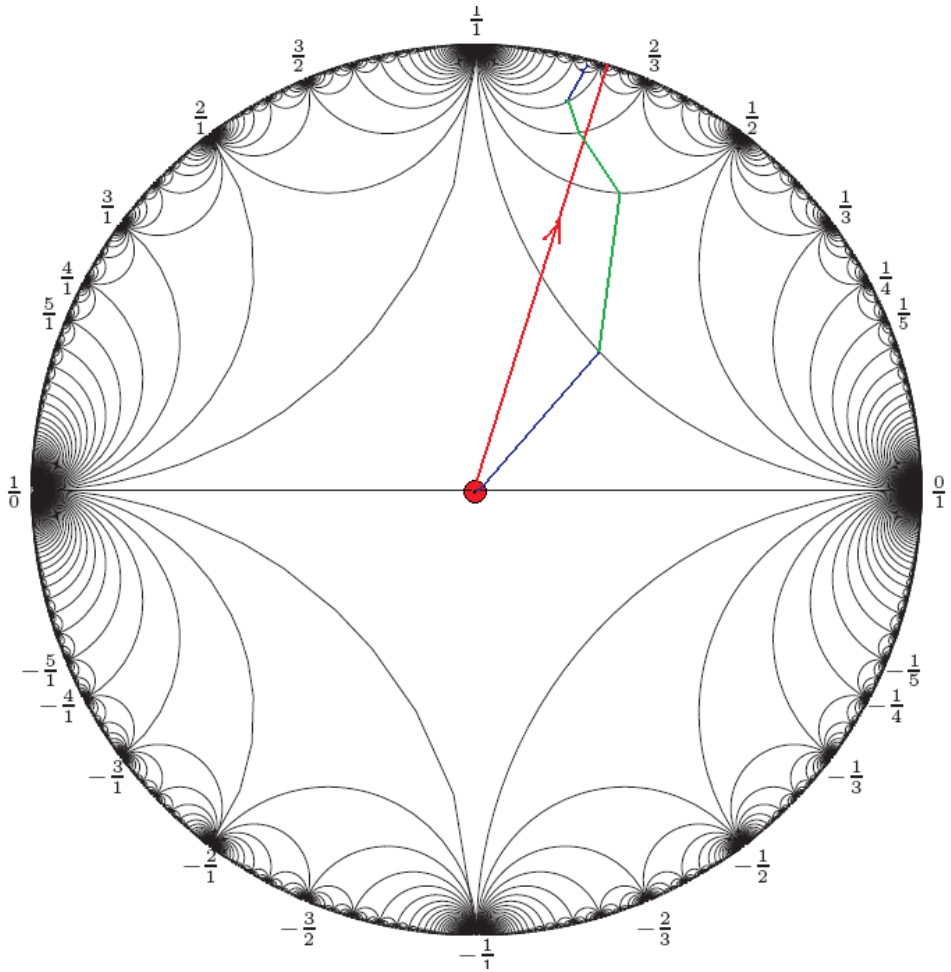

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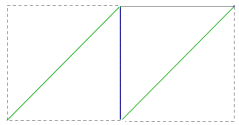

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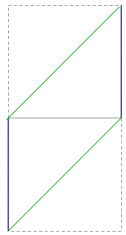

