On the H-triangle of generalised nonnesting partitions

Marko Thiel

Fakultät für Mathematik, Universität Wien, 1090 Vienna, Austria

Classical Catalan objects

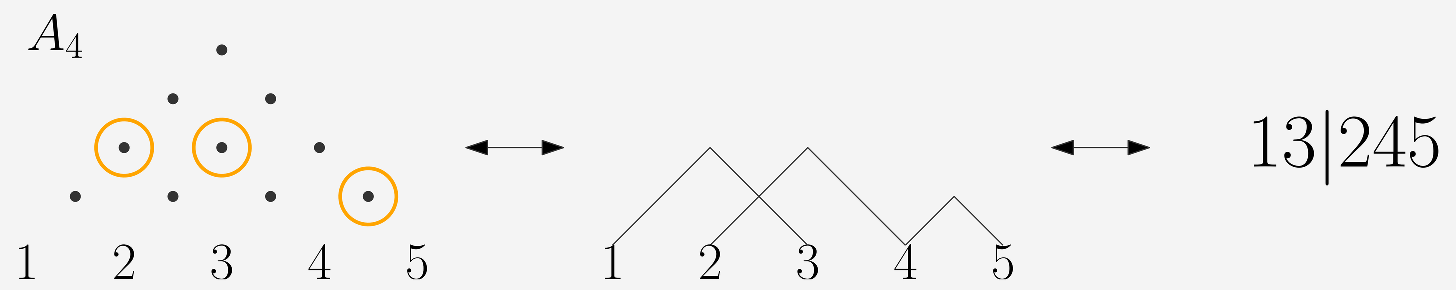
The *Catalan number* $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$ counts many classical objects in combinatorics, for example the set of *nonnesting partitions* of $[n] = \{1, 2, \ldots, n\}$ and the set of triangulations of a convex $(n+2)$-gon.

Nonnesting partitions

The *root poset* of Φ is the set Φ^+ of positive roots with the partial order relation

$$\beta \geq \alpha \text{ if and only if } \beta - \alpha \in \langle S \rangle_{\mathbb{N}}.$$

The set $NN(\Phi)$ of antichains in the root poset is called the set of *nonnesting partitions* of Φ.
If Φ is of type A_{n-1}, it can be identified with the set of *classical nonnesting partitions* of $[n]$:



