§4: Singular integral operators

- In this section, we will study operators which are formally given as:

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y) \cdot f(y)\, dy = (K * f)(x)$$

for a singular convolution potential K which is not integrable near zero.

More generally, we will study operators of the form:

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) \cdot f(y)\, dy$$

for a singular kernel which is not integrable near the diagonal $x=y$ of $\mathbb{R}^n \times \mathbb{R}^n$.

We will show that there exists a class of such operators which are well-defined and which satisfy good boundedness properties. $\rightsquigarrow$ <u>Calderón – Zygmund operators.</u>

4.1. The Hilbert transform

Throughout section 4.1, we work in dimension $n=1$.

We can first define the Hilbert transform on $\mathcal{S}(\mathbb{R})$ by:

$$Hf(x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \cdot \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy$$

We note that the operator is formally given as a convolution with $K(x) = \frac{1}{\pi x}$, which is not integrable near zero.

Let us note that, for fixed $\varepsilon > 0$:

$\frac{1}{\pi} \cdot \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy$ is well-defined and it equals:

$$= \frac{1}{\pi} \cdot \int_{1 \geq |x-y| > \varepsilon} \frac{f(y)}{x-y} dy + \frac{1}{\pi} \cdot \int_{|x-y|>1} \frac{f(y)}{x-y} dy$$

$$= \frac{1}{\pi} \cdot \int_{1 \geq |x-y| > \varepsilon} \frac{f(y)-f(x)}{x-y} dy + \frac{1}{\pi} \cdot \int_{|x-y|>1} \frac{f(y)}{x-y} dy$$

Since $|f(y)-f(x)| \leq C|x-y|$, it follows that the first term has a limit as $\varepsilon \to 0$.
The second term is finite since $f \in L^1$.
Moreover, we can control it in absolute value by $C \, \|(1+x^2) f\|_{L^\infty}$.

- It is possible to rewrite the Hilbert transform in terms of tempered distributions.

The principal value of $\frac{1}{x}$, p.v. $\frac{1}{x}$ is defined as:

$$\text{p.v.} \frac{1}{x}(f) := \lim_{\varepsilon \to 0} \left(\int_{|y|>\varepsilon} \frac{f(y)}{y} dy \right), \text{ for } f \in \mathcal{S}(\mathbb{R}).$$

This object is well-defined by the argument given earlier.
Namely, for $f \in \mathcal{S}(\mathbb{R})$, it is the case that:

$$\text{p.v.} \frac{1}{x}(f) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{f(y)}{y} dy = \lim_{\varepsilon \to 0} \left(\int_{1 \geq |y| > \varepsilon} \frac{f(y)-f(0)}{y} dy \right)$$

$$+ \int_{|y|>1} \frac{f(y)}{y} dy$$

In particular, p.v. $\frac{1}{x}(f)$ makes sense and:

$$\left| \text{p.v.} \frac{1}{x}(f) \right| \leq C \left(\|f'\|_{L^\infty} + \|xf\|_{L^\infty} \right).$$

$\Rightarrow$ p.v. $\frac{1}{x} \in \mathcal{S}'(\mathbb{R})$, i.e. this is a tempered distribution.

We can write:

$$Hf = \frac{1}{\pi} \text{p.v.} \frac{1}{x} * f.$$

It can be shown that, under a suitable definition of the convolution of a tempered distribution and a Schwartz function $(Hf)^\wedge(\xi) = \frac{1}{\pi}\left(\text{p.v.}\frac{1}{x}\right)^\wedge(\xi) \cdot \hat{f}(\xi)$.

[Namely, if $u \in \mathcal{S}'(\mathbb{R})$, $\varphi \in \mathcal{S}(\mathbb{R})$, then $u * \varphi$ makes sense as a function

$(u * \varphi)(x) = $ " $\int u(y)\varphi(x-y)\,dy$ " $= u(\varphi(-\cdot + x))$

The definition $u(\varphi(-\cdot+x))$ makes sense in general.]

Another fact (which we will not prove) is that:

$$\left(\text{p.v.}\frac{1}{x}\right)^\wedge(\xi) = -i\pi \,\text{sign}(\xi)$$

where:

$$\text{sign}(\xi) = \begin{cases} 1 & \text{for } \xi \geq 0 \\ -1 & \text{for } \xi < 0 \end{cases}$$

In particular, it follows that (at least formally), for $f \in \mathcal{S}(\mathbb{R})$, it is the case that

$$\widehat{Hf}(\xi) = -i\,\text{sign}(\xi) \cdot \hat{f}(\xi) \qquad (*)$$

By Plancherel's theorem, it follows that:

$$\|Hf\|_{L^2} = \|f\|_{L^2}.$$

$\Rightarrow$ By density, it is possible to extend the definition of H to all of L^2.

Let us note two more properties of H that follow from $(*)$.

① $H(Hf) = -f$

② $\int Hf \cdot g = -\int f \cdot Hg$

· We prove ① and ② for $f, g \in \mathcal{S}(\mathbb{R})$. By density, the claims then hold for $f, g \in L^2(\mathbb{R})$.

<u>Proof of ①</u>:

$$(H(Hf))^\wedge(\xi) = -i\,\mathrm{sign}(\xi) \cdot \widehat{Hf}(\xi) =$$

$$= (-i\,\mathrm{sign}(\xi)) \cdot (-i\,\mathrm{sign}(\xi)) \cdot \hat{f}(\xi) = -\hat{f}(\xi)$$

$$\Rightarrow \quad H(Hf) = -f \quad \text{i.e. } H^2 = -Id.$$

<u>Proof of ②</u>: Let $\check{\ }$ denote the inverse Fourier transform and let $\tilde{h}(x) := h(-x)$. We note that $\check{h} = \hat{\tilde{h}}$.

$$\int Hf \cdot g = \int Hf \cdot \widehat{\check{g}} = \int \widehat{Hf} \cdot \check{g} = \int -i\,\mathrm{sign}(\xi)\hat{f}(\xi)\hat{\tilde{g}}(\xi)\,d\xi$$

$$= \int \hat{f}(\xi) \cdot \left(-i\,\mathrm{sign}(\xi) \cdot \hat{\tilde{g}}(\xi)\right) d\xi = \int \hat{f}(\xi) \cdot (H\tilde{g})^\wedge(\xi)\,d\xi =$$

$$= \int \hat{\hat{f}}(\xi) \cdot (H\tilde{g})(\xi)\,d\xi = \int \tilde{f}(\xi) \cdot (H\tilde{g})(\xi)\,d\xi$$

$$= \int \tilde{f}(\xi) \cdot \widetilde{Hg}(\xi)\,d\xi = -\int f(\xi) \cdot Hg(\xi)\,d\xi = -\int f \cdot Hg$$

Here, we used the fact that $H\tilde{g} = \widetilde{Hg}$.

In order to see this, we note that, for fixed $\varepsilon > 0$:

$$\int_{|y|>\varepsilon} \frac{\tilde{g}(x-y)}{y}\,dy = \int_{|y|>\varepsilon} \frac{g(y-x)}{y}\,dy = \int_{|z|>\varepsilon} \frac{g(-x-z)}{z}\,dz$$

Letting $\varepsilon \to 0$, we indeed obtain that:

$$(H\tilde{g})(x) = (Hg)(-x) = \widetilde{Hg}(x), \text{ as was claimed.} \quad \square$$

The main result that we will prove about the Hilbert transform is:

Theorem 4.1:

1) (Kolmogorov)

H is weak $(1,1)$, i.e.