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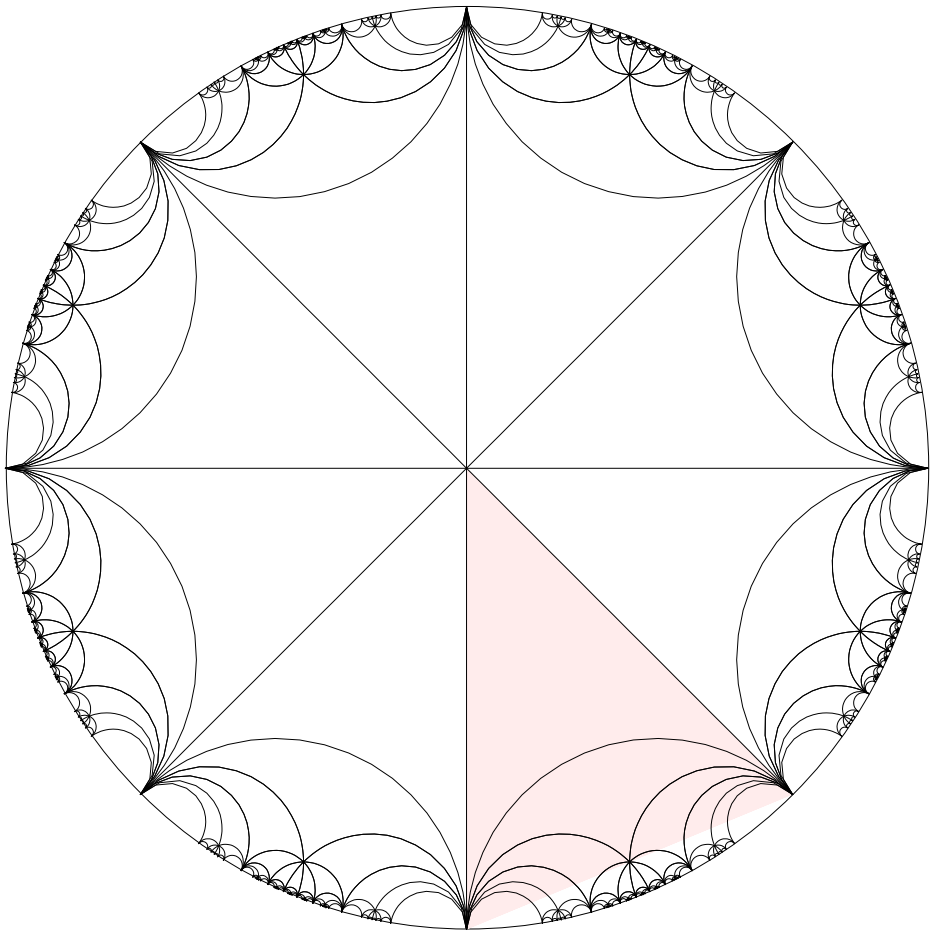

$V(O) \cdot O$ centers of ideal octagons

tree of renormalization moves

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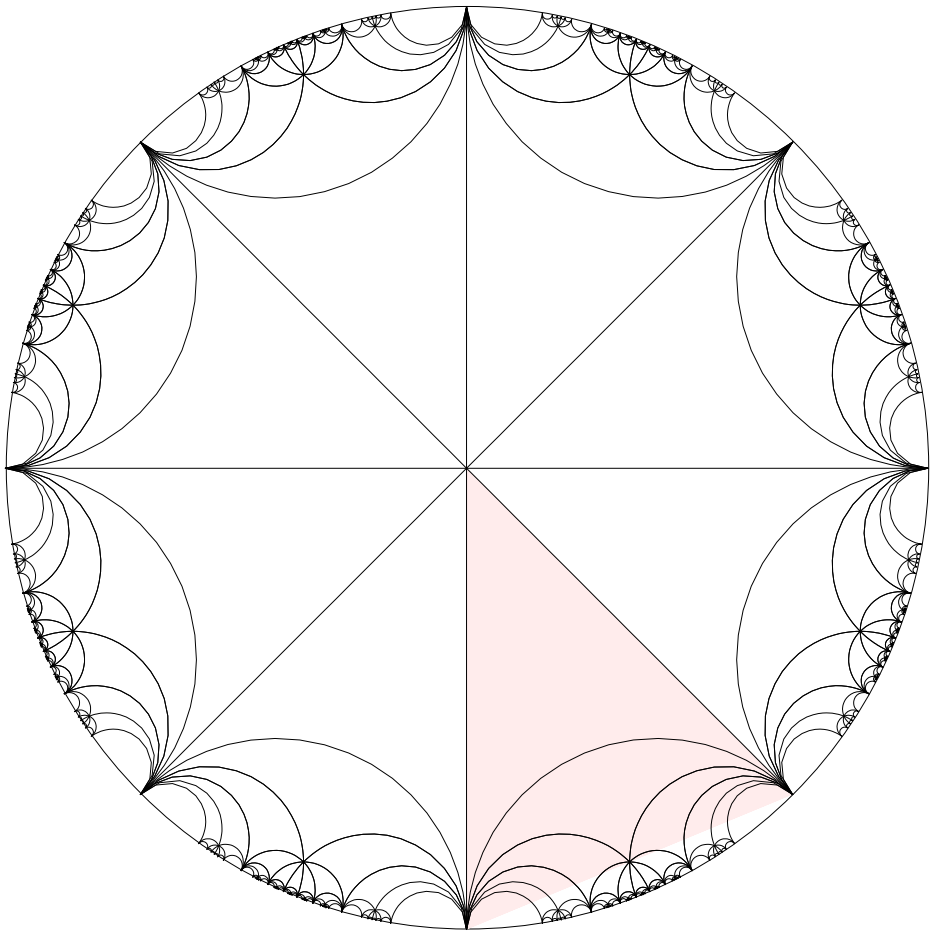

