Shearing and Mixing in Parabolic Flows

Corinna Ulcigrai
University of Bristol

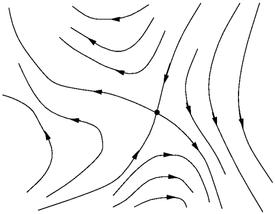
Dynamical systems

Dynamical systems: study of the long-term evolution in chaotic systems.

Let $\phi_t : X \to X$ be a flow, i.e.
a $1-$parameter family of transformations.

E.g. Solutions to a differential equation.

The *trajectory* $\{\phi_t(x), t \geq 0\}$
of a point $x \in X$ is called *orbit*.



Dynamical systems can be roughly diveded into:

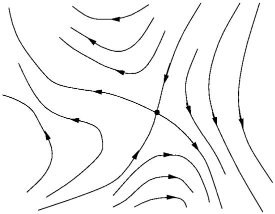
- *Hyperbolic dynamical systems*: nearby orbits diverge exponentially
- *Parabolic dynamical systems*: nearby orbits diverge polynomially
- *Elliptic dynamical systems*: no divergence (or perhaps slower than polynomial)

Dynamical systems

Dynamical systems: study of the long-term evolution in chaotic systems.

Let $\phi_t : X \to X$ be a flow, i.e.
a $1-$parameter family of transformations.

E.g. Solutions to a differential equation.



The *trajectory* $\{\phi_t(x), t \geq 0\}$
of a point $x \in X$ is called *orbit*.

Dynamical systems can be roughly diveded into:

- *Hyperbolic dynamical systems*: nearby orbits diverge exponentially
- *Parabolic dynamical systems*: nearby orbits diverge polynomially
- *Elliptic dynamical systems*: no divergence (or perhaps slower than polynomial)

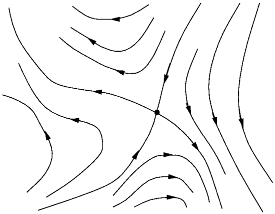
Dynamical systems

Dynamical systems: study of the long-term evolution in chaotic systems.

Let $\phi_t : X \to X$ be a flow, i.e.
a $1-$parameter family of transformations.

E.g. Solutions to a differential equation.

The *trajectory* $\{\phi_t(x), t \geq 0\}$
of a point $x \in X$ is called *orbit*.



Dynamical systems can be roughly diveded into:

- *Hyperbolic dynamical systems*: nearby orbits diverge exponentially
- *Parabolic dynamical systems*: nearby orbits diverge polynomially
- *Elliptic dynamical systems*: no divergence (or perhaps slower than polynomial)

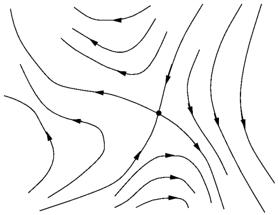
Dynamical systems

Dynamical systems: study of the long-term evolution in chaotic systems.

Let $\phi_t : X \to X$ be a flow, i.e.
a $1-$parameter family of transformations.

E.g. Solutions to a differential equation.

The *trajectory* $\{\phi_t(x), t \geq 0\}$
of a point $x \in X$ is called *orbit*.



Dynamical systems can be roughly diveded into:

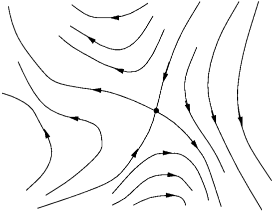
- *Hyperbolic dynamical systems*: nearby orbits diverge exponentially
- *Parabolic dynamical systems*: nearby orbits diverge polynomially
- *Elliptic dynamical systems*: no divergence (or perhaps slower than polynomial)

Dynamical systems

Dynamical systems: study of the long-term evolution in chaotic systems.

Let $\phi_t : X \to X$ be a flow, i.e.
a $1-$parameter family of transformations.
E.g. Solutions to a differential equation.

The *trajectory* $\{\phi_t(x), t \geq 0\}$
of a point $x \in X$ is called *orbit*.



Dynamical systems can be roughly diveded into:

- *Hyperbolic dynamical systems*: nearby orbits diverge exponentially
- *Parabolic dynamical systems*: nearby orbits diverge polynomially
- *Elliptic dynamical systems*: no divergence (or perhaps slower than polynomial)

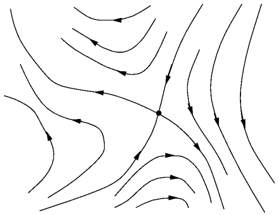
Dynamical systems

Dynamical systems: study of the long-term evolution in chaotic systems.

Let $\phi_t : X \to X$ be a flow, i.e.
a $1-$parameter family of transformations.

E.g. Solutions to a differential equation.

The *trajectory* $\{\phi_t(x), t \geq 0\}$
of a point $x \in X$ is called *orbit*.



Dynamical systems can be roughly diveded into:

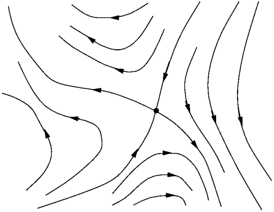
- *Hyperbolic dynamical systems*: nearby orbits diverge exponentially
- *Parabolic dynamical systems*: nearby orbits diverge polynomially
- *Elliptic dynamical systems*: no divergence (or perhaps slower than polynomial)

Dynamical systems

Dynamical systems: study of the long-term evolution in chaotic systems.

Let $\phi_t : X \to X$ be a flow, i.e.
a $1-$parameter family of transformations.
E.g. Solutions to a differential equation.

The *trajectory* $\{\phi_t(x), t \geq 0\}$
of a point $x \in X$ is called *orbit*.



Dynamical systems can be roughly diveded into:

- *Hyperbolic dynamical systems*: nearby orbits diverge exponentially
- *Parabolic dynamical systems*: nearby orbits diverge polynomially
- *Elliptic dynamical systems*: no divergence (or perhaps slower than polynomial)

Parabolic flows

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);

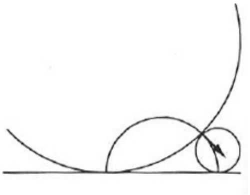
We will be intererested in chaotic properties of parabolic flows. While classical examples as the horocycle flows and nilflows are well understood, not much is known for general smooth parabolic flows, not even for smooth perturbations of these standard examples.

We will focus in particular on the property of mixing and the mechanism of shearing which produces it.

Parabolic flows

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);



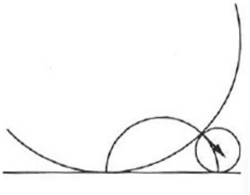
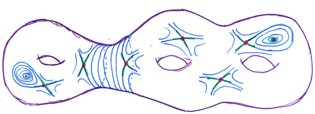
We will be intererested in chaotic properties of parabolic flows. While classical examples as the horocycle flows and nilflows are well understood, not much is known for general smooth parabolic flows, not even for smooth perturbations of these standard examples.

We will focus in particular on the property of mixing and the mechanism of shearing which produces it.

Parabolic flows

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);

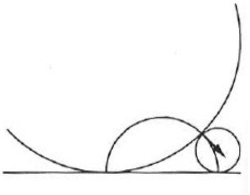
We will be intererested in chaotic properties of parabolic flows.
While classical examples as the horocycle flows and nilflows are well understood, not much is known for general smooth parabolic flows, not even for smooth perturbations of these standard examples.

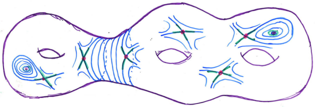
We will focus in particular on the property of mixing and the mechanism of shearing which produces it.

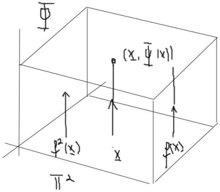
Parabolic flows

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);







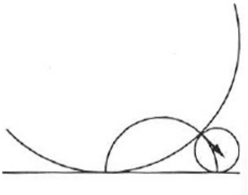
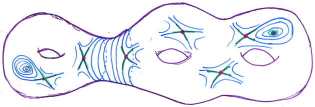
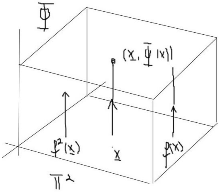
We will be intererested in chaotic properties of parabolic flows. While classical examples as the horocycle flows and nilflows are well understood, not much is known for general smooth parabolic flows, not even for smooth perturbations of these standard examples.

We will focus in particular on the property of mixing and the mechanism of shearing which produces it.

Parabolic flows

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);

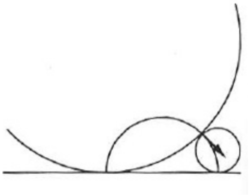
We will be intererested in chaotic properties of parabolic flows.
While classical examples as the horocycle flows and nilflows are well understood, not much is known for general smooth parabolic flows, not even for smooth perturbations of these standard examples.

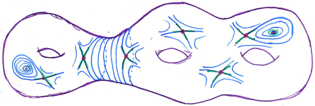
We will focus in particular on the property of mixing and the mechanism of shearing which produces it.

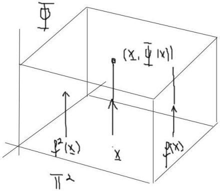
Parabolic flows

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);







We will be intererested in chaotic properties of parabolic flows.
While classical examples as the horocycle flows and nilflows are well understood, not much is known for general smooth parabolic flows, not even for smooth perturbations of these standard examples.

We will focus in particular on the property of mixing and the mechanism of shearing which produces it.

Definition of Mixing

Let φ_t be a flow preserving a probability measure μ (i.e. $\mu(\varphi_t A) = \mu(A)$ for each $t \in \mathbb{R}$ and A measurable). Let A be a measurable set. Flow points in A for time t: does $\varphi_t(A)$ spreads uniformely?

Definition

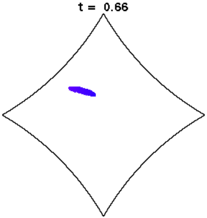
φ_t is *mixing* if for any A, B measurable $\mu(\varphi_t(A) \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B)$.

Equivalently, for any $f, g \in L^2(X, \mu)$, f of zero-average, i.e. $\int f \, \mathrm{d}\mu = 0$,

$$< f \circ \varphi_t, g >_{L^2(M,\mu)} = \int_M f(\varphi_t x)\overline{g(x)} \, \mathrm{d}\mu(x) \xrightarrow{t\to\infty} 0.$$

Definition of Mixing

Let φ_t be a flow preserving a probability measure μ (i.e. $\mu(\varphi_t A) = \mu(A)$ for each $t \in \mathbb{R}$ and A measurable). Let A be a measurable set. Flow points in A for time t: does $\varphi_t(A)$ spreads uniformely?



Definition

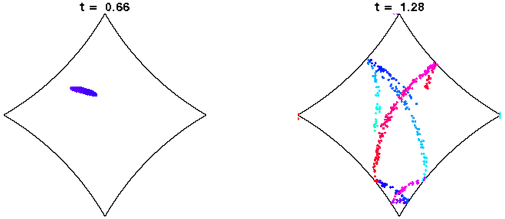
φ_t is *mixing* if for any A, B measurable
$\mu(\varphi_t(A) \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B)$.

Equivalently, for any $f, g \in L^2(X, \mu)$, f of zero-average, i.e. $\int f \, \mathrm{d}\mu = 0$,

$$< f \circ \varphi_t, g >_{L^2(M,\mu)} = \int_M f(\varphi_t x)\overline{g(x)} \, \mathrm{d}\mu(x) \xrightarrow{t\to\infty} 0.$$

Definition of Mixing

Let φ_t be a flow preserving a probability measure μ (i.e. $\mu(\varphi_t A) = \mu(A)$ for each $t \in \mathbb{R}$ and A measurable). Let A be a measurable set. Flow points in A for time t: does $\varphi_t(A)$ spreads uniformely?



Definition

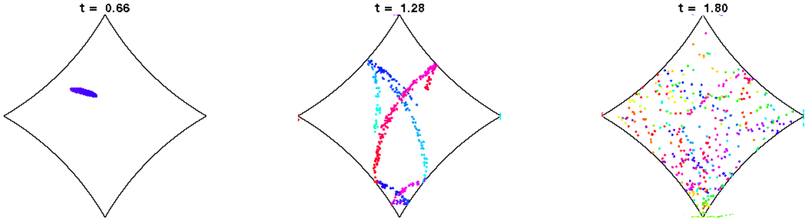
φ_t is *mixing* if for any A, B measurable
$\mu(\varphi_t(A) \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B)$.

Equivalently, for any $f, g \in L^2(X, \mu)$, f of zero-average, i.e. $\int f \,\mathrm{d}\mu = 0$,

$$< f \circ \varphi_t, g >_{L^2(M,\mu)} = \int_M f(\varphi_t x)\overline{g(x)}\,\mathrm{d}\mu(x) \xrightarrow{t\to\infty} 0.$$

Definition of Mixing

Let φ_t be a flow preserving a probability measure μ (i.e. $\mu(\varphi_t A) = \mu(A)$ for each $t \in \mathbb{R}$ and A measurable). Let A be a measurable set. Flow points in A for time t: does $\varphi_t(A)$ spreads uniformely?



Definition

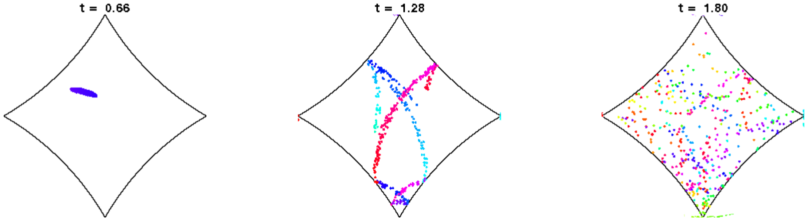
φ_t is *mixing* if for any A, B measurable
$\mu(\varphi_t(A) \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B)$.

