Billiards, Pretzels
... and Chaos

Inaugural lecture
Professor Corinna Ulcigrai
Wednesday 30 November 2016

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Please note the building's fire exits. There are no planned fire alarms taking place today, so if you hear the alarm sound, please leave via the fire exits and gather at the meeting point outside the Merchant Venturers' Building.
Chaotic systems are everywhere:

the weather...

financial markets...

molecules of a gas,

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These systems is **deterministic**, they obey "rules":

- laws of physics,
- mathematical equations...
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Pierre-Simon Laplace (1747-1827)

Laplace, A Philosophical Essay on Probabilities
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“An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, [...] for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.”

Laplace, *A Philosophical Essay on Probabilities*

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Sensitive dependence: the “Butterfly effect”
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A key feature of chaotic systems is the **Butterfly Effect**:

“a butterfly flapping its wings in Brazil can cause a tornado in Texas”
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Fast versus slow Chaos

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Fast versus *slow* Chaos

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Law of Optics: angle of incidence = angle of reflection

Mathematical Idealization: the ball is a point with no mass, there is no friction, consider trajectories that never enter a pocket: ⇒ motion is infinite.
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with barrier
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- with obstacle
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Ehrenfest Model, 1912
Tatjana and Paul Ehrenfest
Periodic version: Hardy-Weber

(image by V. Delecroix)
Sensitive Dependence: circular vs rectangular scatters

- Fast chaos (hyperbolic billiard)
  - Only corners create divergence:
    - Slow chaos (parabolic billiard)

Sinai billiard (Yakov Sinai, Abel Prize 2014)
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defocusing mechanism:
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much studied since the Seventies...

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Feature of chaotic systems: *most “trajectories” explore all space*
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Billiard trajectories "bal" motion) can e.g.

- close up (*periodic motion*)
- get arbitrarily close to any point (*dense*)

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Recent results on the Ehrenfest model

[Fraczek-Ulcigrai, Inventiones, 2014]

on the Ehrenfest model, for almost every direction, NO trajectory is dense (explore all parts of space).

courtesy of V. Delecroix
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Pretzels
From a rectangular billiard to a bagel...

Unfolding:

Don't reflect the trajectory, REFLECT the TABLE!

4 copies are enough; glue opposite sides: surface of a bagel!
From a rectangular billiard to a bagel...

**Unfolding:** don’t reflect the trajectory, REFLECT the TABLE!
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Gain: one can show trajectories are either closed or dense;

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Unfolding, then glueing sides...

...surface of pretzel with 5 holes!
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Unfolding, then gluing sides...

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Surfaces

Unfolding (rational) polygonal billiards one gets surfaces:
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genus 1

...
Surfaces

Unfolding (rational) polygonal billiards one gets surfaces:

genus 1  
genus 2  
genus 3  

...  

Pretezels and bagels in the presentation by T. Hansson of the 2016 Physics Nobel Prize work by Thouless, Haldane and Kosterlitz
Flows on surfaces

Motion of a point $p$ point on the surface:
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after time \( t \), \( p \) "flows" to \( \varphi_t(p) \);
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Motion of a point $p$ point on the surface:
after time $t$, $p$ "flows" to $\varphi_t(p)$;
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H. Poincaré 1854-1912

Electrons in metals in solid state physics (Fermi surfaces)

Novikov model (1990s)
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- motion of bodies in celestial mechanics
- electrons in metals in solid state physics (Fermi surfaces)
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Novikov model (1990s)
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Classification of mixing properties

Answer to Arnold’s question on mixing
(for motion of electrons on metal Fermi surfaces):

- For bagels (genus one):
  Yes (Khanin-Sinai, 1992)

- For pretzels with many holes (genus $\geq 2$):
  it depends, on whether there are traps:
  - no traps: typically not mixing, but weakly mixing
  - traps: typically mixing (outside the traps); also mixing of all orders; mixing speed is subpolynomial

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Mixing Mechanism

How does mixing happen?

- Near a trap, trajectories slow down at different speeds.
- This creates shearing; \( \phi_t(A) \) elongates and wraps around the surface.
- This seems to be a key phenomenon for slow chaos!
- Remark: even stronger chaotic properties can be deduced from shearing! Eg: Katok-Thouvenot conjecture, on Lebesgue spectrum for certain parabolic flows (time-changes of horocycle flows) [Forni-Ulcigrai, JMD 2012].
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Remark: even stronger chaotic properties can be deduced from shearing! Eg: Katok-Thouvenot conjecture, on Lebesgue spectrum for certain parabolic flows (time-changes of horocycle flows) [Forni-Ulcigrai, JMD 2012].
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