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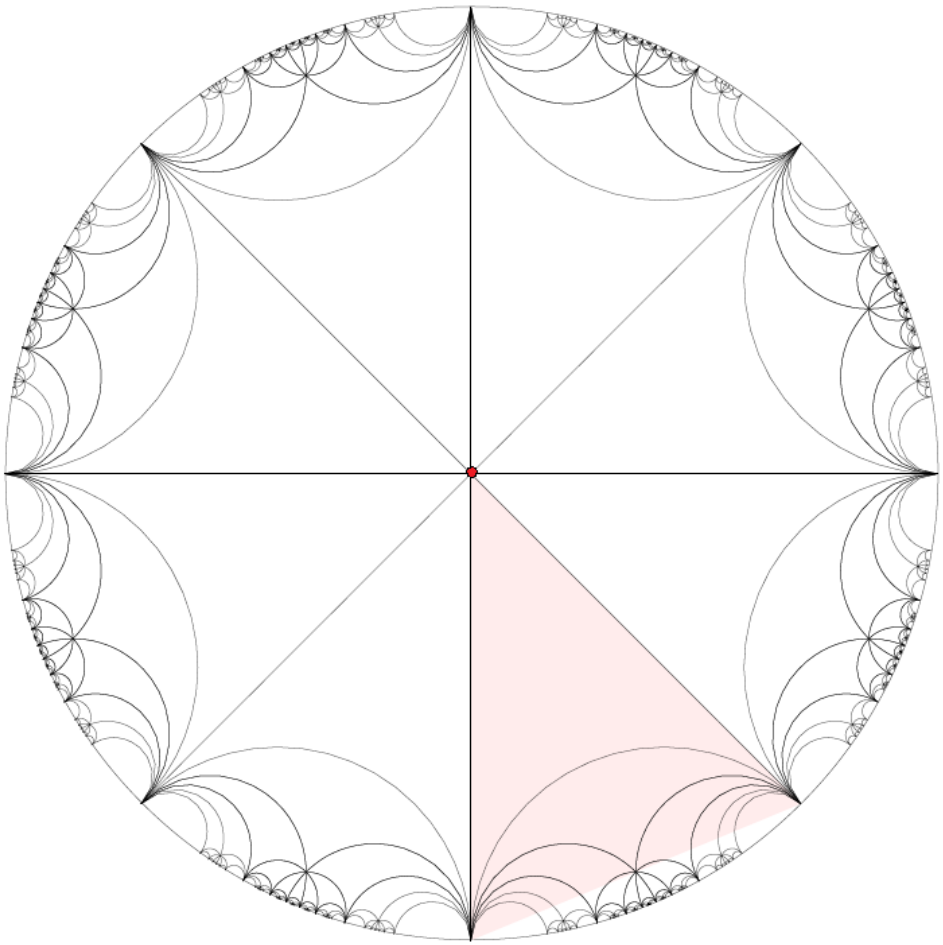