Equivalently, for any $f, g \in L^2(X, \mu)$, f of zero-average, i.e. $\int f \, \mathrm{d}\mu = 0$,

$$< f \circ \varphi_t, g >_{L^2(M,\mu)} = \int_M f(\varphi_t x)\overline{g(x)} \, \mathrm{d}\mu(x) \xrightarrow{t\to\infty} 0.$$

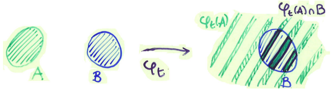
Definition of Mixing

Let φ_t be a flow preserving a probability measure μ (i.e. $\mu(\varphi_t A) = \mu(A)$ for each $t \in \mathbb{R}$ and A measurable). Let A be a measurable set. Flow points in A for time t: does $\varphi_t(A)$ spreads uniformely?



Definition

φ_t is *mixing* if for any A, B measurable
$\mu(\varphi_t(A) \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B)$.

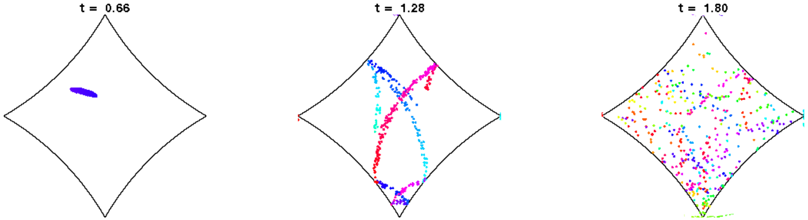


Equivalently, for any $f, g \in L^2(X, \mu)$, f of zero-average, i.e. $\int f \,\mathrm{d}\mu = 0$,

$$< f \circ \varphi_t, g >_{L^2(M,\mu)} = \int_M f(\varphi_t x)\overline{g(x)}\,\mathrm{d}\mu(x) \xrightarrow{t\to\infty} 0.$$

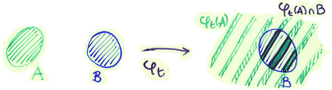
Definition of Mixing

Let φ_t be a flow preserving a probability measure μ (i.e. $\mu(\varphi_t A) = \mu(A)$ for each $t \in \mathbb{R}$ and A measurable). Let A be a measurable set. Flow points in A for time t: does $\varphi_t(A)$ spreads uniformely?



Definition

φ_t is *mixing* if for any A, B measurable
$\mu(\varphi_t(A) \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B)$.



Equivalently, for any $f, g \in L^2(X,\mu)$, f of zero-average, i.e. $\int f \, \mathrm{d}\mu = 0$,

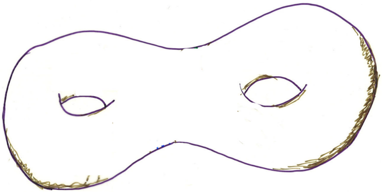
$$< f \circ \varphi_t, g >_{L^2(M,\mu)} = \int_M f(\varphi_t x)\overline{g(x)} \, \mathrm{d}\mu(x) \xrightarrow{t\to\infty} 0.$$

Horocycle Flow

A fundamental example of parabolic flow is the classical horocycle flow on a compact hyperbolic surface S.

The universal cover of S is the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \text{ s.t. } Im(z) > 0\}$ with the hyperbolic metric $(\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$.

- Let g_t be the geodesic flow: geometrically, geodesics are half circles.



- Orbits of the horocycle flow h_t on $\mathbb{H}$ are shown above.

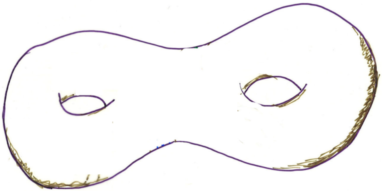
The geodesic and horocycle flow on $\mathbb{H}$ descend to flows on the surface S: g_t is a hyperbolic flow, h_t is a parabolic flow.

Horocycle Flow

A fundamental example of parabolic flow is the classical horocycle flow on a compact hyperbolic surface S.

The universal cover of S is the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \text{ s.t. } Im(z) > 0\}$ with the hyperbolic metric $(\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$.

- Let g_t be the geodesic flow: geometrically, geodesics are half circles.



- Orbits of the horocycle flow h_t on $\mathbb{H}$ are shown above.

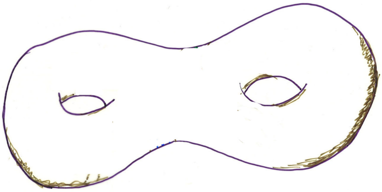
The geodesic and horocycle flow on $\mathbb{H}$ descend to flows on the surface S: g_t is a hyperbolic flow, h_t is a parabolic flow.

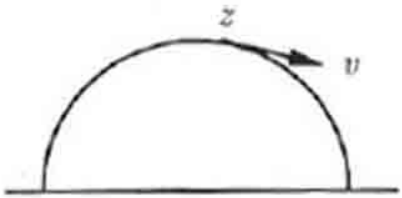
Horocycle Flow

A fundamental example of parabolic flow is the classical horocycle flow on a compact hyperbolic surface S.

The universal cover of S is the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \text{ s.t. } Im(z) > 0\}$ with the hyperbolic metric $(\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$.

- Let g_t be the geodesic flow: geometrically, geodesics are half circles.





- Orbits of the horocycle flow h_t on $\mathbb{H}$ are shown above.

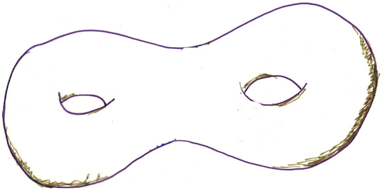
The geodesic and horocycle flow on $\mathbb{H}$ descend to flows on the surface S: g_t is a hyperbolic flow, h_t is a parabolic flow.

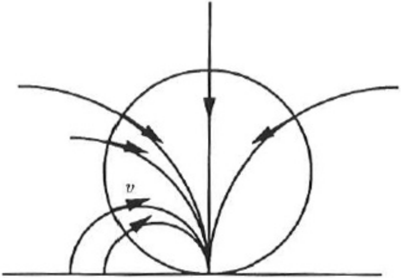
Horocycle Flow

A fundamental example of parabolic flow is the classical horocycle flow on a compact hyperbolic surface S.

The universal cover of S is the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \text{ s.t. } Im(z) > 0\}$ with the hyperbolic metric $(\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$.

- Let g_t be the geodesic flow: geometrically, geodesics are half circles.





- Orbits of the horocycle flow h_t on $\mathbb{H}$ are shown above.

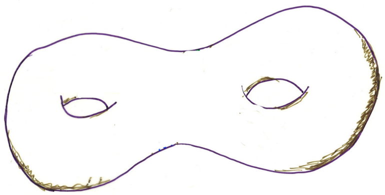
The geodesic and horocycle flow on $\mathbb{H}$ descend to flows on the surface S: g_t is a hyperbolic flow, h_t is a parabolic flow.

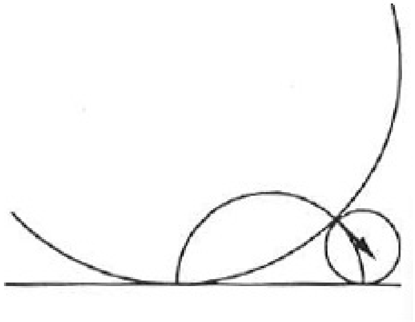
Horocycle Flow

A fundamental example of parabolic flow is the classical horocycle flow on a compact hyperbolic surface S.

The universal cover of S is the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \text{ s.t. } Im(z) > 0\}$ with the hyperbolic metric $(\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$.

- Let g_t be the geodesic flow: geometrically, geodesics are half circles.





- Orbits of the horocycle flow h_t on $\mathbb{H}$ are shown above.

The geodesic and horocycle flow on $\mathbb{H}$ descend to flows on the surface S: g_t is a hyperbolic flow, h_t is a parabolic flow.

Dynamical properties of classical horocycle flows

The dynamical properties of the horocycle flow h_t on a compact (and not only) surface have been studied in great detail.
It is known for example that h_t is:

- minimal, i.e. all orbits are dense (Hedlund, 1936);
- uniquely ergodic (Fursternberg, 1972); in particular, for any smooth observable $f : M \to \mathbb{R}$, one has uniform convergence (in x) of

$$\frac{1}{T}\int_0^T f(h_t x))\,\mathrm{d}t \xrightarrow{T\to\infty} \int_M f\,\mathrm{d}\mu;$$

- is mixing and has countable Lebesgue spectrum (Parasyuk, 1953);
- the speed of mixing is polynomial (Ratner, 1987);
- has the speed of convergence of ergodic integrals is polynomial (Flaminio-Forni, 2003);

Not much is known for smooth perturbations. Even the dynamics of non-trivial smooth time-changes is poorly understood.

Dynamical properties of classical horocycle flows

The dynamical properties of the horocycle flow h_t on a compact (and not only) surface have been studied in great detail.
It is known for example that h_t is:

- minimal, i.e. all orbits are dense (Hedlund, 1936);
- uniquely ergodic (Fursternberg, 1972); in particular, for any smooth observable $f : M \to \mathbb{R}$, one has uniform convergence (in x) of

$$\frac{1}{T}\int_0^T f(h_t x))\,\mathrm{d}t \xrightarrow{T\to\infty} \int_M f\,\mathrm{d}\mu;$$

- is mixing and has countable Lebesgue spectrum (Parasyuk, 1953);
- the speed of mixing is polynomial (Ratner, 1987);
- has the speed of convergence of ergodic integrals is polynomial (Flaminio-Forni, 2003);

Not much is known for smooth perturbations. Even the dynamics of non-trivial smooth time-changes is poorly understood.

Dynamical properties of classical horocycle flows

The dynamical properties of the horocycle flow h_t on a compact (and not only) surface have been studied in great detail.
It is known for example that h_t is:

- minimal, i.e. all orbits are dense (Hedlund, 1936);
- uniquely ergodic (Fursternberg, 1972); in particular, for any smooth observable $f : M \to \mathbb{R}$, one has uniform convergence (in x) of

$$\frac{1}{T}\int_0^T f(h_t x))\,\mathrm{d}t \xrightarrow{T\to\infty} \int_M f\,\mathrm{d}\mu;$$

- is mixing and has countable Lebesgue spectrum (Parasyuk, 1953);
- the speed of mixing is polynomial (Ratner, 1987);
- has the speed of convergence of ergodic integrals is polynomial (Flaminio-Forni, 2003);

Not much is known for smooth perturbations. Even the dynamics of non-trivial smooth time-changes is poorly understood.

Dynamical properties of classical horocycle flows

The dynamical properties of the horocycle flow h_t on a compact (and not only) surface have been studied in great detail.
It is known for example that h_t is:

- minimal, i.e. all orbits are dense (Hedlund, 1936);
- uniquely ergodic (Fursternberg, 1972); in particular, for any smooth observable $f : M \to \mathbb{R}$, one has uniform convergence (in x) of

$$\frac{1}{T}\int_0^T f(h_t x))\,\mathrm{d}t \xrightarrow{T\to\infty} \int_M f\,\mathrm{d}\mu;$$

- is mixing and has countable Lebesgue spectrum (Parasyuk, 1953);
- the speed of mixing is polynomial (Ratner, 1987);
- has the speed of convergence of ergodic integrals is polynomial (Flaminio-Forni, 2003);

Not much is known for smooth perturbations. Even the dynamics of non-trivial smooth time-changes is poorly understood.

Dynamical properties of classical horocycle flows

The dynamical properties of the horocycle flow h_t on a compact (and not only) surface have been studied in great detail.
It is known for example that h_t is:

- minimal, i.e. all orbits are dense (Hedlund, 1936);
- uniquely ergodic (Fursternberg, 1972); in particular, for any smooth observable $f : M \to \mathbb{R}$, one has uniform convergence (in x) of

$$\frac{1}{T} \int_0^T f(h_t x)) \, \mathrm{d}t \xrightarrow{T \to \infty} \int_M f \, \mathrm{d}\mu;$$

- is mixing and has countable Lebesgue spectrum (Parasyuk, 1953);
- the speed of mixing is polynomial (Ratner, 1987);
- has the speed of convergence of ergodic integrals is polynomial (Flaminio-Forni, 2003);

Not much is known for smooth perturbations. Even the dynamics of non-trivial smooth time-changes is poorly understood.

Dynamical properties of classical horocycle flows

The dynamical properties of the horocycle flow h_t on a compact (and not only) surface have been studied in great detail.
It is known for example that h_t is:

- minimal, i.e. all orbits are dense (Hedlund, 1936);
- uniquely ergodic (Fursternberg, 1972); in particular, for any smooth observable $f : M \to \mathbb{R}$, one has uniform convergence (in x) of

$$\frac{1}{T}\int_0^T f(h_t x))\,\mathrm{d}t \xrightarrow{T\to\infty} \int_M f\,\mathrm{d}\mu;$$

- is mixing and has countable Lebesgue spectrum (Parasyuk, 1953);
- the speed of mixing is polynomial (Ratner, 1987);
- has the speed of convergence of ergodic integrals is polynomial (Flaminio-Forni, 2003);

Not much is known for smooth perturbations. Even the dynamics of non-trivial smooth time-changes is poorly understood.

