From Anderson to Zeta

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Universität Zürich

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Overview

Overview

- The combinatorial zeta map ζ_{HL}

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- The uniform zeta map ζ

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- How ζ specializes to ζ_{HL}

The combinatorial zeta map ζ_{HL}

Diagonal Harmonics

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$\mathrm{DH}^{\epsilon} = \{f \in \mathcal{DH}_n : \sigma \cdot f = \mathrm{sgn}(\sigma) f \text{ for all } \sigma \in S_n\}$.

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Theorem (Haglund, Haiman 2002)

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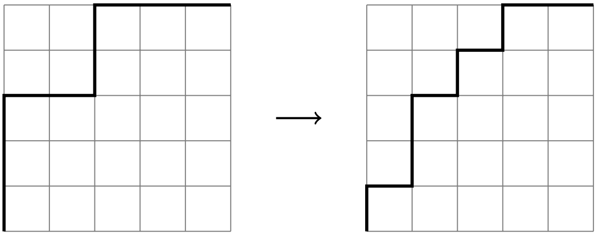

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Similarly,

$$\mathcal{DH}_n(q,t) = \sum_{(P,\sigma) \in \mathcal{PF}_n} q^{\mathrm{dinv'}(P,\sigma)} t^{\mathrm{area}(P,\sigma)}$$
$$= \sum_{(w,D) \in \mathcal{D}_n} q^{\mathrm{area'}(w,D)} t^{\mathrm{bounce}(w,D)}.$$

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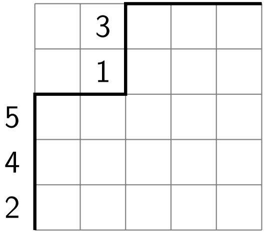

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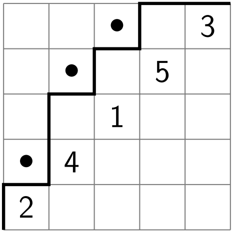

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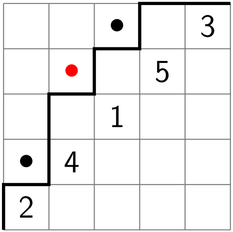

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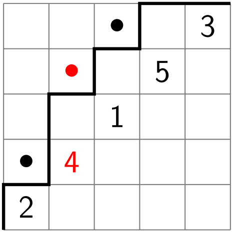

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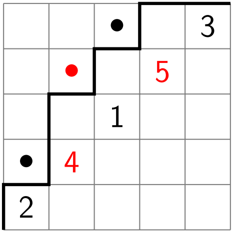

The combinatorial zeta map ζ_{HL}

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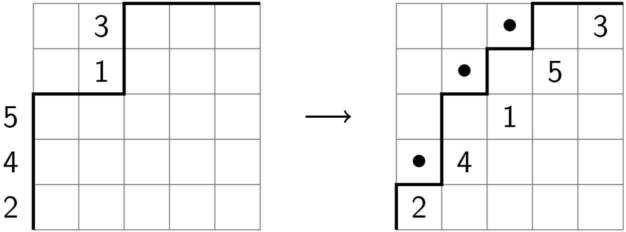

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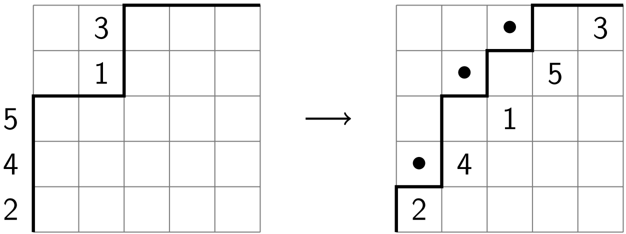

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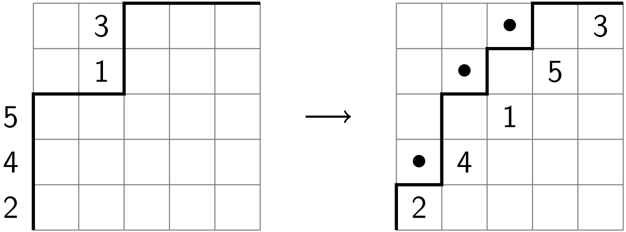

