RESEARCH STATEMENT

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0. Overview

My main interest so far has been the subfield of algebraic combinatorics known as *Coxeter-Catalan combinatorics*, which studies objects counted by *Coxeter-Catalan numbers*. I have proven several conjectures posed in 2005-2009 about refined enumerative properties of Coxeter-Catalan objects and their relationships to each other, focusing in particular on the combinatorics of the *Shi arrangement* of a crystallographic root system Φ. Furthermore, I used the geometry of *affine Weyl groups* to generalize several constructions and results that arose in the study of *diagonal harmonics* and *simultaneous core partitions*. In the future, I intend to prove new connections between Coxeter-Catalan objects via *quiver representation theory*.

1. Introduction to Coxeter-Catalan combinatorics

One of the most famous number sequences in combinatorics is the sequence $1, 2, 5, 14, 42, \ldots$ of *Catalan numbers* defined by $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$. It counts many objects, such as:

(1) *Noncrossing partitions* of $[n]$
(2) Triangulations of a convex $(n+2)$-gon
(3) *Nonnesting partitions* of $[n]$
(4) Increasing parking functions of length n
(5) Simultaneous $(n+1, n)$-core partitions.

All of these may be seen as type A_{n-1} specializations of more general constructions associated to a crystallographic root system Φ. These are, respectively:

(1) Elements in the interval $[e, c]$ of the reflection order on W
(2) Facets of the *cluster complex* $\Delta(\Phi)$
(3) Dominant regions of the Shi arrangement of Φ
(4) Orbits of the W-action on the finite torus $Q^\vee/(h+1)Q^\vee$
(5) Lattice points in $Q^\vee \cap \mathcal{S}_\Phi(h+1)$.

Here W is the Weyl group of Φ, c is a Coxeter element, h is the Coxeter number, $Q^\vee$ is the coroot lattice and $\mathcal{S}_\Phi(h+1)$ is a certain simplex isometric to the $(h+1)$-st dilation of the fundamental alcove. All these objects are Coxeter-Catalan objects, that is they are counted by the Coxeter-Catalan number

$$\mathrm{Cat}(\Phi) = \prod_{i=1}^{r} \frac{h + d_i}{d_i}.$$

Here r is the rank of Φ and $d_1 \leq d_2 \leq \ldots \leq d_r = h$ are its *degrees*. However, for both (1) and (2), every known proof of this fact is non-uniform, that is it uses the classification of irreducible crystallographic root systems. Thus the central problem of Coxeter-Catalan combinatorics is the following.

Problem. *Give a uniform proof of a bijection between one of the* noncrossing *objects (1), (2) and one of the* nonnesting *objects (3), (4), (5).*

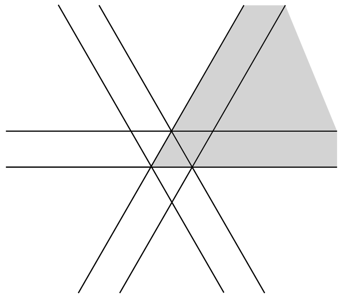

FIGURE 1. The Shi arrangement of type A_2 has $(3+1)^2 = 16$ regions, $\mathrm{Cat}(A_2) = 5$ of which are dominant.

2. The Shi arrangement and the $H = F$ correspondence

My personal favorite Coxeter-Catalan object is the Shi arrangement. It is the hyperplane arrangement in V, the ambient space of Φ, given by the hyperplanes $H_\alpha^d = \{x \in V \mid \langle x, \alpha \rangle = d\}$ for $\alpha \in \Phi^+$ and $d = 0, 1$. It has $(h+1)^r$ regions, $\mathrm{Cat}(\Phi)$ of which are contained in the dominant chamber. A *wall* of a region is a hyperplane supporting one of its facets. A wall of a region R that does not contain the origin is called a *floor* of R if it separates it from the origin, and a *ceiling* otherwise. In 2006, Chapoton defined the *H-triangle* as

$$H_\Phi(x, y) = \sum_R x^{|FL(R)|} y^{|FL(R) \cap S|},$$

where $FL(R) = \{\alpha \in \Phi^+ \mid H_\alpha^1 \text{ is a floor of } R\}$ and the sum is over all dominant regions R of the Shi arrangement. He conjectured a close relationship, called the *$H = F$ correspondence*, with the *F-triangle*, a similar polynomial for the cluster complex $\Delta(\Phi)$ [Cha06, Conjecture 6.1]. In 2009, Armstrong generalized this correspondence to the *m-extended Shi arrangement*, given by the hyperplanes H_α^d for $\alpha \in \Phi^+$ and $-m < d \leq m$, and the generalized cluster complex $\Delta^{(m)}(\Phi)$ [Arm09, Conjecture 5.3.2].

In 2014, I proved Armstrong's conjecture [Thi14b], and thereby also Chapoton's, by combining a case-by-case verification of a conjecture of Fomin and Reading [FR05, Conjecture 10.1] (corresponding to the $y = 1$ specialization of the $H = F$ correspondence) with an otherwise uniform argument. The result has many interesting corollaries, relating the H-triangle also to the M-triangle of noncrossing partitions as well as verifying a conjecture of Athanasiadis and Tzanaki on ceilings of bounded dominant regions of the m-extended Shi arrangement [AT06, Conjecture 1.2].