Time-changes

Intuition: if $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ *is a time-change of* $\{h_t\}_{t\in\mathbb{R}}$*, the trajectories of* $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ *are the same than* $\{h_t\}_{t\in\mathbb{R}}$ *but the speed is different.*

Definition

A flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *time-change* of a flow $\{h_t\}_{t\in\mathbb{R}}$ on M (or a *reparametrization*) if there exists $\tau : X \times \mathbb{R} \to \mathbb{R}$ s.t.

$$\widetilde{h}_t(x) = h_{\tau(x,t)}(x), \qquad \forall x \in X, t \in \mathbb{R}.$$

We say that $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *smooth time-change* if τ is a smooth function.

Equivalently, there exists a positive smooth function $\alpha : M \to \mathbb{R}^+$ such that

$$\frac{\partial h_t}{\partial t}(x,0) = \alpha(x)\frac{\partial \widetilde{h}_t}{\partial t}(x,0).$$

$$[\alpha(x) := \frac{\partial \tau}{\partial t}(x,t)|_{t=0}, \quad \text{for all } (x,t) \in M \times \mathbb{R}.]$$

Time-changes

Intuition: if $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a time-change of $\{h_t\}_{t\in\mathbb{R}}$, the trajectories of $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ are the same than $\{h_t\}_{t\in\mathbb{R}}$ but the speed is different.

Definition

A flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *time-change* of a flow $\{h_t\}_{t\in\mathbb{R}}$ on M (or a *reparametrization*) if there exists $\tau : X \times \mathbb{R} \to \mathbb{R}$ s.t.

$$\widetilde{h}_t(x) = h_{\tau(x,t)}(x), \qquad \forall x \in X, t \in \mathbb{R}.$$

We say that $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *smooth time-change* if τ is a smooth function. Equivalently, there exists a positive smooth function $\alpha : M \to \mathbb{R}^+$ such that

$$\frac{\partial h_t}{\partial t}(x,0) = \alpha(x)\frac{\partial \widetilde{h}_t}{\partial t}(x,0).$$

$$[\alpha(x) := \frac{\partial \tau}{\partial t}(x,t)|_{t=0}, \text{ for all } (x,t) \in M \times \mathbb{R}.]$$

Time-changes

Intuition: if $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a time-change of $\{h_t\}_{t\in\mathbb{R}}$, the trajectories of $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ are the same than $\{h_t\}_{t\in\mathbb{R}}$ but the speed is different.

Definition

A flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *time-change* of a flow $\{h_t\}_{t\in\mathbb{R}}$ on M (or a *reparametrization*) if there exists $\tau : X \times \mathbb{R} \to \mathbb{R}$ s.t.

$$\widetilde{h}_t(x) = h_{\tau(x,t)}(x), \qquad \forall x \in X, t \in \mathbb{R}.$$

We say that $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *smooth time-change* if τ is a smooth function. Equivalently, there exists a positive smooth function $\alpha : M \to \mathbb{R}^+$ such that

$$\frac{\partial h_t}{\partial t}(x,0) = \alpha(x)\frac{\partial \widetilde{h}_t}{\partial t}(x,0).$$

$$[\alpha(x) := \frac{\partial \tau}{\partial t}(x,t)|_{t=0}, \ \text{ for all } (x,t) \in M \times \mathbb{R}.]$$

Smooth-time changes of horocycle flows

Let $\alpha : M \to \mathbb{R}_+$ be a smooth positive function and let h_t^α be the corresponding time change of h_t given by a

$$\frac{\partial h_t}{\partial t}(x,0) = \alpha(x)\frac{\partial \widetilde{h}_t}{\partial t}(x,0).$$

Remark that:

- also h_t^α is parabolic (infinitesimal divergence of trajectories is at most quadratic in time, as for h_t^α);
- h_t^α preserves the (smooth) volume form $\mathrm{vol}_\alpha := \alpha \mathrm{vol}$;
- even a generic smooth time-change is not even measurably conjugate to the horocycle flow, hence its ergodic properties are unrelated to the ones of the original flow.

Theorem (Marcus '77, (Kushnirenko '74 under stronger hyp.))

All time-change of the classical horocycle flow satisfying a mild differentiability conditions are mixing.

Conjecture by Katok-Thouvenot on spectral properties of time-changes.

Smooth-time changes of horocycle flows

Let $\alpha : M \to \mathbb{R}_+$ be a smooth positive function and let h_t^α be the corresponding time change of h_t given by a

$$\frac{\partial h_t}{\partial t}(x,0) = \alpha(x)\frac{\partial \widetilde{h_t}}{\partial t}(x,0).$$

Remark that:

- also h_t^α is parabolic (infinitesimal divergence of trajectories is at most quadratic in time, as for h_t^α);
- h_t^α preserves the (smooth) volume form $\mathrm{vol}_\alpha := \alpha\mathrm{vol}$;
- even a generic smooth time-change is not even measurably conjugate to the horocycle flow, hence its ergodic properties are unrelated to the ones of the original flow.

Theorem (Marcus '77, (Kushnirenko '74 under stronger hyp.))

All time-change of the classical horocycle flow satisfying a mild differentiability conditions are mixing.

Conjecture by Katok-Thouvenot on spectral properties of time-changes.

Smooth-time changes of horocycle flows

Let $\alpha : M \to \mathbb{R}_+$ be a smooth positive function and let h_t^α be the corresponding time change of h_t given by a

$$\frac{\partial h_t}{\partial t}(x,0) = \alpha(x)\frac{\partial \widetilde{h}_t}{\partial t}(x,0).$$

Remark that:

- also h_t^α is parabolic (infinitesimal divergence of trajectories is at most quadratic in time, as for h_t^α);
- h_t^α preserves the (smooth) volume form $\mathrm{vol}_\alpha := \alpha \mathrm{vol}$;
- even a generic smooth time-change is not even measurably conjugate to the horocycle flow, hence its ergodic properties are unrelated to the ones of the original flow.

Theorem (Marcus '77, (Kushnirenko '74 under stronger hyp.))

All time-change of the classical horocycle flow satisfying a mild differentiability conditions are mixing.

Conjecture by Katok-Thouvenot on spectral properties of time-changes.

Smooth-time changes of horocycle flows

Let $\alpha : M \to \mathbb{R}_+$ be a smooth positive function and let h_t^α be the corresponding time change of h_t given by a

$$\frac{\partial h_t}{\partial t}(x,0) = \alpha(x)\frac{\partial \widetilde{h}_t}{\partial t}(x,0).$$

Remark that:

- also h_t^α is parabolic (infinitesimal divergence of trajectories is at most quadratic in time, as for h_t^α);
- h_t^α preserves the (smooth) volume form $\mathrm{vol}_\alpha := \alpha\mathrm{vol}$;
- even a generic smooth time-change is not even measurably conjugate to the horocycle flow, hence its ergodic properties are unrelated to the ones of the original flow.

Theorem (Marcus '77, (Kushnirenko '74 under stronger hyp.))

All time-change of the classical horocycle flow satisfying a mild differentiability conditions are mixing.

Conjecture by Katok-Thouvenot on spectral properties of time-changes.

Smooth-time changes of horocycle flows

Let $\alpha : M \to \mathbb{R}_+$ be a smooth positive function and let h_t^α be the corresponding time change of h_t given by a

$$\frac{\partial h_t}{\partial t}(x,0) = \alpha(x)\frac{\partial \widetilde{h}_t}{\partial t}(x,0).$$

Remark that:

- also h_t^α is parabolic (infinitesimal divergence of trajectories is at most quadratic in time, as for h_t^α);
- h_t^α preserves the (smooth) volume form $\mathrm{vol}_\alpha := \alpha \mathrm{vol}$;
- even a generic smooth time-change is not even measurably conjugate to the horocycle flow, hence its ergodic properties are unrelated to the ones of the original flow.

Theorem (Marcus '77, (Kushnirenko '74 under stronger hyp.))

All time-change of the classical horocycle flow satisfying a mild differentiability conditions are mixing.

Conjecture by Katok-Thouvenot on spectral properties of time-changes.

Smooth-time changes of horocycle flows

Let $\alpha : M \to \mathbb{R}_+$ be a smooth positive function and let h_t^α be the corresponding time change of h_t given by a

$$\frac{\partial h_t}{\partial t}(x,0) = \alpha(x)\frac{\partial \widetilde{h}_t}{\partial t}(x,0).$$

Remark that:

- also h_t^α is parabolic (infinitesimal divergence of trajectories is at most quadratic in time, as for h_t^α);
- h_t^α preserves the (smooth) volume form $\mathrm{vol}_\alpha := \alpha\mathrm{vol}$;
- even a generic smooth time-change is not even measurably conjugate to the horocycle flow, hence its ergodic properties are unrelated to the ones of the original flow.

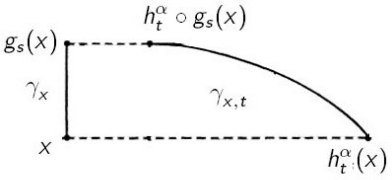
Theorem (Marcus '77, (Kushnirenko '74 under stronger hyp.))

All time-change of the classical horocycle flow satisfying a mild differentiability conditions are mixing.

Conjecture by Katok-Thouvenot on spectral properties of time-changes.

Shearing for the horocycle flow

Key idea used by Marcus to prove mixing: consider the evolution $\gamma_{x,t}$ of a small geodesic arc γ_x. As t grows, $\gamma^\sigma_{x,t}$ is sheared in the horocycle direction:



$$\gamma_x = \{g_s(x),\ s \in [0, \epsilon]\}$$

$$\gamma_{x,t} = \{h_t^\alpha \circ g_s(x),\ s \in [0, \epsilon]\}$$

Integrals over $\gamma_{x,t}$ are close to integrals along flow arcs. Recall the h_t (and h_t^α) are uniquely ergodic, so long trajectories of h_t are equidistributed.

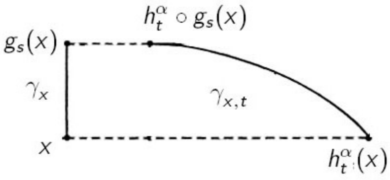
Thus, if f is continuous, one can show that for any $x \in M$

$$\frac{1}{\epsilon}\int_0^\epsilon f\left(h_t^\alpha \circ g_s(x)\right)\,\mathrm{d}s \xrightarrow{t\to\infty} \int_M f\,\mathrm{d}vol_\alpha .$$

Mixing then follows from a Fubini argument (integrate over small geodesic arcs first).

Shearing for the horocycle flow

Key idea used by Marcus to prove mixing: consider the evolution $\gamma_{x,t}$ of a small geodesic arc γ_x. As t grows, $\gamma^{\sigma}_{x,t}$ is sheared in the horocycle direction:



$$\gamma_x = \{g_s(x),\ s \in [0,\epsilon]\}$$

$$\gamma_{x,t} = \{h_t^\alpha \circ g_s(x),\ s \in [0,\epsilon]\}$$

Integrals over $\gamma_{x,t}$ are close to integrals along flow arcs. Recall the h_t (and h_t^α) are uniquely ergodic, so long trajectories of h_t are equidistributed.

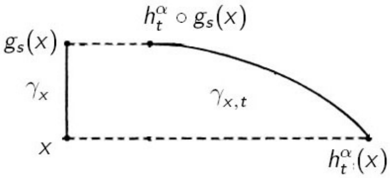
Thus, if f is continuous, one can show that for any $x \in M$

$$\frac{1}{\epsilon}\int_0^\epsilon f\left(h_t^\alpha \circ g_s(x)\right)\mathrm{d}s \xrightarrow{t\to\infty} \int_M f\,\mathrm{d}vol_\alpha .$$

Mixing then follows from a Fubini argument (integrate over small geodesic arcs first).

Shearing for the horocycle flow

Key idea used by Marcus to prove mixing: consider the evolution $\gamma_{x,t}$ of a small geodesic arc γ_x. As t grows, $\gamma^{\sigma}_{x,t}$ is sheared in the horocycle direction:



$$\gamma_x = \{g_s(x),\, s \in [0,\epsilon]\}$$

$$\gamma_{x,t} = \{h^{\alpha}_t \circ g_s(x),\, s \in [0,\epsilon]\}$$

Integrals over $\gamma_{x,t}$ are close to integrals along flow arcs. Recall the h_t (and h^{α}_t) are uniquely ergodic, so long trajectories of h_t are equidistributed.

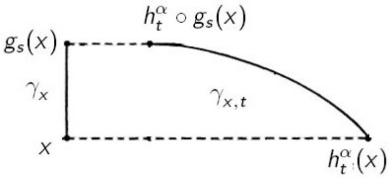
Thus, if f is continuous, one can show that for any $x \in M$

$$\frac{1}{\epsilon}\int_0^{\epsilon} f\left(h^{\alpha}_t \circ g_s(x)\right) \mathrm{d}s \xrightarrow{t\to\infty} \int_M f \,\mathrm{d}vol_{\alpha}.$$

Mixing then follows from a Fubini argument (integrate over small geodesic arcs first).