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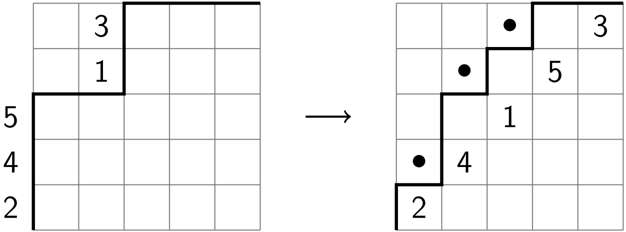

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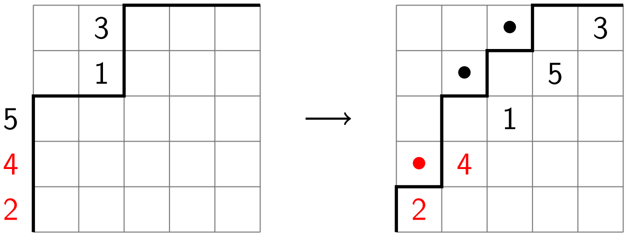

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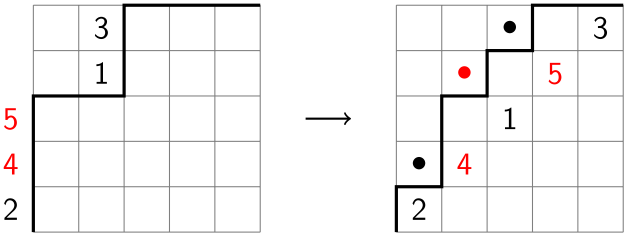

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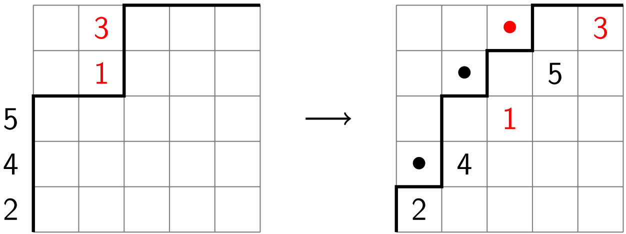

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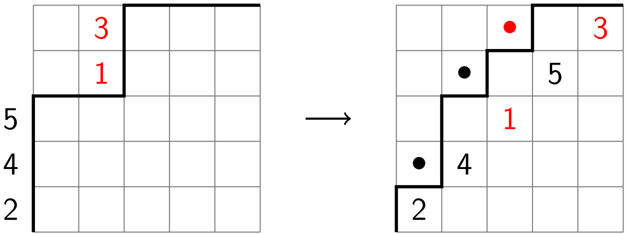

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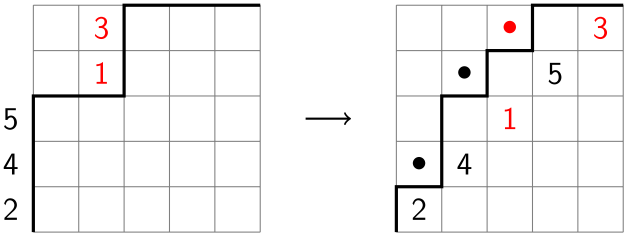

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ζ_{HL} maps rises to valleys, preserving labels!
We can define natural S_n-actions on $\mathcal{PF}_n$ and $\mathcal{D}_n$ such that this is equivalent to ζ_{HL} being S_n-equivariant.

The uniform zeta map ζ

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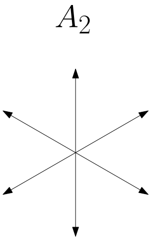
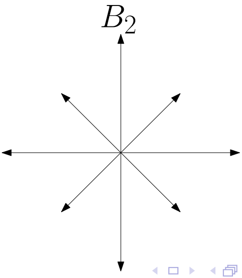
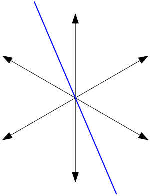


