Simultaneous core partitions for affine Weyl groups

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Joint work with Nathan Williams (LACIM, Montréal)

Outline

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– a-core partitions

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– $\widetilde{S}_a$-action on a-cores

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- Other affine Weyl groups

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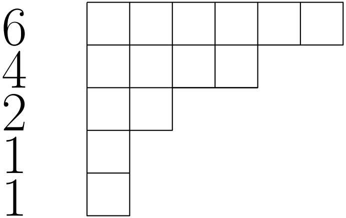

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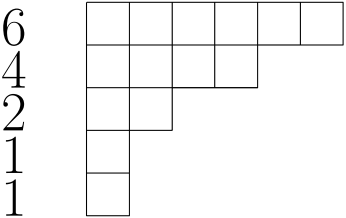

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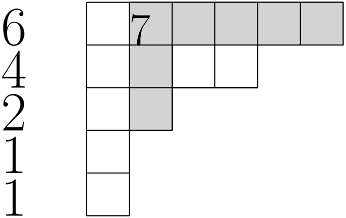

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Originally introduced by Nakayama in the study of the representation theory of the symmetric group in prime characteristic.

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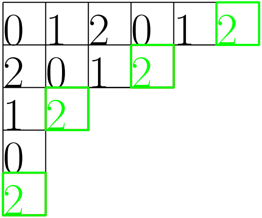

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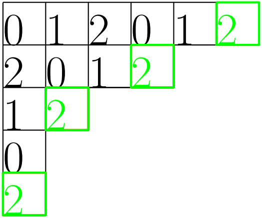

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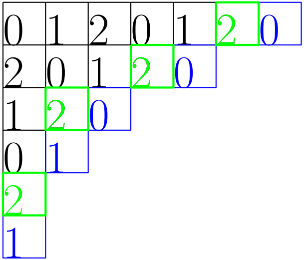

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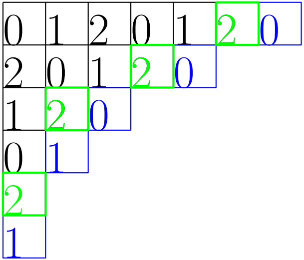

removable box: Can remove this box and still have a Ferrers shape.
addable box: Can add this box and still have a Ferrers shape.
Note: There are never both addable and removable boxes of the same residue.

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$$\widetilde{S}_a = \langle s_0, s_1, \ldots, s_{a-1} \mid \text{Relations} \rangle$$

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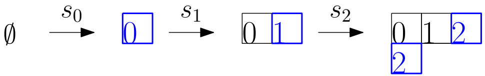

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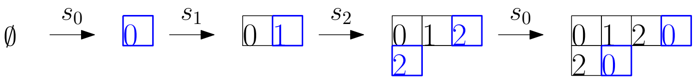

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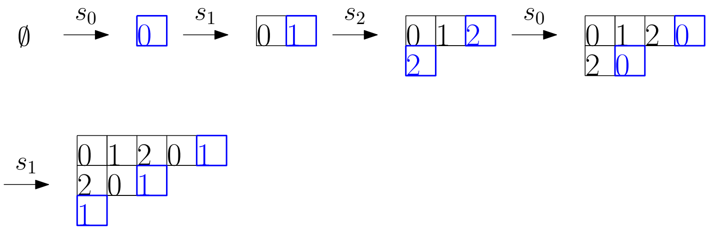

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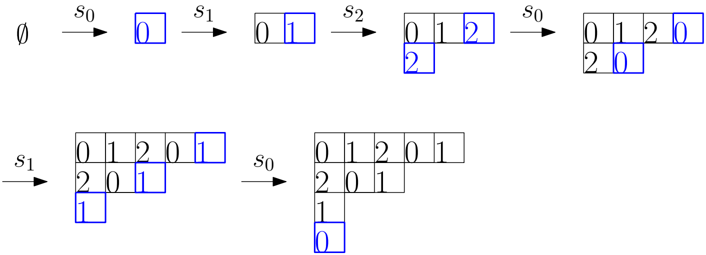

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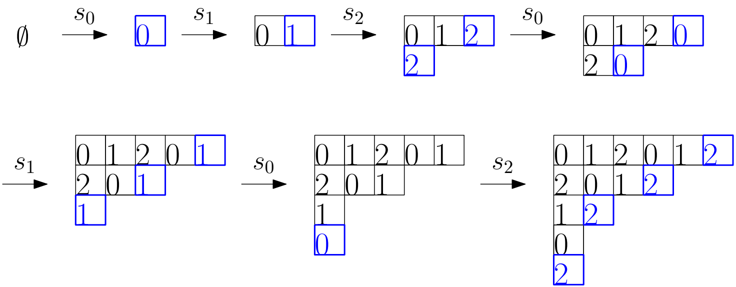