$\mu(|Hf|>\lambda) \leq \frac{C}{\lambda} \cdot \|f\|_{L^1}$.

2) (M. Riesz)

H is strong (p,p) for $1<p<\infty$, i.e.

$\|Hf\|_{L^p} \leq C_p \|f\|_{L^p}$.

Proof: 1) Let $\lambda>0$ be given. Let us for now consider the case when $f \in \mathcal{S}(\mathbb{R})$ and $f \geq 0$. Then Hf is well-defined.

Let us start by forming the Calderón-Zygmund decomposition of f at height λ.

We get a collection of disjoint dyadic intervals such that:

- $f \leq \lambda$ on $(\bigcup_j I_j)^c$
- $\mu(\bigcup_j I_j) \leq \frac{1}{\lambda} \cdot \|f\|_{L^1}$
- $\lambda < \fint_{I_j} f \leq 2\lambda$.

· Let us define:

$$g := \begin{cases} f & \text{on } (\bigcup_j I_j)^c \\ \fint_{I_j} f & \text{on } I_j \end{cases}$$

$$b := f - g.$$

Moreover, given j, we define $b_j := b \cdot \chi_{I_j}$.

We observe that $\int_{-\infty}^{+\infty} b_j = \int_{I_j} b = \int_{I_j} (f-g) = \int_{I_j} (f - \fint_{I_j} f) = 0$.

Since $f \in \mathcal{S}(\mathbb{R})$, it follows that $g = f$ outside of a compact set.

Furthermore, if $g(x) \neq f(x)$, then $x \in I_j$ for some j and so $g(x) \in [0, 2\lambda]$.

It follows that $g \in L^2(\mathbb{R}) \Rightarrow b = f - g \in L^2(\mathbb{R})$.

Moreover, $g \leq 2\lambda$. Namely, this holds on $\bigcup_j I_j$ by construction, but it also holds on $(\bigcup_j I_j)^c$ since $f \leq \lambda$ there.

- In particular, Hg and Hb are well-defined.

We observe that:

$\mu(|Hf| > \lambda) \leq \mu(|Hg| > \frac{\lambda}{2}) + \mu(|Hb| > \frac{\lambda}{2})$

We will estimate each term separately.

- Let us first estimate $\mu(|Hg| > \frac{\lambda}{2})$. This is, by Markov's inequality

$$\leq \left(\frac{2}{\lambda}\right)^2 \cdot \int_{-\infty}^{+\infty} |Hg(x)|^2 dx = \left(\frac{2}{\lambda}\right)^2 \cdot \int_{-\infty}^{+\infty} |g(x)|^2 dx$$

$$\leq \left(\frac{2}{\lambda}\right)^2 \cdot 2\lambda \cdot \int_{-\infty}^{+\infty} |g(x)| dx = \frac{8}{\lambda} \cdot \int_{-\infty}^{+\infty} |g(x)| dx = \frac{8}{\lambda} \cdot \|g\|_{L^1}$$

We note that $\|g\|_{L^1} = \|f\|_{L^1}$ because $g = f$ on $(\bigcup_j I_j)^c$ and $\int_{I_j} |g| = |\int_{I_j} f| = \int_{I_j} |f|$ since $f \geq 0$.

In conclusion:

$$\mu(|Hg| > \tfrac{\lambda}{2}) \leq \frac{8}{\lambda} \cdot \|f\|_{L^1} \quad (1)$$

- Let us now estimate $\mu(|Hb| > \frac{\lambda}{2})$.

We first define:

$\Omega^* := \bigcup_j \widehat{I_j}$, where $\widehat{I_j}$ is the interval with same center as I_j of twice the length.

Then:

$$\mu(\Omega^*) \leq \sum_j \mu(\widehat{I}_j) = \sum_j 2\mu(I_j) =$$

$$= 2\mu\left(\bigcup_j I_j\right) \leq \frac{2}{\lambda}\|f\|_{L^1} \quad (2)$$

from the construction of the Calderón - Zygmund decomposition.

So, $\mu(|Hb| > \frac{\lambda}{2}) = \mu(\{x \in \Omega^*, |Hb(x)| > \frac{\lambda}{2}\}) + \mu(\{x \notin \Omega^*, |Hb(x)| > \frac{\lambda}{2}\})$

Now, $\mu(\{x \in \Omega^*, |Hb(x)| > \frac{\lambda}{2}\}) \leq \mu(\Omega^*) \leq \frac{2}{\lambda}\|f\|_{L^1}$.

· We now estimate $\mu(\{x \notin \Omega^*, |Hb(x)| > \frac{\lambda}{2}\})$

$$\leq \frac{2}{\lambda} \int_{\mathbb{R}\setminus\Omega^*} |Hb(x)|\,dx$$

· By the Dominated Convergence theorem, we know that

$$\sum_{j=1}^{N} b_j \to b \text{ in } L^2 \text{ as } N \to \infty$$

By L^2-unitarity of H, it follows that

$$\sum_{j=1}^{N} Hb_j \to Hb \text{ in } L^2 \text{ as } N \to \infty.$$

Hence, up to a subsequence:

$$\sum_j Hb_j \to Hb \text{ almost everywhere.}$$

In particular:

$$\sum_j |Hb_j| \geq |Hb|, \text{ almost everywhere.}$$

So:

$$\mu(\{x \notin \Omega^*, |Hb(x)| > \tfrac{\lambda}{2}\}) \leq \mu(\{x \notin \Omega^*, \sum_j |Hb_j(x)| > \tfrac{\lambda}{2}\})$$

$$\leq \frac{2}{\lambda} \int_{\mathbb{R}\setminus\Omega^*} \left(\sum_j |Hb_j(x)|\right) dx$$

$$\leq \sum_j \left(\frac{2}{\lambda} \cdot \int_{\mathbb{R}\setminus\widehat{I_j}} |Hb_j(x)|\, dx \right).$$

- Let $c_j :=$ the center of I_j.

Then, for $x \in \mathbb{R}\setminus\widehat{I_j}$, we know that:

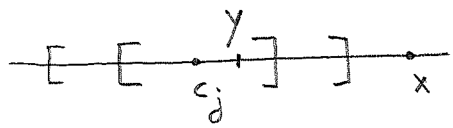
$$Hb_j(x) = \frac{1}{\pi} \cdot \int_{I_j} \frac{b_j(y)}{x-y}\, dy.$$ We note that $|x-y|$ is bounded away from 0 uniformly in $y \in I_j$

Since $\int_{I_j} b_j(y)\, dy = 0$, we deduce that:

$$Hb_j(x) = \frac{1}{\pi} \cdot \int_{I_j} b_j(y) \cdot \left(\frac{1}{x-y} - \frac{1}{x-c_j} \right) dy =$$

$$= \frac{1}{\pi} \cdot \int_{I_j} b_j(y) \cdot \frac{y-c_j}{(x-y)\cdot(x-c_j)}\, dy$$

