

Renewal-type limit theorem for the Gauss map and continued fractions

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Dedicated to the memory of Bill Parry

Abstract. In this paper we prove a renewal-type limit theorem. Given $\alpha \in (0, 1) \setminus \mathbb{Q}$ and $R > 0$, let q_{n_R} be the first denominator of the convergents of α which exceeds R . The main result in the paper is that the ratio q_{n_R}/R has a limiting distribution as R tends to infinity. The existence of the limiting distribution uses mixing of a special flow over the natural extension of the Gauss map.

1. Introduction

1.1. *Main result.* For $\alpha \in (0, 1) \setminus \mathbb{Q}$, denote the continued fraction expansion of α by

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}} = [a_1, a_2, \dots, a_n, \dots]$$

where $a_n \in \mathbb{N}_+$ are the entries of the continued fraction and $\{p_n/q_n\}_{n \in \mathbb{N}_+}$ are the convergents of α , i.e. $p_n/q_n = [a_1, a_2, \dots, a_n]$ with $(p_n, q_n) = 1$.

We prove the following theorem.

THEOREM 1.1. *Given $R > 0$, introduce*

$$n_R = \min\{n \in \mathbb{N} \mid q_n > R\}.$$

Also fix $N \geq 0$. Then the ratio q_{n_R}/R and the entries a_{n_R-k} for $0 \leq k < N$ have a joint limiting distribution, as R tends to infinity, with respect to the Gauss measure μ_1 given by the density $d\mu_1/d\alpha = (\ln 2(1 + \alpha))^{-1}$.

Theorem 1.1 means that for each $N \geq 0$ there exists a probability measure P_N on $(1, \infty) \times \mathbb{N}_+^N$ such that, for all $a, b > 1$, $c_k \in \mathbb{N}_+$, $0 \leq k < N$,

$$\mu_1 \left\{ \alpha : a < \frac{q_{n_R}}{R} < b, a_{n_R-k} = c_k, 0 \leq k < N \right\} \xrightarrow{R \rightarrow \infty} P_N((a, b) \times \{c_0\} \times \dots \times \{c_{N-1}\}). \quad (1)$$

In Theorem 1.1, instead of μ_1 , one can consider any absolutely continuous measure, but we do not dwell on this.

1.2. *Applications.* Theorem 1.1 is useful in many applications. As an example, we refer to the following two papers. In [1], the authors consider the problem of the limiting behavior of large Frobenius numbers, initially investigated by Arnold [8]. If $a = (a_1, \dots, a_n)$ is an n -tuple of positive integers which are coprime, the Frobenius number $F(a)$ is the smallest F such that any integer $t \geq F$ can be written in the form $t = \sum_{j=1}^n x_j a_j$ where x_j are non-negative integers. Let Ω_N be the ensemble of all coprime n -tuples with entries less than N with uniform probability distribution. In the case $n = 3$, the existence of the limiting distribution of $(1/N^{3/2})F(a)$ as N tends to infinity is proved using a discrete version of Theorem 1.1.

In [7], the following trigonometric sums are considered:

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{1 - e^{2\pi i(n\alpha+x)}}, \quad (x, \alpha) \in (0, 1) \times (0, 1),$$

where $(0, 1) \times (0, 1)$ is endowed with uniform probability distribution. The authors prove that such trigonometric sums (and, more generally, the Birkhoff sums of a function with a singularity of type $1/x$ over a rotation) have a non-trivial joint limiting distribution in x and α as N tends to infinity. Also in this case the proof of the existence of the limiting distribution is based on the existence of the limiting distribution in Theorem 1.1.

1.3. *Outline.* The main idea of the proof of Theorem 1.1 is to reformulate the problem in terms of a certain special flow over the natural extension of the Gauss map and to exploit mixing of the flow to prove the existence of the limiting distribution. The definitions of the natural extension of the Gauss map and of special flows are recalled in §2. The reduction to a special flow is shown in §3 and the existence of the limiting distribution is proved in §4. The same special flow was also considered in [3] and the proof that the special flow is mixing is recalled in §5.

2. *Definitions*

2.1. *Gauss map.* Let \mathcal{G} be the Gauss map, i.e. the transformation on $(0, 1)$ given by $\alpha \mapsto \mathcal{G}(\alpha) = \{1/\alpha\}$, where $\{\cdot\}$ denotes the fractional part. The Gauss measure μ_1 is invariant under \mathcal{G} . The sequence $\{a_n\}_{n \in \mathbb{N}_+}$ can be seen as a symbolic coding for the orbit $\{\mathcal{G}^n \alpha\}_{n \in \mathbb{N}}$, since $a_n = [(\mathcal{G}^{n-1}(\alpha))^{-1}]$ where $[\cdot]$ denotes the integer part. A point $\alpha \in (0, 1) \setminus \mathbb{Q}$ will often be identified with the infinite sequence $\{a_n\}_{n \in \mathbb{N}_+}$ in $\mathbb{N}_+^{\mathbb{N}_+}$.

For convenience, we will use also the following notation:

$$[a_0; a_1, \dots, a_n, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{\ddots}}, \quad [a_0; a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_n}}}.$$

2.2. *Natural extension of the Gauss map.* The natural extension $\hat{\mathcal{G}}$ of the Gauss map \mathcal{G} acts on infinite bi-sided sequences $\{a_n\}_{n \in \mathbb{Z}} \in \mathbb{N}_+^{\mathbb{Z}}$ as the two-sided shift, i.e. $\hat{\mathcal{G}}\{a_n\}_{n \in \mathbb{Z}} = \{a'_n\}_{n \in \mathbb{Z}}$ where $a'_n = a_{n+1}$. The map $\hat{\mathcal{G}}$ admits the following geometric interpretation.

Consider the domain $D(\hat{\mathcal{G}}) = (0, 1) \setminus \mathbb{Q} \times (0, 1) \setminus \mathbb{Q}$. Let us identify the sequence $\{a_n\}_{\mathbb{Z}}$ with the point $\hat{\alpha} = (\hat{\alpha}^-, \hat{\alpha}^+) \in D(\hat{\mathcal{G}})$, which is given by

$$\hat{\alpha}^+ = [a_1, a_2, \dots, a_n, \dots]; \quad \hat{\alpha}^- = [a_0, a_{-1}, \dots, a_{-n}, \dots]. \tag{2}$$

Then $\hat{\mathcal{G}}(\hat{\alpha}) = \hat{\beta}$ where $\hat{\beta} = (\hat{\beta}^-, \hat{\beta}^+)$ and

$$\hat{\beta}^+ = \mathcal{G}(\hat{\alpha}^+) = \left\{ \frac{1}{\hat{\alpha}^+} \right\} = \frac{1}{\hat{\alpha}^+} - a_1; \quad \hat{\beta}^- = \frac{1}{[1/\hat{\alpha}^+] + \hat{\alpha}^-} = \frac{1}{a_1 + \hat{\alpha}^-}.$$

Clearly, denoting by π the projection $\pi(\hat{\alpha}) = \hat{\alpha}^+$ or equivalently $\pi(\{a_n\}_{n \in \mathbb{Z}}) = \{a_n\}_{n \in \mathbb{N}_+}$, we have $\pi \hat{\mathcal{G}} = \mathcal{G} \pi$. The sequence $\{a_n\}_{n \in \mathbb{Z}}$ is the symbolic coding of $\hat{\alpha} \in D(\hat{\mathcal{G}})$ under $\hat{\mathcal{G}}$ in the sense that $a_n = [(\pi \hat{\mathcal{G}}^{n-1} \hat{\alpha})^{-1}]$.

The map $\hat{\mathcal{G}}$ admits a natural invariant probability measure μ_2 on $D(\hat{\mathcal{G}})$ which is given by the density

$$\rho_2(\alpha^-, \alpha^+) = \frac{1}{\ln 2(1 + \alpha^- \alpha^+)^2}.$$

Remark 2.1. The Gauss measure μ_1 can be recovered as $\pi_* \mu_2$, i.e. for each measurable set $A \subset (0, 1)$, we have $\mu_1(A) = \mu_2(\pi^{-1}A)$.