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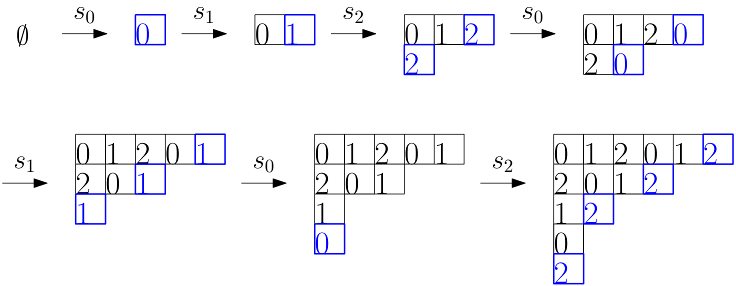

Can get all a-cores by acting on the empty a-core $\emptyset$ by some element $\widetilde{w}$ of $\widetilde{S}_a$ (transitive action).

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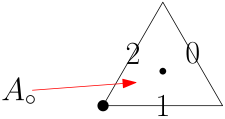

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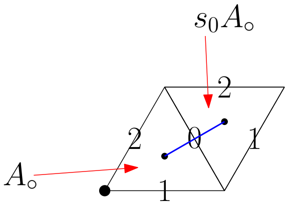

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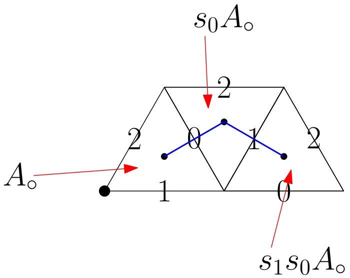

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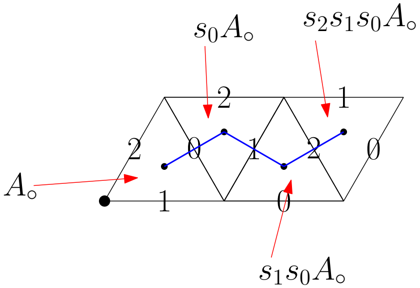

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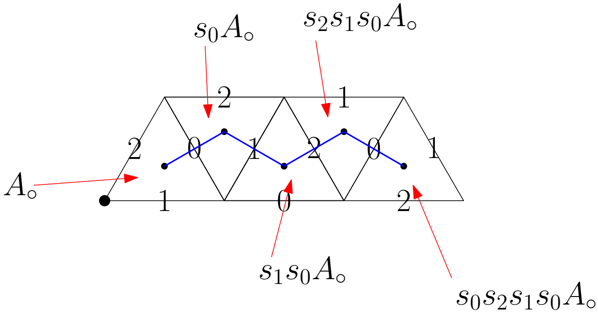

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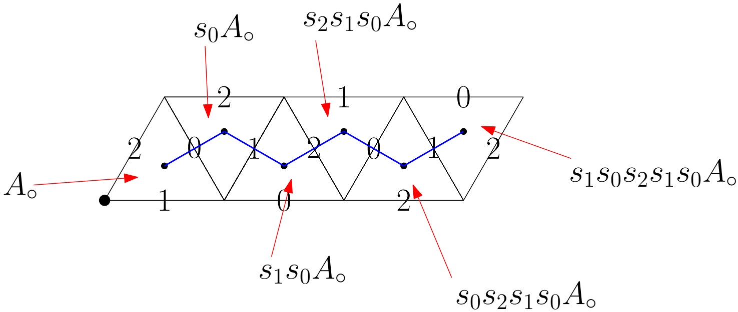

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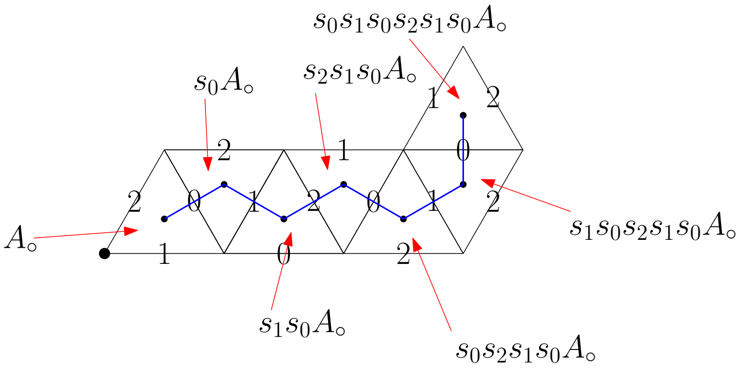