Shearing for the horocycle flow

Key idea used by Marcus to prove mixing: consider the evolution $\gamma_{x,t}$ of a small geodesic arc γ_x. As t grows, $\gamma^{\sigma}_{x,t}$ is sheared in the horocycle direction:



$$\gamma_x = \{g_s(x),\ s \in [0,\epsilon]\}$$

$$\gamma_{x,t} = \{h_t^\alpha \circ g_s(x),\ s \in [0,\epsilon]\}$$

Integrals over $\gamma_{x,t}$ are close to integrals along flow arcs. Recall the h_t (and h_t^α) are uniquely ergodic, so long trajectories of h_t are equidistributed.

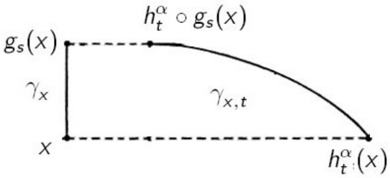
Thus, if f is continuous, one can show that for any $x \in M$

$$\frac{1}{\epsilon}\int_0^\epsilon f\left(h_t^\alpha \circ g_s(x)\right)\,\mathrm{d}s \xrightarrow{t\to\infty} \int_M f\,\mathrm{d}vol_\alpha.$$

Mixing then follows from a Fubini argument (integrate over small geodesic arcs first).

Shearing for the horocycle flow

Key idea used by Marcus to prove mixing: consider the evolution $\gamma_{x,t}$ of a small geodesic arc γ_x. As t grows, $\gamma^{\sigma}_{x,t}$ is sheared in the horocycle direction:



$$\gamma_x = \{g_s(x),\ s \in [0,\epsilon]\}$$

$$\gamma_{x,t} = \{h^{\alpha}_t \circ g_s(x),\ s \in [0,\epsilon]\}$$

Integrals over $\gamma_{x,t}$ are close to integrals along flow arcs. Recall the h_t (and h^{α}_t) are uniquely ergodic, so long trajectories of h_t are equidistributed.

Thus, if f is continuous, one can show that for any $x \in M$

$$\frac{1}{\epsilon}\int_0^{\epsilon} f\left(h^{\alpha}_t \circ g_s(x)\right) \mathrm{d}s \xrightarrow{t\to\infty} \int_M f \, \mathrm{d}vol_{\alpha}.$$

Mixing then follows from a Fubini argument (integrate over small geodesic arcs first).

Quantitative mixing for horocycle time-changes

Marcus' proof of mixing can be made quantitative.

Theorem (Forni-U. '12)

For any sufficiently regular time change h_t^α,
for any two sufficiently smooth observables f, g, f with zero average,

$$| < f \circ h_t^\alpha, g >_{L^2(M, \mathrm{vol}_\alpha)} | \leq C\, t^{-\frac{1-\nu_0}{2}} (1 + \log t)^{\epsilon_0} ,$$

where $C = C(f, g, \alpha) > 0$ and $\nu_0 \in [0, 1)$, $\epsilon_0 \in \{0, 1\}$ are defined in terms of the smallest positive eigenvalue $\mu_0 > 0$ for the hyperbolic Laplacian on the compact surface S

Idea: refine Marcus' proof of mixing with a key bootstrap trick and combine it with quantitative equidistribution results/techniques for horocycle flows (from Forni-Flaminio) and their time changes (Bufetov-Forni).

Quantitative mixing for horocycle time-changes

Marcus' proof of mixing can be made quantitative.

Theorem (Forni-U. '12)

For any sufficiently regular time change h_t^α,
for any two sufficiently smooth observables f, g, f with zero average,

$$| < f \circ h_t^\alpha, g >_{L^2(M,\mathrm{vol}_\alpha)} | \leq C\, t^{-\frac{1-\nu_0}{2}} (1 + \log t)^{\epsilon_0} ,$$

where $C = C(f, g, \alpha) > 0$ and $\nu_0 \in [0, 1)$, $\epsilon_0 \in \{0, 1\}$ are defined in terms of the smallest positive eigenvalue $\mu_0 > 0$ for the hyperbolic Laplacian on the compact surface S

Idea: refine Marcus' proof of mixing with a key bootstrap trick and combine it with quantitative equidistribution results/techniques for horocycle flows (from Forni-Flaminio) and their time changes (Bufetov-Forni).

Quantitative mixing for horocycle time-changes

Marcus' proof of mixing can be made quantitative.

Theorem (Forni-U. '12)

For any sufficiently regular time change h_t^α,
for any two sufficiently smooth observables f, g, f with zero average,

$$| < f \circ h_t^\alpha, g >_{L^2(M, \mathrm{vol}_\alpha)} | \leq C\, t^{-\frac{1-\nu_0}{2}} (1 + \log t)^{\epsilon_0} ,$$

where $C = C(f, g, \alpha) > 0$ and $\nu_0 \in [0, 1)$, $\epsilon_0 \in \{0, 1\}$ are defined in terms of the smallest positive eigenvalue $\mu_0 > 0$ for the hyperbolic Laplacian on the compact surface S

Idea: refine Marcus' proof of mixing with a key bootstrap trick and combine it with quantitative equidistribution results/techniques for horocycle flows (from Forni-Flaminio) and their time changes (Bufetov-Forni).

Spectral results

This results on decay of correlations of time-changes was the starting point to address the

Conjecture (Katok-Thouvenot)

Any flow obtained by a sufficiently smooth time change from a horocycle flow has absolutely continuous (countable Lebesgue) spectrum.

Theorem (Forni-U. '12)

Any sufficiently smooth time-change h_t^α of the horocycle flow h_t on a compact hyperbolic surface has purely absolutely continuous (Lebesgue) spectrum.

[See also: Tiedra de Aldecoa, partial result by different techniques.]

Recall: a flow is said to have *absolutely continuous spectrum* if all spectral measures μ_f, $f \in L^2$, i.e. all measures on $\mathbb{R}$ with Fourier coefficients given by

$$\hat{\mu}_f(t) = < f \circ h_t^\alpha, f >_{L^2(M, \mathrm{vol}_\alpha)},$$

are absolutely continous with respect to the Lebesgue measure on the real line.

Idea: prove square-integrable quantitative estimates for *coboundaries*.

Spectral results

This results on decay of correlations of time-changes was the starting point to address the

Conjecture (Katok-Thouvenot)

Any flow obtained by a sufficiently smooth time change from a horocycle flow has absolutely continuous (countable Lebesgue) spectrum.

Theorem (Forni-U. '12)

Any sufficiently smooth time-change h_t^α of the horocycle flow h_t on a compact hyperbolic surface has purely absolutely continuous (Lebesgue) spectrum.

[See also: Tiedra de Aldecoa, partial result by different techniques.]

Recall: a flow is said to have *absolutely continuous spectrum* if all spectral measures μ_f, $f \in L^2$, i.e. all measures on $\mathbb{R}$ with Fourier coefficients given by

$$\hat{\mu}_f(t) = < f \circ h_t^\alpha, f >_{L^2(M, \mathrm{vol}_\alpha)},$$

are absolutely continous with respect to the Lebesgue measure on the real line.

Idea: prove square-integrable quantitative estimates for *coboundaries*.

Spectral results

This results on decay of correlations of time-changes was the starting point to address the

Conjecture (Katok-Thouvenot)

Any flow obtained by a sufficiently smooth time change from a horocycle flow has absolutely continuous (countable Lebesgue) spectrum.

Theorem (Forni-U. '12)

Any sufficiently smooth time-change h_t^α of the horocycle flow h_t on a compact hyperbolic surface has purely absolutely continuous (Lebesgue) spectrum.

[See also: Tiedra de Aldecoa, partial result by different techniques.]

Recall: a flow is said to have *absolutely continuous spectrum* if all spectral measures μ_f, $f \in L^2$, i.e. all measures on $\mathbb{R}$ with Fourier coefficients given by

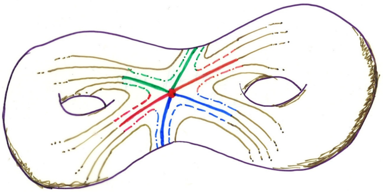
$$\hat{\mu}_f(t) = < f \circ h_t^\alpha, f >_{L^2(M,\mathrm{vol}_\alpha)},$$

are absolutely continous with respect to the Lebesgue measure on the real line.

Idea: prove square-integrable quantitative estimates for *coboundaries*.

Locally Hamiltonian flows on surfaces

Consider *flows on surfaces* which preserve a *smooth area form* ω (symplectic). Let φ_t be the flow associated to an area-preserving vector field X (given by a smooth closed 1-form η by $i_X\omega = \omega(X, \cdot) = \eta$).



Locally, on $U \subset S$, $\omega = \mathrm{d}x \wedge \mathrm{d}y$ and the *trajectories* of φ_t are solutions to *Hamiltonian equations*

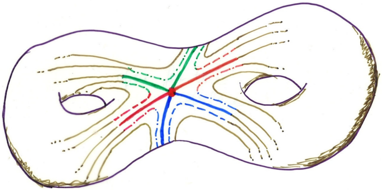
$$\begin{cases} \frac{\partial x}{\partial t} = & \frac{\partial H}{\partial y} \\ \frac{\partial y}{\partial t} = & -\frac{\partial H}{\partial x} \end{cases}$$

for $H : U \to \mathbb{R}$.
Thus the flow h_t is known as *locally Hamiltonian flow*.

- Initial motivation from *solid state physics*: motion of electrons under a magnetic field on Fermi energy surfaces (Novikov).
- *Conjecture* by Arnold (1990s) on mixing in locally Hamiltonian flows.

Locally Hamiltonian flows on surfaces

Consider *flows on surfaces* which preserve a *smooth area form* ω (symplectic). Let φ_t be the flow associated to an area-preserving vector field X (given by a smooth closed 1-form η by $i_X\omega = \omega(X, \cdot) = \eta$).



Locally, on $U \subset S$, $\omega = \mathrm{d}x \wedge \mathrm{d}y$ and the *trajectories* of φ_t are solutions to *Hamiltonian equations*

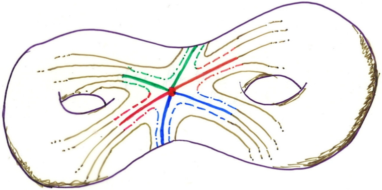
$$\begin{cases} \frac{\partial x}{\partial t} = & \frac{\partial H}{\partial y} \\ \frac{\partial y}{\partial t} = & -\frac{\partial H}{\partial x} \end{cases}$$

for $H : U \to \mathbb{R}$.
Thus the flow h_t is known as *locally Hamiltonian flow*.

- Initial motivation from *solid state physics*: motion of electrons under a magnetic field on Fermi energy surfaces (Novikov).
- *Conjecture* by Arnold (1990s) on mixing in locally Hamiltonian flows.

Locally Hamiltonian flows on surfaces

Consider *flows on surfaces* which preserve a *smooth area form* ω (symplectic). Let φ_t be the flow associated to an area-preserving vector field X (given by a smooth closed 1-form η by $i_X\omega = \omega(X, \cdot) = \eta$).



Locally, on $U \subset S$, $\omega = \mathrm{d}x \wedge \mathrm{d}y$ and the *trajectories* of φ_t are solutions to *Hamiltonian equations*

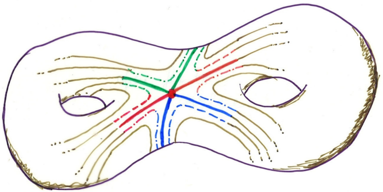
$$\begin{cases} \frac{\partial x}{\partial t} = \frac{\partial H}{\partial y} \\ \frac{\partial y}{\partial t} = -\frac{\partial H}{\partial x} \end{cases}$$

for $H : U \to \mathbb{R}$.
Thus the flow h_t is known as *locally Hamiltonian flow*.

- Initial motivation from *solid state physics*: motion of electrons under a magnetic field on Fermi energy surfaces (Novikov).
- *Conjecture* by Arnold (1990s) on mixing in locally Hamiltonian flows.

Locally Hamiltonian flows on surfaces

Consider *flows on surfaces* which preserve a *smooth area form* ω (symplectic). Let φ_t be the flow associated to an area-preserving vector field X (given by a smooth closed 1-form η by $i_X\omega = \omega(X, \cdot) = \eta$).



Locally, on $U \subset S$, $\omega = \mathrm{d}x \wedge \mathrm{d}y$ and the *trajectories* of φ_t are solutions to *Hamiltonian equations*

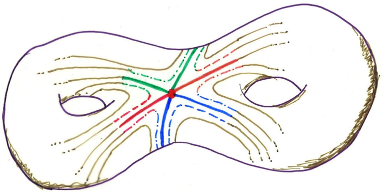
$$\begin{cases} \frac{\partial x}{\partial t} = & \frac{\partial H}{\partial y} \\ \frac{\partial y}{\partial t} = & -\frac{\partial H}{\partial x} \end{cases}$$

for $H : U \to \mathbb{R}$.
Thus the flow h_t is known as *locally Hamiltonian flow*.

- Initial motivation from *solid state physics*: motion of electrons under a magnetic field on Fermi energy surfaces (Novikov).
- *Conjecture* by Arnold (1990s) on mixing in locally Hamiltonian flows.

Locally Hamiltonian flows on surfaces

Consider *flows on surfaces* which preserve a *smooth area form* ω (symplectic). Let φ_t be the flow associated to an area-preserving vector field X (given by a smooth closed 1-form η by $i_X\omega = \omega(X, \cdot) = \eta$).