Given any $\hat{\alpha} = \{a_n\}_{n \in \mathbb{Z}} \in D(\hat{\mathcal{G}})$, $\hat{\alpha}^-$ and $\hat{\alpha}^+$ will always denote the two components of $\hat{\alpha} \in D(\hat{\mathcal{G}})$ which are given explicitly in terms of the a_n by (2).

Let $q_n = q_n(\hat{\alpha}) = q_n(\hat{\alpha}^+)$, $n \in \mathbb{N}_+$, be the sequence of denominators of the convergents of $\hat{\alpha}^+$. Also, given $R > 0$, $n_R(\hat{\alpha})$ and $q_{n_R}(\hat{\alpha})$ are set equal to the analogous quantities defined for $\alpha = \hat{\alpha}^+$.

Remark 2.2. By construction, the functions q_n (for any $n \in \mathbb{N}_+$), n_R and q_{n_R} (for any $R > 0$) on $D(\hat{\mathcal{G}})$ are constant on fibers $\pi^{-1}\alpha$, $\alpha \in (0, 1) \setminus \mathbb{Q}$.

2.3. *Cylinders.* For $b_k \in \mathbb{N}_+$, $k = 1, \dots, n$, denote by $\mathcal{C}([b_1, \dots, b_n])$ the cylinder

$$\mathcal{C}([b_1, \dots, b_n]) = \{\alpha = \{a_n\}_{n \in \mathbb{N}_+} \in (0, 1) \setminus \mathbb{Q} : a_k = b_k, 1 \leq k \leq n\}.$$

We will denote by \mathcal{C}_n^+ the set of all cylinders of length n , i.e. the set of all $\mathcal{C}([b_1, \dots, b_n])$ with $b_k \in \mathbb{N}_+$ for $1 \leq k \leq n$. Moreover, if $\mathcal{C} \in \mathcal{C}_n^+$, we will denote by $\hat{\mathcal{C}}$ the set $\pi^{-1}\mathcal{C} \subset D(\hat{\mathcal{G}})$.

More generally, given $b_k \in \mathbb{N}_+$, $k = 0, \pm 1, \dots, \pm n$, let

$$\mathcal{C}([b_{-n}, \dots, b_0; b_1, \dots, b_n]) = \{\alpha = \{a_n\}_{n \in \mathbb{Z}} \in D(\hat{\mathcal{G}}) : a_k = b_k, -n \leq k \leq n\}$$

and let \mathcal{C}_n be the set of all bi-sided cylinders of length n , i.e. the set, as $b_k \in \mathbb{N}_+$ for $-n \leq k \leq n$, of all $\mathcal{C}([b_{-n}, \dots, b_0; b_1, \dots, b_n])$.

Remark 2.3. From the expression of the Gauss density and Remark 2.1, we get that $\mu_2(\hat{\mathcal{C}}([n])) = \mu_1(\mathcal{C}([n])) = \mu_1(1/(n+1), 1/n) = O(1/n^2)$.

2.4. *Special flows.* Consider a probability space (D, \mathcal{B}, μ_2) and an invertible map $F : D \rightarrow D$ which preserves μ_2 . Let $\varphi : D \rightarrow \mathbb{R}_+$ be a strictly positive function such that $\int_D \varphi(\alpha) d\mu_2 < \infty$. The phase space D_Φ of the special flow is the subset of $D \times \mathbb{R}$ given by

$$D_\Phi = \{(x, y) \mid x \in D : 0 \leq y < \varphi(x)\}$$

and can be depicted as the set of points below the graph of the roof function φ . Consider the normalized measure μ_3 which is the restriction to D_Φ of the product measure $(\int_D \varphi(\alpha) d\mu_2)^{-1} \mu_2 \times \lambda$, where λ denotes the Lebesgue measure on \mathbb{R} .

The *special flow* $\{\Phi_t\}_{t \in \mathbb{R}}$ built over F with the help of the roof function φ is a one-parameter group of μ_3 -measure-preserving transformations of D_Φ whose action is generated by the following two relations:

$$\begin{cases} \Phi_t(x, y) = (x, y + t) & \text{if } 0 \leq y + t < \varphi(x); \\ \Phi_{\varphi(x)}(x, 0) = (F(x), 0). \end{cases} \tag{3}$$

Under the action of the flow, a point of $(x, y) \in D_\Phi$ moves with unit velocity along the vertical line up to the point $(x, \varphi(x))$, then jumps instantly to the point $(F(x), 0)$, according to the base transformation. Afterwards it continues its motion along the vertical line until the next jump and so on (see e.g. [2]). Abusing the notation, we will often identify any set $C \subset D$ with $C \times \{0\} \subset D \times \{0\} \subset D_\Phi$.

We will denote by

$$S_0(\varphi, F)(x) := 0; \quad S_r(\varphi, F)(x) = S_r(\varphi)(x) := \sum_{i=0}^{r-1} \varphi(F^i(x)), \quad x \in D, r \in \mathbb{N}^+,$$

the r th non-renormalized *Birkhoff sum* of φ along the trajectory of x under F .

Let $t > 0$. Given $x \in D$ denote by $r(x, t)$ the integer uniquely defined by

$$r(x, t) := \min\{r \in \mathbb{N} \mid S_r(\varphi)(x) > t\}. \tag{4}$$

Then $r(x, t) - 1$ gives the number of discrete iterations of F which the point $(x, 0)$ undergoes before time t . According to this notation, the flow Φ_t defined by (3) acts as

$$\Phi_t(x, 0) = (F^{r(x,t)-1}(x), t - S_{r(x,t)-1}(\varphi)(x)). \tag{5}$$

For $t < 0$, the action of the flow is defined as the inverse map.

3. Reduction to a special flow

Let us first define the special flow that we are going to consider.

3.1. *Roof function.* Consider the following positive real-valued function on $D(\hat{\mathcal{G}})$:

$$\varphi(\hat{\alpha}) = \ln \left(a_1 + \frac{1}{a_0 + \frac{1}{a_{-1} + \dots}} \right) = -\ln(\hat{\mathcal{G}}\hat{\alpha})^-. \tag{6}$$

The reason for this definition will be clear after Lemma 3.1. It is easy to see, from Remark 2.3, that the function φ is integrable with respect to μ_2 .

Remark 3.1. Let $\mathcal{C} \in \mathcal{C}_n$, $n \geq 1$, be any cylinder. Then there exists $\delta = \delta(\mathcal{C}) > 0$ and $M = M(\mathcal{C}) > 0$ such that $\inf_{\hat{\alpha} \in \mathcal{C}} \varphi(\hat{\alpha}) \geq \delta$ and $\sup_{\hat{\alpha} \in \mathcal{C}} \varphi(\hat{\alpha}) \leq M$. It follows by noting that $\varphi(\hat{\alpha}) \geq \ln(1 + (1/(a_0 + 1)))$, and $\varphi(\hat{\alpha}) \leq \ln(a_1 + 1)$.

Let us consider the special flow $\{\Phi_t\}_{t \in \mathbb{R}}$ built over $\hat{\mathcal{G}}$ under the function φ and let $\mu_3 = (\int \varphi d\mu_2)^{-1} \mu_2 \times \lambda$ be the Φ_t -invariant probability measure. Let us recall that $\{\Phi_t\}_{t \in \mathbb{R}}$ is said to be mixing if, for all Borel subsets A, B of D_Φ , we have

$$\lim_{t \rightarrow \infty} \mu_3(\Phi_{-t}(A) \cap B) = \mu_3(A)\mu_3(B).$$

PROPOSITION 3.1. *The flow $\{\Phi_t\}_{t \in \mathbb{R}}$ is mixing.*

Proposition 3.1 was proved in [3]. We recall the proof in §5.