- We know that, for all $y \in I_j$:



$$\begin{cases} |y-c_j| \leq \frac{1}{2}|I_j| \\ |x-y| \geq \frac{1}{2}|x-c_j| \end{cases}$$ (since $x \in \mathbb{R}\setminus\widehat{I_j}$)

So: $$|Hb_j(x)| \leq \frac{1}{\pi} \cdot \int_{I_j} |b_j(y)| \cdot \frac{\frac{1}{2}|I_j|}{\frac{1}{2}|x-c_j|\cdot|x-c_j|}\, dy$$

$$= \frac{1}{\pi} \cdot \frac{|I_j|}{|x-c_j|^2} \cdot \int_{I_j} |b_j(y)|\, dy$$

$$\Rightarrow \sum_j \int_{\mathbb{R}\setminus\widehat{I_j}} |Hb_j(x)|\, dx \leq \sum_j \frac{1}{\pi} \cdot \left(\int_{\mathbb{R}\setminus\widehat{I_j}} \frac{|I_j|}{|x-c_j|^2}\, dx \right) \cdot \left(\int_{I_j} |b_j(y)|\, dy \right)$$

Let us note that:

$$\int_{\mathbb{R}\setminus\widehat{I}_j} \frac{|I_j|}{|x-c_j|^2}\,dx = 2\cdot\int_{r>|I_j|} \frac{|I_j|}{r^2}\,dr = 2$$

So:
$$\sum_j \int_{\mathbb{R}\setminus\widehat{I}_j} |Hb_j(x)|\,dx \leq \sum_j \frac{2}{\pi}\cdot\underbrace{\left(\int_{I_j} |b_j(y)|\,dy\right)}_{\leq 2\int_{I_j} f(y)\,dy}$$

$$\leq \frac{4}{\pi}\cdot\sum_j \int_{I_j} |f(y)|\,dy \leq \frac{4}{\pi}\|f\|_{L^1}.$$

Hence:

$$\mu\left(\{x\notin\Omega^*,\ |Hb(x)|>\tfrac{\lambda}{2}\}\right) \leq \frac{8/\pi}{\lambda}\cdot\|f\|_{L^1} \quad (3)$$

(1), (2), (3) then imply that:

$$\mu(\{x : |Hf(x)|>\lambda\}) \leq \frac{C}{\lambda}\cdot\|f\|_{L^1} \quad (\Delta)$$

We have shown (Δ) for $f\in\mathcal{S}(\mathbb{R})$ and $f\geq 0$.
A standard decomposition argument allows us to deduce (Δ) for general $f\in\mathcal{S}(\mathbb{R})$.
We can rewrite this as

$$\|Hf\|_{L^{1,\infty}} \leq C\|f\|_{L^1} \quad \text{for all } f\in\mathcal{S}(\mathbb{R})$$

By density of $\mathcal{S}(\mathbb{R})$ in $L^1(\mathbb{R})$ and by the completeness properties of $L^1(\mathbb{R})$ and $L^{1,\infty}(\mathbb{R})$, it follows that H extends uniquely to a linear operator on $L^1(\mathbb{R})$ which satisfies $\|Hf\|_{L^{1,\infty}}\leq C\|f\|_{L^1}$ for all $f\in L^1(\mathbb{R})$. In other words,

$$\mu(|Hf|>\lambda) \leq \frac{C}{\lambda}\cdot\|f\|_{L^1}, \text{ for all } f\in L^1(\mathbb{R}).\ (\Delta\Delta).$$

• Let us elaborate a bit on the last step.

Suppose that (f_n) is a sequence of Schwartz functions that converge in $L^1(\mathbb{R})$. Let $g_n := Hf_n$. It follows from (Δ) that (g_n) is a Cauchy sequence in $L^{1,\infty}(\mathbb{R})$ in the sense that for all $\varepsilon>0$, there exists $M \in \mathbb{N}$ s.t. for $n, m \geq M$ we have:

$$\|g_n - g_m\|_{L^{1,\infty}} \leq \varepsilon. \quad (*)$$

In particular: $\sup_{\lambda>0} \lambda \mu(|g_n-g_m|>\lambda) \leq \varepsilon$, for such n,m.

So, for all $\lambda>0$:

$$\mu(|g_n-g_m|>\lambda) \leq \frac{\varepsilon}{\lambda}.$$

From the above line, it follows that (g_n) is Cauchy in measure. In particular, up to a subsequence, which we relabel to be (g_n) this sequence converges to some function g in measure. We want to show that $\|g_n-g\|_{L^{1,\infty}} \to 0$ as $n\to\infty$. Let $\varepsilon>0$ be fixed. We find $M_0 \in \mathbb{N}$ such that, for $n,m \geq M_0$ we have: $\|g_n-g_m\|_{L^{1,\infty}} < \frac{\varepsilon}{4}$.

For fixed $\lambda>0$ and $n,m\geq 0$ we estimate:

$$\lambda\mu(|g-g_n|>\lambda) \leq \lambda\mu(|g-g_m|>\tfrac{\lambda}{2}) + \lambda\mu(|g_n-g_m|>\tfrac{\lambda}{2})$$
$$< \lambda\mu(|g-g_m|>\tfrac{\lambda}{2}) + \frac{\varepsilon}{2}$$

Choosing m even larger (depending on λ), we can arrange so that $\lambda\mu(|g-g_m|>\frac{\lambda}{2}) < \frac{\varepsilon}{2}$ (here, we used that $g_m \to g$ in measure).

Therefore $\lambda\mu(|g-g_n|>\lambda) < \varepsilon$ for all $n \geq M_0$ and $\lambda>0$.

$\Rightarrow \|g_n - g\|_{L^{1,\infty}} < \varepsilon$ for all $n \geq M_0$. So $\|g_n-g\|_{L^{1,\infty}} \to 0$ as $n\to\infty$.

• We now note that $\|g\|_{L^{1,\infty}} \leq C\|f\|_{L^1}$, where f is the limit in L^1 of the sequence (f_n). $C =$ the constant from (Δ).

- Let $\varepsilon, \delta > 0$ be given.

For fixed $\lambda > 0$, we estimate:

$$\lambda \mu(|g| > \lambda) \leq \lambda \mu(|g_m| > \lambda(1-\varepsilon)) + \lambda \mu(|g - g_m| > \lambda \varepsilon)$$

$$\leq \frac{1}{1-\varepsilon} \|g_m\|_{L^{1,\infty}} + \frac{1}{\varepsilon} \|g - g_m\|_{L^{1,\infty}}$$

$$\leq \frac{C}{1-\varepsilon} \|f_m\|_{L^1} + \frac{1}{\varepsilon} \|g - g_m\|_{L^{1,\infty}}.$$

In the last inequality, we used (Δ) in the first term.

For m sufficiently large (independent of λ), this quantity is:

$$\leq \frac{C}{(1-\varepsilon)} \|f\|_{L^1} + \delta + \varepsilon.$$

Since $\varepsilon, \delta > 0$ were arbitrary, we deduce that, for all $\lambda > 0$:

$$\lambda \mu(|g| > \lambda) \leq C \|f\|_{L^1}$$

Taking suprema over $\lambda > 0$, it follows that:

$\|g\|_{L^{1,\infty}} \leq C \|f\|_{L^1}$.