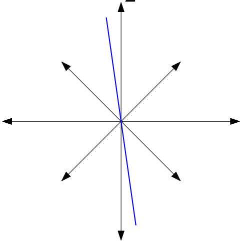


Root systems

A_2



B_2



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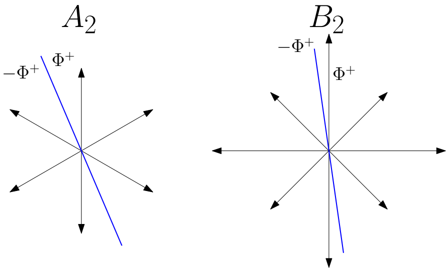
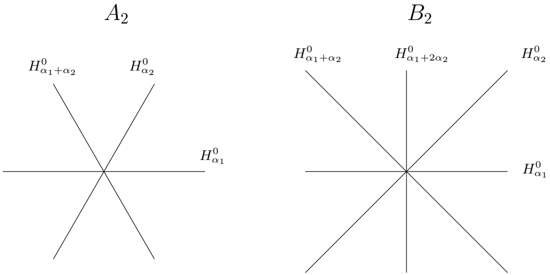


Set of positive roots Φ^+ such that $\Phi = \Phi^+ \sqcup -\Phi^+$.

The Coxeter arrangement

Hyperplane arrangement with hyperplanes
$H_\alpha^0 := \{x \in V \mid \langle x, \alpha \rangle = 0\}$ for $\alpha \in \Phi$.



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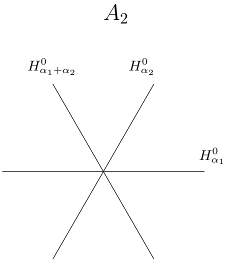

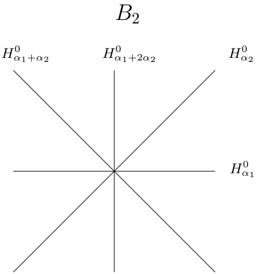

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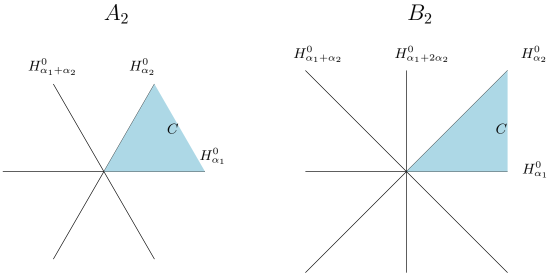

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The chamber $C = \{x \in V : \langle x, \alpha \rangle > 0 \text{ for all } \alpha \in \Phi^+\}$ is called dominant.

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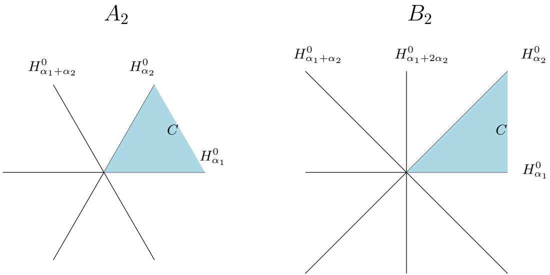

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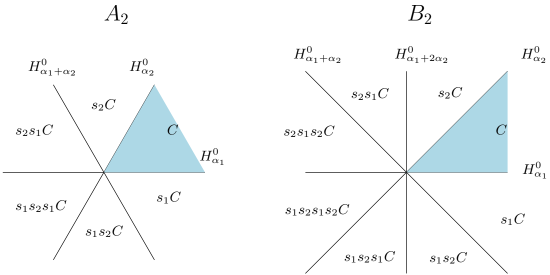

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The Weyl group $W := \langle \{s_\alpha : \alpha \in \Phi\} \rangle$ acts simply transitively on the chambers, so we can write each chamber as wC for a unique $w \in W$.