3.2. *Denominators growth and Birkhoff sums.* Let us show that $\ln q_n$ can be approximated by Birkhoff sums of φ . Indeed, put

$$f_n(\hat{\alpha}) = \ln q_n(\hat{\alpha}) - S_n(\varphi)(\hat{\alpha}). \tag{7}$$

LEMMA 3.1. *There exists a function f on $D(\hat{\mathcal{G}})$ such that f_n converges to f uniformly in $\hat{\alpha}$ and exponentially fast in n , i.e.*

$$\ln q_n(\hat{\alpha}) = S_n(\varphi)(\hat{\alpha}) + f(\hat{\alpha}) + \epsilon_n(\hat{\alpha}), \quad \sup_{\hat{\alpha} \in D(\hat{\mathcal{G}})} |\epsilon_n(\hat{\alpha})| = O(2^{-n}). \tag{8}$$

Proof. Let $q_0 = 1, q_{-1} = 0$ and $r_n = q_n/q_{n-1}$ for $n \in \mathbb{N}_+$, so that $q_n = \prod_{k=1}^n r_k$. From the well-known relation $q_{k+1} = a_{k+1}q_k + q_{k-1}$ which holds for $k \geq 0$ (see e.g. [5]), we get by recursion that, for $k \geq 1$,

$$r_{k+1} = a_{k+1} + \frac{1}{r_k} = [a_{k+1}; a_k, \dots, a_1] \tag{9}$$

and hence

$$\log q_n = \sum_{k=1}^n \ln[a_k; a_{k-1}, \dots, a_1]. \tag{10}$$

Since $(\hat{\mathcal{G}}^k \hat{\alpha})^- = [a_k, a_{k-1}, \dots]$, from the definition (6) of φ we get

$$S_n(\varphi)(\hat{\alpha}) = \sum_{k=1}^n \ln \frac{1}{(\hat{\mathcal{G}}^k \hat{\alpha})^-} = \sum_{k=1}^n \ln[a_k; a_{k-1}, \dots]. \tag{11}$$

Thus, from (7) and (9),

$$(f_{k+1} - f_k)(\hat{\alpha}) = \ln r_{k+1} - \varphi(\hat{\mathcal{G}}^k \hat{\alpha}) = \ln r_{k+1} - \ln \frac{1}{(\hat{\mathcal{G}}^{k+1} \hat{\alpha})^-} = \ln \frac{[a_{k+1}; a_k, \dots, a_1]}{[a_{k+1}; a_k, \dots]}.$$

In order to estimate the last term, consider $\beta_k = [a_k, a_{k-1}, \dots]$ and let $\{p_{k,m}/q_{k,m}\}_m$ be the convergents of β_k , so that in particular $p_{k,k}/q_{k,k} = [a_k, \dots, a_1]$. Recalling the well-known formula (see [5])

$$\left| \beta_k - \frac{p_{k,m}}{q_{k,m}} \right| \leq \frac{1}{(q_{k,m})^2} \tag{12}$$

and using the fact that for all sequences of denominators of convergents $q_{k,m} \geq 2^{(m-1)/2}$ [5, Theorem 12] and the fact that $[a_{k+1}; a_k, \dots] \geq 1$, we get

$$\left| \ln \frac{[a_{k+1}; a_k, \dots, a_1]}{[a_{k+1}; a_k, \dots]} \right| \leq \left| \ln \left(1 + \frac{(p_{k,k}/q_{k,k}) - \beta_k}{[a_{k+1}; a_k, \dots]} \right) \right| \leq 2 \left| \beta_k - \frac{p_{k,k}}{q_{k,k}} \right| \leq 2^{2-k}. \tag{13}$$

Hence, (13) shows that we can well define

$$f(\hat{\alpha}) = \sum_{k=0}^{\infty} (f_{k+1} - f_k)(\hat{\alpha}) \tag{14}$$

and (8) clearly follows from the geometric bound of the series, with $\epsilon_n = 2^{3-n}$. □

LEMMA 3.2. *If $\alpha_1, \alpha_2 \in \mathcal{C}([a_{-n}, \dots, a_0; a_1, \dots, a_n])$, then*

$$|f(\hat{\alpha}_1) - f(\hat{\alpha}_2)| \leq C2^{-n}, \tag{15}$$

where $C > 0$ is an absolute constant.

Proof. First, by Lemma 3.1, we have, for some $C_1 > 0$,

$$|f(\hat{\alpha}_1) - f(\hat{\alpha}_2)| \leq C_1 2^{-n} + |f_n(\hat{\alpha}_1) - f_n(\hat{\alpha}_2)|. \tag{16}$$

Let us estimate the second term of the right-hand side. Let $\hat{\alpha}_1 = \{a'_m\}_{m \in \mathbb{Z}}$, $\hat{\alpha}_2 = \{a''_m\}_{m \in \mathbb{Z}}$, where by assumption $a'_m = a''_m = a_m$ for $-n \leq m \leq n$. From (11),

$$S_n(\varphi)(\hat{\alpha}_1) - S_n(\varphi)(\hat{\alpha}_2) = \sum_{k=1}^n \ln \frac{[a_k; a_{k-1}, \dots, a_{-n}, a'_{-n-1}, a'_{-n-2}, \dots]}{[a_k; a_{k-1}, \dots, a_{-n}, a''_{-n-1}, a''_{-n-2}, \dots]}. \tag{17}$$

Arguing as in Lemma 3.1, for $k = 1, \dots, n$, consider

$$\beta'_k = [a_{k-1}, a_{k-2}, \dots, a_{-n}, a'_{-n-1}, \dots] \quad \text{and} \quad \beta''_k = [a_{k-1}, a_{k-2}, \dots, a_{-n}, a''_{-n-1}, \dots]$$

and denote by $\{p'_{k,n}/q'_{k,n}\}_n$ and $\{p''_{k,n}/q''_{k,n}\}_n$ their respective convergents. Since $p'_{k,k+n}/q'_{k,k+n} = p''_{k,k+n}/q''_{k,k+n}$, from (12) and $q_m \geq 2^{(m-1)/2}$ we get

$$|\beta'_k - \beta''_k| \leq \left| \beta'_k - \frac{p'_{k,k+n}}{q'_{k,k+n}} \right| + \left| \beta''_k - \frac{p''_{k,k+n}}{q''_{k,k+n}} \right| \leq \frac{1}{(q'_{k,k+n})^2} + \frac{1}{(q''_{k,k+n})^2} \leq 2^{2-k-n}.$$

Hence, as in (13), for some $C_2 > 0$,

$$\left| \sum_{k=1}^n \ln \frac{[a_k; \dots, a_{-n}, a'_{-n-1}, \dots]}{[a_k; \dots, a_{-n}, a''_{-n-1}, \dots]} \right| \leq \sum_{k=1}^n 2|\beta'_k - \beta''_k| \leq \sum_{k=1}^n 2^{3-k-n} \leq C_2 2^{-n}.$$

Hence, by (7), noting that we also have $q_n(\hat{\alpha}_1^+) = q_n(\hat{\alpha}_2^+)$, this implies that $|f_n(\hat{\alpha}_1) - f_n(\hat{\alpha}_2)| \leq C_2 2^{-n}$ and gives the desired estimate of (16). □

3.3. *Comparing renewal times.* Given $\hat{\alpha}$ and $R > 0$, we want to choose T as a function of R so that we can compare $n_R(\hat{\alpha})$ and $r(\hat{\alpha}, T)$, where $r(\hat{\alpha}, T) - 1$ is the discrete number of iterations undergone by $\Phi_t(\hat{\alpha}, 0)$ for $t \leq T$, see (4). Let us recall that $n_R(\hat{\alpha})$ is uniquely determined by

$$\ln q_{n_R(\hat{\alpha})-1} \leq \ln R < \ln q_{n_R(\hat{\alpha})}.$$

By (7), the previous inequality can be rewritten as

$$S_{n_R(\hat{\alpha})-1}(\varphi)(\hat{\alpha}) + f_{n_R(\hat{\alpha})-1}(\hat{\alpha}) \leq \ln R < S_{n_R(\hat{\alpha})}(\varphi)(\hat{\alpha}) + f_{n_R(\hat{\alpha})}(\hat{\alpha}). \tag{18}$$

To avoid the dependence of the time on $\hat{\alpha}$, let us localize to a set of $\mathcal{C} \subset D(\hat{\mathcal{G}})$ and denote $f_{\mathcal{C}} = \sup_{\hat{\alpha} \in \mathcal{C}} f(\hat{\alpha})$. Assume that for all $\hat{\alpha} \in \mathcal{C}$ we have $|f(\hat{\alpha}) - f_{\mathcal{C}}| \leq \epsilon/2$ (such sets will be constructed in the Proof of Theorem 1.1).