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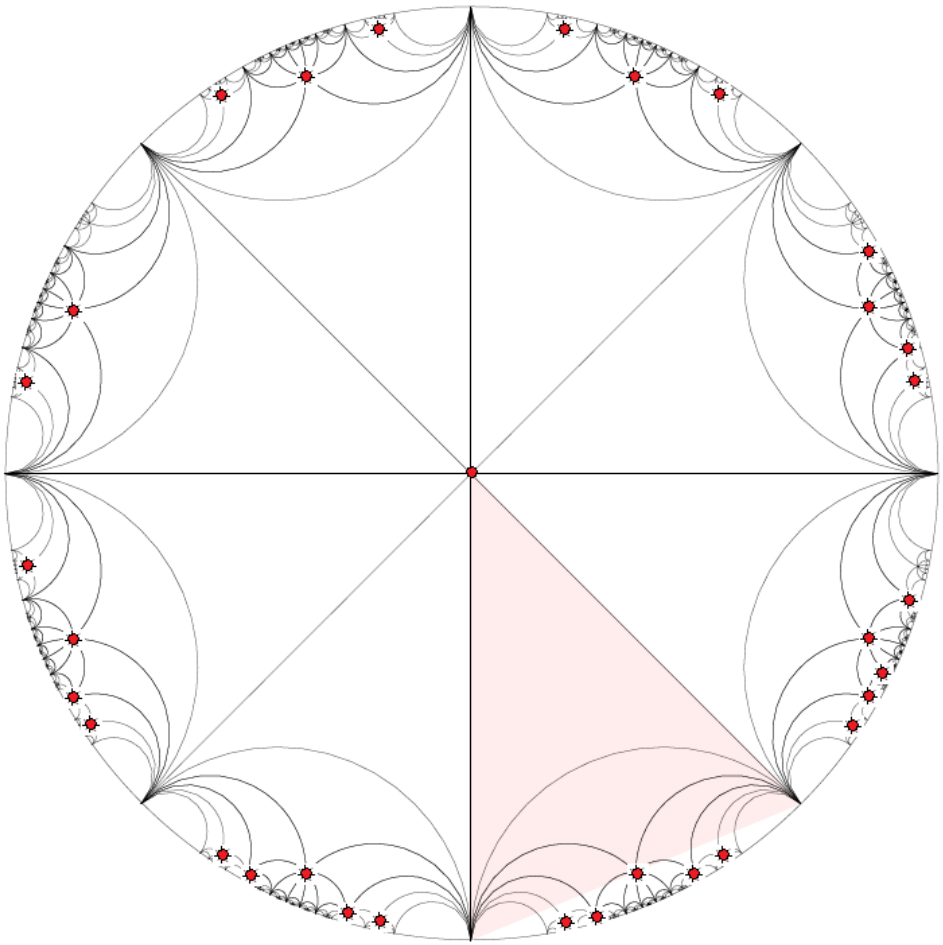