The affine Coxeter arrangement

Hyperplane arrangement with hyperplanes
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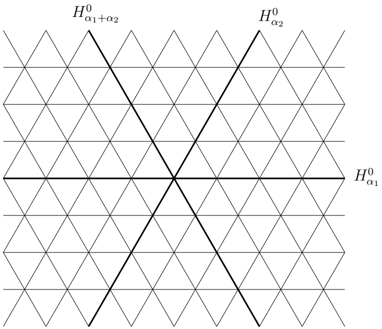

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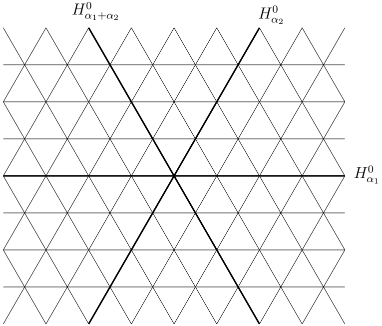

They divide the affine space V into alcoves.

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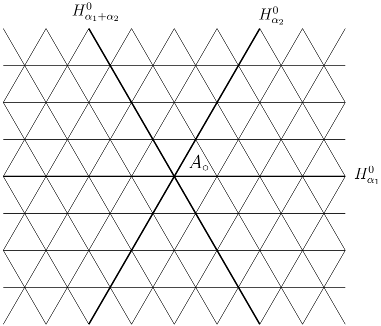

They divide the affine space V into alcoves. The alcove $A_\circ = \{x \in V : 0 < \langle x, \alpha \rangle < 1 \text{ for all } \alpha \in \Phi^+\}$ is called the fundamental alcove.

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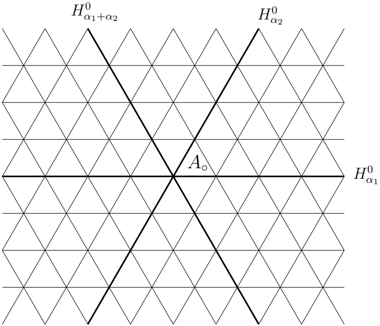

Define the affine Weyl group $\widetilde{W}$ as the group generated by all affine reflections through the hyperplanes of the affine Coxeter arrangement.

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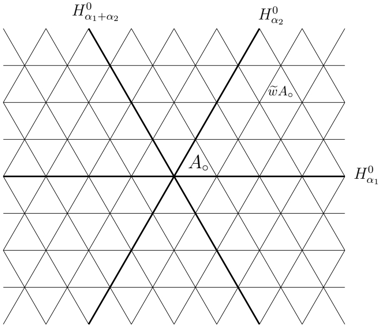

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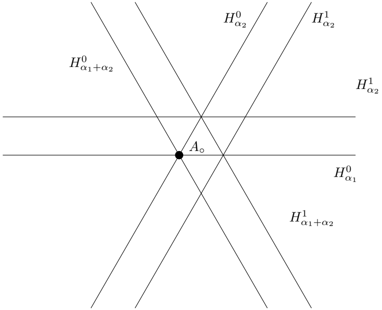
It acts simply transitively on the set of alcoves, so any alcove may be written as $\widetilde{w}A_\circ$ for a unique $\widetilde{w} \in \widetilde{W}$.

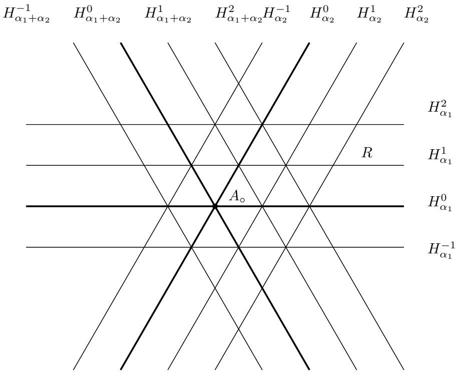
The Shi arrangement

Hyperplane arrangement with hyperplanes
$H_\alpha^k := \{x \in V \mid \langle x, \alpha \rangle = k\}$ for $\alpha \in \Phi^+$ and $k \in \{0, 1\}$.



