Dynamical properties of billiards and flows on surfaces

Corinna Ulcigrai

University of BRISTOL

Mathematical Billiards

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a *billiard table*: $P \subset \mathbb{R}^2$;

a *ball*: point particle with no-mass;

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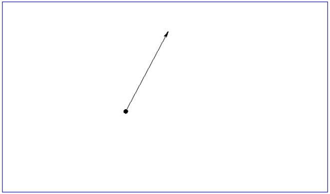

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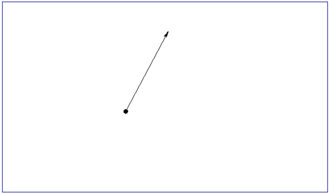

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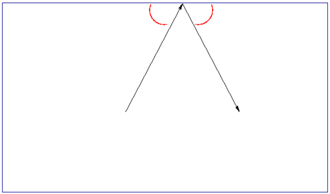

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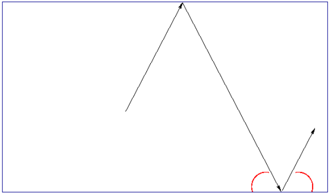

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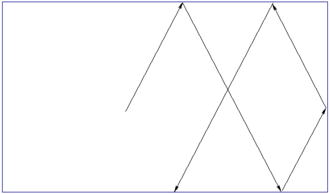

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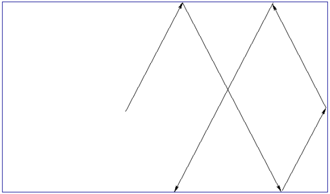

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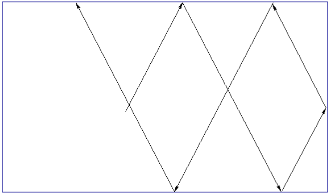

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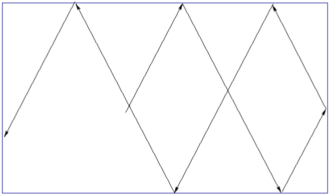

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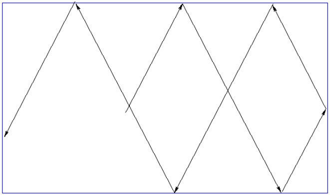

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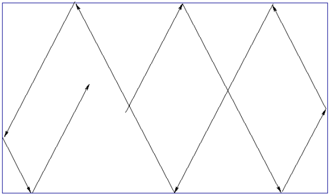

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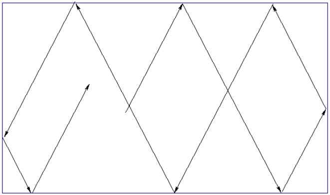

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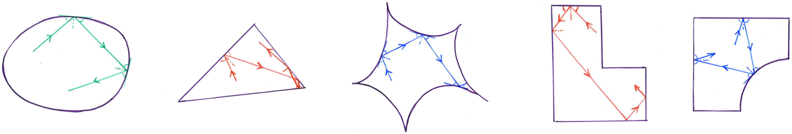

An *L*-shape billiard in real life:

[Credit: Photo courtesy of Moon Duchin]

Why to study mathematical billiards?

Mathematical billiards arise naturally in many problems in physics, e.g.:

- Lorentz Gas $\leftrightarrow$ Sinai billiard
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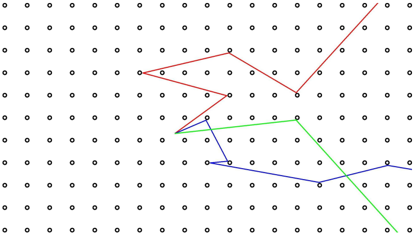

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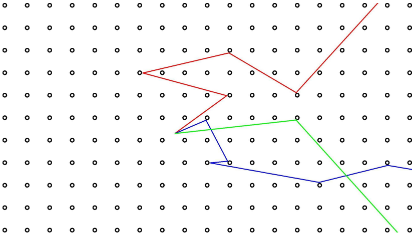

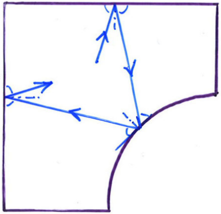

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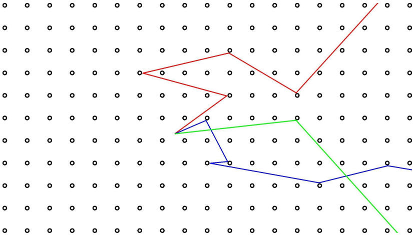

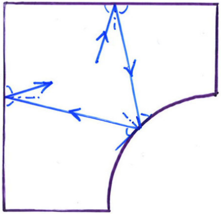

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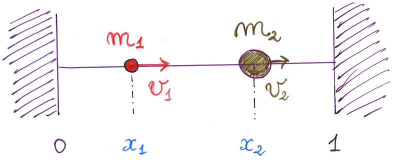

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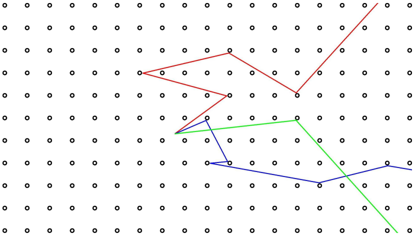

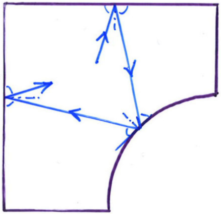

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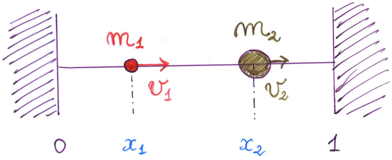

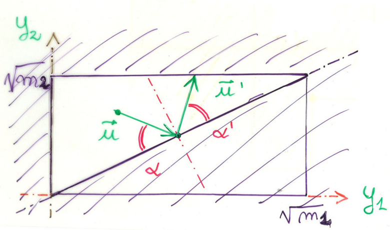

Dynamical Systems

Mathematical billiards are an example of a *dynamical system*, that is a system that evolves in time.

Usually dynamical systems are *chaotic* and one is interested in determining the *asymptotic behaviour*, or long-time evolution of the system.

Basic questions are:

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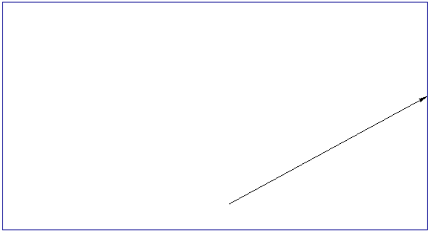

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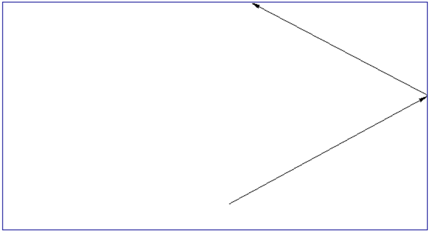

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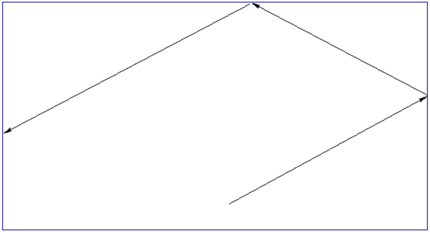

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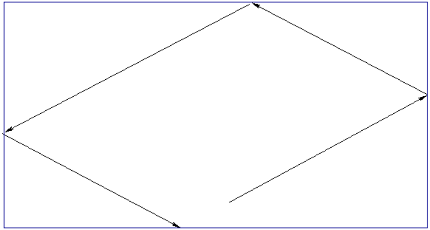