In particular, given (f_n) in $\mathcal{S}(\mathbb{R})$ s.t. $f_n \to f$ in L^1, we have found $g \in L^{1,\infty}(\mathbb{R})$ s.t. $\|H f_n - g\|_{L^{1,\infty}} \to 0$ as $n \to \infty$ and $\|g\|_{L^{1,\infty}} \leq C \|f\|_{L^1}$, for the constant C from (Δ).

We define $Hf := g$.

This is well-defined. Namely if $(\tilde{f}_n)$ is a sequence in $\mathcal{S}(\mathbb{R})$ s.t. $\tilde{f}_n \to f$ in L^1, then by (Δ) $\|H\tilde{f}_n - Hf_n\|_{L^{1,\infty}} \to 0$, and so $H\tilde{f}_n - Hf_n \to 0$ in measure as $n \to \infty$. $\Rightarrow \tilde{g} = g$, where $\tilde{g} := \lim H\tilde{f}_n$ in measure. This defines H on L^1 satisfying $(\Delta\Delta)$.

Furthermore, H is linear on L^1, by construction.

• Let us now prove part 2).

We know that H is strong $(2,2)$.

By part 1) and the Marcinkiewicz interpolation theorem, it follows that H is strong (p,p) for all $1<p<2$.

• Suppose now that $p \in (2,\infty)$ is given. We will deduce the claim by using duality.

Let $f \in \mathcal{S}(\mathbb{R})$ be given.

Then:

$$\|Hf\|_{L^p} = \sup_{\substack{g\in\mathcal{S}(\mathbb{R})\\ \|g\|_{L^{p'}}\leq 1}} \left\{ \left|\int Hf\cdot g\right| \right\} = \sup_{\substack{g\in\mathcal{S}(\mathbb{R})\\ \|g\|_{L^{p'}}\leq 1}} \left\{ \left|-\int f\cdot Hg\right| \right\} =$$

$$= \sup_{\substack{g\in\mathcal{S}(\mathbb{R})\\ \|g\|_{L^{p'}}\leq 1}} \left\{ \left|\int f\cdot Hg\right| \right\}$$

We note that for all $g\in\mathcal{S}(\mathbb{R})$ with $\|g\|_{L^{p'}}\leq 1$, it is the case that

$$\left|\int Hf\cdot g\right| \leq \|f\|_{L^p}\cdot\|Hg\|_{L^{p'}}$$
$$\leq C_{p'}\cdot\|f\|_{L^p}\cdot\|g\|_{L^{p'}} = C_{p'}\cdot\|f\|_{L^p}$$

by using the strong boundedness of H on $L^{p'}$, $p'\in(1,2)$.

$\Rightarrow \|Hf\|_{L^p} \leq C\|f\|_{L^p}$ for all $f\in\mathcal{S}(\mathbb{R})$.

By density, we obtain the bound for all $f\in L^p(\mathbb{R})$.

The claim now follows. $\square$

<u>Remark:</u> Using (Δ), we extended H to L^1 by using density.
Using unitarity, we extended H to L^2 using density. These extensions coincide on $L^1\cap L^2$. This follows from the fact that, given $f\in L^1\cap L^2$, there exists (φ_n) in $\mathcal{S}(\mathbb{R})$ s.t. $\varphi_n\to f$ in L^1 and $\varphi_n\to f$ in L^2. cf. Page 19.

<u>Remarks:</u> ① H is <u>not</u> strong $(1,1)$ nor strong (∞,∞).

It can be shown that, for $f = \chi_{[0,1]}$, it is the case that $Hf(x) = \frac{1}{\pi} \log \left| \frac{x}{x-1} \right|$.

We note that $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, but $Hf \notin L^1(\mathbb{R})$, $Hf \notin L^\infty(\mathbb{R})$

② Given $\varphi \in \mathcal{S}(\mathbb{R})$, the following holds:

$$H\varphi \in L^1 \text{ if and only if } \int \varphi = 0.$$

4.2: Singular integral operators in n dimensions

Definition: A Singular Integral Operator is an operator T given by:

$$Tf(x) := \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} \cdot f(x-y)\,dy \quad \text{for some}$$

function

$\Omega \in L^1(S^{n-1})$ with $\int_{S^{n-1}} \Omega = 0$.

Here, $y' := \frac{y}{|y|}$ for $y \in \mathbb{R}^n \setminus \{0\}$. Note that $y' \in S^{n-1}$.

- Let us observe that Tf is well-defined if $f \in \mathcal{S}(\mathbb{R}^n)$:

$$\int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} \cdot f(x-y)\,dy = \left\{ \begin{array}{l} \text{we use polar coordinates} \\ \text{and the fact that } \int_{S^{n-1}} \Omega = 0 \end{array} \right\}$$

$$= \int_{\varepsilon<|y|<1} \frac{\Omega(y')}{|y|^n} \cdot (f(x-y)-f(x))\,dy + \int_{|y|\geq 1} \frac{\Omega(y')}{|y|^n} \cdot f(x-y)\,dy$$

This quantity has a limit as $\varepsilon \to 0$.

- We note that, when $n=1$ and $\Omega(y') = \frac{1}{\pi} \cdot \operatorname{sign}(y)$, then T is the Hilbert transform.
- Let us note that the condition that $\int_{S^{n-1}} \Omega = 0$ is necessary.

Proposition 4.2: A necessary condition for Tf to be defined when $f \in \mathcal{S}(\mathbb{R}^n)$ is that $\int_{S^{n-1}} \Omega = 0$.

Proof: Let us choose $f \in \mathcal{S}(\mathbb{R}^n)$ such that $f(x) = 1$ for all $|x| \leq 2$.

Then: for $|x|<1$:

$$Tf(x) = \int_{|y|\geq 1} \frac{\Omega(y')}{|y|^n} f(y)\,dy + \lim_{\varepsilon \to 0} \int_{\varepsilon<|y|<1} \frac{\Omega(y')}{|y|^n}\,dy.$$

By polar coordinates, the second summand is:

$\lim_{\varepsilon\to 0}\left[\left(\int_{S^{n-1}} \Omega(y')\,d\sigma(y')\right) \cdot \log(1/\varepsilon)\right]$; $d\sigma$ = surface measure on S^{n-1}

· The first summand is finite since $f \in \mathcal{S}(\mathbb{R}^n)$.

The limit as $\varepsilon \to 0$ exists if and only if $\int_{S^{n-1}} \Omega = 0$. $\square$

Remark: Here, $\int_{S^{n-1}} (\dots)$ denotes $\int_{S^{n-1}} (\dots)\, d\sigma$.

· In particular, in one dimension, any singular integral operator is a multiple of the Hilbert transform.

An example in higher dimensions: (due to A. P. Calderón and A. Zygmund)

Let $f = f(x_1, x_2)$ denote the density of a mass distribution in the plane.

The Newtonian potential in half-space is given by:

$$u(x_1, x_2, x_3) = \int_{\mathbb{R}^2} \frac{f(y_1, y_2)}{\left((x_1-y_1)^2+(x_2-y_2)^2+x_3^2\right)^{\frac{1}{2}}}\, dy_1\, dy_2 \quad (\text{for } x_3 > 0)$$

The gravitational field is given by the gradient of u.