Let us first show that on large measure sets the growth of n_R is guaranteed by the growth of R .

LEMMA 3.3. *For each measurable $\mathcal{C} \subset D(\hat{\mathcal{G}})$ and $\epsilon > 0$ there exists a measurable $\mathcal{C}' \subset \mathcal{C}$ such that $\mu_2(\mathcal{C} \setminus \mathcal{C}') \leq \epsilon \mu_2(\mathcal{C})$ and $\min_{\hat{\alpha} \in \mathcal{C}'} n_R(\hat{\alpha})$ tends to infinity uniformly as R tends to infinity.*

Similarly, given $\epsilon > 0$ there exists a measurable $\mathcal{C}_\epsilon \subset (0, 1)$ such that $\mu_1((0, 1) \setminus \mathcal{C}_\epsilon) \leq \epsilon$ and $\min_{\alpha \in \mathcal{C}_\epsilon} n_R(\alpha)$ tends to infinity uniformly as R tends to infinity.

Proof. By the Lévy–Khinchin theorem, for μ_1 -a.e. $\alpha \in (0, 1)$, there exists an absolute constant $l > 0$ such that $\lim_{n \rightarrow \infty} (\ln q_n/n) = l$. By Remarks 2.1 and 2.2, the same holds for μ_2 -a.e. $\hat{\alpha} \in D(\hat{\mathcal{G}})$. By Egorov’s theorem, we can find for each $\epsilon > 0$ a measurable subset $\mathcal{C}_1 \subset \mathcal{C}$ with $\mu_2(\mathcal{C} \setminus \mathcal{C}_1) \leq (\epsilon/2)\mu_2(\mathcal{C})$ on which the convergence is uniform, so that for some \bar{n} , $(\ln q_n)/n \leq 2l$ for each $n \geq \bar{n}$. Moreover, there exists $\mathcal{C}_2 \subset \mathcal{C}$ with $\mu_2(\mathcal{C} \setminus \mathcal{C}_2) \leq (\epsilon/2)\mu_2(\mathcal{C})$ such that, on \mathcal{C}_2 , the functions $(\ln q_n)/n$ for $n = 1, \dots, \bar{n}$ are uniformly bounded. Hence, setting $\mathcal{C}' = \mathcal{C}_1 \cap \mathcal{C}_2$, $\mu_2(\mathcal{C} \setminus \mathcal{C}') \leq \epsilon \mu_2(\mathcal{C})$ and there exists a constant $c = c(\mathcal{C}, \epsilon) > 0$ such that for all $\hat{\alpha} \in \mathcal{C}'$ and all $n \in \mathbb{N}_+$ we have $\ln q_n \leq cn$. Since by definition $q_{n_R(\hat{\alpha})} > R$, this implies that $\min_{\hat{\alpha} \in \mathcal{C}'} n_R(\hat{\alpha}) \geq (c)^{-1} \ln q_{n_R(\hat{\alpha})} \geq (c)^{-1} \ln R$, from which the lemma follows. The proof of the second part proceeds in exactly the same way. □

LEMMA 3.4. *Assume that $\mathcal{C} \in \mathcal{C}_n$ and for all $\hat{\alpha} \in \mathcal{C}$ we have $|f(\hat{\alpha}) - f_{\mathcal{C}}| \leq \epsilon/2$. There exists $R_0 = R_0(\mathcal{C}) > 0$ such that, whenever $R \geq R_0$, if we set $T = T(R, \mathcal{C}) = \ln R - f_{\mathcal{C}}$ and consider*

$$U = U_{\mathcal{C}} := \{\hat{\alpha} \in \mathcal{C} : n_R(\hat{\alpha}) \neq r(\hat{\alpha}, T)\},$$

we have $\mu_2(U) \leq 7\epsilon \mu_2(\mathcal{C})$.

Hence, outside of a subset of \mathcal{C} of arbitrarily small proportion and for large R , the function $T = T(R, \mathcal{C})$ is such that $n_R(\cdot) = r(\cdot, T)$.

Proof. Let $T = T(R, \mathcal{C}) = \ln R - f_{\mathcal{C}}$. By definition, we have

$$S_{r(\hat{\alpha}, T)-1}(\varphi)(\hat{\alpha}) \leq T = \ln R - f_{\mathcal{C}} < S_{n_R(\hat{\alpha})}(\varphi)(\hat{\alpha}). \tag{19}$$

Let $\mathcal{C}' \subset \mathcal{C}$ be given by Lemma 3.3. Let us show, by comparing (19) to (18), that, as long as $R \geq R_0$ for some R_0 defined below, we have

$$U \cap \mathcal{C}' \subset U_\epsilon \cup U_{-\epsilon} \cap \mathcal{C}' \tag{20}$$

where $U_{\pm\epsilon} \subset D(\hat{\mathcal{G}})$ are defined as

$$U_{-\epsilon} = \{\hat{\alpha} : T < S_{r(\hat{\alpha}, T)}(\varphi)(\hat{\alpha}) \leq T + \epsilon\}, \quad U_\epsilon = \{\hat{\alpha} : T - \epsilon < S_{r(\hat{\alpha}, T)-1}(\varphi)(\hat{\alpha}) \leq T\}. \tag{21}$$

To show the inclusion (20), assume that $\hat{\alpha} \in \mathcal{C}'$, but $\hat{\alpha} \notin U_\epsilon \cup U_{-\epsilon}$. Then, by (21) and (19), we have

$$S_{r(\hat{\alpha}, T)-1}(\varphi)(\hat{\alpha}) \leq T - \epsilon, \quad S_{r(\hat{\alpha}, T)}(\varphi)(\hat{\alpha}) > T + \epsilon. \tag{22}$$

Choose $n_0 \gg 1$ so that by Lemma 3.1, for all $n \geq n_0 - 1$, $|f_n - f| \leq \epsilon/2$. By Lemma 3.3 we can choose R_0 so that, if $R \geq R_0$, then $n_R(\hat{\alpha}) \geq n_0$ for all $\hat{\alpha} \in \mathcal{C}'$. This, together with the assumptions on \mathcal{C} , implies that, since $\hat{\alpha} \in \mathcal{C}'$, $|f_i(\hat{\alpha}) - f_{\mathcal{C}}| < \epsilon$ for $i = n_R(\hat{\alpha})$, $i = n_R(\hat{\alpha}) - 1$. Hence, (18) gives

$$\begin{aligned} S_{n_R(\hat{\alpha})}(\varphi)(\hat{\alpha}) &> \ln R - f_{n_R(\hat{\alpha})}(\hat{\alpha}) > T - \epsilon, \\ S_{n_R(\hat{\alpha})-1}(\varphi)(\hat{\alpha}) &\leq \ln R - f_{n_R(\hat{\alpha})-1}(\hat{\alpha}) < T + \epsilon, \end{aligned}$$

which, compared with (22), since $S_n(\varphi)(\hat{\alpha})$ is increasing in n , implies that

$$r(\hat{\alpha}, T) - 1 < n_R(\hat{\alpha}) \quad \text{and} \quad n_R(\hat{\alpha}) - 1 < r(\hat{\alpha}, T).$$

Since both $n_R(\hat{\alpha})$ and $r(\hat{\alpha}, T)$ are integers, these inequalities imply $n_R(\hat{\alpha}) = r(\hat{\alpha}, T)$, i.e. $\hat{\alpha} \notin U$, proving (20) as desired.