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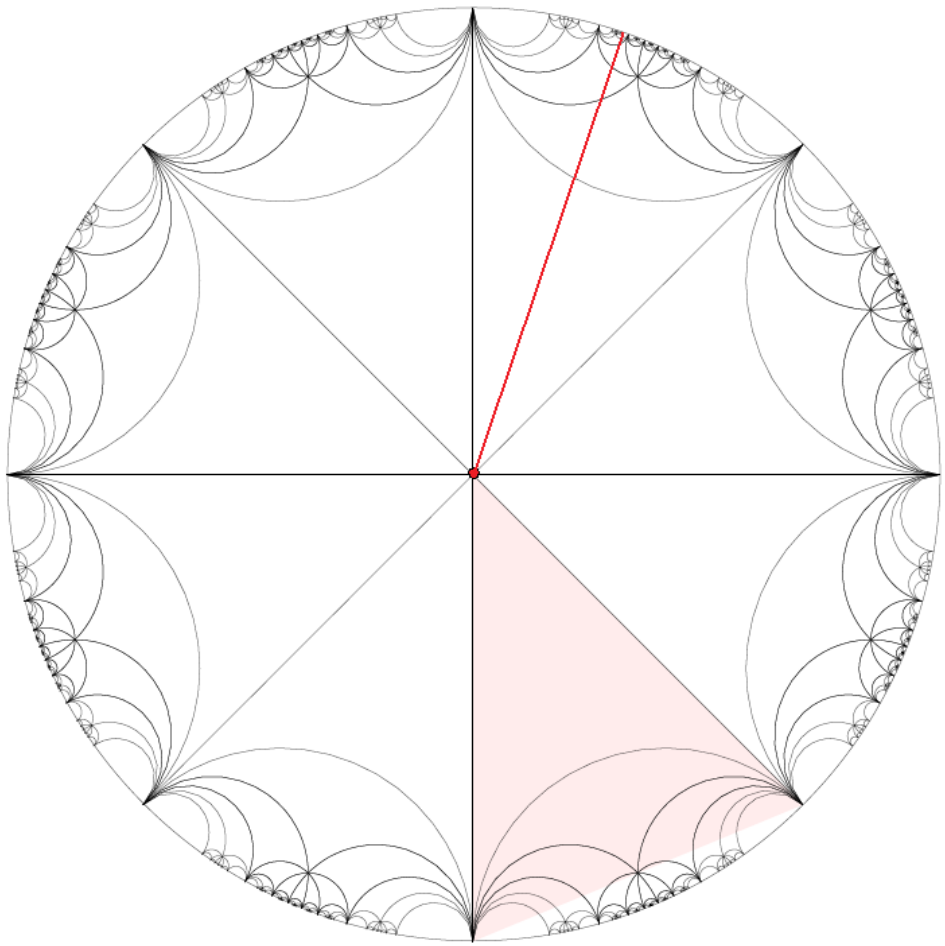

$V(O) \cdot O$ centers of ideal octagons

tree of renormalization moves

Given θ, g_t^θ geodesics is approximated by a sequence of renormalization moves

the Farey map of the octagon is given by the action on $\partial\mathbb{D}$.