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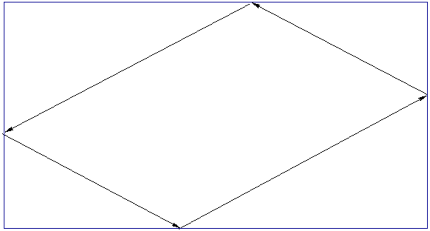

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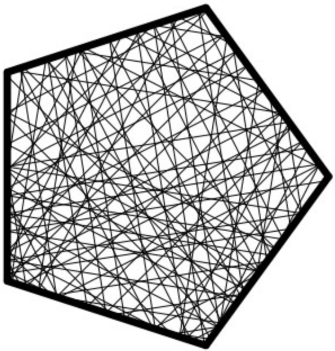

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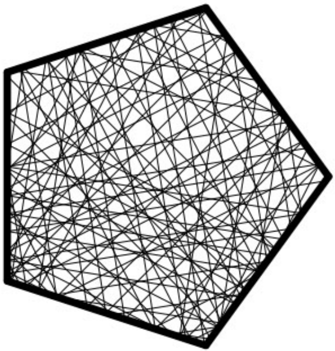

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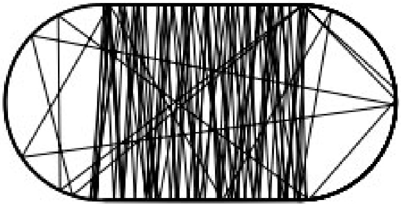
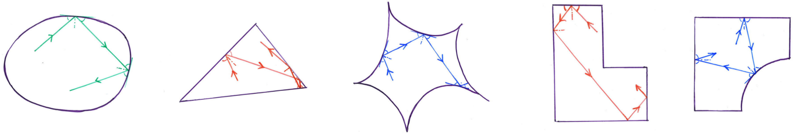


Shape of billiards and areas of dynamics



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(convex boundary)

Polygonal billiards
(flat boundary)

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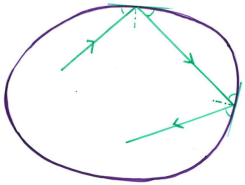

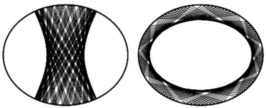

Hamiltonian Dynamics
variational methods
many periodic orbits

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(flat boundary)

rational billiards
(angles π-multiples):
Teichmüller Dynamics
very active area

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very chaotic

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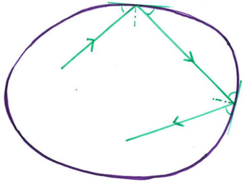

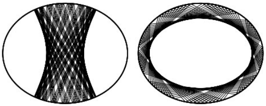

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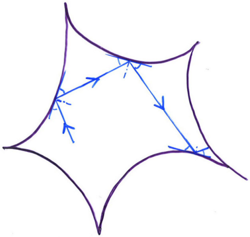

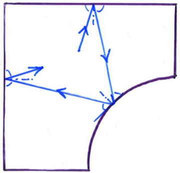

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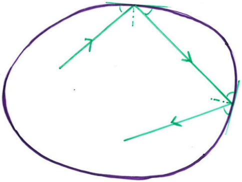

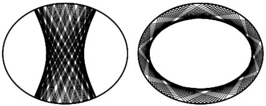

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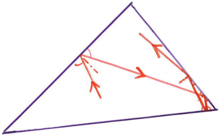

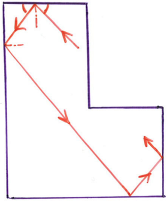

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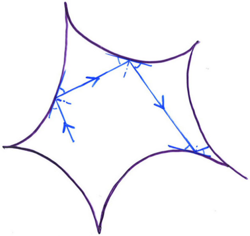

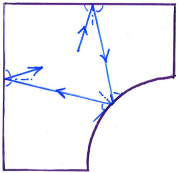

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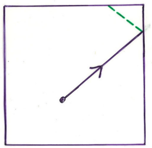
Unfolding a billiard trajectory: the square

From a billiard to a surface (Katok-Zemlyakov construction):



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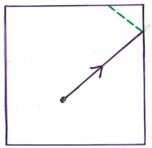

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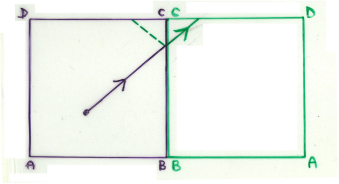
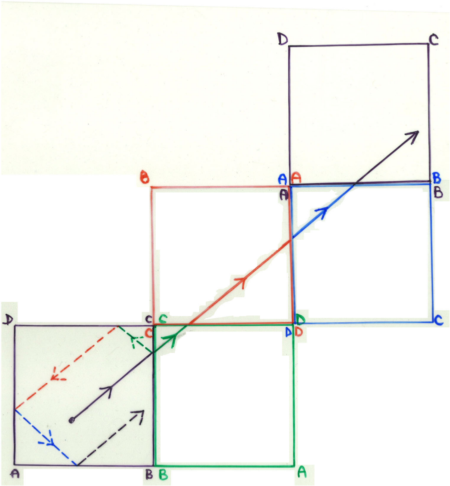


Unfolding a billiard trajectory: the square

From a billiard to a surface (Katok-Zemlyakov construction):



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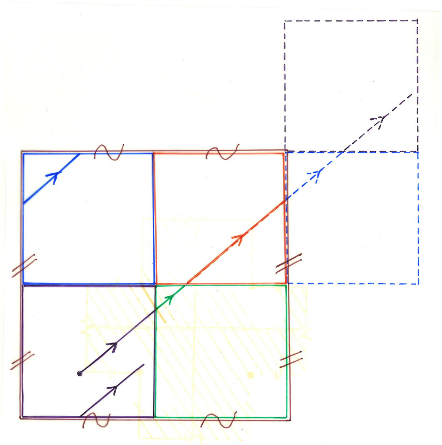

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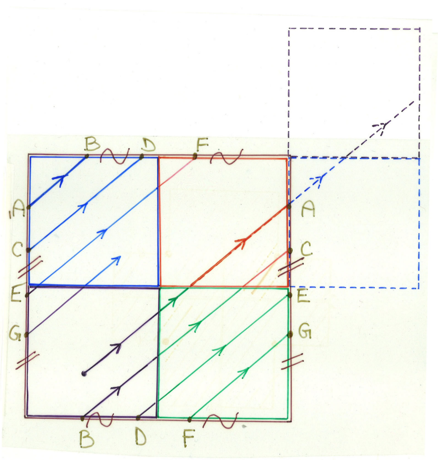

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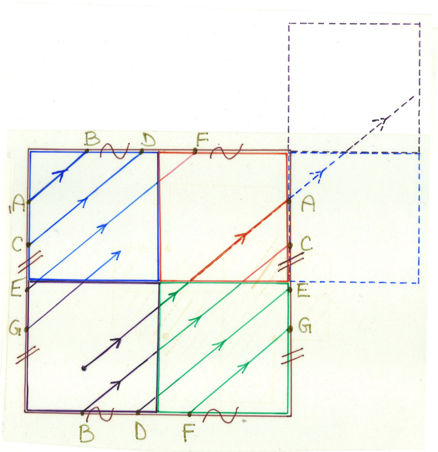

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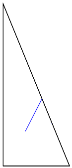

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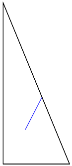

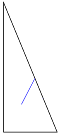

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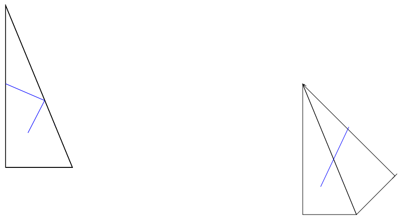

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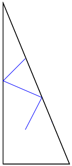

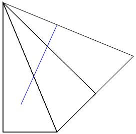

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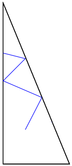