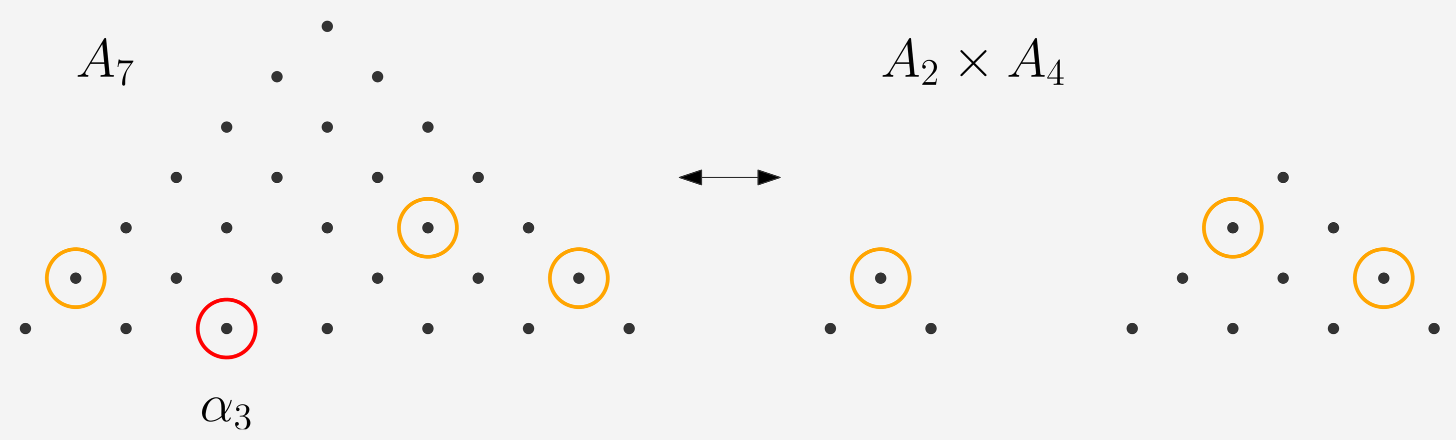
The H-triangle

The H-*triangle* of Φ is the polynomial

$$H_\Phi(x, y) = \sum_{A \in NN(\Phi)} x^{|A|} y^{|A \cap S|}.$$

A decomposition

For $\alpha \in S$, the map $A \mapsto A \backslash \{\alpha\}$ is a bijection from the set of nonnesting partitions of Φ containing α to the set of nonnesting partitions of $\Phi(S \backslash \{\alpha\}) = \Phi \cap \langle S \backslash \{\alpha\} \rangle_{\mathbb{R}}$.



A differential equation

The decomposition gives us the differential equation

$$\frac{\partial}{\partial y} H_\Phi(x, y) = x \sum_{\alpha \in S} H_{\Phi(S \backslash \{\alpha\})}(x, y).$$

The $H = F$ correspondence

Theorem: If n is the rank of Φ, we have that

$$H_\Phi(x, y) = (x-1)^n F_\Phi \left(\frac{1}{x-1}, \frac{1 + (y-1)x}{x-1} \right).$$

Proof: Induction on n.

$$\begin{aligned} \frac{\partial}{\partial y} H_\Phi(x, y) &= x \sum_{\alpha \in S} H_{\Phi(S \backslash \{\alpha\})}(x, y) \\ &= x \sum_{\alpha \in S} (x-1)^{n-1} F_{\Phi(S \backslash \{\alpha\})} \left(\frac{1}{x-1}, \frac{1 + (y-1)x}{x-1} \right) \\ &= \frac{\partial}{\partial y} (x-1)^n F_\Phi \left(\frac{1}{x-1}, \frac{1 + (y-1)x}{x-1} \right). \end{aligned}$$

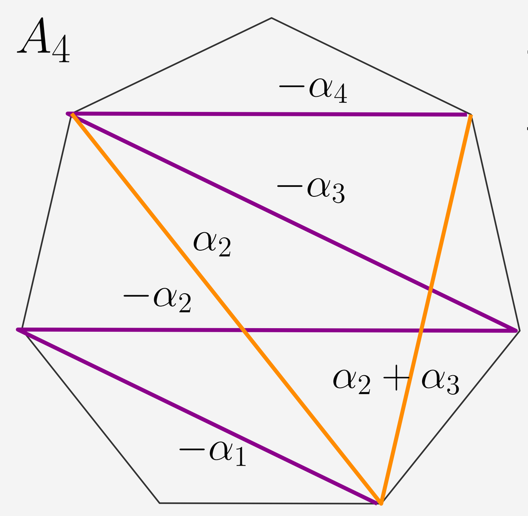
The specialisation at $y = 1$ is verified case-by-case. □

Coxeter-Catalan objects

In Coxeter-Catalan combinatorics, both of these are seen as the "type A_{n-1}" cases of more general objects associated to a crystallographic root system Φ. Fix a positive system Φ^+ and simple system S for Φ.

The cluster complex

There is a binary relation called *compatibility* on the set $\Phi_{\geq -1} = \Phi^+ \sqcup -S$ of *almost positive roots* of Φ such that $-\alpha \in -S$ is compatible with $\beta \in \Phi_{\geq -1}$ if and only if $\beta \in \langle S \backslash \{\alpha\} \rangle_{\mathbb{R}}$. The simplicial complex $\Delta(\Phi)$ of pairwise compatible subsets of $\Phi_{\geq -1}$ is called the *cluster complex* of Φ.
If Φ is of type A_{n-1}, one may label the diagonals of a convex $(n+2)$-gon by the almost positive roots in such a way that two almost positive roots are compatible if and only if the corresponding diagonals do not cross.