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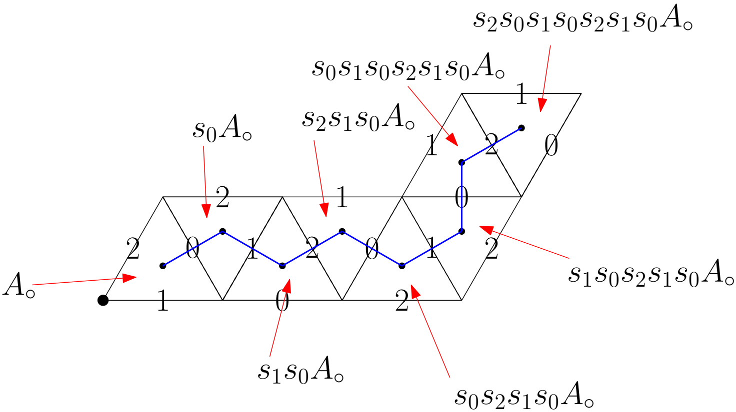

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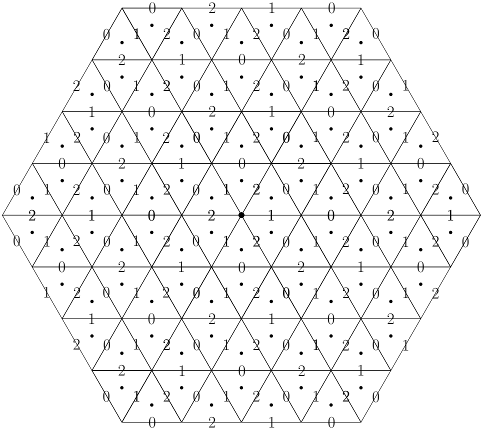

The affine reflection group $\widetilde{S}_a$ acting on a-cores

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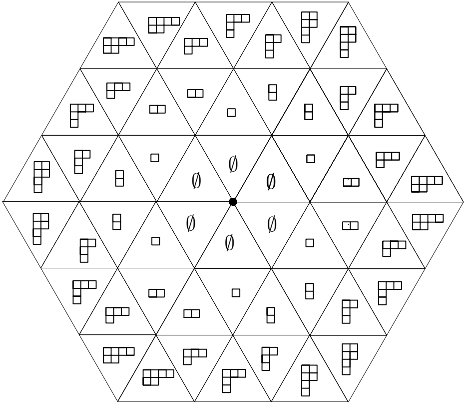

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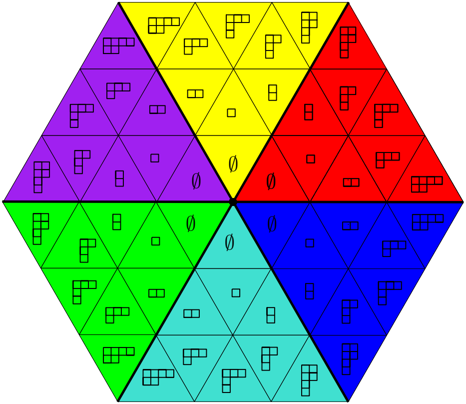

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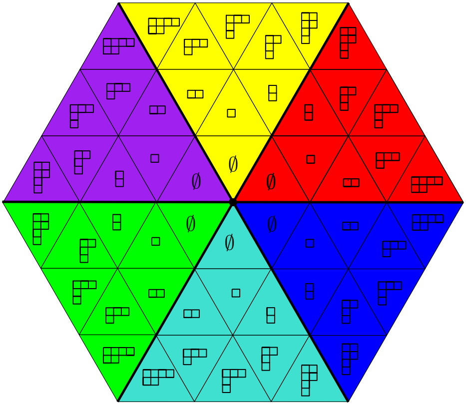

In every chamber, have every a-core exactly once.