Recalling the definition (5) of the special flow action, we can rewrite the sets $U_{\pm\epsilon}$ as

$$\begin{aligned} U_\epsilon &= \{\hat{\alpha} : 0 \leq T - S_{r(\hat{\alpha}, T)-1}(\varphi)(\hat{\alpha}) < \epsilon\} = \{(\hat{\alpha}, 0) : \Phi_T(\hat{\alpha}, 0) \in D_\Phi^\epsilon\}, \\ U_{-\epsilon} &= \{\hat{\alpha} : \varphi(\hat{\mathcal{G}}^{r(\hat{\alpha}, T)-1}\hat{\alpha}) - \epsilon \leq T - S_{r(\hat{\alpha}, T)-1}(\varphi)(\hat{\alpha}) < \varphi(\hat{\mathcal{G}}^{r(\hat{\alpha}, T)-1}\hat{\alpha})\} \\ &= \{(\hat{\alpha}, 0) : \Phi_T(\hat{\alpha}, 0) \in D_\Phi^{-\epsilon}\}, \end{aligned}$$

where $D_\Phi^\epsilon = D(\hat{\mathcal{G}}) \times [0, \epsilon)$ and $D_\Phi^{-\epsilon} = \{(\hat{\alpha}, y) : \varphi(\hat{\alpha}) - \epsilon \leq y < \varphi(\hat{\alpha})\}$.

We want to use mixing of $\{\Phi_t\}_{t \in \mathbb{R}}$ to estimate the measures of the last two sets. In order to do this, we need to ‘thicken’ them as follows. Choose $0 < \delta \leq \epsilon$ such that, by Remark 3.1, $\delta < \min_{\hat{\alpha} \in \mathcal{C}} \varphi(\hat{\alpha})$ and consider the following two subsets of D_Φ :

$$U_{\pm\epsilon}^\delta = \{(\hat{\alpha}, z) : 0 \leq z < \delta; \Phi_T(\hat{\alpha}, z) \in D_\Phi^{\pm\epsilon}\} = D_\Phi^\delta \cap \Phi_{-T} D_\Phi^{\pm\epsilon}.$$

Let us show that if $\hat{\alpha} \in U_\epsilon \cap \mathcal{C}$, then, for each $0 \leq z < \delta$, we have $(\hat{\alpha}, z) \in U_{\epsilon+\delta}^\delta$. Indeed, by choice of δ ,

$$\begin{aligned} \Phi_T(\hat{\alpha}, z) &= \Phi_{T+z}(\hat{\alpha}, 0) = \Phi_z(\hat{\mathcal{G}}^{r(\hat{\alpha}, T)-1}\hat{\alpha}, T - S_{r(\hat{\alpha}, T)-1}(\varphi)(\hat{\alpha})) \\ &= \begin{cases} (\hat{\mathcal{G}}^{r(\hat{\alpha}, T)-1}\hat{\alpha}, T + z - S_{r(\hat{\alpha}, T)-1}(\varphi)(\hat{\alpha})) & \begin{cases} \text{if } S_{r(\hat{\alpha}, T)}(\varphi)(\hat{\alpha}) > T + z \\ \text{and } r(\hat{\alpha}, T) = r(\hat{\alpha}, T + z), \end{cases} \\ (\hat{\mathcal{G}}^{r-1}\hat{\alpha}, T + z - S_{r-1}(\varphi)(\hat{\alpha})) & \begin{cases} \text{if } S_{r(\hat{\alpha}, T)}(\varphi)(\hat{\alpha}) \leq T + z \\ \text{and } r = r(\hat{\alpha}, T + z) > r(\hat{\alpha}, T). \end{cases} \end{cases} \tag{23} \end{aligned}$$

Since $r > r(\hat{\alpha}, T)$ and $S_r(\varphi)$ is increasing in r and by definition of U_ϵ , both $T + z - S_{r-1}(\varphi) \leq T + \delta - S_{r(\hat{\alpha}, T)-1}(\varphi)(\hat{\alpha}) \leq \epsilon + \delta$, and hence $(\hat{\alpha}, z) \in U_{\epsilon+\delta}^\delta$.

Reasoning in a similar way, let us also show that, if $\hat{\alpha} \in U_{-\epsilon} \cap \mathcal{C}$, then for each $0 \leq z < \delta$, we have $(\hat{\alpha}, z) \in U_{-\epsilon}^\delta \cup U_\delta^\delta$. Indeed, from (23), in the first case $T + z - S_{r(\hat{\alpha}, T)-1}(\varphi)(\hat{\alpha}) < \varphi(\hat{\mathcal{G}}^{r(\hat{\alpha}, T)-1})(\hat{\alpha})$ and since $\hat{\alpha} \in U_{-\epsilon}$, also $T + z - S_{r(\hat{\alpha}, T)-1}(\varphi)(\hat{\alpha}) \geq T - S_{r(\hat{\alpha}, T)-1}(\varphi)(\hat{\alpha}) \geq \varphi(\hat{\mathcal{G}}^{r(\hat{\alpha}, T)-1})(\hat{\alpha}) - \epsilon$, so $(\hat{\alpha}, z) \in U_{-\epsilon}^\delta$; in the second case, since $r - 1 \geq r(\hat{\alpha}, T)$, we have $S_{r-1}(\varphi)(\hat{\alpha}) \geq S_{r(\hat{\alpha}, T)}(\varphi)(\hat{\alpha}) > T$ and hence $0 \leq T + z - S_{r-1}(\varphi)(\hat{\alpha}) < z < \delta$, so $(\hat{\alpha}, z) \in U_\delta^\delta$.

Let $\mathcal{C}'_\delta = \mathcal{C}' \times [0, \delta)$. Hence, recalling also (20), we have proved that

$$\begin{aligned} (U \cap \mathcal{C}') \times [0, \delta) &\subset ((U_\epsilon \cup U_{-\epsilon}) \cap \mathcal{C}') \times [0, \delta) \subset \mathcal{C}'_\delta \cap (U_{-\epsilon}^\delta \cup U_{\delta+\epsilon}^\delta) \\ &= \mathcal{C}'_\delta \cap \Phi_{-T}(D_\Phi^{-\epsilon} \cup D_\Phi^{\delta+\epsilon}). \end{aligned}$$

Considering the measures of the above sets and using mixing (Proposition 3.1), one can find T_0 such that, as soon as $T \geq T_0$, we have

$$\begin{aligned} \mu_2(U \cap \mathcal{C}')\delta &= \mu_3(U \cap \mathcal{C}' \times [0, \delta)) \\ &\leq \mu_3(\mathcal{C}'_\delta \cap \Phi_{-T}(D_\Phi^{-\epsilon} \cup D_\Phi^{\delta+\epsilon})) \\ &\leq 2\mu_2(\mathcal{C}')\delta\mu_3(D_\Phi^{-\epsilon} \cup D_\Phi^{\delta+\epsilon}) \leq 2\mu_2(\mathcal{C}')\delta(3\epsilon), \end{aligned} \tag{24}$$

where the last inequality follows from the fact that $\mu_3(D_\Phi^{\pm\epsilon}) \leq \epsilon$ and gives $\mu_2(U \cap \mathcal{C}') \leq 6\epsilon\mu_2(\mathcal{C}')$. Enlarging R_0 if necessary so that $\ln R_0 - f_{\mathcal{C}} \geq T_0$, if $R \geq R_0$ then also $T \geq T_0$ and (24) holds. Hence $\mu_2(U) \leq \mu_2(U \setminus \mathcal{C}') + 6\epsilon\mu_2(\mathcal{C}') \leq 7\epsilon\mu_2(\mathcal{C}')$, concluding the proof of the lemma. \square