The negative simple roots form a snake.

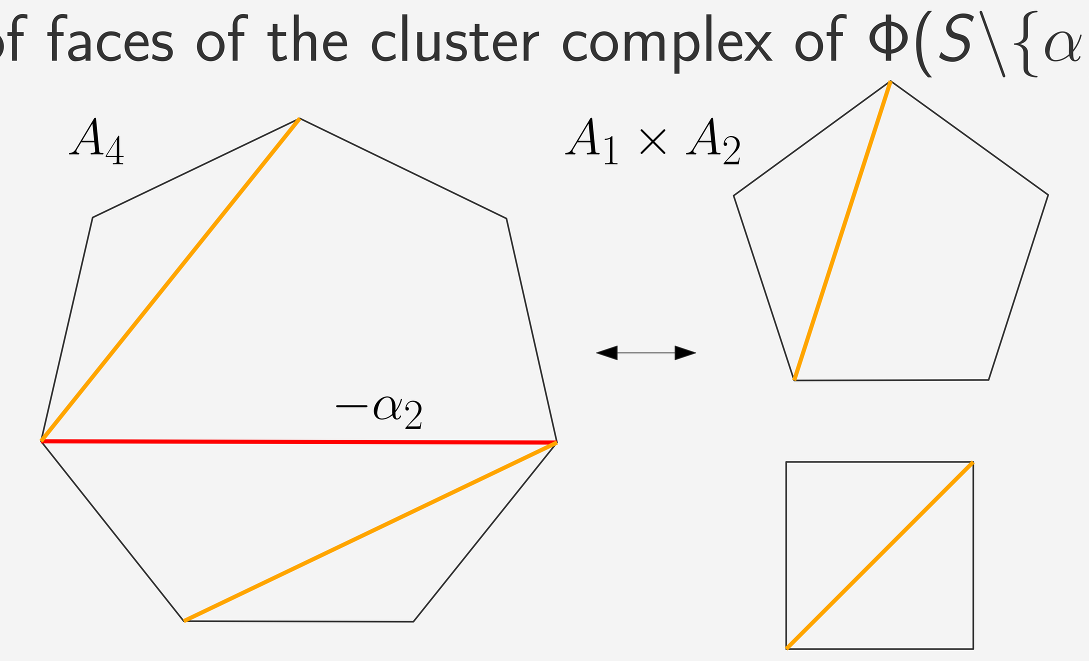
The F-triangle

The F-*triangle* of Φ is the polynomial

$$F_\Phi(x, y) = \sum_{F \in \Delta(\Phi)} x^{|F \cap \Phi^+|} y^{|F \cap -S|}.$$

A decomposition

For $\alpha \in -S$, the map $F \mapsto F \backslash \{-\alpha\}$ is a bijection from the set of faces of the cluster complex of Φ containing $-\alpha$ to the set of faces of the cluster complex of $\Phi(S \backslash \{\alpha\}) = \Phi \cap \langle S \backslash \{\alpha\} \rangle_{\mathbb{R}}$.



A differential equation

The decomposition gives us the differential equation

$$\frac{\partial}{\partial y} F_\Phi(x, y) = \sum_{\alpha \in S} F_{\Phi(S \backslash \{\alpha\})}(x, y).$$

The Fuss-Catalan generalisation

For a positive integer k, one can define two Fuß-Catalan objects: the set of k-generalised nonnesting partitions $NN^{(k)}(\Phi)$ and the generalised cluster complex $\Delta^{(k)}(\Phi)$. These also have an H-triangle and an F-triangle respectively and the same $H = F$ correspondence holds. They specialise to the corresponding Coxeter-Catalan objects when $k = 1$.

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marko.thiel@univie.ac.at