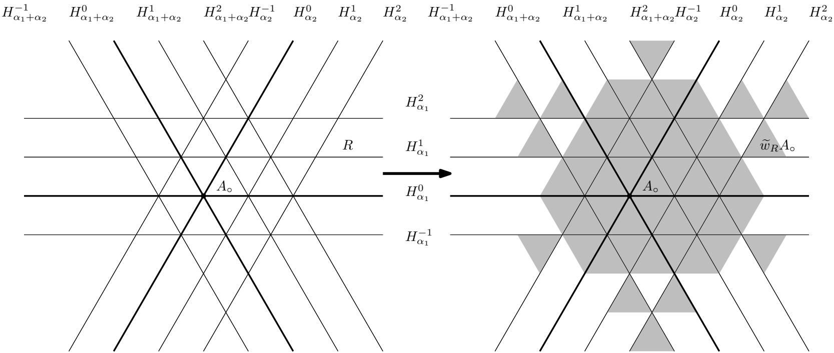
The m-Shi arrangement

Fix $m \in \mathbb{N}$. Hyperplane arrangement with hyperplanes $H_\alpha^k := \{x \in V \mid \langle x, \alpha \rangle = k\}$ for $\alpha \in \Phi^+$ and $-m < k \leq m$.



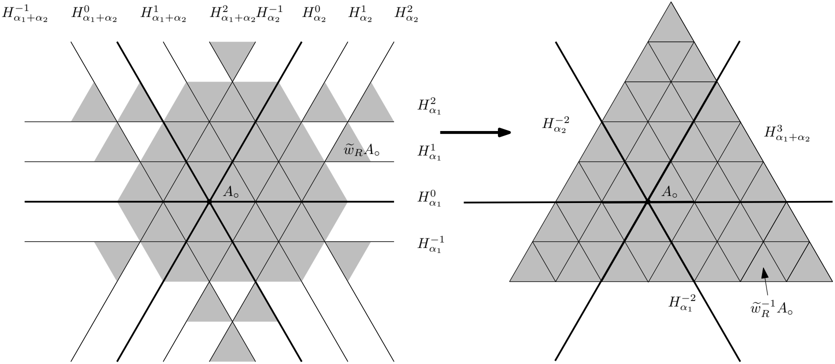
Minimal alcoves of the m-Shi arrangement

Every m-Shi region R contains a unique alcove closest to the origin called its minimal alcove $\widetilde{w}_R A_\circ$.



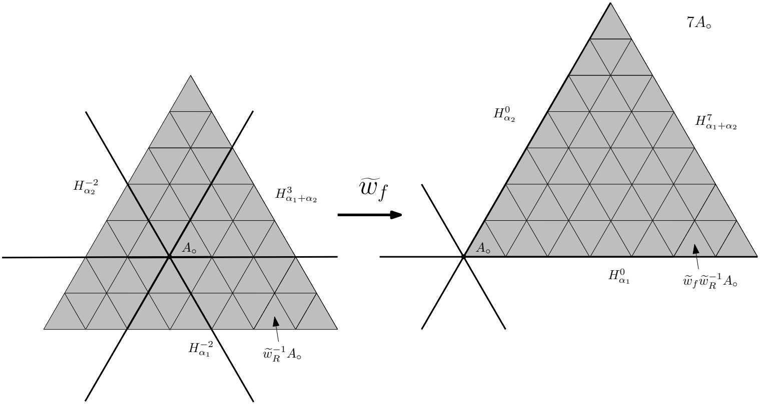
The Sommers region

The inverses of the minimal alcoves coalesce into a simplex called the Sommers region: $\widetilde{w}_R A_\circ \mapsto \widetilde{w}_R^{-1} A_\circ$



The dilated fundamental alcove

There is an element $\widetilde{w}_f \in \widetilde{W}$ that maps the Sommers region to $(mh+1)A_\circ$. Here h is the Coxeter number of the root system.



The finite torus

Apply the inverses to 0.