3. Floors and Ceilings

In 2014, I gave a construction that for any set M of hyperplanes of the m-extended Shi arrangement provides a bijection between the set $U(M)$ of dominant regions R such that all hyperplanes in M are floors of R and the set $L(M)$ of dominant regions R' such that all hyperplanes in M are ceilings of R' [Thi14a]. This allowed me to deduce some enumerative corollaries, including another conjecture by Armstrong [Arm09, Conjecture 5.1.24].

4. Diagonal Harmonics

The Hilbert series of the space of diagonal harmonics of the symmetric group S_n has two combinatorial interpretations [CM15]:

$$\mathcal{DH}(n;q,t) = \sum_{P \in \mathcal{P}_n} q^{\mathrm{dinv'}(P)} t^{\mathrm{area}(P)} = \sum_{R \in \mathcal{D}_n} q^{\mathrm{area'}(R)} t^{\mathrm{bounce}(R)},$$

where $\mathcal{P}_n$ is the set of *parking functions* of length n, viewed as Dyck paths with labelled north steps, and $\mathcal{D}_n$ is the set of *diagonally labelled* Dyck paths. There is a bijection ζ_{HL} due to Haglund and Loehr [HL05] that maps $\mathcal{P}_n$ to $\mathcal{D}_n$ and sends the bistatistic $(\mathrm{dinv'}, \mathrm{area})$ to $(\mathrm{area'}, \mathrm{bounce})$, demonstrating the second equality. One may naturally identify $\mathcal{P}_n$ with the finite torus $Q^\vee/(h+1)Q^\vee$ and $\mathcal{D}_n$ with the set $\mathsf{Shi}(A_{n-1})$ of regions of the Shi arrangement of type A_{n-1}.

In 2016, I gave a uniform bijection ζ from the finite torus $Q^\vee/(h+1)Q^\vee$ to the set $\mathsf{Shi}(\Phi)$ of regions of the Shi arrangement for any crystallographic root system Φ [Thi16]. This map generalizes ζ_{HL}. In the process, I also generalized the *Anderson map* $\mathcal{A}_{GMV}$ that was defined by Gorsky, Mazin and Vazirani in their study of rational parking functions [GMV16].

Robin Sulzgruber and I have found combinatorial descriptions of ζ for the classical types B, C and D [ST16].

5. Simultaneous Cores

When a and b are relatively prime positive integers, the set of simultaneous (a,b)-core partitions is counted by the *rational Catalan number* $\frac{1}{a+b}\binom{a+b}{a}$. Furthermore the maximal and average size of an (a,b)-core have nice product formulae [OS07] [Joh15]:

$$\max_{\kappa\ (a,b)\text{-core}} \mathrm{size}(\kappa) = \frac{(a^2-1)(b^2-1)}{24} \tag{1}$$

$$\mathop{\mathbb{E}}_{\kappa\ (a,b)\text{-core}} \mathrm{size}(\kappa) = \frac{(a-1)(b-1)(a+b+1)}{24} \tag{2}$$

In 2015, Nathan Williams and I noticed that (a,b)-core partitions can be interpreted as coroot lattice points in a certain simplex $\mathcal{S}(b)$ associated to the affine symmetric group $\widetilde{S}_a$, and size as a quadratic functional [TW15]. This allowed us to use Ehrhart theory to give uniform formulae generalizing (1) and (2) for a simply-laced root system Φ:

$$\max_{\mu \in Q^\vee \cap \mathcal{S}_\Phi(b)} \mathrm{size}(\mu) = \frac{r(h+1)(b^2-1)}{24} \tag{3}$$

$$\mathop{\mathbb{E}}_{\mu \in Q^\vee \cap \mathcal{S}_\Phi(b)} \mathrm{size}(\mu) = \frac{r(b-1)(h+b+1)}{24} \tag{4}$$

for b relatively prime to the Coxeter number h of Φ.

Problem. *Find a uniform proof of (4).*

6. Quiver Representations

The indecomposable representations of a *Dynkin quiver* Q are in canonical bijection with the positive roots in Φ^+ of the corresponding root system Φ. We call a subset $A \subseteq \mathsf{Ind}(Q)$ a *subfactor antichain* if no element of A is a subfactor of another. We call a subset $B \subseteq \mathsf{Ind}(Q)$ a Hom-*antichain* if there are no non-trivial homomorphisms between elements of B.

Theorem ([DR79, IT09]). *The subfactor antichains in* $\mathsf{Ind}(Q)$ *are nonnesting Coxeter-Catalan objects. The* Hom*-antichains in* $\mathsf{Ind}(Q)$ *are noncrossing Coxeter-Catalan objects.*

In particular both are counted by $\mathrm{Cat}(\Phi)$, but in the case of the latter no uniform proof is known. Claus Michael Ringel has proposed to seek a uniform bijection between subfactor antichains and Hom antichains using certain pullback-pushout squares he calls *crossovers*. Starting in 2017, I am working with him on this project.

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