4. Existence of the limiting distribution

Proof of Theorem 1.1. Assume $b > a > 1$ and $c_k \in \mathbb{N}_+$, $0 \leq k < N$. We want to estimate the expression (1). Recalling Remark 2.2, as soon as $n_R(\alpha) > N$ we can rewrite the condition $a_{n_R(\alpha)-k} = c_k$, $0 \leq k < N$, in an equivalent way as

$$\hat{\mathcal{G}}^{n_R(\hat{\alpha})-1}\hat{\alpha} \in C_N \quad \text{where } \hat{\alpha}^+ = \alpha, \quad C_N := \hat{\mathcal{G}}^{N-1}\hat{\mathcal{C}}([c_{N-1}, c_{N-2}, \dots, c_0]), \tag{25}$$

since, if (25) holds, $\{a'_j\}_{j \in \mathbb{Z}} := \hat{\mathcal{G}}^{n_R(\hat{\alpha})-N}\hat{\alpha} = \hat{\mathcal{G}}^{-(N-1)}\hat{\mathcal{G}}^{n_R(\alpha)-1}\hat{\alpha} \in \hat{\mathcal{C}}([c_{N-1}, \dots, c_0])$, so $a'_k = c_{N-k}$ for $1 \leq j \leq N$ and $a'_j = a_{n_R(\alpha)-N+j}$ by definition of $\hat{\mathcal{G}}$; hence $a_{n_R(\alpha)-N+j} = c_{N-k}$ for $1 \leq j \leq N$ gives the desired set of equalities for $k = N - l$.

Given two functions g_1, g_2 on $D(\hat{\mathcal{G}})$, $g_1 \leq g_2$, let us denote by $D_\Phi(g_1, g_2)$ the following subsets:

$$D_\Phi(g_1, g_2) = \{(\hat{\alpha}, y) \in D_\Phi : \varphi(\hat{\alpha}) - g_2(\hat{\alpha}) < y < \varphi(\hat{\alpha}) - g_1(\hat{\alpha})\}.$$

Notice that for some values of $g_1(\hat{\alpha}), g_2(\hat{\alpha})$, the corresponding set of y can be empty. Also, let $p(x, y) = x$ be the projection to the base of the special flow.

Remark 4.1. If $g'_1 \leq g_1$ and $g'_2 \geq g_2$, then $D_\Phi(g_1, g_2) \subset D_\Phi(g'_1, g'_2)$.

We will show that the limiting distribution (1) exists and is given by

$$P_N((a, b) \times \{c_0\} \times \dots \times \{c_{N-1}\}) = \mu_3(D_\Phi(\ln a, \ln b) \cap p^{-1}C_N). \tag{26}$$

Take $\epsilon > 0$. For each $n \in \mathbb{N}$, the cylinders $\{\mathcal{C} : \mathcal{C} \in \mathcal{C}_n\}$ constitute a countable partition of $D(\hat{\mathcal{G}})$. Choose n so large that, by Lemma 3.2, we have $|f(\hat{\alpha}_1) - f(\hat{\alpha}_2)| \leq \epsilon/2$ for all $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{C}$. Let

$$A_{\mathcal{C}} := \left\{ \hat{\alpha} \in \mathcal{C} : a < \frac{q_{n_R(\hat{\alpha})}(\hat{\alpha})}{R} < b, \hat{\mathcal{G}}^{n_R(\hat{\alpha})-1} \hat{\alpha} \in C_N \right\}.$$

By the second part of Lemma 3.3, there exists $R_1 > 0$ such that if $R \geq R_1$ we have $n_R(\alpha) > N$ for any α outside a set of μ_1 -measure less than ϵ . Hence, by (25) (and Remarks 2.1 and 2.2), we have

$$\left| \mu_1 \left(\alpha : a < \frac{q_{n_R(\alpha)}}{R} < b, a_{n_R(\alpha)-k} = c_k, 0 \leq k < N \right) - \sum_{\mathcal{C} \in \mathcal{C}_n} \mu_2(A_{\mathcal{C}}) \right| \leq 2\epsilon. \tag{27}$$

Let us first reduce to a finite sum. Consider the finite subset \mathcal{C}_n^m of cylinders $\mathcal{C} = \mathcal{C}([a_{-n}, \dots, a_n])$ such that $a_i < m$ for $-n \leq i \leq n$. Since, if $\mathcal{C} \in \mathcal{C}_n \setminus \mathcal{C}_n^m$, there exists $-n \leq i \leq n$ and $k \geq m$ such that $a_i = k$, we have, using Remark 2.3 and invariance of μ_2 ,

$$\sum_{\mathcal{C} \in \mathcal{C}_n \setminus \mathcal{C}_n^m} \mu_2(\mathcal{C}) \leq \sum_{i=-n}^n \sum_{k=m}^{\infty} \mu_2(\hat{\mathcal{G}}^i \hat{\mathcal{C}}([k])) \leq (2n+1) \sum_{k=m}^{\infty} O\left(\frac{1}{k^2}\right) = O\left(\frac{1}{m}\right). \tag{28}$$

Hence, choosing m large enough, we can make (28) less than ϵ . To each $\mathcal{C} \in \mathcal{C}_n^m$ we can apply Lemma 3.4 and, hence, for $R \geq \max_{\mathcal{C} \in \mathcal{C}_n^m} R_0(\mathcal{C})$ (where $R_0(\mathcal{C})$ and $U_{\mathcal{C}}$ are as in Lemma 3.4) we get, from (27),

$$\begin{aligned} & \left| \mu_1 \left(\alpha : a < \frac{q_{n_R(\alpha)}}{R} < b, a_{n_R(\alpha)-k} = c_k, 0 \leq k < N \right) - \sum_{\mathcal{C} \in \mathcal{C}_n^m} \mu_2(A_{\mathcal{C} \setminus U_{\mathcal{C}}}) \right| \\ & \leq 2\epsilon + \left| \sum_{\mathcal{C} \in \mathcal{C}_n \setminus \mathcal{C}_n^m} \mu_2(A_{\mathcal{C}}) + \sum_{\mathcal{C} \in \mathcal{C}_n^m} \mu_2(A_{\mathcal{C} \cap U_{\mathcal{C}}}) \right| \leq 3\epsilon + 7\epsilon \sum_{\mathcal{C} \in \mathcal{C}_n^m} \mu_2(\mathcal{C}) \leq 10\epsilon, \end{aligned}$$

where the second but last inequality follows from the observation that $A_{\mathcal{C}} \subset \mathcal{C}$, (28) and Lemma 3.4.

To conclude the proof and to get (26), it is enough to prove that, for each $\mathcal{C} \in \mathcal{C}_n^m$, as long as R is sufficiently large, we have, for some $C > 0$,

$$\left| \frac{\mu_2(A_{\mathcal{C} \setminus U_{\mathcal{C}}})}{\mu_2(\mathcal{C} \setminus U_{\mathcal{C}})} - \mu_3(D_{\Phi}(\ln a, \ln b) \cap p^{-1}C_N) \right| \leq C\epsilon. \tag{29}$$