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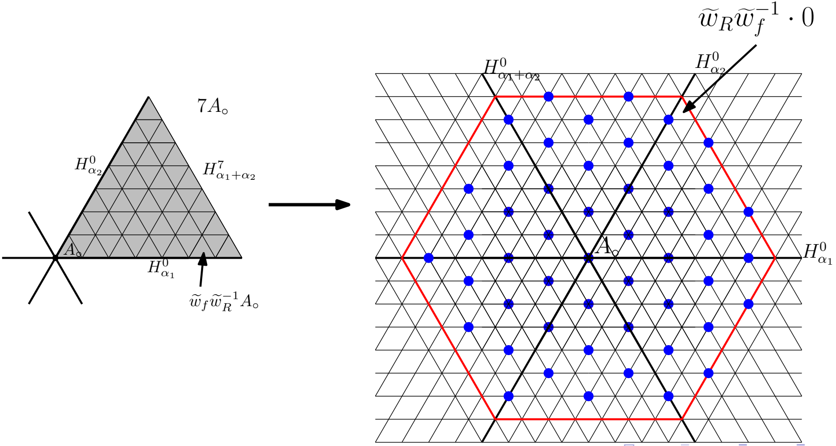
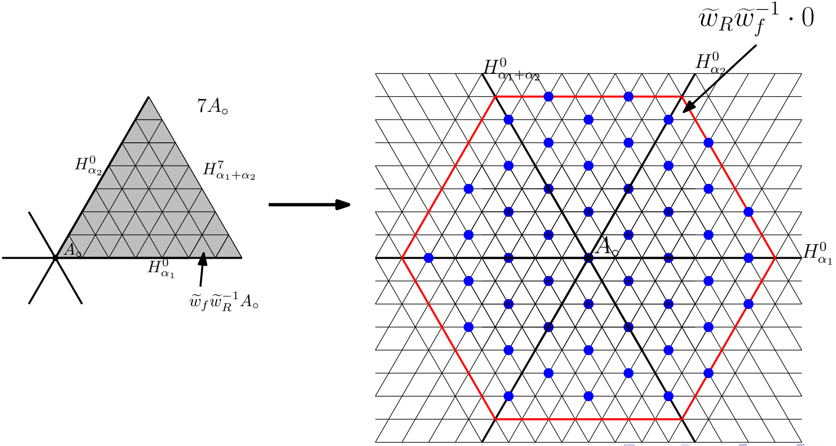


The finite torus

Apply the inverses to 0. Get a set of points in the coroot lattice $\check{Q}$. They are a set of representatives for the finite torus $\check{Q}/(mh+1)\check{Q}$.



The uniform zeta map

Theorem (P. Cellini and P. Papi '00, E. Sommers '03, C. Athanasiadis '05, B. Rhoades '12, M. Thiel '16)

There is a natural bijection ζ^{-1} from the set of m-Shi regions to the finite torus $\check{Q}/(mh+1)\check{Q}$.

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m-Shi regions	$\rightarrow$	m-Shi alcoves	$\rightarrow$	Sommers region	$\rightarrow$	$(mh+1)A_\circ$	$\rightarrow$	$\check{Q}/(mh+1)\check{Q}$
R	$\rightarrow$	$\widetilde{w}_R A_\circ$	$\rightarrow$	$\widetilde{w}_R^{-1} A_\circ$	$\rightarrow$	$\widetilde{w}_f \widetilde{w}_R^{-1} A_\circ$	$\rightarrow$	$\widetilde{w}_R \widetilde{w}_f^{-1} \cdot 0$

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m-Shi regions	→	m-Shi alcoves	→	Sommers region	→	$(mh+1)A_\circ$	→	$\check{Q}/(mh+1)\check{Q}$
R	→	$\widetilde{w}_R A_\circ$	→	$\widetilde{w}_R^{-1} A_\circ$	→	$\widetilde{w}_f \widetilde{w}_R^{-1} A_\circ$	→	$\widetilde{w}_R \widetilde{w}_f^{-1} \cdot 0$

One can define natural actions of the Weyl group W on m-Shi regions and on $\check{Q}/(mh+1)\check{Q}$ that make ζ^{-1} equivariant.

How ζ specializes to ζ_{HL}

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Take $m = 1$ and Φ of type A_{n-1}. The following square commutes:

$$\begin{array}{ccc} \text{Shi regions} & \xrightarrow{\zeta^{-1}} & \check{Q}/(n+1)\check{Q} \\ \updownarrow & & \updownarrow \\ \mathcal{D}_n & \xleftarrow{\zeta_{HL}} & \mathcal{PF}_n \end{array}$$

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\end{array}
$$

Every arrow is an S_n-equivariant bijection.

Thanks for your attention!