Renormalization and dynamics on the Teichmüller disk

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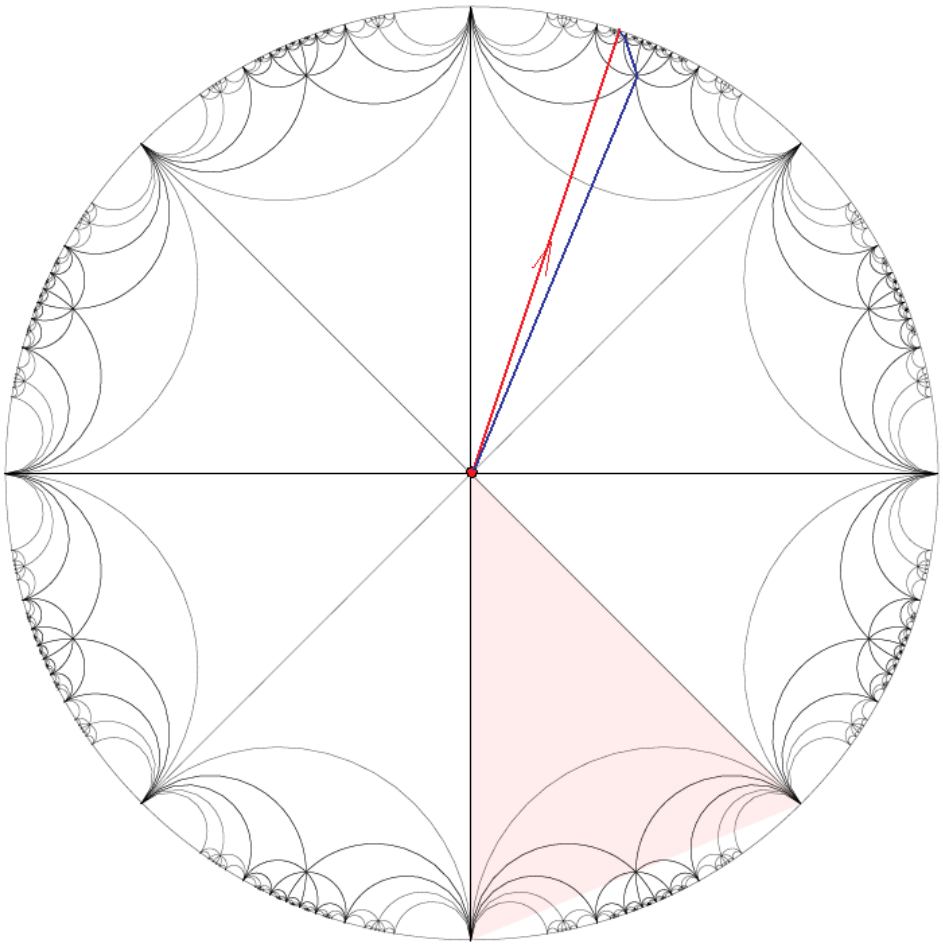

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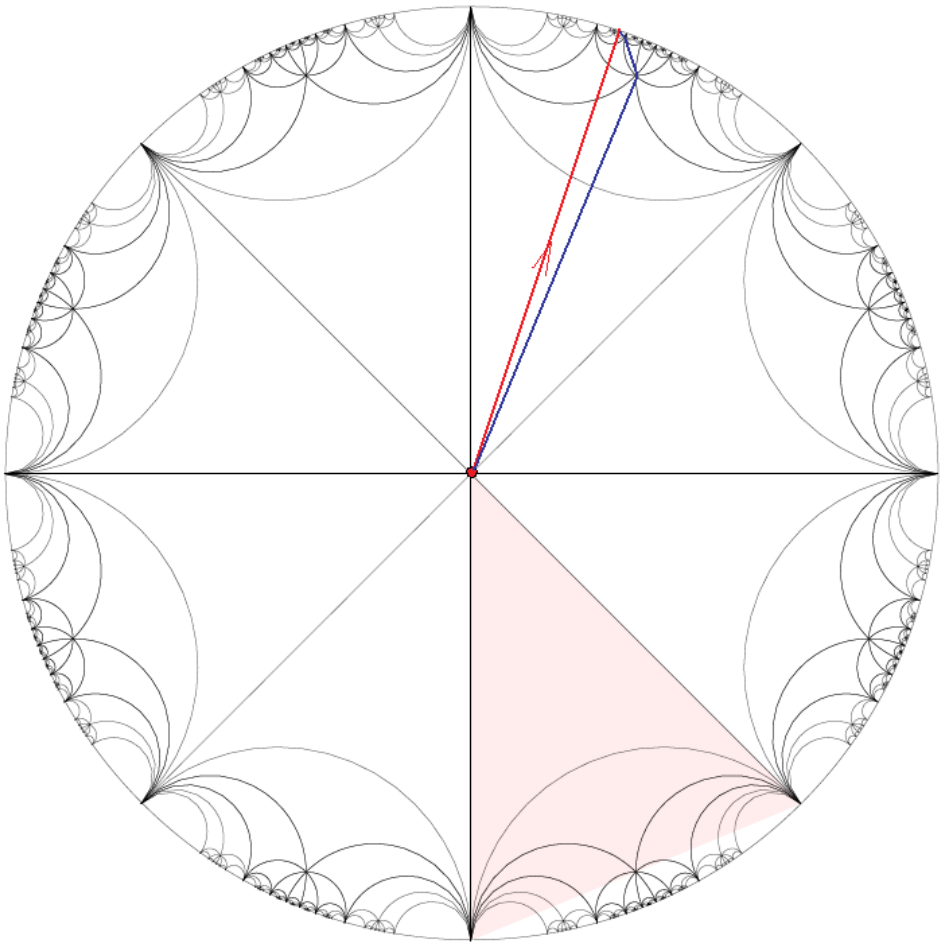