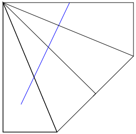

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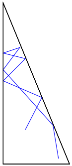

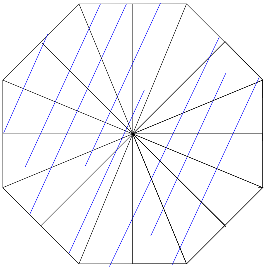

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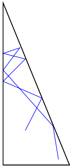

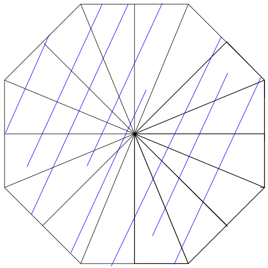

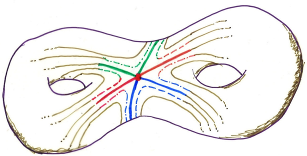

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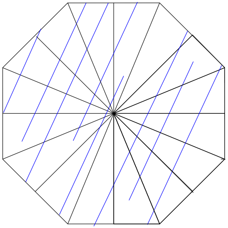
Translation surfaces

The surfaces one obtains by unfolding are known as *translation surfaces*:

Definition

A translation surface S is a closed dimension two manifold with a *locally Euclidean structure*, apart from finitely many points $\Sigma \subset S$, called singularities: each point outside Σ has a neighbourbood isomorphic to $\mathbb{R}^2$; changes of coordinates between neighbourhoods are translations; points in Σ are *conical singularities* of cone angle $2\pi k$, $k \in \mathbb{N}$.



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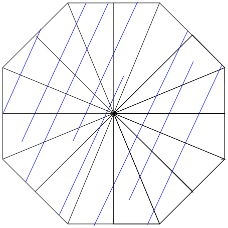

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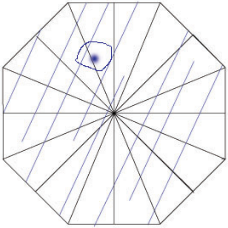

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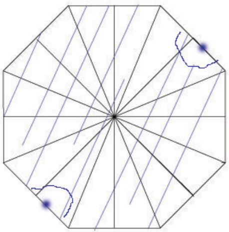

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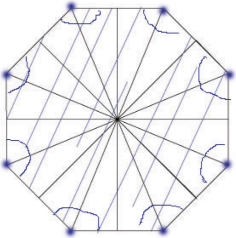

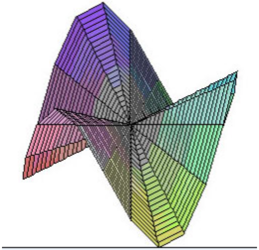

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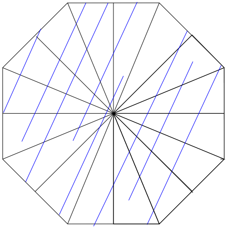

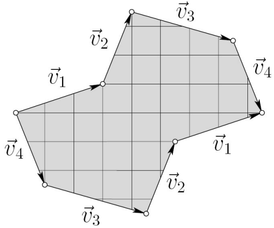

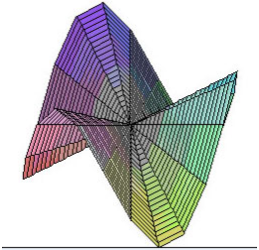

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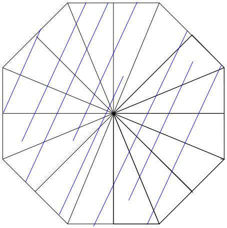

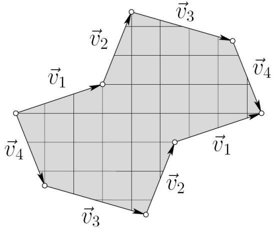

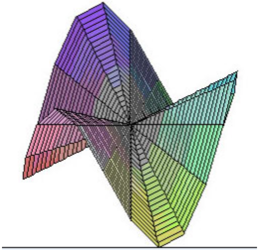

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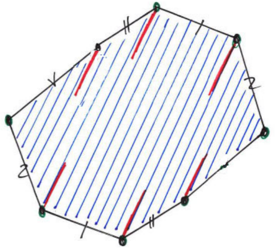
Linear flows on translation surfaces

On a translation surface, outside Σ, there is a well defined notion of *direction* and we can talk of *lines in direction* θ.



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The linear flow in direction φ_t^θ sends a point $p \in S$ to the point $\varphi_t^\theta(p)$ reached moving with unit speed along the line in direction θ.
Remark: φ_t^θ is *not* defined at singularity points: on a singularity there are many lines in direction θ.

- Directional flows are *area preserving*: for any measurable set $A \subset S$ $Area(\varphi_t^\theta(A)) = Area(A), \quad \forall t \in \mathbb{R}$.
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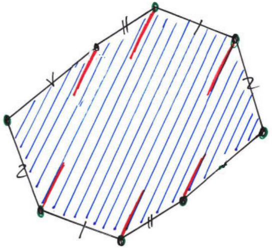

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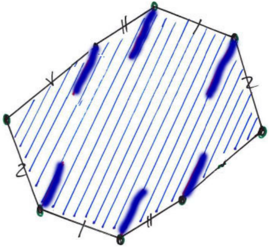

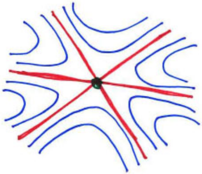

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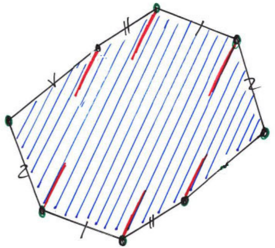

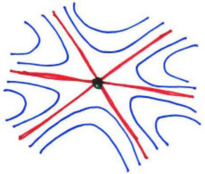

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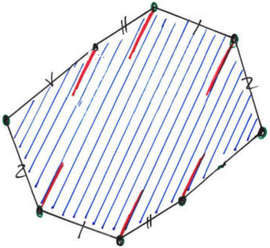

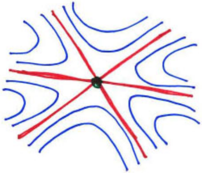

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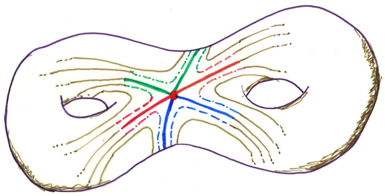

The *same trajectories* of φ_t^θ are also local solutions of *Hamiltonian equations*: for $U \subset S$ and $H : U \to \mathbb{R}$

$$\begin{cases} \frac{\partial x}{\partial t} = & \frac{\partial H}{\partial y} \\ \frac{\partial y}{\partial t} = & -\frac{\partial H}{\partial x} \end{cases}$$

solutions give a *locally Hamiltonian flow* h_t, $\{h_t(p)\}_{t\geq 0}$ trajectory of p

- More precisely: let ω be a smooth area form, the *locally Hamiltonian flow* h_t given by a closed smooth 1-form η is the flow generated by the vector field X determined by $i_X\omega = \eta$ (i_X contraction);
- *Initial motivation from solid state physics: locally Hamiltonian flows describe the motion of an electron under a magnetic filed on the Fermi energy level surface in the semi-classical limit* (Novikov).
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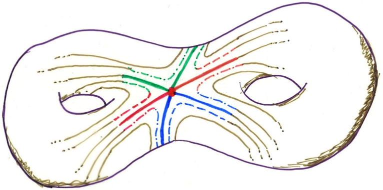

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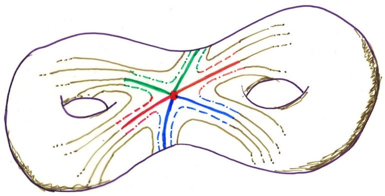