Locally, on $U \subset S$, $\omega = \mathrm{d}x \wedge \mathrm{d}y$ and the *trajectories* of φ_t are solutions to *Hamiltonian equations*

$$\begin{cases} \frac{\partial x}{\partial t} = \frac{\partial H}{\partial y} \\ \frac{\partial y}{\partial t} = -\frac{\partial H}{\partial x} \end{cases}$$

for $H : U \to \mathbb{R}$.
Thus the flow h_t is known as *locally Hamiltonian flow*.

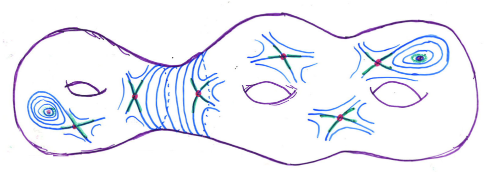
- Initial motivation from *solid state physics*: motion of electrons under a magnetic field on Fermi energy surfaces (Novikov).
- *Conjecture* by Arnold (1990s) on mixing in locally Hamiltonian flows.

Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.

Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

Question: Is a *typical* φ_t restricted to each minimal component *mixing*?

[*Typical* means for almost every choice of *periods*.]

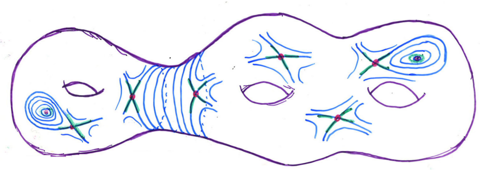
Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.

Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

Question: Is a *typical* φ_t restricted to each minimal component *mixing*?

[*Typical* means for almost every choice of *periods*.]

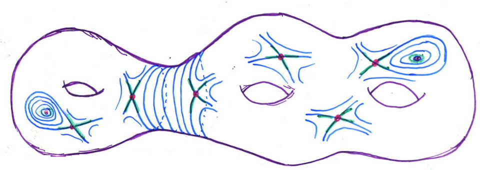
Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.

Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

Question: Is a *typical* φ_t restricted to each minimal component *mixing*?

[*Typical* means for almost every choice of *periods*.]

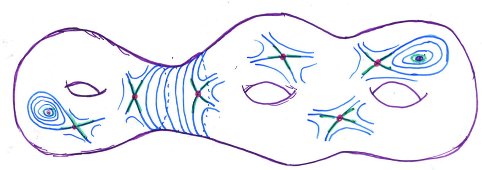
Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.

Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

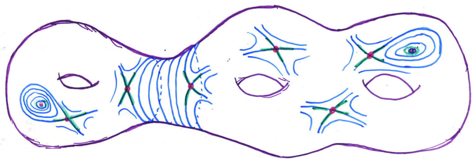
Question: Is a *typical* φ_t restricted to each minimal component *mixing*?
[*Typical* means for almost every choice of *periods*.]
Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.

Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

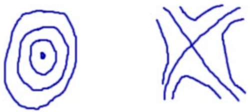
Question: Is a *typical* φ_t restricted to each minimal component *mixing*?
[*Typical* means for almost every choice of *periods*.]

Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

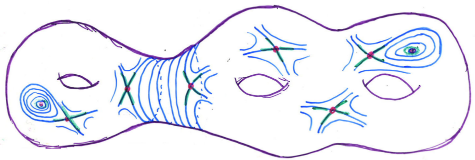
Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.



Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

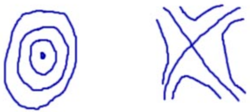
- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

Question: Is a *typical* φ_t restricted to each minimal component *mixing*?
[*Typical* means for almost every choice of *periods*.]
Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

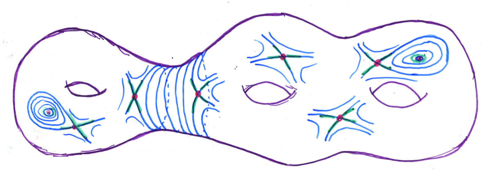
Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.



Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

Question: Is a *typical* φ_t restricted to each minimal component *mixing*?

[*Typical* means for almost every choice of *periods*.]

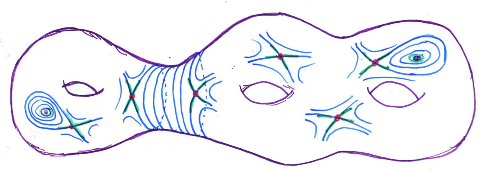
Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.

Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

Question: Is a *typical* φ_t restricted to each minimal component *mixing*?
[*Typical* means for almost every choice of *periods*.]

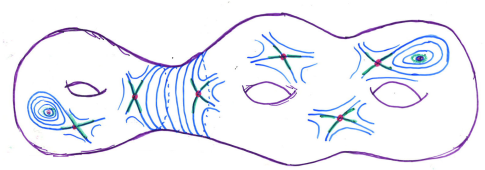
Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.

Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

Question: Is a *typical* φ_t restricted to each minimal component *mixing*?

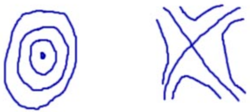
[*Typical* means for almost every choice of *periods*.]

Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

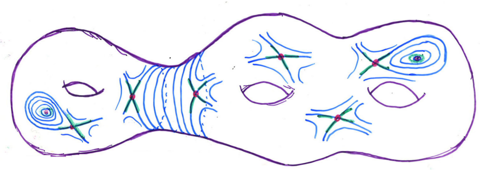
Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.



Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

Question: Is a *typical* φ_t restricted to each minimal component *mixing*?
[*Typical* means for almost every choice of *periods*.]

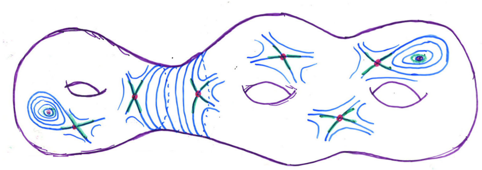
Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.

Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

Question: Is a *typical* φ_t restricted to each minimal component *mixing*?

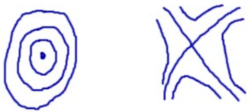
[*Typical* means for almost every choice of *periods*.]

Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

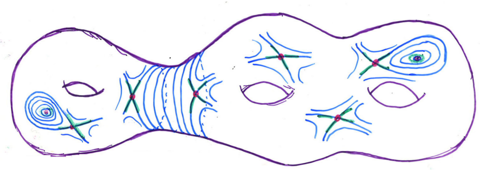
Minimal components

Let us assume that the flow is *Morse* and has fixed points, that hence are:

- centers and
- *simple saddles*.



Let us say that φ_t is *minimal* if all orbits which are not fixed points are dense.



Decompose S in:

- islands of periodic orbits,
- cylinders of periodic orbits,
- minimal components.

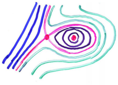
Question: Is a *typical* φ_t restricted to each minimal component *mixing*?

[*Typical* means for almost every choice of *periods*.]

Arnold's Conjecture: Yes for $g = 1$ (Khanin-Sinai). For $g \geq 2$?

Mixing in locally Hamiltonian flows on surfaces

Mixing crucially depends on the presence of *saddle loops*:



Theorem (U'07)

A typical locally Hamiltonian flow on a surface that has saddle loops is mixing in each minimal component.

Theorem (U'11)

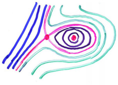
A typical locally Hamiltonian flow on a surface that has only saddles (in particular, no saddle loops) is not mixing.

Theorem (U'10)

Typical locally Hamiltonian flows with only saddles are weakly mixing.

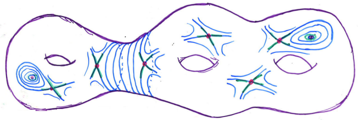
Mixing in locally Hamiltonian flows on surfaces

Mixing crucially depends on the presence of *saddle loops*:



Theorem (U'07)

A typical locally Hamiltonian flow on a surface that has saddle loops is mixing in each minimal component.



Theorem (U'11)

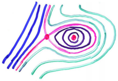
A typical locally Hamiltonian flow on a surface that has only saddles (in particular, no saddle loops) is not mixing.

Theorem (U'10)

Typical locally Hamiltonian flows with only saddles are weakly mixing.

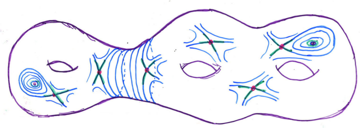
Mixing in locally Hamiltonian flows on surfaces

Mixing crucially depends on the presence of *saddle loops*:



Theorem (U'07)

A typical locally Hamiltonian flow on a surface that has saddle loops is mixing in each minimal component.



[*Recall:* Mixing if $\forall A, B$ measurable $\mu(h_t A \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B)$]

Theorem (U'11)

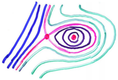
A typical locally Hamiltonian flow on a surface that has only saddles (in particular, no saddle loops) is not mixing.

Theorem (U'10)

Typical locally Hamiltonian flows with only saddles are weakly mixing.

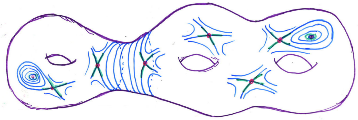
Mixing in locally Hamiltonian flows on surfaces

Mixing crucially depends on the presence of *saddle loops*:

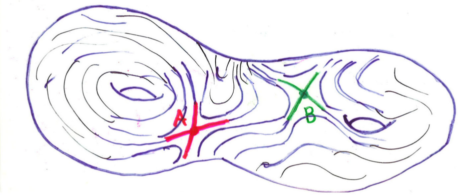


Theorem (U'07)

A typical locally Hamiltonian flow on a surface that has saddle loops is mixing in each minimal component.



[*Recall:* Mixing if $\forall A, B$ measurable $\mu(h_t A \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B)$]



Theorem (U'11)

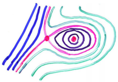
A typical locally Hamiltonian flow on a surface that has only saddles (in particular, no saddle loops) is not mixing.

Theorem (U'10)

Typical locally Hamiltonian flows with only saddles are weakly mixing.

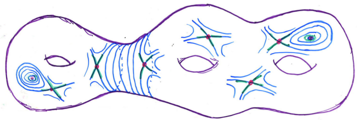
Mixing in locally Hamiltonian flows on surfaces

Mixing crucially depends on the presence of *saddle loops*:

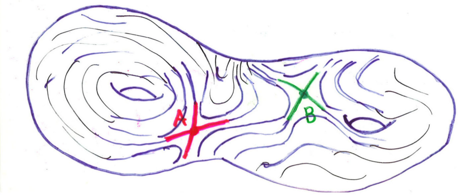


Theorem (U'07)

A typical locally Hamiltonian flow on a surface that has saddle loops is mixing in each minimal component.



[*Recall:* Mixing if $\forall A, B$ measurable $\mu(h_t A \cap B) \xrightarrow{t \to \infty} \mu(A)\mu(B)$]



Theorem (U'11)

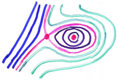
A typical locally Hamiltonian flow on a surface that has only saddles (in particular, no saddle loops) is not mixing.

Theorem (U'10)

Typical locally Hamiltonian flows with only saddles are weakly mixing.

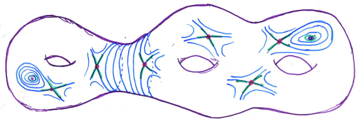
Mixing in locally Hamiltonian flows on surfaces

Mixing crucially depends on the presence of *saddle loops*:

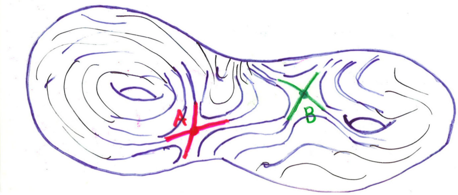


Theorem (U'07)

A typical locally Hamiltonian flow on a surface that has saddle loops is mixing in each minimal component.



[*Recall:* Mixing if $\forall A, B$ measurable $\mu(h_t A \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B)$]



Theorem (U'11)

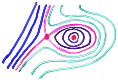
A typical locally Hamiltonian flow on a surface that has only saddles (in particular, no saddle loops) is not mixing.

Theorem (U'10)

Typical locally Hamiltonian flows with only saddles are weakly mixing.

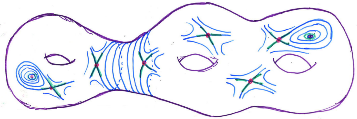
Mixing in locally Hamiltonian flows on surfaces

Mixing crucially depends on the presence of *saddle loops*:

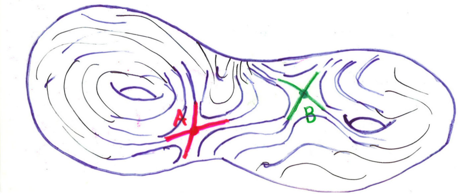


Theorem (U'07)

A typical locally Hamiltonian flow on a surface that has saddle loops is mixing in each minimal component.



[*Recall:* Mixing if $\forall A, B$ measurable $\mu(h_t A \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B)$]



[Weak mixing: $\forall A, B$ measurable $\frac{1}{T}\int_0^T |\mu(h_t A \cap B) - \mu(A)\mu(B)| \,\mathrm{d}T \xrightarrow{t\to\infty} 0$]

Theorem (U'11)

A typical locally Hamiltonian flow on a surface that has only saddles (in particular, no saddle loops) is not mixing.

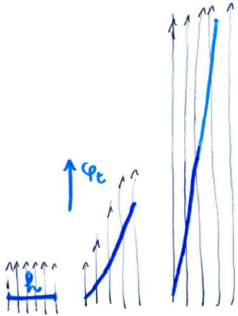
Theorem (U'10)

Typical locally Hamiltonian flows with only saddles are weakly mixing.