Fix $\mathcal{C} \in \mathcal{C}_n^m$ and consider on \mathcal{C} the function $T = T(R) = \ln R - f_{\mathcal{C}}$ (recall that $f_{\mathcal{C}} = \sup_{\mathcal{C}} f$) and let $U = U_{\mathcal{C}}$ be given by Lemma 3.4. Since by Lemma 3.4, for $R \geq R_0(\mathcal{C})$, on $\mathcal{C} \setminus U$ we have $n_R(\hat{\alpha}) = r(\hat{\alpha}, T)$, applying (8) of Lemma 3.1 we get

$$\begin{aligned} \left\{ \hat{\alpha} \in \mathcal{C} \setminus U : a < \frac{q_{n_R(\hat{\alpha})}}{R} < b \right\} &= \{ \hat{\alpha} \in \mathcal{C} \setminus U : \ln a < \ln q_{r(\hat{\alpha}, T)}(\hat{\alpha}) - \ln R < \ln b \} \\ &= \{ \hat{\alpha} \in \mathcal{C} \setminus U : \ln a < S_{r(\hat{\alpha}, T)}(\varphi)(\hat{\alpha}) - T + \epsilon_{R, \mathcal{C}}(\hat{\alpha}) < \ln b \}, \end{aligned}$$

where we denote $\epsilon_{R, \mathcal{C}}(\hat{\alpha}) = \epsilon_{n_R(\hat{\alpha})}(\hat{\alpha}) - f_{\mathcal{C}} + f(\hat{\alpha})$. Let us show that $|\epsilon_{R, \mathcal{C}}| \leq 2\epsilon$ uniformly on $\mathcal{C} \setminus U$. Indeed, by construction of \mathcal{C} , $|f(\hat{\alpha}) - f_{\mathcal{C}}| \leq \epsilon$; moreover, since by (19) and Remark 3.1, $\ln R - f_{\mathcal{C}} < S_{r(\hat{\alpha}, T)}(\varphi)(\hat{\alpha}) \leq M(\mathcal{C})r(\hat{\alpha}, T)$, on $\mathcal{C} \setminus U$ we have

$n_R(\hat{\alpha}) = r(\hat{\alpha}, T) \geq (\ln R - f_{\mathcal{C}})/M(\mathcal{C})$. So enlarging $R_0(\mathcal{C})$ we can ensure that, for each $\hat{\alpha} \in \mathcal{C} \setminus U$ and $R \geq R_0$, $n_R(\hat{\alpha})$ is so large that, by Lemma 3.1, also $|\epsilon_{n_R(\hat{\alpha})}(\hat{\alpha})| \leq \epsilon$.

Let us denote by $(\Phi_t(x, y))^v$ the vertical component y' of $\Phi_t(x, y) = (x', y')$. Using the definition of the flow action (5) and the equality $n_R(\hat{\alpha}) = r(\hat{\alpha}, T)$, we can rewrite

$$S_{r(\hat{\alpha}, T)}(\varphi)(\hat{\alpha}) - T = \varphi(\hat{\mathcal{G}}^{n_R(\hat{\alpha})-1}\hat{\alpha}) - (\Phi_T(\hat{\alpha}, 0))^v. \tag{30}$$

Note that $\hat{\mathcal{G}}^{n_R(\hat{\alpha})-1}\hat{\alpha} = p(\Phi_T(\hat{\alpha}, 0))$ and that condition (25) can be expressed as $p(\Phi_T(\hat{\alpha}, 0)) \in C_N$. The quantity (30) represents geometrically the vertical distance of $\Phi_T(\hat{\alpha}, 0)$ from the roof function.

Hence, recalling the definitions of the sets $D_\Phi(g_1, g_2)$ and C_N given at the beginning of the proof, using also Remark 4.1 combined with $|\epsilon_{R, \mathcal{C}}| \leq 2\epsilon$, we have shown that

$$\begin{aligned} A_{\mathcal{C} \setminus U} &\subset \mathcal{C} \setminus U \times \{0\} \cap \Phi_{-T}(D_\Phi(\ln a - 2\epsilon, \ln b + 2\epsilon) \cap p^{-1}C_N); \\ A_{\mathcal{C} \setminus U} &\supset \mathcal{C} \setminus U \times \{0\} \cap \Phi_{-T}(D_\Phi(\ln a + 2\epsilon, \ln b - 2\epsilon) \cap p^{-1}C_N). \end{aligned}$$

Let us ‘thicken’ the sets in order to apply mixing of $\{\Phi_t\}_{t \in \mathbb{R}}$ as in the proof of Lemma 3.4. If we choose $0 < \delta < \min\{\min_{\hat{\alpha} \in \mathcal{C}} \varphi(\hat{\alpha}), \epsilon\}$ (well defined by Remark 3.1), for each $\hat{\alpha} \in A_{\mathcal{C} \setminus U}$ and $0 \leq z < \delta$, reasoning as in the proof of Lemma 3.4 (see e.g. (23)), we get $(\hat{\alpha}, z) \in \Phi_{-T}(D_\Phi(\ln a - 2\epsilon - \delta, \ln b + 2\epsilon) \cap p^{-1}C_N \cup D_\Phi^\delta)$. Hence, using also Remark 4.1, combined with $\delta \leq \epsilon$,

$$\delta\mu_2(A_{\mathcal{C} \setminus U}) \leq \mu_3(\mathcal{C} \setminus U \times [0, \delta) \cap \Phi_{-T}(D_\Phi(\ln a - 3\epsilon, \ln b + 2\epsilon) \cap p^{-1}C_N \cup D_\Phi^\delta)).$$

Note that $\mu_3(D_\Phi^\delta) \leq \epsilon$. Enlarging R_0 , so that if $R \geq R_0$ then also $T(R)$ is sufficiently large, we can use mixing (see Proposition 3.1) to get

$$\delta\mu_2(A_{\mathcal{C} \setminus U}) \leq \delta\mu_2(\mathcal{C} \setminus U)(\mu_3(D_\Phi(\ln a - 3\epsilon, \ln b + 2\epsilon) \cap p^{-1}C_N) + 2\epsilon). \tag{31}$$

In order to get the opposite inequality, one can show, reasoning again as in the proof of Lemma 3.4, that if $(\hat{\alpha}, z) \in \mathcal{C} \setminus U \times [0, \delta)$ is such that

$$(\hat{\alpha}, z) \in \Phi_{-T}(D_\Phi(\ln a + 2\epsilon, \ln b - 2\epsilon - \delta) \cap p^{-1}C_N \setminus D_\Phi^\delta),$$

we have that $\hat{\alpha} \in A_{\mathcal{C} \setminus U}$. This means, also using again Remark 4.1, that

$$\mathcal{C} \setminus U \times [0, \delta) \cap \Phi_{-T}(D_\Phi(\ln a + 2\epsilon, \ln b - 3\epsilon) \cap p^{-1}C_N \setminus D_\Phi^\delta) \subset A_{\mathcal{C} \setminus U} \times [0, \delta).$$

Again applying mixing, again enlarging R_0 if necessary, for $R \geq R_0$, and using the fact that for any measurable $D \subset D_\Phi$ we have $\mu_3(D \setminus D_\Phi^\delta) \geq \mu_3(D) - \epsilon$, we get

$$\delta\mu_2(A_{\mathcal{C} \setminus U}) \geq \delta\mu_2(\mathcal{C} \setminus U)(\mu_3(D_\Phi(\ln a + 2\epsilon, \ln b - 3\epsilon) \cap p^{-1}C_N) - 2\epsilon). \tag{32}$$

Since, moreover, by the Fubini theorem

$$|\mu_3(D_\Phi(\ln a \pm \epsilon_1, \ln b \mp \epsilon_1) \cap p^{-1}C_N) - \mu_3(D_\Phi(\ln a, \ln b) \cap p^{-1}C_N)| \leq 2\epsilon_1,$$

combining (31) and (32) we get (29) and hence conclude the proof of the existence of the limiting distribution. □

5. Mixing of the special flow

In what follows we briefly outline the proof of Proposition 3.1 given in [3].

Proof of Proposition 3.1. Given a point $(\hat{\alpha}_0, y_0) \in D_\Phi$, let us construct the local stable and unstable leaves through it, denoted by $\Gamma_{\text{loc}}^{(s)}(\hat{\alpha}_0, y_0)$ and $\Gamma_{\text{loc}}^{(u)}(\hat{\alpha}_0, y_0)$ respectively (as a general reference, see e.g. [6]).