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Generation of cutting sequences

Let us define operators g_i^j that invert derivation.

The operator g_i^j *interpolates* a sequence w admissible in $\mathscr{D}_j$ producing a sequence ammissible in $\mathscr{D}_i$ and such that $(g_i^j(w))' = w$. For example:

Definition (Generation operator g_0^2)

Let w be ammissible in $\mathscr{D}_0$. The sequence $g_0^2 w$ is obtained interpolating the letters corresponding to vertices in Figure with the words on the arrows:

Example

$w = \ldots\ D\ C\ A\ B\ B\ B\ A\ C\ \ldots$

$g_0^2 w = \ldots\ D\ B\ C\ \ldots$

The only sandwiched letters are the coloured ones, thus $(g_0^2 w)' = w$.

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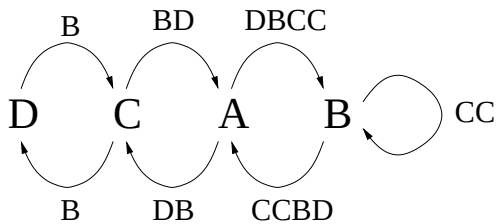

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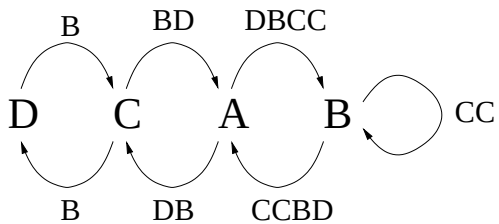

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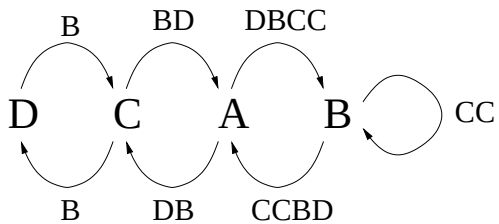

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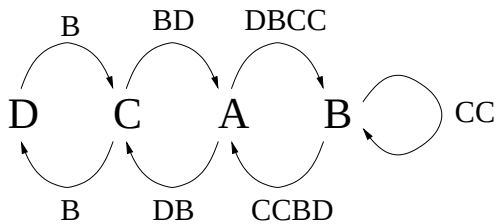

Example

$$\begin{aligned} w &= \ldots \; D \quad C \quad\quad A \;\; B\, B\, B\, A\, C \ldots \\ g_0^2 w &= \ldots \; D\, B\, C\, BD\, A \;\; \ldots \end{aligned}$$

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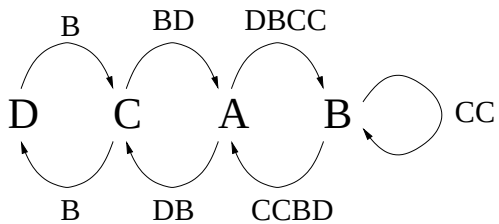

Example

w $= \ldots$ D C A B B B A C $\ldots$

$g_0^2 w$ $= \ldots$ D B C BD A DBCC B $\ldots$

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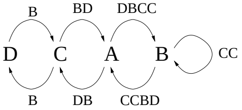