Shearing in area-preserving flows

Mixing (when it happens) is again produced by a *shearing phenomenon*:

near a Hamiltonian saddles, trajectories are decelerated: thus a trasversal arc flowing near a saddle is sheared.



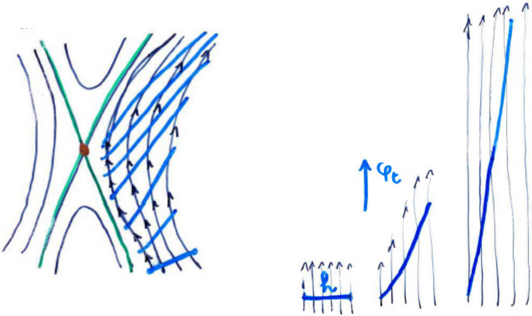
but... the other side of the saddle shears in the opposite direction... thus the shearing effect of one side of the saddle can be *cancelled*.

It is the presence of saddle loops creates an *asymmetry*, so that shearing is prevalently in one direction.

Shearing in area-preserving flows

Mixing (when it happens) is again produced by a *shearing phenomenon*:

near a Hamiltonian saddles, trajectories are decelerated: thus a trasversal arc flowing near a saddle is sheared.



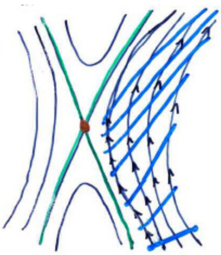
but... the other side of the saddle shears in the opposite direction... thus the shearing effect of one side of the saddle can be *cancelled*.

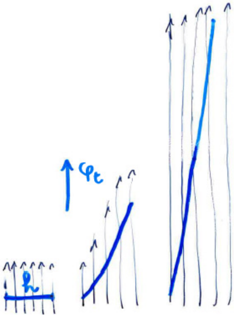
It is the presence of saddle loops creates an *asymmetry*, so that shearing is prevalently in one direction.

Shearing in area-preserving flows

Mixing (when it happens) is again produced by a *shearing phenomenon*:

near a Hamiltonian saddles, trajectories are decelerated: thus a trasversal arc flowing near a saddle is sheared.





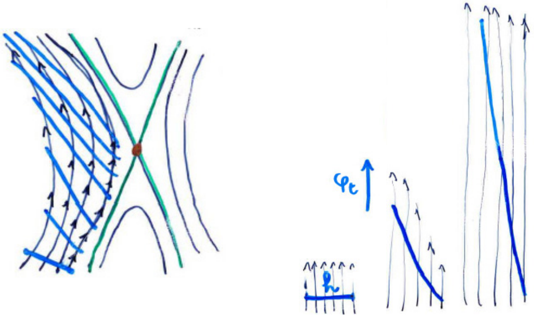
but... the other side of the saddle shears in the opposite direction... thus the shearing effect of one side of the saddle can be *cancelled*.

It is the presence of saddle loops creates an *asymmetry*, so that shearing is prevalently in one direction.

Shearing in area-preserving flows

Mixing (when it happens) is again produced by a *shearing phenomenon*:

near a Hamiltonian saddles,
trajectories are decelerated:
thus a trasversal arc flowing near a saddle is sheared.



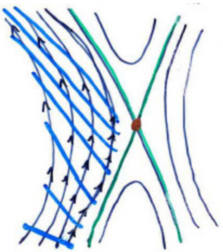
but... the other side of the saddle shears in the opposite direction... thus the shearing effect of one side of the saddle can be *cancelled*.

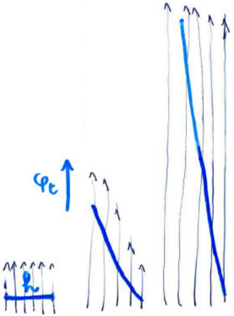
It is the presence of saddle loops creates an *asymmetry*, so that shearing is prevalently in one direction.

Shearing in area-preserving flows

Mixing (when it happens) is again produced by a *shearing phenomenon*:

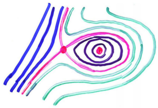
near a Hamiltonian saddles, trajectories are decelerated: thus a trasversal arc flowing near a saddle is sheared.





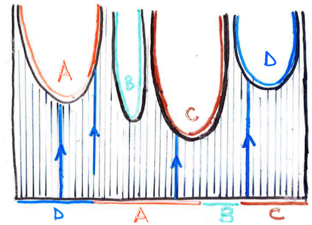
but... the other side of the saddle shears in the opposite direction... thus the shearing effect of one side of the saddle can be *cancelled*.

It is the presence of saddle loops creates an *asymmetry*, so that shearing is prevalently in one direction.



Special flows and Birkhoff sums

Use: representation as *special flow* over T under f, where:



- $f : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ has logarithmic singularities;
- $T : [0,1] \to [0,1]$ is an interval exchange map (piecewise isometry);
- h_t is the vertical unit speed flow modulo $(x, f(x)) \sim (T(x), 0)$;

The evolution of a horizontal segment is described by *Birkhoff sums*

$$S_n f(x) = \sum_{i=0}^{n-1} f(T^i x)$$

.

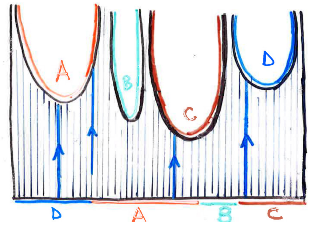
The shear is given by $S_n f'$. Prove limit theorems on the growth of $S_n f'$.

- *Key tool:* Diophantine conditions on interval exchanges, using Teichmueller dynamics.

If the singularities are *asymmetric*, there is shear. Use it to prove mixing.

Special flows and Birkhoff sums

Use: representation as *special flow* over T under f, where:



- $f : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ has logarithmic singularities;
- $T : [0,1] \to [0,1]$ is an interval exchange map (piecewise isometry);
- h_t is the vertical unit speed flow modulo $(x, f(x)) \sim (T(x), 0)$;

The evolution of a horizontal segment is described by *Birkhoff sums*

$$S_n f(x) = \sum_{i=0}^{n-1} f(T^i x)$$

.

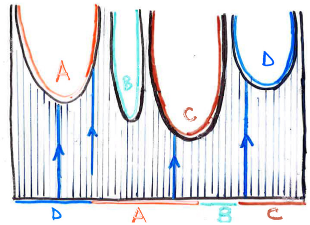
The shear is given by $S_n f'$. Prove limit theorems on the growth of $S_n f'$.

- *Key tool:* Diophantine conditions on interval exchanges, using Teichmueller dynamics.

If the singularities are *asymmetric*, there is shear. Use it to prove mixing.

Special flows and Birkhoff sums

Use: representation as *special flow* over T under f, where:



- $f : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ has logarithmic singularities;
- $T : [0,1] \to [0,1]$ is an interval exchange map (piecewise isometry);
- h_t is the vertical unit speed flow modulo $(x, f(x)) \sim (T(x), 0)$;

The evolution of a horizontal segment is described by *Birkhoff sums*

$$S_n f(x) = \sum_{i=0}^{n-1} f(T^i x)$$

.

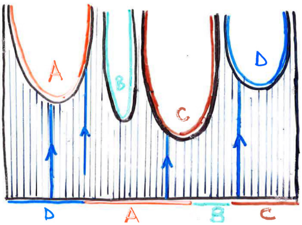
The shear is given by $S_n f'$. Prove limit theorems on the growth of $S_n f'$.

- *Key tool:* Diophantine conditions on interval exchanges, using Teichmueller dynamics.

If the singularities are *asymmetric*, there is shear. Use it to prove mixing.

Special flows and Birkhoff sums

Use: representation as *special flow* over T under f, where:



- $f : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ has logarithmic singularities;
- $T : [0,1] \to [0,1]$ is an interval exchange map (piecewise isometry);
- h_t is the vertical unit speed flow modulo $(x, f(x)) \sim (T(x), 0)$;

The evolution of a horizontal segment is described by *Birkhoff sums*

$$S_n f(x) = \sum_{i=0}^{n-1} f(T^i x)$$

.

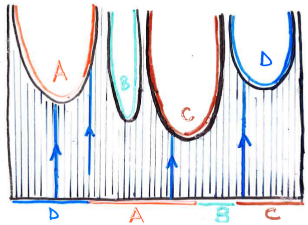
The shear is given by $S_n f'$. Prove limit theorems on the growth of $S_n f'$.

- *Key tool:* Diophantine conditions on interval exchanges, using Teichmueller dynamics.

If the singularities are *asymmetric*, there is shear. Use it to prove mixing.

Special flows and Birkhoff sums

Use: representation as *special flow* over T under f, where:



- $f : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ has logarithmic singularities;
- $T : [0,1] \to [0,1]$ is an interval exchange map (piecewise isometry);
- h_t is the vertical unit speed flow modulo $(x, f(x)) \sim (T(x), 0)$;

The evolution of a horizontal segment is described by *Birkhoff sums*

$$S_n f(x) = \sum_{i=0}^{n-1} f(T^i x)$$

.

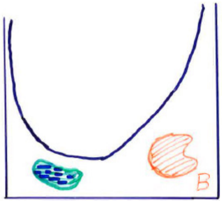
The shear is given by $S_n f'$. Prove limit theorems on the growth of $S_n f'$.

- *Key tool:* Diophantine conditions on interval exchanges, using Teichmueller dynamics.

If the singularities are *asymmetric*, there is shear. Use it to prove mixing.

Special flows and Birkhoff sums

Use: representation as *special flow* over T under f, where:



E.g. $f(x) = |\log x| + 2|\log(1-x)|$

The evolution of a horizontal segment is described by *Birkhoff sums*

$$S_n f(x) = \sum_{i=0}^{n-1} f(T^i x)$$

.

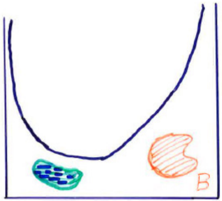
The shear is given by $S_n f'$. Prove limit theorems on the growth of $S_n f'$.

- *Key tool:* Diophantine conditions on interval exchanges, using Teichmueller dynamics.

If the singularities are *asymmetric*, there is shear. Use it to prove mixing.

Special flows and Birkhoff sums

Use: representation as *special flow* over T under f, where:



E.g. $f(x) = |\log x| + 2|\log(1-x)|$

The evolution of a horizontal segment is described by *Birkhoff sums*

$$S_n f(x) = \sum_{i=0}^{n-1} f(T^i x)$$

.

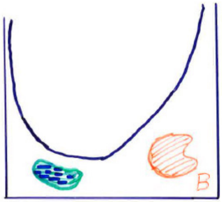
The shear is given by $S_n f'$. Prove limit theorems on the growth of $S_n f'$.

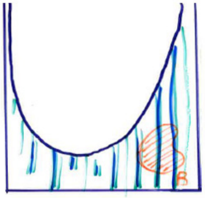
- *Key tool:* Diophantine conditions on interval exchanges, using Teichmueller dynamics.

If the singularities are *asymmetric*, there is shear. Use it to prove mixing.

Special flows and Birkhoff sums

Use: representation as *special flow* over T under f, where:





The evolution of a horizontal segment is described by *Birkhoff sums*

$$S_n f(x) = \sum_{i=0}^{n-1} f(T^i x)$$

.

The shear is given by $S_n f'$. Prove limit theorems on the growth of $S_n f'$.

- *Key tool:* Diophantine conditions on interval exchanges, using Teichmueller dynamics.

If the singularities are *asymmetric*, there is shear. Use it to prove mixing.

Nilflows on nilmanifolds

- A *nilmanifold* is the quotient $M := \Gamma\backslash G$ of a nilpotent Lie group over lattice $\Gamma < G$.
- *Example:* a *Heisenberg nilfmanifold* $M := \Gamma\backslash N$, is the quotient of the *Heisenberg group* N, which, up to isomorphisms, is the group of upper triangular unipotent matrices, by a lattice:

$$N = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y, z \in \mathbb{R} \right\},$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : \qquad x, y, z \in \mathbb{Z} \right\}.$$

- The group N acts on on M by right multiplication. *Heisenberg nilflows* H_t are obtained by the restriction of this action to a one-parameter subgroups of N.
- On any nilmanifold M the probability measure μ induced by the *Haar measure* is invariant under all nilflows on M.

Nilflows on nilmanifolds

- A *nilmanifold* is the quotient $M := \Gamma\backslash G$ of a nilpotent Lie group over lattice $\Gamma < G$.
- *Example:* a *Heisenberg nilfmanifold* $M := \Gamma\backslash N$, is the quotient of the *Heisenberg group* N, which, up to isomorphisms, is the group of upper triangular unipotent matrices, by a lattice:

$$N = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y, z \in \mathbb{R} \right\},$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : \qquad x, y, z \in \mathbb{Z} \right\}.$$

- The group N acts on on M by right multiplication. *Heisenberg nilflows* H_t are obtained by the restriction of this action to a one-parameter subgroups of N.
- On any nilmanifold M the probability measure μ induced by the *Haar measure* is invariant under all nilflows on M.

Nilflows on nilmanifolds

- A *nilmanifold* is the quotient $M := \Gamma\backslash G$ of a nilpotent Lie group over lattice $\Gamma < G$.
- *Example:* a *Heisenberg nilfmanifold* $M := \Gamma\backslash N$, is the quotient of the *Heisenberg group* N, which, up to isomorphisms, is the group of upper triangular unipotent matrices, by a lattice:

$$N = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y, z \in \mathbb{R} \right\},$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : \qquad x, y, z \in \mathbb{Z} \right\}.$$

- The group N acts on on M by right multiplication. *Heisenberg nilflows* H_t are obtained by the restriction of this action to a one-parameter subgroups of N.
- On any nilmanifold M the probability measure μ induced by the *Haar measure* is invariant under all nilflows on M.