$\frac{\partial u}{\partial x_3} \rightsquigarrow$ an approximation of the identity which can be analyzed as $x_3 \to 0$.

$$\frac{\partial u}{\partial x_1} = - \int_{\mathbb{R}^2} \frac{f(y_1, y_2) \cdot (x_1 - y_1)}{\left((x_1-y_1)^2+(x_2-y_2)^2+x_3^2\right)^{\frac{3}{2}}}\, dy_1\, dy_2$$

As $x_3 \to 0$, we would expect this to converge to:

$$- \int_{\mathbb{R}^2} \frac{f(y_1, y_2) \cdot (x_1 - y_1)}{\left((x_1-y_1)^2+(x_2-y_2)^2\right)^{\frac{3}{2}}}\, dy_1\, dy_2$$

$\rightsquigarrow$ This is a singular integral operator with $\Omega(y') = -\frac{y_1}{|y|} = -y_1'$.

· Let us now observe several facts about the Fourier transform and homogeneity:

A function f is homogeneous of degree a if:

$$f(\lambda x) = \lambda^a f(x), \quad \text{for all } x \in \mathbb{R}, \lambda > 0.$$

Given a function φ and $\lambda > 0$, we define:

$$\varphi_\lambda(x) := \frac{1}{\lambda^n} \varphi\left(\frac{x}{\lambda}\right).$$

· Let f be homogeneous of degree a. Then:

$$\int_{\mathbb{R}^n} f(x)\cdot \varphi_\lambda(x) = \int_{\mathbb{R}^n} f(x)\cdot \frac{1}{\lambda^n}\varphi\left(\frac{x}{\lambda}\right) dx = \left\{ \begin{array}{l} y = \frac{x}{\lambda} \\ dx = \lambda^n dy \end{array} \right\} =$$

$$= \int_{\mathbb{R}^n} f(\lambda y)\cdot \varphi(y)\, dy = \lambda^a \cdot \int_{\mathbb{R}^n} f(y)\cdot \varphi(y)\, dy$$

for all $\lambda > 0$.

Consequently, we say that a (tempered) distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree a if :

$$u(\varphi_\lambda) = \lambda^a u(\varphi)$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(This notion makes sense for more general objects, but we will work only in $\mathcal{S}'(\mathbb{R}^n)$ here.)

Proposition 4.3: Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be homogeneous of degree a. Then $\widehat{u}$ is homogeneous of degree $-n-a$.

Proof: Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda > 0$ be given. Then:

$$\widehat{u}(\varphi_\lambda) = u(\widehat{\varphi_\lambda}) = u(\widehat{\varphi}(\lambda\cdot)) =$$

$$= \lambda^{-n}\cdot u(\lambda^n\cdot \widehat{\varphi}(\lambda\cdot)) =$$

$$= \lambda^{-n}\cdot u(\widehat{\varphi}_{\frac{1}{\lambda}}) = \{\text{by the homogeneity of } u\}$$

$$= \lambda^{-n-a}\cdot u(\widehat{\varphi}) =$$

$$= \lambda^{-n-a}\cdot \widehat{u}(\varphi)$$

$\Rightarrow \widehat{u}$ is homogeneous of degree $-n-a$. $\square$

· Let us observe that p.v. $\frac{\Omega(x')}{|x|^n}$ is homogeneous of degree $-n$.
We consider $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda > 0$
Then:

$$\text{p.v.} \frac{\Omega(x')}{|x|^n}(\varphi_\lambda) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} \cdot \varphi_\lambda(y)\, dy =$$

$$= \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} \cdot \frac{1}{\lambda^n} \cdot \varphi\left(\frac{y}{\lambda}\right) dy = \left\{ \begin{array}{l} w = \frac{y}{\lambda} \\ dy = \lambda^n dw \end{array} \right\}$$

$$= \lim_{\varepsilon \to 0} \int_{|w|>\frac{\varepsilon}{\lambda}} \frac{\Omega(w')}{\lambda^n |w|^n} \cdot \varphi(w)\, dw =$$

$$= \frac{1}{\lambda^n} \cdot \lim_{\varepsilon \to 0} \int_{|w|>\frac{\varepsilon}{\lambda}} \frac{\Omega(w')}{|w|^n} \cdot \varphi(w)\, dw =$$

$$= \lambda^{-n} \cdot \text{p.v.} \frac{\Omega(x')}{|x|^n}(\varphi).$$

<u>Theorem 4.4:</u> Let $\Omega \in L^1(S^{n-1})$ be a function of zero average. Suppose furthermore that $\Omega = \Omega^{(1)} + \Omega^{(2)}$, where $\Omega^{(1)}$ is an odd L^1 function and $\Omega^{(2)} \in L^q(S^{n-1})$ for some $q > 1$. (If $n=1$, we assume that $\Omega^{(2)} \equiv 0$.)
Then $m(\xi) := \left(\text{p.v.} \frac{\Omega(x')}{|x|^n}\right)^\wedge(\xi)$ is a homogenous <u>function</u> of degree 0 given by:

$$m(\xi) = \int_{S^{n-1}} \Omega(u) \cdot \left(\log\left(\frac{1}{|u \cdot \xi'|}\right) - i \cdot \frac{\pi}{2} \cdot \text{sign}(u \cdot \xi') \right) d\sigma(u)$$

<u>Remarks:</u>

1. Since $\Omega^{(1)}$ is odd, it follows that $\int_{S^{n-1}} \Omega^{(1)}(u) \cdot \frac{1}{\log(|u \cdot \xi'|)}\, d\sigma(u) = 0$
(this integral should be interpreted as a principal value).
2. It can be shown, when $n \geq 2$, $\log(|u \cdot \xi'|) \in L^p(S^{n-1})$ for all $1 \leq p < \infty$. Hence $\Omega^{(2)}(u) \cdot \log\left(\frac{1}{|u \cdot \xi'|}\right) \in L^1(S^{n-1})$. (cf. Page 91)
3. Since $\Omega \in L^1(S^{n-1})$, it follows that $\Omega(u) \cdot \text{sign}(u \cdot \xi') \in L^1(S^{n-1})$

Proof: We note that $\left(\text{p.v. } \frac{\Omega(x')}{|x|^n}\right)^{\wedge} \in \mathcal{S}'(\mathbb{R}^n)$ is a homogeneous distribution of order 0.

If we can show that it is given as a function which satisfies the stated formula when $|\xi| = 1$, we will be done.

When $|\xi|=1$, we note that $\xi' = \xi$.