The *same trajectories* of φ_t^θ are also local solutions of *Hamiltonian equations*: for $U \subset S$ and $H : U \to \mathbb{R}$

$$\begin{cases} \frac{\partial x}{\partial t} = \frac{\partial H}{\partial y} \\ \frac{\partial y}{\partial t} = -\frac{\partial H}{\partial x} \end{cases}$$

solutions give a *locally Hamiltonian flow* h_t, $\{h_t(p)\}_{t\geq 0}$ trajectory of p

- More precisely: let ω be a smooth area form, the *locally Hamiltonian flow* h_t given by a closed smooth 1-form η is the flow generated by the vector field X determined by $i_X\omega = \eta$ (i_X contraction);
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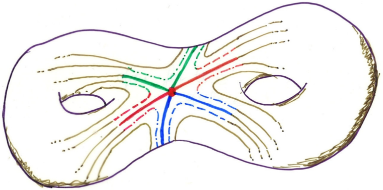

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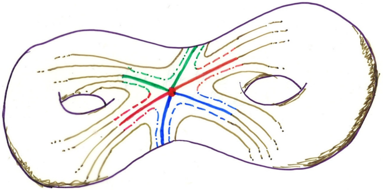

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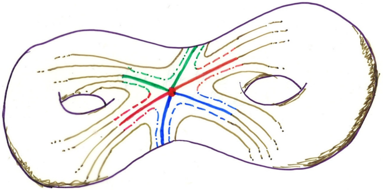

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Definition of Mixing

An important property studied in ergodic theory/dynamical systems is *mixing*. Assume $Area(S) = 1$. Let $A \subset S$ be a measurable set.

Flow points in A for time t: does $h_t(A)$ spreads uniformely?

Definition

The flow h_t is *mixing* if for any two measurable A, B on the surface,
$Area(h_t(A) \cap B) \xrightarrow{t \to \infty} Area(A)Area(B)$.

- *Question*: (by Arnold, 1980s): are locally Hamiltonian flows on surfaces *mixing*? Conjecture: yes on the torus (genus 1).

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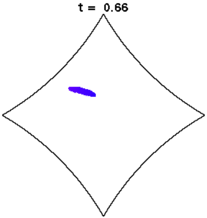

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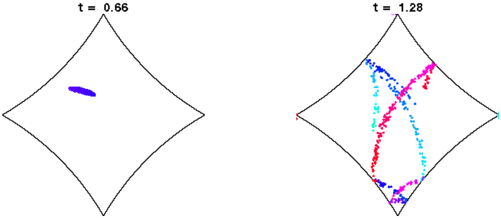

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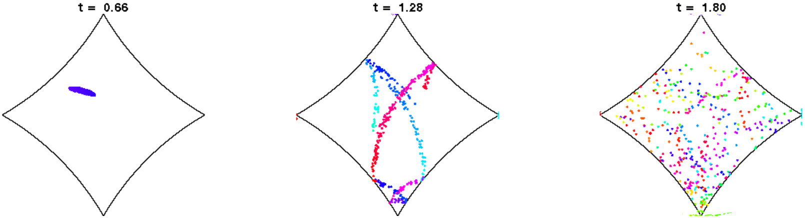

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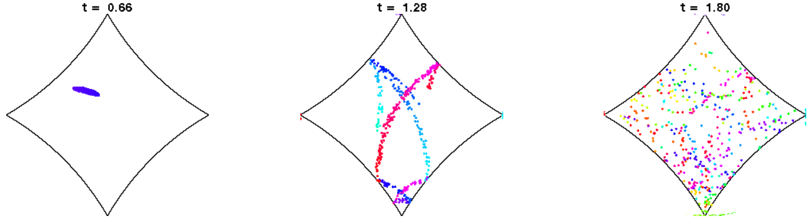

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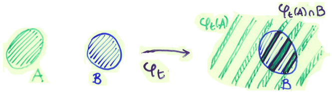

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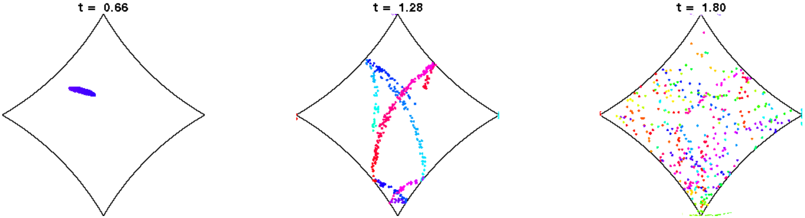

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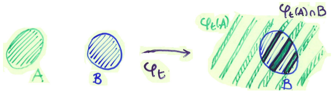

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A typical locally Hamiltonian flow on a surface that has only saddles (in particular, no traps) is not mixing.

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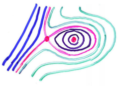

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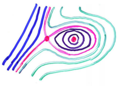

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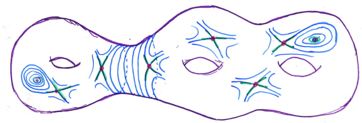

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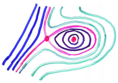

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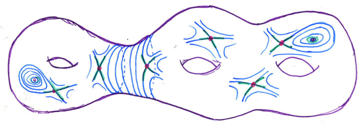

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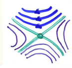

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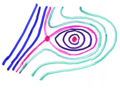

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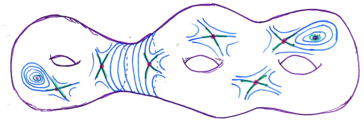

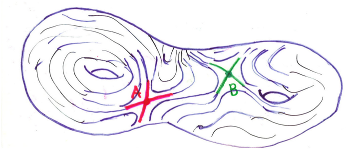

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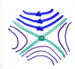

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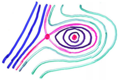

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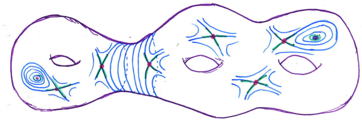

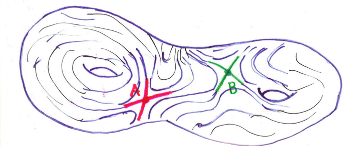

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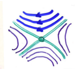

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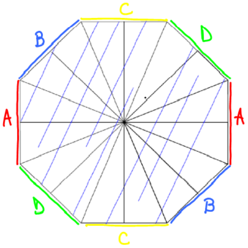
Symbolic coding of trajectories in the octagon

Consider a regular octagon (or more in general regular polygon with $2n$ sides).
Glue opposite sides.
Label pairs of sides by $\{A, B, C, D\}$.



Let φ_t^θ be the *linear flow* in direction θ: trajectories which do not hit singularities are straight lines in direction θ.

Definition (Cutting sequence)

The *cutting sequence* in $\{A, B, C, D\}^{\mathbb{Z}}$ that codes a bi-infinite linear trajectory of φ_t^θ consists of the sequence of labels of the sides hit by the trajectory.