Example

$$
\begin{aligned}
\mathrm{w} &= \ldots\ D\quad C\quad A\qquad\quad B\quad B\quad B\qquad A\quad C\ \ldots \\
g_0^2 w &= \ldots\ D\ B\ C\ BD\ A\ DBCC\ B\ CC\ B\ CC\ B\ CCBD\ A\ DB\ C\ \ldots
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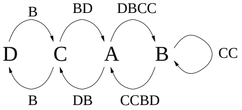

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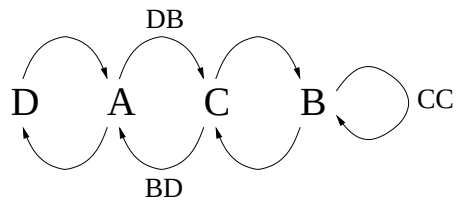
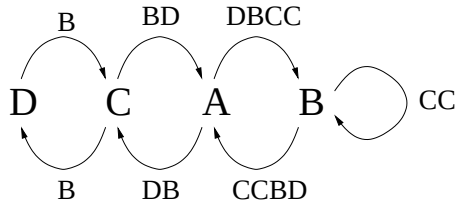
The interpolation operators

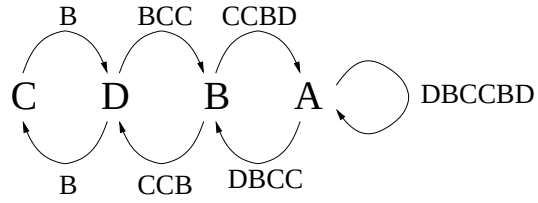
g_1^0



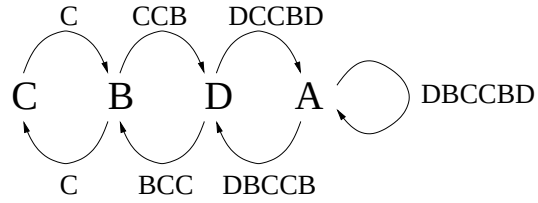
g_2^0



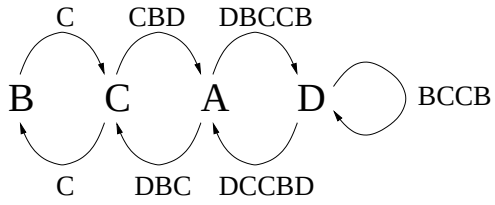
g_3^0



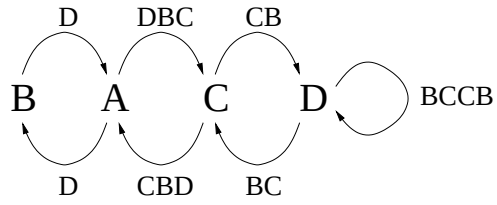
g_4^0



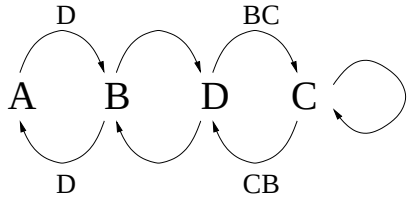
g_5^0



g_6^0



g_7^0



The other operators are obtained from these ones by permuting the letters.

Characterization of the closure of cutting sequences

Lemma

If w is a cutting sequence and $\{s_n\}_{n\in\mathbb{N}}$ a sequence of sectors, we have

$$w \in \cap_n g_{s_1}^{s_0} g_{s_2}^{s_1} \dots g_{s_n}^{s_{n-1}} \{A, B, C, D\}^{\mathbb{Z}}.$$

This condition, together with infinite derivability, is *necessary* and *sufficient* to characterize the *closure* of octagon cutting sequences in $\{A, B, C, D\}^{\mathbb{Z}}$:

Theorem (Smillie-U)

A sequence $w \in \{A, B, C, D\}^{\mathbb{Z}}$ belongs to the closure of cutting sequences in the octagon iff there exists a squence $\{s_n\}_{n\in\mathbb{N}} \in \{0, \dots, 7\}^{\mathbb{N}}$ such that

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An analogous theorem holds for every regular polygon.

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