For fixed $\varepsilon > 0$ and $|\xi| = 1$, we compute:

$$\int_{|x|>\varepsilon} \frac{\Omega(x')}{|x|^n} e^{-2\pi i x\cdot\xi}\, dx \underset{\substack{\uparrow \\ \text{since } \int_{S^{n-1}} \Omega = 0}}{=} \int_{\varepsilon<|x|<1} \frac{\Omega(x')}{|x|^n}\cdot\left(e^{-2\pi i x\cdot\xi} - 1\right) dx$$

$$+ \int_{|x|\geq 1} \frac{\Omega(x')}{|x|^n}\cdot e^{-2\pi i x\cdot \xi}\, dx$$

$$= \int_{S^{n-1}} \int_{\varepsilon}^{1} \frac{\Omega(u)}{r^n}\cdot\left(e^{-2\pi i r u\cdot\xi} - 1\right) r^{n-1}\, dr\, d\sigma(u)$$

$$+ \int_{S^{n-1}} \int_{1}^{+\infty} \frac{\Omega(u)}{r^n}\cdot e^{-2\pi i r u\cdot\xi}\cdot r^{n-1}\, dr\, d\sigma(u)$$

$$= \int_{S^{n-1}} \Omega(u) \left(\int_{\varepsilon}^{1} \frac{\cos(2\pi r u\cdot\xi) - 1}{r}\, dr + \int_{1}^{+\infty} \frac{\cos(2\pi r u\cdot\xi)}{r}\, dr \right) d\sigma(u)$$

$$- i\cdot \int_{S^{n-1}} \Omega(u)\cdot \left(\int_{\varepsilon}^{+\infty} \frac{\sin(2\pi r u\cdot\xi)}{r}\, dr\right) d\sigma(u)$$

$$= \int_{S^{n-1}} \Omega^{(2)}(u)\cdot \left(\int_{\varepsilon}^{1} \frac{\cos(2\pi r u\cdot\xi) - 1}{r}\, dr + \int_{1}^{+\infty} \frac{\cos(2\pi r u\cdot\xi)}{r}\, dr\right) d\sigma(u)$$

$$- i\cdot \int_{S^{n-1}} \Omega(u)\cdot \left(\int_{\varepsilon}^{+\infty} \frac{\sin(2\pi r u\cdot\xi)}{r}\, dr\right) d\sigma(u) \qquad (*)$$

- Let us note that, for fixed $\xi \in S^{n-1}$, it is the case that $|u\cdot\xi| \neq 0$ for a.e. $u \in S^{n-1}$. We henceforth consider the contribution from $|u\cdot\xi| \neq 0$.

* (We note that:
$\cos(2\pi r\, u\cdot\xi) = \cos(2\pi r|u\cdot\xi|)$)

• For fixed $\varepsilon > 0$:

$$\int_{\varepsilon}^{1} \frac{\cos(2\pi r\, u\cdot\xi) - 1}{r}\, dr + \int_{1}^{+\infty} \frac{\cos(2\pi r\, u\cdot\xi)}{r}\, dr$$

$$\overset{*}{=} \left\{ \begin{array}{l} s = 2\pi r|u\cdot\xi| \\ \frac{ds}{s} = \frac{dr}{r} \end{array} \right\} = \int_{2\pi\varepsilon|u\cdot\xi|}^{2\pi|u\cdot\xi|} \frac{\cos(s)-1}{s}\, ds + \int_{2\pi|u\cdot\xi|}^{+\infty} \frac{\cos(s)}{s}\, ds$$

$$= \int_{2\pi\varepsilon|u\cdot\xi|}^{1} \frac{\cos(s)-1}{s}\, ds + \int_{1}^{+\infty} \frac{\cos(s)}{s}\, ds - \log(2\pi) - \log(|u\cdot\xi|)$$

Since $\int_{S^{n-1}} \Omega^{(2)}(u)\, d\sigma(u) = 0$, the limit as $\varepsilon \to 0$ of the first expression in $(*)$ equals

$$-\int_{S^{n-1}} \Omega^{(2)}(u)\cdot \log(|u\cdot\xi|)\, d\sigma(u) = \int_{S^{n-1}} \Omega^{(2)}(u)\cdot \log\left(\frac{1}{|u\cdot\xi|}\right) d\sigma(u)$$

Here, we also used the Dominated Convergence Theorem and the fact that $\Omega^{(2)} \in L^q(S^{n-1})$ for some $q > 1$.

This equals:

$$\int_{S^{n-1}} \Omega(u)\cdot \log\left(\frac{1}{|u\cdot\xi|}\right) d\sigma(u)$$

(with the suitable interpretation of the integral.)

• Furthermore, for fixed $\varepsilon > 0$:

$$\int_{\varepsilon}^{+\infty} \frac{\sin(2\pi r\, u\cdot\xi)}{r}\, dr = \left\{ \begin{array}{l} s = 2\pi r|u\cdot\xi| \\ \frac{ds}{s} = \frac{dr}{r} \end{array} \right\} =$$

$$= \int_{2\pi\varepsilon|u\cdot\xi|}^{+\infty} \frac{\operatorname{sign}(u\cdot\xi)\cdot \sin(s)}{s}\, ds \to \frac{\pi}{2}\operatorname{sign}(u\cdot\xi) \text{ as } \varepsilon \to 0.$$

So, the limit as $\varepsilon \to 0$ of the second term in $(*)$ equals:

$$\int_{S^{n-1}} \Omega(u) \cdot \left(-i \cdot \frac{\pi}{2} \operatorname{sign}(u \cdot \xi) \right) d\sigma(u)$$

$$= -\frac{i\pi}{2} \int_{S^{n-1}} \Omega(u) \cdot \operatorname{sign}(u \cdot \xi') \, d\sigma(u)$$

Here, we again used the Dominated Convergence Theorem and the fact that $\Omega \in L^1(S^{n-1})$, $\operatorname{sign}(u \cdot \xi') \in L^\infty(S^{n-1})$.
Hence $m(\xi)$ is indeed a function which is given by the stated formula. $\square$

Notes:

① If $n=1$, we note that for the Hilbert transform, we consider Ω defined on $S^{n-1} = \{-1, 1\}$ as

$\Omega(u) = \frac{1}{\pi} u$.

Then, since Ω is odd, it follows that:

$$m(\xi) = \int_{S^{n-1}} \Omega(u) \cdot \left(\frac{-i\pi}{2} \right) \cdot \operatorname{sign}(u \cdot \xi') \, d\sigma(u) =$$

$$= -\frac{i}{2} \cdot \left[1 \cdot \operatorname{sign}(\xi') + (-1) \cdot \operatorname{sign}(-\xi') \right] =$$

$$= -\frac{i}{2} \cdot 2 \operatorname{sign}(\xi') = -i \operatorname{sign}(\xi') = -i \operatorname{sign}(\xi).$$

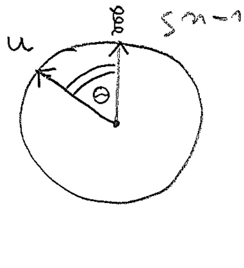
This is the formula that we had before.

② Let us show that, for fixed $\xi \in S^{n-1}$, the function $\varphi(u) := \log(|u \cdot \xi|) \in L^p(S^{n-1})$ for all $p \in [1, \infty)$, when $n \geq 2$

· We will use spherical coordinates.