E.g.: The cutting sequence of the example trajectory is:

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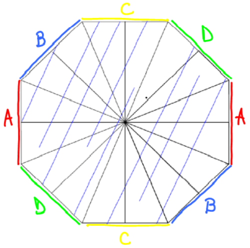

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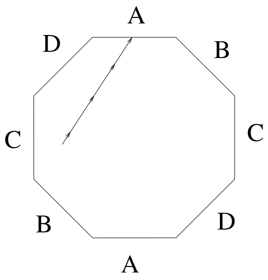

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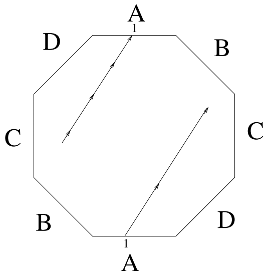

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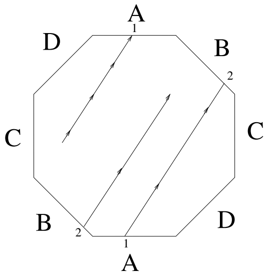

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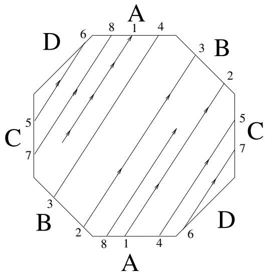

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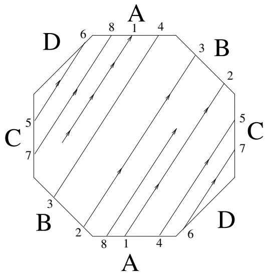

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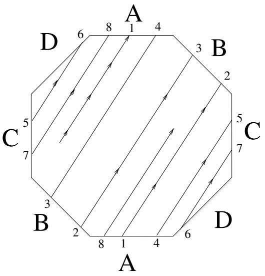

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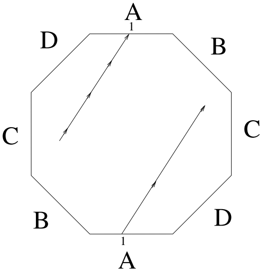

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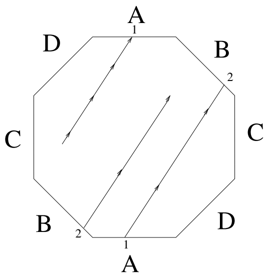
Which sequences in $\{A, B, C, D\}^{\mathbb{Z}}$ are cutting sequences?

Symbolic coding of trajectories in the octagon

Consider a regular octagon (or more in general regular polygon with $2n$ sides).
Glue opposite sides.
Label pairs of sides by $\{A, B, C, D\}$.

Let φ_t^θ be the *linear flow* in direction θ: trajectories which do not hit singularities are straight lines in direction θ.



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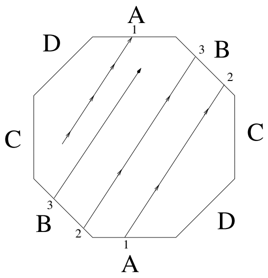

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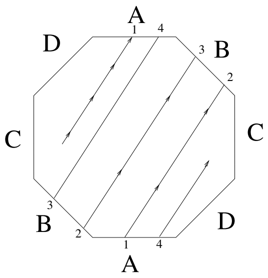

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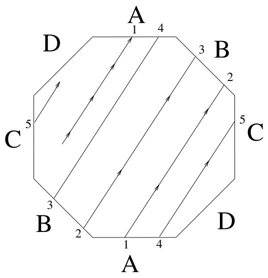

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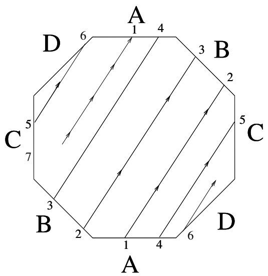

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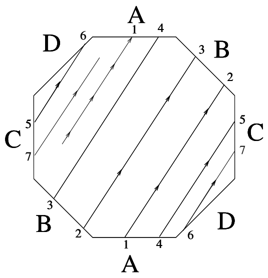

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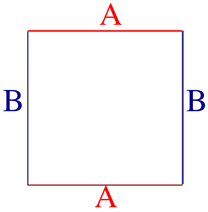
A classical case: Sturmian sequences

Consider the special case in which the polygon is a square.



In this case the cutting sequences are *Sturmian sequences*:

- Sturmian sequences correspond to the sequence of horizontal (letter A) and vertical (letter B) sides crossed by a line in direction θ in a *square grid*: ... A B A B B AB...

- Sturmian sequences are non-periodic sequences with the smallest possible *complexity* and are intimately related to continued fractions

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

Reference: Caroline Series, *Mathematical Intelligencer*

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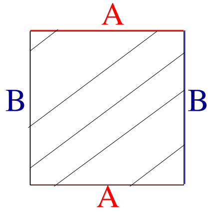

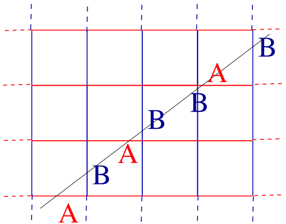

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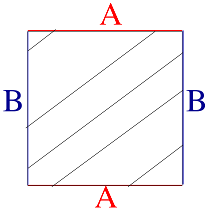

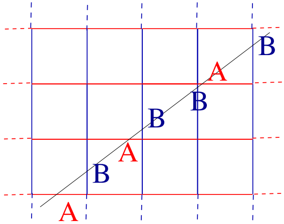

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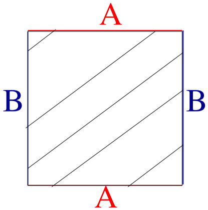

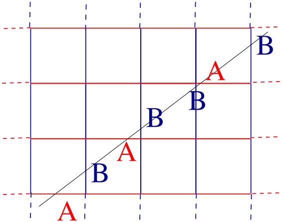

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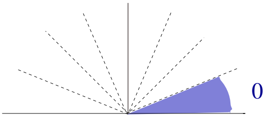
The octagon: allowed transitions in Σ_0

Let $\theta \in \Sigma_0 := \left[0, \frac{\pi}{8}\right]$.



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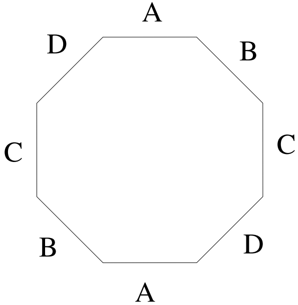

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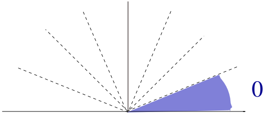

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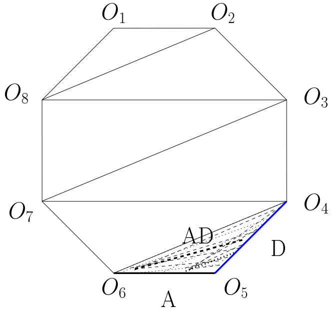

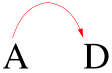

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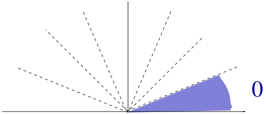

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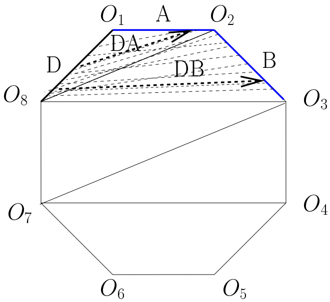

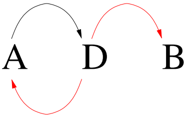

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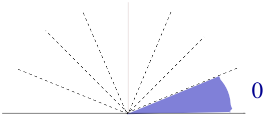

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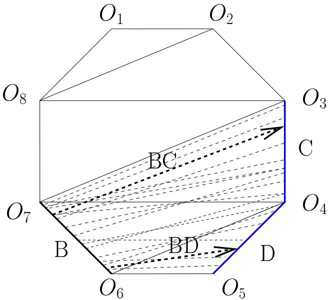

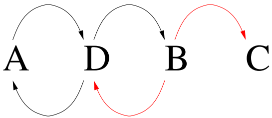

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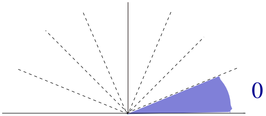

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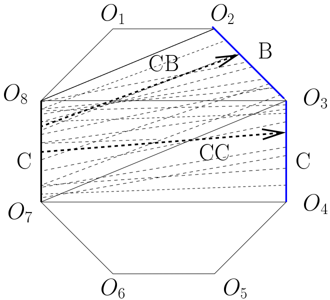