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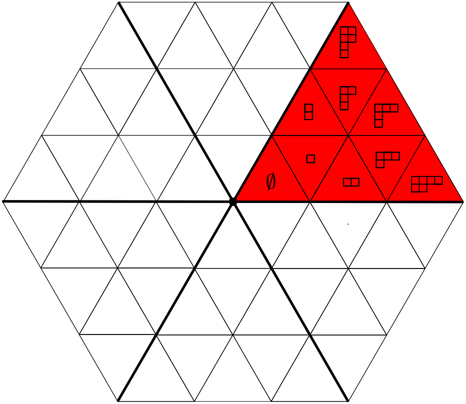

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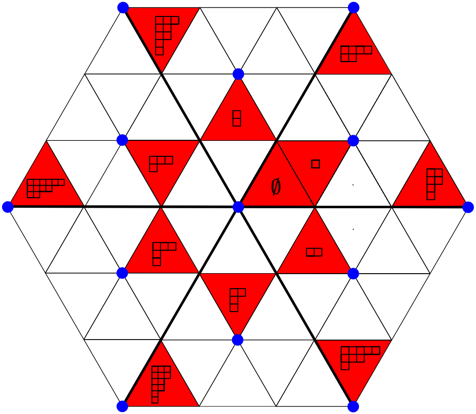
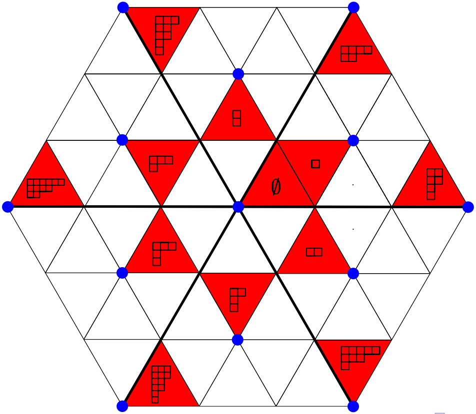


The lattice of cores

The coroot lattice $\check{Q}$ is the lattice of cores. Bijection:

$$\text{core}_{\check{Q}} : \{a\text{-cores}\} \to \check{Q}$$



The size of cores

For an a-core λ and $\mu = \text{core}_{\check{Q}}(\lambda)$, we have ($\rho$ is the Weyl vector)

$$\text{size}(\lambda) = \frac{a}{2}\left(\left\|\mu - \frac{\rho}{a}\right\|^2 - \left\|\frac{\rho}{a}\right\|^2\right)$$

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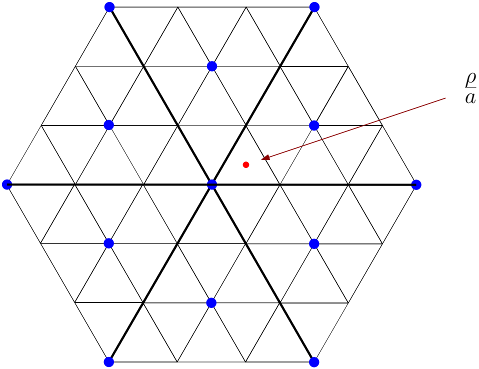

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using a formula of Macdonald. The classical proof of this is more direct.

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So we restrict to that case.

Simultaneous core partitions

A simultaneous (a, b)-core is an a-core that is also a b-core.
That is, an (a, b)-core is an integer partition with no hooklength equal to either a or b.
There are finitely many (a, b)-cores if and only if a and b are relatively prime.
So we restrict to that case.
Our first aim is to count the number of (a, b)-cores.

Simultaneous core partitions

The simultaneous $(3, 4)$-cores in green:

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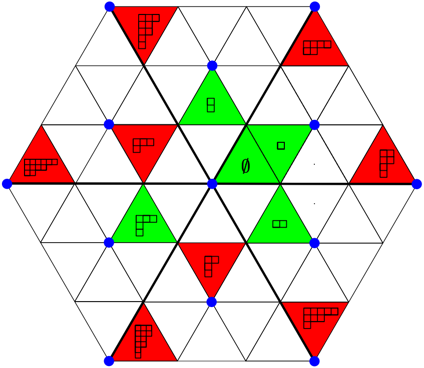

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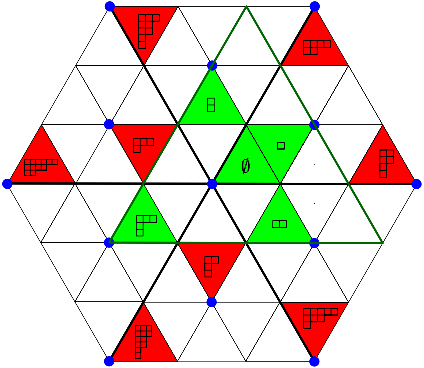
They are the lattice points in a simplex $\mathcal{S}_3(4)$!