By symmetry, it suffices to consider $\xi = (0, 0, \dots, 1)$ (north pole)

Then $u \cdot \xi = \cos\theta$



So: $\|\varphi\|_{L^p}^p = \underbrace{C_n}_{>0} \cdot \int_0^{\pi} |\log|\cos\theta||^p \cdot (\sin\theta)^{n-2}\, d\theta$

So, we look at:

$$\int_0^{\pi} |\log|\cos\theta||^p \cdot (\sin\theta)^{n-2}\, d\theta =$$

$$= 2 \cdot \int_0^{\frac{\pi}{2}} |\log|\cos\theta||^p \cdot (\sin\theta)^{n-2}\, d\theta =$$

$$= \left\{ \begin{array}{l} t = \cos\theta \\ dt = -\sin\theta\, d\theta \end{array} \right\} = 2 \int_0^1 |\log|t||^p \cdot (1-t^2)^{\frac{n-3}{2}} dt$$

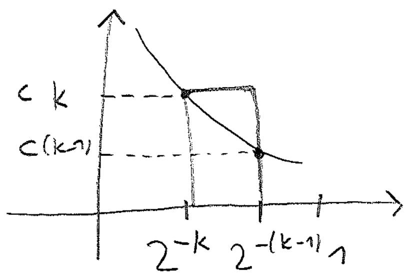
$$= 2 \int_0^1 |\log t|^p \cdot (1-t^2)^{\frac{n-3}{2}}\, dt .$$

Let us note that, for $n \geq 2$ and $0 \leq t \leq 1$, it is the case that:

$$(1-t^2)^{\frac{n-3}{2}} = \left[(1-t)(1+t)\right]^{\frac{n-3}{2}} \leq C(1-t)^{-\frac{1}{2}},$$

which is integrable near 1.

Hence, we just need to estimate $\int_0^1 |\log t|^p dt$. By looking at Riemann sums:



$$\int_0^1 |\log t|^p dt \leq \sum_{k=1}^{+\infty} (Ck)^p \cdot \left(2^{-k} - 2^{-(k-1)}\right)$$

$$= \sum_{k=1}^{+\infty} \frac{(Ck)^p}{2^k} < +\infty .$$

· The following result can be deduced from Theorem 4.4:

<u>Corollary 4.5:</u> Let $n \geq 2$ be given. Given Ω, a function on S^{n-1} with zero average, suppose that $\Omega_o(u) := \frac{1}{2} \cdot (\Omega(u) - \Omega(-u)) \in L^1(S^{n-1})$ and $\Omega_e(u) := \frac{1}{2}(\Omega(u) + \Omega(-u)) \in L^q(S^{n-1})$ for some $q > 1$. Then, the Fourier transform of p.v. $\frac{\Omega(x')}{|x|^n}$ is bounded.

In particular, the associated singular integral operator

$$Tf := \text{p.v.} \frac{\Omega(x')}{|x|^n} * f \text{ is bounded on } L^2.$$

4.3: Calderón - Zygmund Operators

. A way to generalize the Hilbert transform to higher dimensions taking into account the L^2 boundedness, without explicitly using the symmetry properties of the kernel is given by:

Theorem 4.6 (Calderón - Zygmund)

Let $K \in \mathcal{S}'(\mathbb{R}^n)$ be such that K coincides with a locally integrable function on $\mathbb{R}^n \setminus \{0\}$.
Suppose that:

i) $\exists A > 0$ s.t. $|\widehat{K}(\xi)| \leq A$, for all $\xi \in \mathbb{R}^n$.

ii) $\exists M > 0$ s.t. $\displaystyle\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \leq M$, for all $y \in \mathbb{R}^n$.

Define for $f \in \mathcal{S}(\mathbb{R}^n)$ $Tf := K * f$.
(for instance, using i) and the Fourier transform.)
Then, T extends to a linear operator which is weak $(1,1)$ and strong (p,p) for all $1 < p < \infty$.

Remark: Suppose that $|\nabla K(x)| \lesssim \dfrac{1}{|x|^{n+1}}$ for $x \neq 0$.
Then condition ii) holds. In particular, ii) is satisfied for the Hilbert transform.
Condition ii) is called the "Hörmander condition."

Proof of Remark:
Suppose $|\nabla K(x)| \lesssim \dfrac{1}{|x|^{n+1}}$. Let $|x| > 2|y|$ be given.
Then, by the Mean Value Theorem:

$$|K(x-y) - K(x)| \lesssim \frac{|y|}{|x|^{n+1}}$$

$$\int_{|x|>2|y|} \frac{|y|}{|x|^{n+1}}\, dx \lesssim \frac{|y|}{|y|} \lesssim 1$$ (note that, if $y=0$, the condition ii) automatically holds)

So, ii) holds. $\square$

Sketch of Proof of Theorem 4.6:

The proof is very similar to that of Theorem 4.1.
We will just outline the main differences.

It suffices to show the weak (1,1) claim.
Namely T^t is given by convolution with $\widetilde{K}(x) := K(-x)$ which satisfies the same properties as K. Hence, it is possible to deduce the general claim from the Marcinkiewicz Interpolation Theorem and duality.
As in Theorem 4.1, we consider $f \in \mathcal{S}(\mathbb{R}^n)$, $f \geq 0$, and we form the Calderón-Zygmund decomposition of f at height λ.
In other words, we obtain a collection of disjoint dyadic cubes $(Q_j)_j$ in $\mathbb{R}^n$ s.t. $\mu(\bigcup_j Q_j) \leq \frac{1}{\lambda} \cdot \|f\|_{L^1}$; $\lambda < \fint_{Q_j} f \leq 2^n \lambda$ $\forall j$.

$$g(x) := \begin{cases} f(x) & \text{if } x \notin \bigcup_j Q_j \\ \fint_{Q_j} f & \text{if } x \in Q_j \end{cases}$$

$b := f - g$, $b_j := b \cdot \chi_{Q_j}$

Given Q_j, let $Q_j^* :=$ cube with same center as Q_j of $2\sqrt{n}$ times the sidelength
$c_j :=$ center of Q_j.

As in the proof of Theorem 4.1, we can reduce to showing that:

$$\int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)|\, dx \leq C \cdot \int_{Q_j} |b_j(x)|\, dx.$$

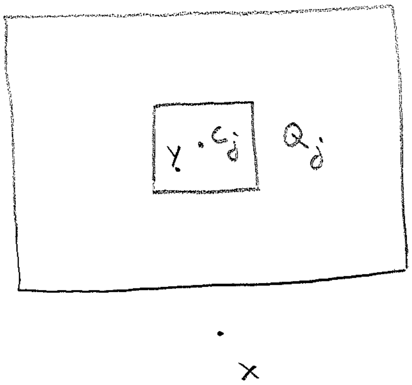
- Let us fix j and $x \in \mathbb{R}^n \setminus Q_j^*$. Then:

$$Tb_j(x) = \int_{Q_j} K(x-y) b_j(y)\, dy = \int_{Q_j} [K(x-y) - K(x-c_j)] \cdot b_j(y)\, dy,$$

since $\int_{Q_j} b_j(y)\, dy = 0$.

Let us note that: for $y \in Q_j$:

$$\mathbb{R}^n \setminus Q_j^* \subseteq \{x \in \mathbb{R}^n, |x-c_j| > 2|y-c_j|\}$$



$$|y-c_j| \leq \frac{\ell(Q_j)\sqrt{n}}{2}, \quad |x-c_j| > \ell(Q_j)\cdot\sqrt{n} \geq 2|y-c_j|$$

In particular:

$$\int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)|\,dx \leq \int_{Q_j} |b_j(y)| \cdot \left(\int_{\mathbb{R}^n\setminus Q_j^*} |K(x-y) - K(x-c_j)|\,dx \right) dy$$

$$\leq \int_{Q_j} |b_j(y)| \cdot \left(\underbrace{\int_{|x-c_j|>2|y-c_j|} |K(x-y)-K(x-c_j)|\,dx}_{\leq\ M, \text{ if we take } \tilde{x} := x-c_j,\ \tilde{y} := y-c_j} \right) dy$$

$$\leq M \cdot \int_{Q_j} |b_j(y)|\,dy.$$

The claim now follows as in Theorem 4.1. $\square$

<u>Remark:</u> We note that for $f \in L^2(\mathbb{R}^n)$, it is the case that $Tf \in L^2(\mathbb{R}^n)$.