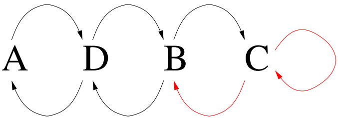

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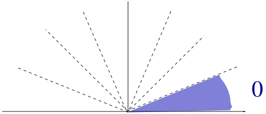

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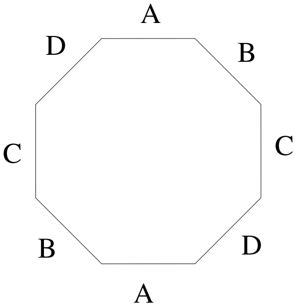

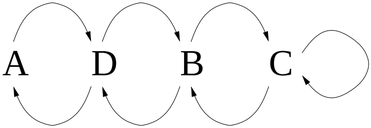

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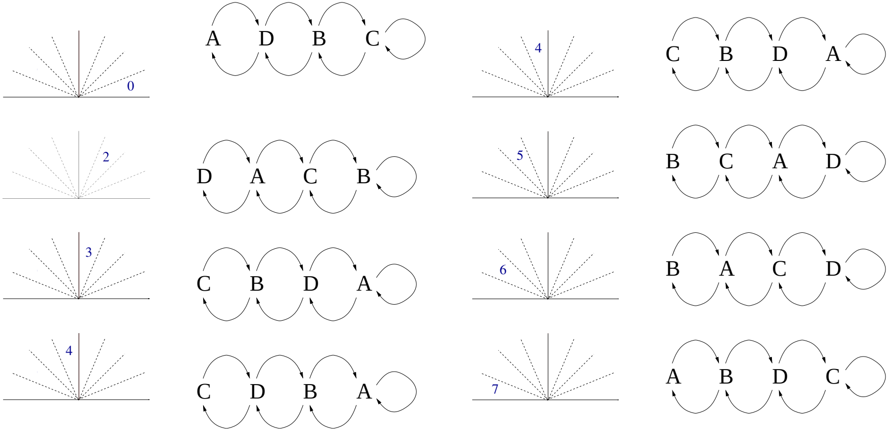
The octagon: possible transitions

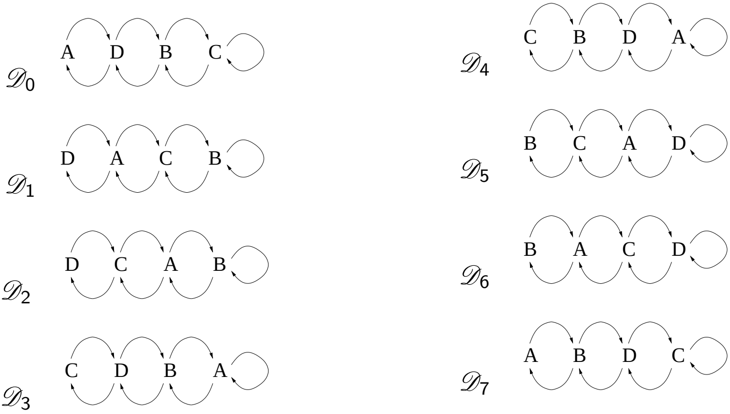
Permuting the letters we obtain the diagrams corresponding to the other sectors:



Admissible sequences

Definition

A sequence $w \in \{A, B, C, D\}^{\mathbb{Z}}$ is *admissible* if it gives an infinite path on one of the following diagrams:



Lemma

An octagon cutting sequence is admissible.

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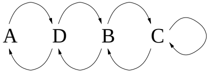

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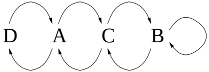

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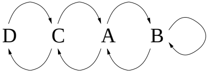

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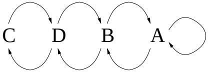
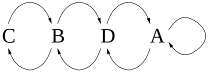
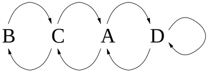


$\mathscr{D}_4$



$\mathscr{D}_5$



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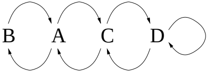
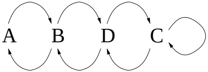


$\mathscr{D}_7$



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Derived sequences

Definition

A letter in $\{A, B, C, D\}$ is *sandwiched* if it is preceeded and followed by the same letter.

Example

In D B B C B A A D the letter C is *sandwitched* between to Bs.

Definition (Derived sequence)

If w is an octagon cutting sequence, the derived sequence w' is obtained erasing all letters which are *NOT sandwitched*.

Example

If w $= \dots$ D $\underline{\text{A}}$ D B C C B C C B D A D B C B D B D B C B D $\dots$,
w' $= \dots$ A

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w' = . . . A B A C D B D C

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A sequence $w \in \{A, B, C, D\}^{\mathbb{Z}}$ is *derivable* if it is admissible and its derivative is still admissible. The sequence w is *infinitely derivable* if each of its derivatives is derivable.

Derived sequences

Definition

A letter in $\{A, B, C, D\}$ is *sandwiched* if it is preceeded and followed by the same letter.

Example

In D B B C B A A D the letter C is *sandwitched* between to Bs.

Definition (Derived sequence)

If w is an octagon cutting sequence, the derived sequence w' is obtained erasing all letters which are *NOT sandwitched*.

Example

$$\begin{array}{lllllllllllllllllllllllllll}
\text{If w} & = \dots & D & \underline{A} & D & B & C & C & \underline{B} & C & C & B & D & \underline{A} & D & B & \underline{C} & B & \underline{D} & \underline{B} & \underline{D} & B & \underline{C} & B & D & \dots, \\
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A characterization of octagon cutting sequences

Theorem (Smillie-U '11)

An octagon cutting sequence is infinitely derivable.

Rk: This is still a only a sufficient condition, but becomes a necessary one by adding some information, i.e. the letters that *interpolate* two sandwiched letters.

The closure of cutting sequences of the octagon consists exactly of sequences obtained by any sequence of the following 7 interpolations rules:

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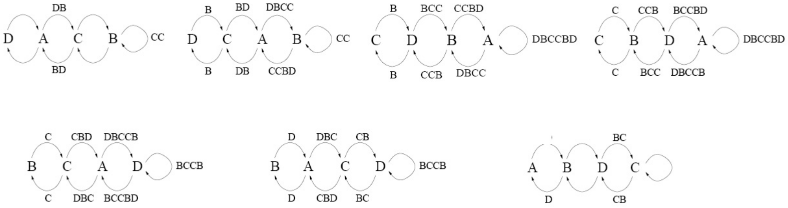

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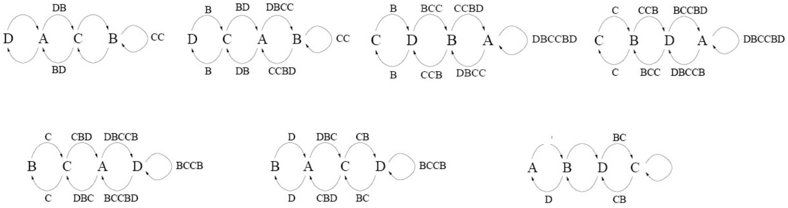

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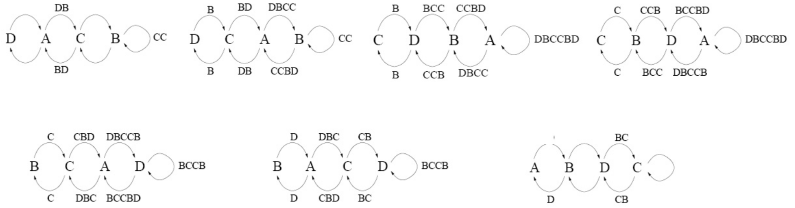

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Tools from Teichmueller dynamics

The proofs of these results (and many other dynamical results on translation surfaces) exploits tools from Teichmueller dynamics i.e.