Since the roof function $\varphi(\hat{\alpha})$ depends only on $(\mathcal{G}\hat{\alpha})^-$, it is easy to construct the local unstable leaf, which is given by a piece of a ‘horizontal’ segment:

$$\Gamma_{\text{loc}}^{(u)}(\hat{\alpha}_0, y_0) \subset \{(\hat{\alpha}, y) : \hat{\alpha}^- = \hat{\alpha}_0^-, y = y_0\}. \tag{33}$$

The local stable leaf through $(\hat{\alpha}_0, y_0)$ is given locally by the following curve parametrized by α^- :

$$\Gamma_{\text{loc}}^{(s)}(\hat{\alpha}_0, y_0) \subset \left\{ (\hat{\alpha}, y) : \hat{\alpha}^+ = \hat{\alpha}_0^+, y = y_0 + \ln \frac{1 + \hat{\alpha}^- \hat{\alpha}_0^+}{1 + \hat{\alpha}_0^- \hat{\alpha}_0^+} \right\}. \tag{34}$$

In order to see it, one can construct it as follows. Let us denote $(\hat{\alpha}_t, y_t) = \Phi_t(\hat{\alpha}_0, y_0)$. Consider a small ‘vertical’ segment at $(\hat{\alpha}_t, y_t)$, i.e.

$$\Gamma_{\delta_t}^t = \{(\hat{\alpha}, y_t) : \hat{\alpha}^+ = \hat{\alpha}_t^+, |\hat{\alpha}^- - \hat{\alpha}_t^-| < \delta_t\},$$

where δ_t is chosen sufficiently small so that, for some $\delta > 0$,

$$\Phi_{-t}(\Gamma_{\delta_t}^t) \subset \{(\hat{\alpha}, y) : \hat{\alpha}^+ = \hat{\alpha}_0^+, |y - y_0| < \delta, 0 < y_0 - \delta < y < \varphi(\hat{\alpha}_0) - \delta\}.$$

Then, if $(\hat{\alpha}, y) \in \Phi_{-t}(\Gamma_{\delta_t}^t)$, by definition of a special flow, since $r(\hat{\alpha}, t) = r(\hat{\alpha}_0, t) = r(t)$ by construction, we have $y - S_{r(t)-1}(\varphi)(\hat{\alpha}) = y_t - t = y_0 - S_{r(t)-1}(\varphi)(\hat{\alpha}_0)$. Denote $\alpha_0^+ = \{a_k^0\}_{k \in \mathbb{N}_+}$ and p_n/q_n its convergents. Let

$$\begin{aligned} \beta' &:= [a_1^0 + \hat{\alpha}_0^-, a_2^0, \dots, a_{r(t)-1}^0] = \frac{1}{\hat{\alpha}_0^- + (q_{r(t)-1}/p_{r(t)-1})}, \\ \beta'' &:= [a_1^0 + \hat{\alpha}^-, a_2^0, \dots, a_{r(t)-1}^0] = \frac{1}{\hat{\alpha}^- + (q_{r(t)-1}/p_{r(t)-1})}, \end{aligned} \tag{35}$$

and p'_n/q'_n and p''_n/q''_n be their respective convergents. Note that $p'_n = p''_n = p_n$ for $1 \leq n \leq r(t)$ since they satisfy the same recursive equations $p_{k+1} = a_{k+1}p_k + p_{k-1}$ for $2 \leq k+1 \leq r(t)$ with initial data $p_0 = 0, p_1 = 1$. Hence, $\beta' = p_{r(t)-1}/q'_{r(t)-1}$ and $\beta'' = p_{r(t)-1}/q''_{r(t)-1}$. Using (10), (11) and (35), one gets

$$\begin{aligned} y &= y_0 + S_{r(t)-1}(\varphi)(\hat{\alpha}) - S_{r(t)-1}(\varphi)(\hat{\alpha}_0) = y_0 + \ln \frac{q''_{r(t)-1}}{q'_{r(t)-1}} \\ &= y_0 + \ln \frac{1 + \hat{\alpha}^- (p_{r(t)-1}/q_{r(t)-1})}{1 + \hat{\alpha}_0^- (p_{r(t)-1}/q_{r(t)-1})}. \end{aligned} \tag{36}$$

As t , and hence $r(t)$, tend to infinity, $p_{r(t)-1}/q_{r(t)-1}$ converge to α_0^+ and we get (34).

The global unstable and stable leaves can be obtained as

$$\Gamma^{(u)}(\hat{\alpha}_0, y_0) = \bigcup_t \Phi_t \Gamma_{\text{loc}}^{(u)}(\hat{\alpha}_{-t}, y_{-t}), \quad \Gamma^{(s)}(\hat{\alpha}_0, y_0) = \bigcup_t \Phi_{-t} \Gamma_{\text{loc}}^{(s)}(\hat{\alpha}_t, y_t).$$

To prove mixing, it is enough to show that the stable and unstable foliations form a non-integrable pair. From their non-integrability, it follows from the general theory (see [6]) that the Pinsker partition is trivial and hence that $\{\Phi_t\}_{t \in \mathbb{R}}$ is a K -flow and, in particular, is mixing.

Consider a sufficiently small neighborhood $\mathcal{U}(\hat{\alpha}_0, y_0) \subset D_\Phi$ of $(\hat{\alpha}_0, y_0)$. It is enough to show that, for a positive measure set of $(\hat{\alpha}, y) \in \mathcal{U}(\hat{\alpha}_0, y_0)$, $(\hat{\alpha}, y)$ can be connected

to $(\hat{\alpha}_0, y_0)$ through segments of local stable and unstable leaves, in particular it is enough to show that there exists $(\hat{\alpha}_i, y_i) \in \mathcal{U}(\hat{\alpha}_0, y_0)$, $i = 1, 2$, such that $(\hat{\alpha}_1, y_1) \in \Gamma^{(s)}(\hat{\alpha}_0, y_0)$, $(\hat{\alpha}_2, y_2) \in \Gamma^{(u)}(\hat{\alpha}_1, y_1)$ and $(\hat{\alpha}, y) \in \Gamma^{(s)}(\hat{\alpha}_2, y_2)$.

Using explicitly equations (33) and (34), one can check that these points exist as soon as we can find y_1 and $\hat{\alpha}_1^-$ such that $((\hat{\alpha}_1^-, \hat{\alpha}_0^+), y_1) \in \mathcal{U}(\hat{\alpha}_0, y_0)$ and

$$y_1 = y_0 + \ln \frac{1 + \hat{\alpha}_1^- \hat{\alpha}_0^+}{1 + \hat{\alpha}_0^- \hat{\alpha}_0^+}, \quad y = y_1 + \ln \frac{1 + \hat{\alpha}^- \hat{\alpha}^+}{1 + \hat{\alpha}_1^- \hat{\alpha}^+}, \tag{37}$$

since in this case we can take $(\hat{\alpha}_1, y_1) = ((\hat{\alpha}_1^-, \hat{\alpha}_0^+), y_1)$ and $(\hat{\alpha}_2, y_2) = ((\hat{\alpha}_1^-, \hat{\alpha}^+), y_1)$. Equations (37) can be solved if

$$\frac{\hat{\alpha}^+}{1 + \hat{\alpha}^- \hat{\alpha}^+} e^y \neq \frac{\hat{\alpha}_0^+}{1 + \hat{\alpha}_0^- \hat{\alpha}_0^+} e^{y_0}. \tag{38}$$

The points $(\hat{\alpha}, y)$ for which there is an equality in (38) lie on a surface in D_Φ and hence have measure zero. This concludes the proof of the non-integrability. □

6. Concluding remarks

Let T be an ergodic automorphism of the measure space (M, \mathcal{M}, μ) and $f \in L^1(M, \mathcal{M}, \mu)$, $\int f dx > 0$. The following problem is a generalization of a classical renewal problem in probability theory. Take $R > 0$ and consider the first n_R such that

$$f(x) + f(Tx) + \dots + f(T^{n_R}x) > R.$$

What will be the limiting distribution of $f(x) + f(Tx) + \dots + f(T^{n_R}x) - R$ as R tends to infinity? The answer can be given in terms of a special flow which is similar to the one considered above. Interesting aspects of this problem appear when $\int |f| dx = \infty$.

Results concerning the limiting distribution when considering the Gauss map and the sum of the entries of the continued fraction expansion can be found in [4].

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