Nilflows on nilmanifolds

- A *nilmanifold* is the quotient $M := \Gamma \backslash G$ of a nilpotent Lie group over lattice $\Gamma < G$.
- *Example:* a *Heisenberg nilfmanifold* $M := \Gamma \backslash N$, is the quotient of the *Heisenberg group* N, which, up to isomorphisms, is the group of upper triangular unipotent matrices, by a lattice:

$$N = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} , \qquad x, y, z \in \mathbb{R} \right\},$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : \qquad x, y, z \in \mathbb{Z} \right\}.$$

- The group N acts on on M by right multiplication. *Heisenberg nilflows* H_t are obtained by the restriction of this action to a one-parameter subgroups of N.
- On any nilmanifold M the probability measure μ induced by the *Haar measure* is invariant under all nilflows on M.

Special flow representation of Heisenberg nilflows

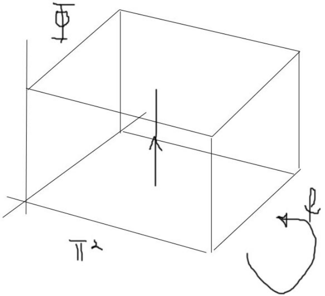
- Heisenberg nilflows have a section isomorphic to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, on which the Poincaré first return map to Σ is the *linear skew shift*:

$$f(x,y) := (x+\alpha, y+x+\beta), \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

- the Heisenberg nilflow is smoothly isomorphic to a *special flow* over f under a constant roof function Φ.

M is isomorphic to $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$;
identify the base torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$
with the top via f;

the *special flow* is the quotient of a constant speed vertical flow $\dot{z} = 1$ by $((x,y),\Phi(x,y)) \sim_\Phi (f(x,y),0)$.



- H_t minimal iff ergodic (Auslander-Green-Hahn '63) iff $\alpha \notin \mathbb{Q}$;
- Heisenberg nilflows are never (weak) mixing.

Special flow representation of Heisenberg nilflows

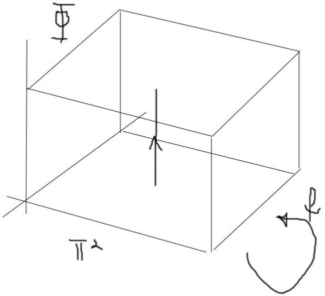
- Heisenberg nilflows have a section isomorphic to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, on which the Poincaré first return map to Σ is the *linear skew shift*:

$$f(x, y) := (x + \alpha, y + x + \beta), \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

- the Heisenberg nilflow is smoothly isomorphic to a *special flow* over f under a constant roof function Φ.

M is isomorphic to $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$;
identify the base torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the top via f;

the *special flow* is the quotient of a constant speed vertical flow $\dot{z} = 1$ by $((x, y), \Phi(x, y)) \sim_\Phi (f(x, y), 0)$.



- H_t minimal iff ergodic (Auslander-Green-Hahn '63) iff $\alpha \notin \mathbb{Q}$;
- Heisenberg nilflows are never (weak) mixing.

Special flow representation of Heisenberg nilflows

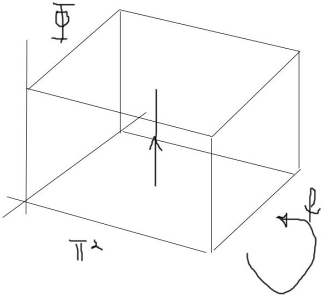
- Heisenberg nilflows have a section isomorphic to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, on which the Poincaré first return map to Σ is the *linear skew shift*:

$$f(x,y) := (x+\alpha, y+x+\beta), \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

- the Heisenberg nilflow is smoothly isomorphic to a *special flow* over f under a constant roof function Φ.

M is isomorphic to $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$;
identify the base torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the top via f;

the *special flow* is the quotient of a constant speed vertical flow $\dot{z} = 1$ by $((x,y), \Phi(x,y)) \sim_\Phi (f(x,y), 0)$.



- H_t minimal iff ergodic (Auslander-Green-Hahn '63) iff $\alpha \notin \mathbb{Q}$;
- Heisenberg nilflows are never (weak) mixing.

Special flow representation of Heisenberg nilflows

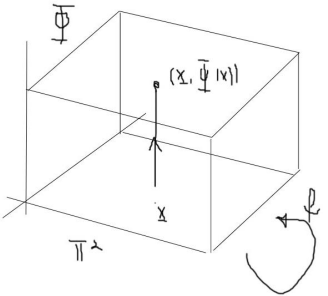
- Heisenberg nilflows have a section isomorphic to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, on which the Poincaré first return map to Σ is the *linear skew shift*:

$$f(x, y) := (x + \alpha, y + x + \beta), \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

- the Heisenberg nilflow is smoothly isomorphic to a *special flow* over f under a constant roof function Φ.

M is isomorphic to $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$;
identify the base torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the top via f;

the *special flow* is the quotient of a constant speed vertical flow $\dot{z} = 1$ by $((x, y), \Phi(x, y)) \sim_\Phi (f(x, y), 0)$.



- H_t minimal iff ergodic (Auslander-Green-Hahn '63) iff $\alpha \notin \mathbb{Q}$;
- Heisenberg nilflows are never (weak) mixing.

Special flow representation of Heisenberg nilflows

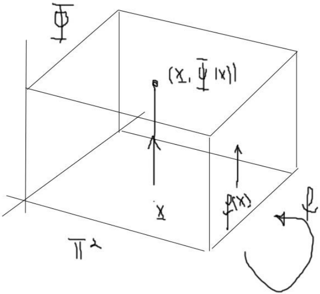
- Heisenberg nilflows have a section isomorphic to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, on which the Poincaré first return map to Σ is the *linear skew shift*:

$$f(x, y) := (x + \alpha, y + x + \beta), \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

- the Heisenberg nilflow is smoothly isomorphic to a *special flow* over f under a constant roof function Φ.

M is isomorphic to $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$; identify the base torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the top via f;

the *special flow* is the quotient of a constant speed vertical flow $\dot{z} = 1$ by $((x, y), \Phi(x, y)) \sim_\Phi (f(x, y), 0)$.



- H_t minimal iff ergodic (Auslander-Green-Hahn '63) iff $\alpha \notin \mathbb{Q}$;
- Heisenberg nilflows are never (weak) mixing.

Special flow representation of Heisenberg nilflows

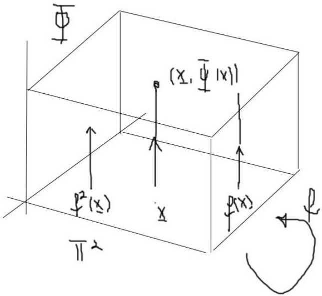
- Heisenberg nilflows have a section isomorphic to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, on which the Poincaré first return map to Σ is the *linear skew shift*:

$$f(x,y) := (x+\alpha, y+x+\beta), \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

- the Heisenberg nilflow is smoothly isomorphic to a *special flow* over f under a constant roof function Φ.

M is isomorphic to $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$; identify the base torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the top via f;

the *special flow* is the quotient of a constant speed vertical flow $\dot{z} = 1$ by $((x,y), \Phi(x,y)) \sim_\Phi (f(x,y), 0)$.



- H_t minimal iff ergodic (Auslander-Green-Hahn '63) iff $\alpha \notin \mathbb{Q}$;
- Heisenberg nilflows are never (weak) mixing.

Special flow representation of Heisenberg nilflows

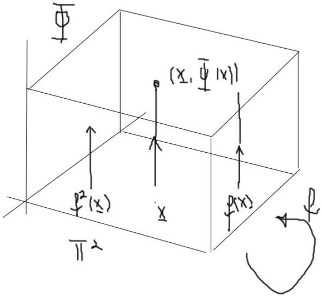
- Heisenberg nilflows have a section isomorphic to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, on which the Poincaré first return map to Σ is the *linear skew shift*:

$$f(x, y) := (x + \alpha, y + x + \beta), \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

- the Heisenberg nilflow is smoothly isomorphic to a *special flow* over f under a constant roof function Φ.

M is isomorphic to $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$;
identify the base torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$
with the top via f;

the *special flow* is the quotient of a constant speed vertical flow $\dot{z} = 1$ by $((x, y), \Phi(x, y)) \sim_\Phi (f(x, y), 0)$.



- H_t minimal iff ergodic (Auslander-Green-Hahn '63) iff $\alpha \notin \mathbb{Q}$;
- Heisenberg nilflows are never (weak) mixing.

Special flow representation of Heisenberg nilflows

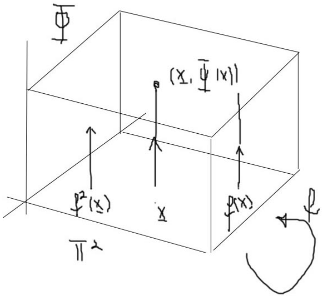
- Heisenberg nilflows have a section isomorphic to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, on which the Poincaré first return map to Σ is the *linear skew shift*:

$$f(x, y) := (x + \alpha, y + x + \beta), \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

- the Heisenberg nilflow is smoothly isomorphic to a *special flow* over f under a constant roof function Φ.

M is isomorphic to $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$;
identify the base torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$
with the top via f;

the *special flow* is the quotient of a constant speed vertical flow $\dot{z} = 1$ by $((x, y), \Phi(x, y)) \sim_\Phi (f(x, y), 0)$.

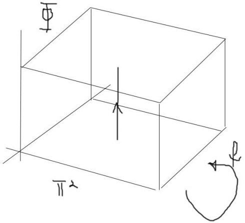


- H_t minimal iff ergodic (Auslander-Green-Hahn '63) iff $\alpha \notin \mathbb{Q}$;
- Heisenberg nilflows are never (weak) mixing.

Mixing time-changes for Heisenberg niflows

What about time changes of Heisenberg nilflows? Let H_t be minimal.

flow H_t	$\leftrightarrow$	special flow under Φ
$\widetilde{H_t}$ (smooth) time-change	$\leftrightarrow$	new (smooth) $\widetilde{\Phi}$ roof
$\widetilde{H_t}$ (smoothly) *trivial* time change	$\leftrightarrow$	cohomologus $\widetilde{\Phi} =$ $= \Phi + h \circ f - h$ for (smooth) h



Theorem (Avila, Forni, U. 2011)

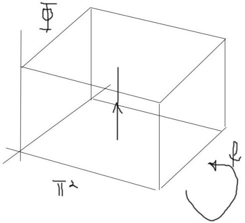
There exists a dense subspace $\mathcal{R}$ in $C^\infty(\mathbb{T}^2)$ (trigonometric polynomials) such that, for any positive function $\Phi \in \mathcal{R}$, the corresponding time-change is mixing iff it is not trivial iff not smoothly trivial.

- Mixing is typical (smoothly trivial roofs have countable codim).
- Mixing roofs are concretely described (by invariant distributions) $\Rightarrow$ explicit examples (e.g. $\Phi(x, y) = \sin(2\pi y) + 2$.)

Mixing time-changes for Heisenberg niflows

What about time changes of Heisenberg nilflows? Let H_t be minimal.

flow H_t	$\leftrightarrow$	special flow under Φ
$\widetilde{H_t}$ (smooth) time-change	$\leftrightarrow$	new (smooth) $\widetilde{\Phi}$ roof
$\widetilde{H_t}$ (smoothly) *trivial* time change	$\leftrightarrow$	cohomologus $\widetilde{\Phi} = \Phi + h \circ f - h$ for (smooth) h



Theorem (Avila, Forni, U. 2011)

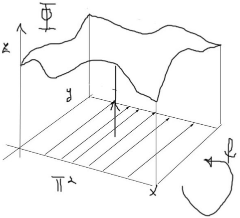
There exists a dense subspace $\mathcal{R}$ in $C^\infty(\mathbb{T}^2)$ (trigonometric polynomials) such that, for any positive function $\Phi \in \mathcal{R}$, the corresponding time-change is mixing iff it is not trivial iff not smoothly trivial.

- Mixing is typical (smoothly trivial roofs have countable codim).
- Mixing roofs are concretely described (by invariant distributions) $\Rightarrow$ explicit examples (e.g. $\Phi(x, y) = \sin(2\pi y) + 2$.)

Mixing time-changes for Heisenberg niflows

What about time changes of Heisenberg nilflows? Let H_t be minimal.

flow H_t	$\leftrightarrow$	special flow under Φ
$\widetilde{H_t}$ (smooth) time-change	$\leftrightarrow$	new (smooth) $\widetilde{\Phi}$ roof
$\widetilde{H_t}$ (smoothly) *trivial* time change	$\leftrightarrow$	cohomologus $\widetilde{\Phi} = \Phi + h \circ f - h$ for (smooth) h



Theorem (Avila, Forni, U. 2011)

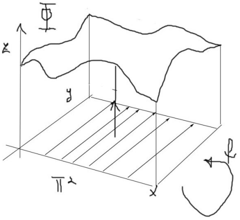
There exists a dense subspace $\mathcal{R}$ in $C^\infty(\mathbb{T}^2)$ (trigonometric polynomials) such that, for any positive function $\Phi \in \mathcal{R}$, the corresponding time-change is mixing iff it is not trivial iff not smoothly trivial.