- the *moduli space* of translation surfaces $\mathscr{M}_g$, i.e. the space of *all* translation surfaces of genus g;
  $\mathscr{M}_g$ is a non-compact orbifold of dimenstion $4g + 2k - 2$ (k number of conical singularities) with the Teichmueller distance.
- The geodesic flow g_t on $\mathscr{M}_g$ with respect to this metric is called the *Teichmueller geodesic flow*. Geometrically, g_t acts on a translation surface by contracting the flow direction by e^{-t} and stretching the orthogonal direction by e^t (modulo equivalence of translation surfaces).

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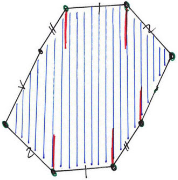

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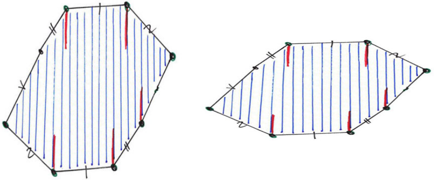

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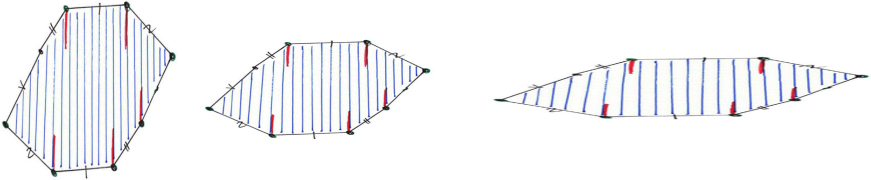

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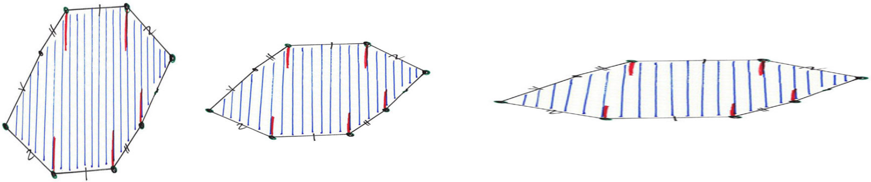

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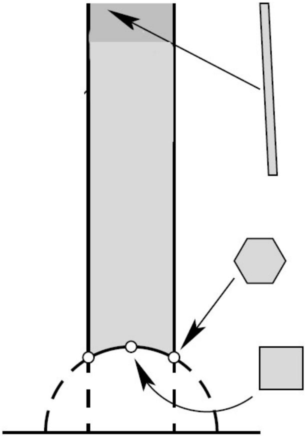
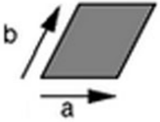
An example: moduli space of flat tori

For $g = 1$: $\mathscr{M}_1$ be the space of *all* translation surfaces on a torus (*moduli space of flat tori*) is given by the unit tangent bundle to the *modular surface* $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$:



Idea of proof:
normalize shortest flat geodesic to be of lenght 1;
the second shortest geodesic gives a vector on $\mathbb{H}$ upper half plane, which has to lie in the fundamental domain in Figure.



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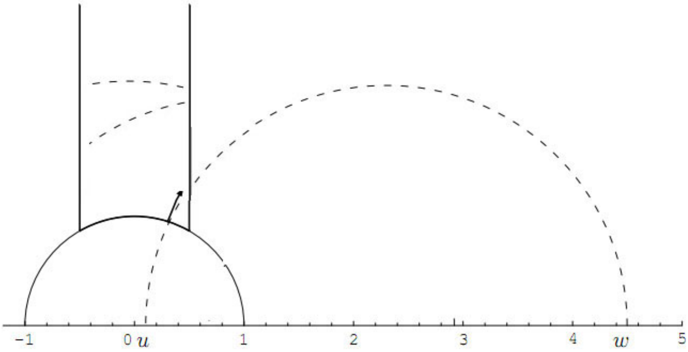

Moreover, when $g = 1$, the *Teichmueller flow* g_t coincides with the geodesic flow on $\mathscr{M}_1$.

Role of Teichmueller geodesic flow in the proofs

Guiding Philosophy:

good dynamical properties of the linear flow φ_t^θ on a given translation surface S (which is a point in $\mathscr{M}_g$)	correspond to	properties of the Teichmueller geodesic through S (deformations of S via the flow g_t) in the space $\mathscr{M}_g$

Examples:

- trajectories of a linear flow are dense and uniformely distributed if the corresponding Teichmueller geodesic returns infinitely often to a compact set (*Masur's criterium*);
- a locally Hamiltonian flows is mixing if the corresponding Teichmueller geodesic has good recurrence properties (visits with controlled frequencies certain special subsets in $\mathscr{M}_g$);
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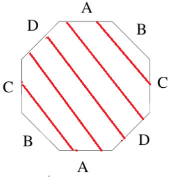

This periodic trajectory has cutting sequence $\cdots$ AC A D D A C A $\cdots$
Apply the *Teichmueller geodesic flow*. Up to equivalence of translation surfaces (*cut and paste by parallel translation*) this surface is the same than to the original regular octagon surface.
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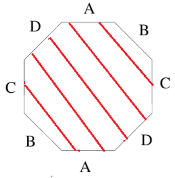

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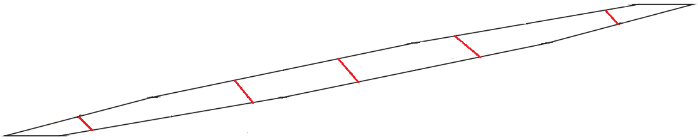

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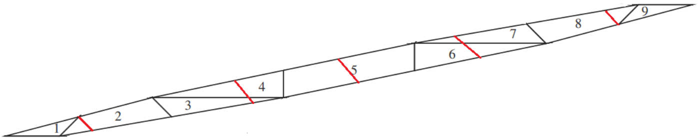

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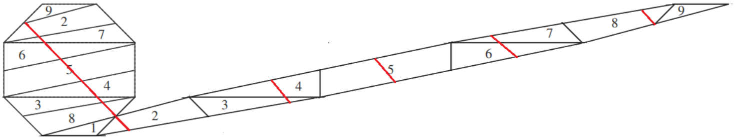

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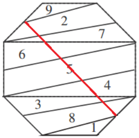

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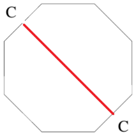

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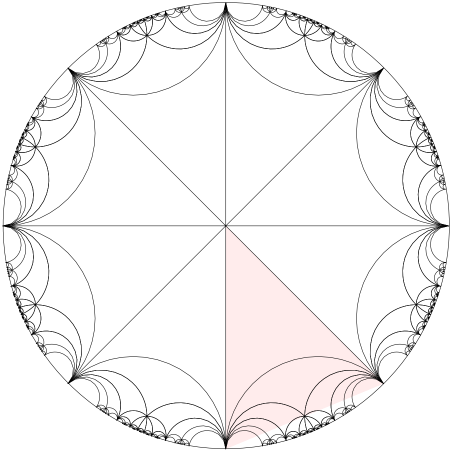
Renormalization for the regular octagon

The moduli space for the regular octagon is a subset of the moduli space $\mathscr{M}_2$, given by $SL(2,\mathbb{R})\backslash\Gamma(O)$, where $\Gamma(O)$ is a Fuchsian group.