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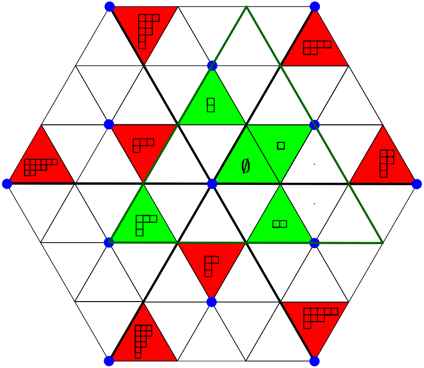

(a, b)-cores $\leftrightarrow$ $\check{Q} \cap \mathcal{S}_a(b)$.

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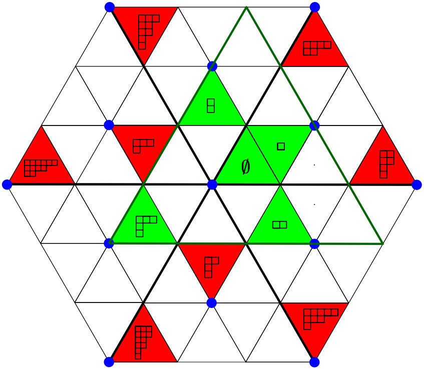

(a, b)-cores $\leftrightarrow$ $\check{Q} \cap \mathcal{S}_a(b)$. So we want to count $\check{Q} \cap \mathcal{S}_a(b)$.

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1. *If V is an n-dimensional vector space, L is a lattice and P is a polytope with vertices in L, then for positive integers b*

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Next goal: average size of an (a, b)-core.

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It is determined by its evaluation at 3 points.

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No simple proof of this fact is known.

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D. Zeilberger has computed some higher moments and conjectured a limiting distribution for the size of an $(a, a+k)$-core for fixed k and $a \to \infty$.

Other affine reflection groups

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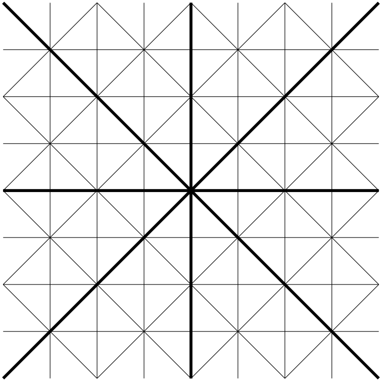
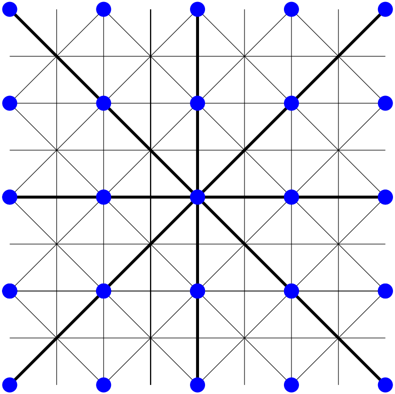


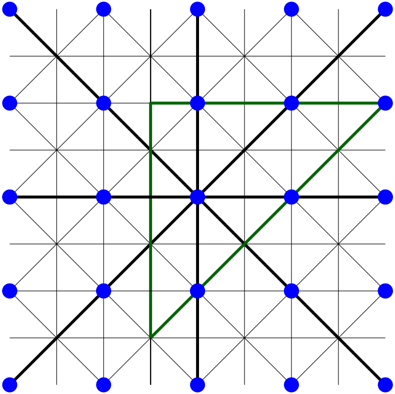
Other affine reflection groups

There is a lattice of cores (coroot lattice) $\check{Q}$. Example:



Other affine reflection groups

For b relatively prime to the Coxeter number h of the affine Weyl group $\widetilde{W}$, there is a simplex $\mathcal{S}_{\widetilde{W}}(b)$. Example: $\mathcal{S}_{\widetilde{W}}(5)$



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Then we can interpret them as special cases of statements that hold for any simply-laced affine Weyl group $\widetilde{W}$.
Simply-laced: All angles between walls of the fundamental alcove $A_\circ$ are 60 or 90 degrees.

The maximal size of an (a, b)-core

In terms of cores:

$$\text{The unique largest } (a,b)\text{-core has size } \frac{(a^2-1)(b^2-1)}{24}.$$

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Thanks for your attention!