- Mixing is typical (smoothly trivial roofs have countable codim).
- Mixing roofs are concretely described (by invariant distributions) $\Rightarrow$ explicit examples (e.g. $\Phi(x, y) = \sin(2\pi y) + 2$.)

Mixing time-changes for Heisenberg niflows

What about time changes of Heisenberg nilflows? Let H_t be minimal.

flow H_t	$\leftrightarrow$	special flow under Φ
$\widetilde{H}_t$ (smooth) time-change	$\leftrightarrow$	new (smooth) $\widetilde{\Phi}$ roof
$\widetilde{H}_t$ (smoothly) *trivial* time change	$\leftrightarrow$	cohomologus $\widetilde{\Phi} =$ $= \Phi + h \circ f - h$ for (smooth) h



Theorem (Avila, Forni, U. 2011)

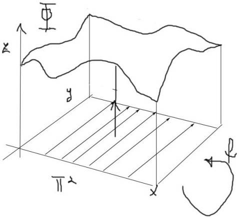
There exists a dense subspace $\mathcal{R}$ in $C^\infty(\mathbb{T}^2)$ (trigonometric polynomials) such that, for any positive function $\Phi \in \mathcal{R}$, the corresponding time-change is mixing iff it is not trivial iff not smoothly trivial.

- Mixing is typical (smoothly trivial roofs have countable codim).
- Mixing roofs are concretely described (by invariant distributions) $\Rightarrow$ explicit examples (e.g. $\Phi(x, y) = \sin(2\pi y) + 2$.)

Mixing time-changes for Heisenberg niflows

What about time changes of Heisenberg nilflows? Let H_t be minimal.

flow H_t	$\leftrightarrow$	special flow under Φ
$\widetilde{H}_t$ (smooth) time-change	$\leftrightarrow$	new (smooth) $\widetilde{\Phi}$ roof
$\widetilde{H}_t$ (smoothly) *trivial* time change	$\leftrightarrow$	cohomologus $\widetilde{\Phi} = \Phi + h \circ f - h$ for (smooth) h



Theorem (Avila, Forni, U. 2011)

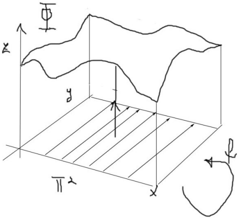
There exists a dense subspace $\mathcal{R}$ in $C^\infty(\mathbb{T}^2)$ (trigonometric polynomials) such that, for any positive function $\Phi \in \mathcal{R}$, the corresponding time-change is mixing iff it is not trivial iff not smoothly trivial.

- Mixing is typical (smoothly trivial roofs have countable codim).
- Mixing roofs are concretely described (by invariant distributions) $\Rightarrow$ explicit examples (e.g. $\Phi(x, y) = \sin(2\pi y) + 2$.)

Mixing time-changes for Heisenberg niflows

What about time changes of Heisenberg nilflows? Let H_t be minimal.

flow H_t	$\leftrightarrow$	special flow under Φ
$\widetilde{H_t}$ (smooth) time-change	$\leftrightarrow$	new (smooth) $\widetilde{\Phi}$ roof
$\widetilde{H_t}$ (smoothly) *trivial* time change	$\leftrightarrow$	cohomologus $\widetilde{\Phi} =$ $= \Phi + h \circ f - h$ for (smooth) h



Theorem (Avila, Forni, U. 2011)

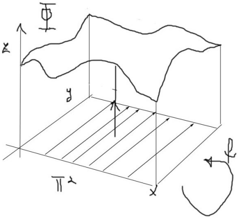
There exists a dense subspace $\mathcal{R}$ in $C^\infty(\mathbb{T}^2)$ (trigonometric polynomials) such that, for any positive function $\Phi \in \mathcal{R}$, the corresponding time-change is mixing iff it is not trivial iff not smoothly trivial.

- Mixing is typical (smoothly trivial roofs have countable codim).
- Mixing roofs are concretely described (by invariant distributions) $\Rightarrow$ explicit examples (e.g. $\Phi(x, y) = \sin(2\pi y) + 2$.)

Mixing time-changes for Heisenberg niflows

What about time changes of Heisenberg nilflows? Let H_t be minimal.

flow H_t	$\leftrightarrow$	special flow under Φ
$\widetilde{H}_t$ (smooth) time-change	$\leftrightarrow$	new (smooth) $\widetilde{\Phi}$ roof
$\widetilde{H}_t$ (smoothly) *trivial* time change	$\leftrightarrow$	cohomologus $\widetilde{\Phi} = = \Phi + h \circ f - h$ for (smooth) h

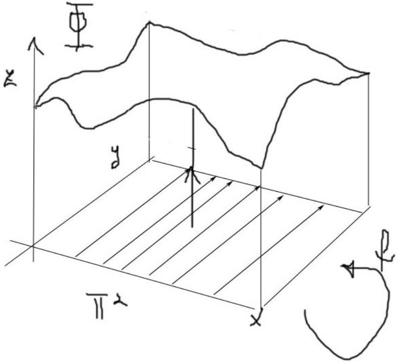


Theorem (Avila, Forni, U. 2011)

There exists a dense subspace $\mathcal{R}$ in $C^\infty(\mathbb{T}^2)$ (trigonometric polynomials) such that, for any positive function $\Phi \in \mathcal{R}$, the corresponding time-change is mixing iff it is not trivial iff not smoothly trivial.

- Mixing is typical (smoothly trivial roofs have countable codim).
- Mixing roofs are concretely described (by invariant distributions) $\Rightarrow$ explicit examples (e.g. $\Phi(x, y) = \sin(2\pi y) + 2$.)

Shearing for mixing time-changes of Heisenberg nilflows



Consider y-fibers $[0,1] \times \{y\} \subset \mathbb{T}^2$.

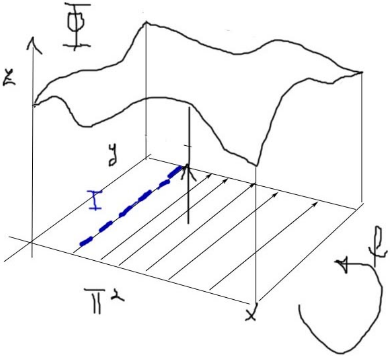
For each $t > 0$ Cover large set of each fiber for large set of y with intervals I s.t.

estimate shear via Birkhoff sums of $\phi(x,y) = \Phi(x,y) - \int \Phi(x,y)\,\mathrm{d}y$:

ϕ not couboundary (Gottshalk-Hedlund type argument) $\Rightarrow \forall C > 1$

$\mathrm{Leb}((x,y)\, s.t. \quad |\phi_n(x,y)| > C) \xrightarrow{n\to\infty} 1\,.$

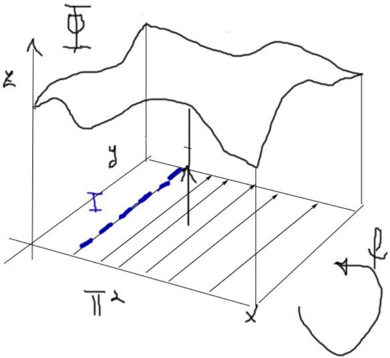
Shearing for mixing time-changes of Heisenberg nilflows

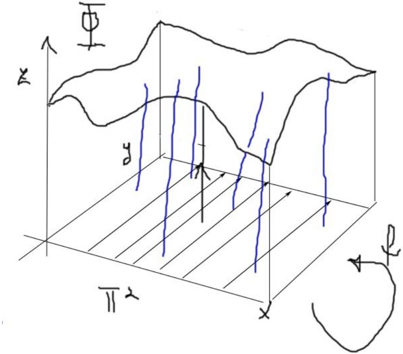


Consider y-fibers $[0,1] \times \{y\} \subset \mathbb{T}^2$. For each $t > 0$ Cover large set of each fiber for large set of y with intervals I s.t.

estimate shear via Birkhoff sums of $\phi(x,y) = \Phi(x,y) - \int \Phi(x,y)\, \mathrm{d}y$: ϕ not couboundary (Gottshalk-Hedlund type argument) $\Rightarrow \forall C > 1$ $\mathrm{Leb}((x,y)\, s.t. \quad |\phi_n(x,y)| > C) \xrightarrow{n\to\infty} 1\,.$

Shearing for mixing time-changes of Heisenberg nilflows



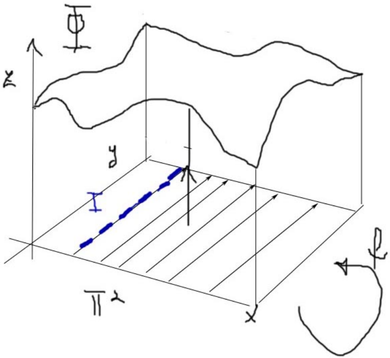


Consider y-fibers $[0,1] \times \{y\} \subset \mathbb{T}^2$. For each $t > 0$ Cover large set of each fiber for large set of y with intervals I s.t.

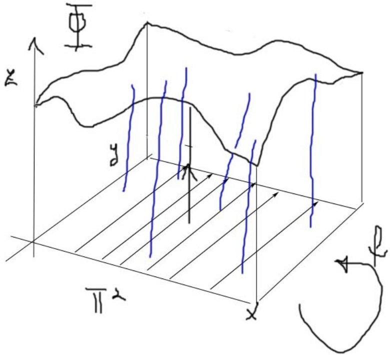
the image $f_t^{\Phi}(I)$ for $t >> 1$ each interval I looks as above (stretched in the z direction and shadows a long orbit of f)

estimate shear via Birkhoff sums of $\phi(x,y) = \Phi(x,y) - \int \Phi(x,y)\,\mathrm{d}y$:
ϕ not couboundary (Gottshalk-Hedlund type argument) $\Rightarrow \forall C > 1$
$\mathrm{Leb}((x,y)\, s.t. \quad |\phi_n(x,y)| > C) \xrightarrow{n\to\infty} 1$.

Shearing for mixing time-changes of Heisenberg nilflows



Consider y-fibers $[0,1] \times \{y\} \subset \mathbb{T}^2$. For each $t > 0$ Cover large set of each fiber for large set of y with intervals I s.t.



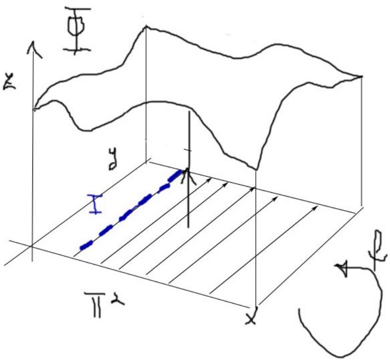
the image $f_t^{\Phi}(I)$ for $t >> 1$ each interval I looks as above (stretched in the z direction and shadows a long orbit of f)

estimate shear via Birkhoff sums of $\phi(x,y) = \Phi(x,y) - \int \Phi(x,y)\,\mathrm{d}y$:

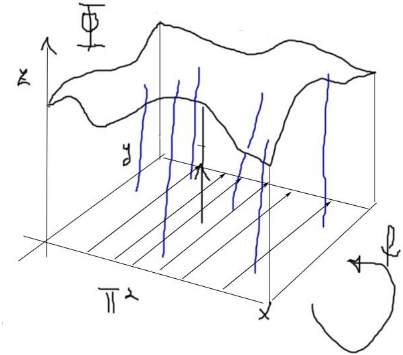
ϕ not couboundary (Gottshalk-Hedlund type argument) $\Rightarrow \forall C > 1$

$\mathrm{Leb}((x,y)\, s.t. \quad |\phi_n(x,y)| > C) \xrightarrow{n\to\infty} 1$.

Shearing for mixing time-changes of Heisenberg nilflows



Consider y-fibers $[0,1] \times \{y\} \subset \mathbb{T}^2$. For each $t > 0$ Cover large set of each fiber for large set of y with intervals I s.t.



the image $f_t^{\Phi}(I)$ for $t >> 1$ each interval I looks as above (stretched in the z direction and shadows a long orbit of f)

estimate shear via Birkhoff sums of $\phi(x,y) = \Phi(x,y) - \int \Phi(x,y)\,\mathrm{d}y$:
ϕ not couboundary (Gottshalk-Hedlund type argument) $\Rightarrow \forall C > 1$
$\mathrm{Leb}((x,y)\, s.t. \quad |\phi_n(x,y)| > C) \xrightarrow{n\to\infty} 1\,.$

Summary

We discussed mixing and choatic properties of some examples of parabolic flows:

- The *horocycle flows* is mixing (Marcus) with polynomial speed (Ratner); time-changes are still mixing (Marcus) with polynomial speed and have absolutely continuous spectrum (Forni-U')
- *Area preserving flows on surfaces*: mixing is delicate and depends on singularities type (Sinai-Khanin, U'07, U'10, U'11 ...)
- *Nilflows on nilmanifolds* are never (weak) mixing, but time-changes of Heisenberg nilflows are typically mixing (Avila-Forni-U').

Philosophy: If a parabolic flow is not mixing, can one reparametrize it (find a time-change) such that it becomes mixing? mixing with polynomial decay of correlations?

Yes for Heisenberg nilflows (other smooth roofs? other nilflows? speed of mixing?), no for locally Hamiltonian flows (elliptic flows with singularities).

... general theory for parabolic flows?