The Teichmueller flow g_t coincides with the hyperbolic geodesic flow.



the dots are the orbit $\Gamma(O) \cdot O$, which are stretched octagons which can be cut and pasted to the regular octagon O (i.e. give the same translation surface)

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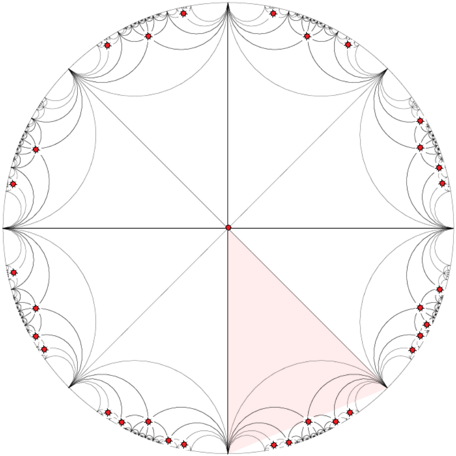

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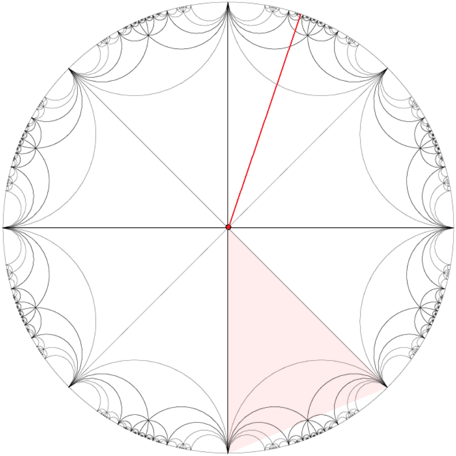

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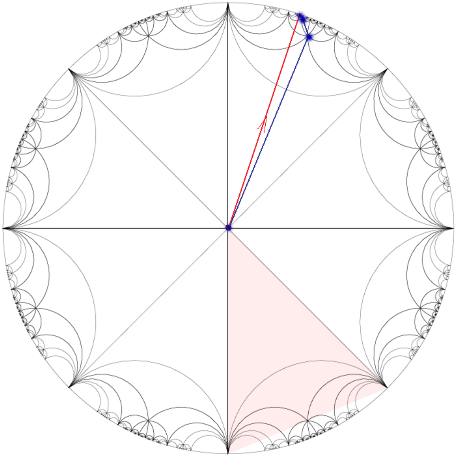

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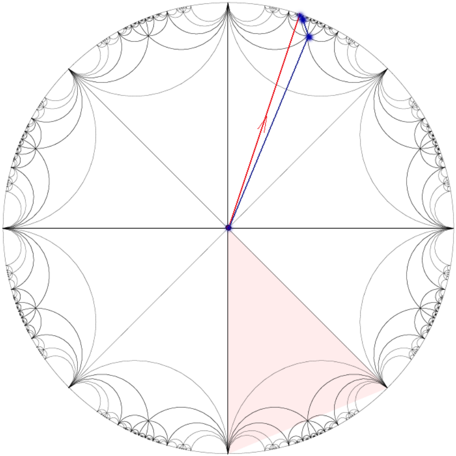

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Teichmueller dynamics

Teichmueller dynamics, in particular the study of the geometry of the moduli space of translation surfaces and the dynamics of the Teichmueller geodesic flow, have connections with many other areas of research:

- *algebraic geometry* (moduli spaces of Riemann surfaces);
- *hyperbolic dynamics* (the Teichmueller geodesic flow is a flow on a non-negatively curved space and shares many similarities with other hyperbolic flows, for example with the geodesic flow on a surface of negative curvature);
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Teichmueller dynamics allowed to prove many dynamical properties of billiards in rational polygons. What about billiards in *irrational* polygons (i.e. polygons with angles that are irrational multiples of π)?

Teichmueller dynamics

Teichmueller dynamics, in particular the study of the geometry of the moduli space of translation surfaces and the dynamics of the Teichmueller geodesic flow, have connections with many other areas of research:

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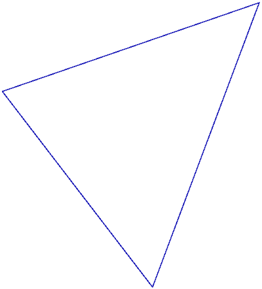
Irrational billiards: a "simple" open question

Back to billiards: Teichmueller dynamics helped answering many dynamical questions about billiards with angles rational multiples of π.
If the angles are *irrational* to π, there is no known "'*renormalization*"' and many basic open questions are open, e.g.



Take an acute triangle,
there is a periodic trajectory
(the Fagnano trajectory)

Take an obtuse triangle:
is there a periodic trajectory?
99% YES, computer assisted proof
(*McBilliard*, by R. Shwartz)

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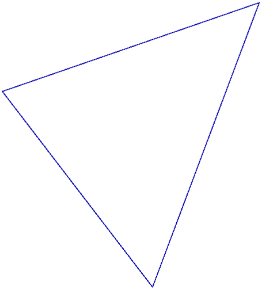

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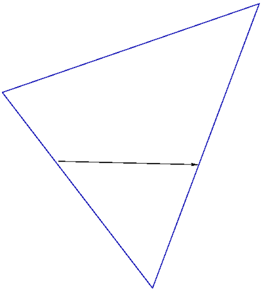

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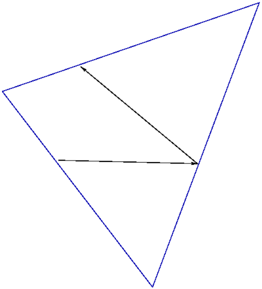

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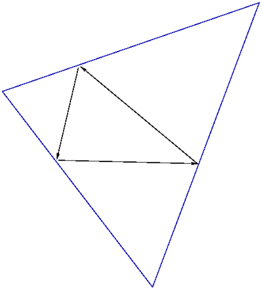

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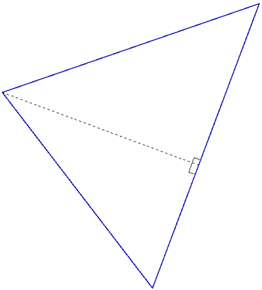

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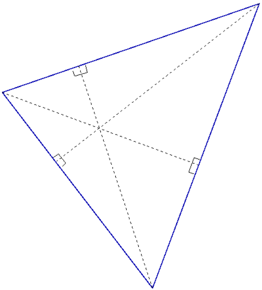

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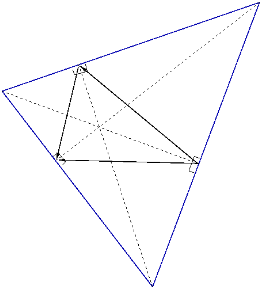

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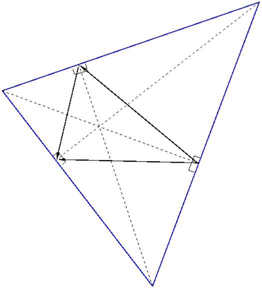

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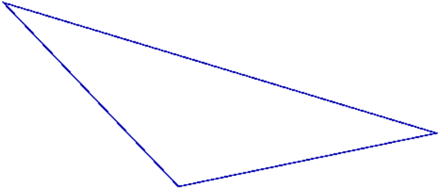

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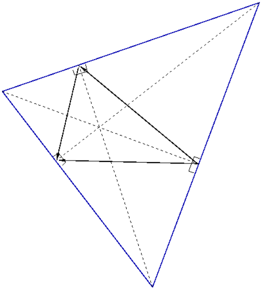

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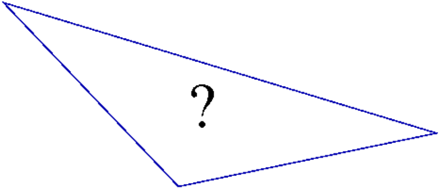

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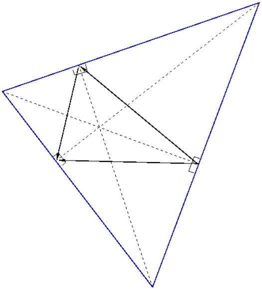

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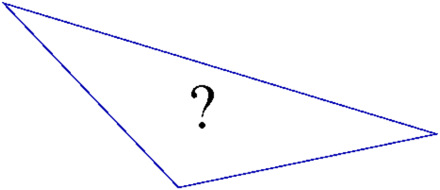

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