Moreover, if f is compactly supported and if $x \notin \operatorname{supp} f$, then

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)\cdot f(y)\,dy.$$ (Here, we interpret the support in the measure theoretic sense.)

We were implicitly using this formula in the above proof when we wrote:

$$T b_j(x) = \int_{Q_j} K(x-y) b_j(y)\, dy.$$

The above expressions make sense since we are taking the convolution of two L^1 functions.

<u>Remark:</u> It is a natural question to ask when a singular integral operator T as defined in the previous section satisfies the assumptions of Theorem 4.5, i.e. when is T a Calderón–Zygmund operator.

- From Corollary 4.4, we know that condition i) is satisfied whenever $\Omega_o(u) = \frac{1}{2}(\Omega(u) - \Omega(-u)) \in L^1(S^{n-1})$ and $\Omega_e(u) = \frac{1}{2}(\Omega(u) + \Omega(-u)) \in L^q(S^{n-1})$ for some $q > 1$.
- Given $t \in [0, +\infty)$, we define $w_\infty(t) := \sup_{\substack{x,y \in S^{n-1} \\ |x-y| \le t}} |\Omega(x) - \Omega(y)|$.

It can be shown that $K(x) = \frac{\Omega(x')}{|x|^n}$ satisfies the Hörmander condition ii) provided that:

$$\int_0^1 \frac{w_\infty(t)}{t}\, dt < \infty.$$

This is called a <u>Dini-type condition</u>.

4.4: Fourier multiplier operators

- Given a function $m: \mathbb{R}^n \to \mathbb{C}$, we can define an operator T by using the Fourier transform:

$$(Tf)^\wedge(\xi) := m(\xi)\cdot \widehat{f}(\xi). \quad \left(\begin{smallmatrix}\text{FREQUENCY}\\ \text{SPACE}\end{smallmatrix}\right)$$

This is well-defined for $f \in \mathcal{S}(\mathbb{R})$ provided that $m \in \mathcal{S}(\mathbb{R})$.

By using the Fourier inversion formula, one can deduce:

$$Tf = \check{m} * f \quad \left(\begin{smallmatrix}\text{PHYSICAL}\\ \text{SPACE}\end{smallmatrix}\right)$$

The first form will sometimes be more useful.

<u>Notation</u>: For T defined as above, we will usually write $m(D)f$,

i.e. $(m(D)f)^\wedge(\xi) = m(\xi)\cdot\widehat{f}(\xi)$.

The motivation is that "$D \sim$ derivative":

$$\left(\frac{\partial}{\partial x} f\right)^\wedge(\xi) \sim \xi \cdot \widehat{f}(\xi).$$

$m(D)$ is called a <u>Fourier multiplier operator</u>.

An important result concerning Fourier multiplier operators is:

<u>Theorem 4.7 (Hörmander – Mikhlin)</u>

Let $m: \mathbb{R}^n \to \mathbb{C}$ be a function such that:

$|\nabla^k m(\xi)| \lesssim |\xi|^{-k}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, whenever $0 \leq k \leq n+2$.

Then, $m(D)$ is weak $(1,1)$ and strong (p,p) for all $1 < p < \infty$.

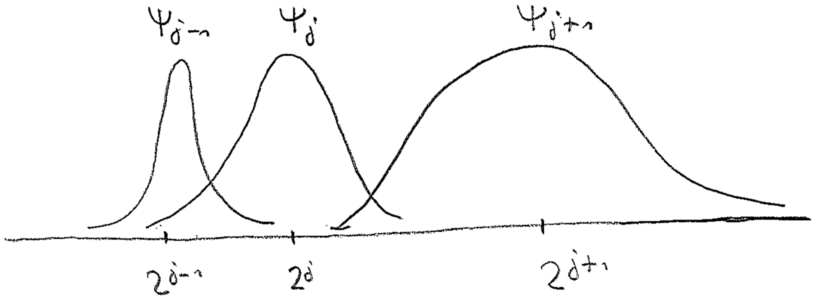
<u>Proof:</u> We choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \equiv 1$ on $B(0,1)$ and $\varphi \equiv 0$ outside $B(0,2)$.

For $j \in \mathbb{Z}$, we define $\psi_j(x) := \varphi\left(\frac{x}{2^j}\right) - \varphi\left(\frac{x}{2^{j-1}}\right)$.

Then $\psi_j \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \psi_j \subseteq \{x : 2^{j-1} \leq |x| \leq 2^{j+1}\}$.

Moreover $\psi_j(x) = \psi_0\left(\frac{x}{2^j}\right)$, for all j.

Moreover, $\sum_j \psi_j \equiv 1$ on $\mathbb{R}^n \setminus \{0\}$. *



* More precisely, given $x \neq 0$, we can choose $j_0 < j_1 \in \mathbb{Z}$ such that $\left|\frac{x}{2^{j_1}}\right| < 1$ and $\left|\frac{x}{2^{j_0-1}}\right| > 2$. For such j_0, j_1, we have $\sum_{j=j_0}^{j_1} \psi_j(x) = 1$ (by telescoping) and $\sum_{j<j_0} \psi_j(x) = \sum_{j>j_1} \psi_j(x) = 0$ (since all the summands are 0)

If $n=1$ and $\varphi \geq 0$, the picture looks as above.

- We note that <u>each ψ_j is supported on a dyadic annulus of size 2^j</u>.

Given $j \in \mathbb{Z}$, we let $m_j(\xi) := \psi_j(\xi) \cdot m(\xi)$.
The corresponding Fourier multiplier operator is $m_j(D)$.
By using the Plancherel Theorem and the Dominated Convergence Theorem, it follows that, for all $f \in L^2(\mathbb{R}^n)$:

$$m(D)f = \sum_j m_j(D) f \quad (\text{in } L^2).$$

- We can write $m_j(D)f$ as $K_j * f$ where:

$$K_j(x) = \check{m}_j(x) = \int_{\mathbb{R}^n} m_j(\xi) \cdot e^{2\pi i x \cdot \xi}\, d\xi. \quad (\text{this makes sense since } m_j \in L^1)$$

We will now derive bounds on $|K_j|$ and $|\nabla K_j|$.

<u>Claim:</u> For all $j \in \mathbb{Z}$:

$$\begin{cases} |K_j(x)| \lesssim_n |x|^{-n} \cdot \min\left\{(2^j|x|)^n, (2^j|x|)^{-2}\right\} \\ |\nabla K_j(x)| \lesssim_n |x|^{-n-1} \cdot \min\left\{(2^j|x|)^{n+1}, (2^j|x|)^{-1}\right\} \end{cases}$$

<u>Proof of Claim:</u>

- Let us first prove the claim for K_j.

We know that $\|m_j\|_{L^\infty} \lesssim 1$ and the support of m_j has measure $\lesssim_n 2^{jn}$. So: $|K_j(x)| \lesssim_n 2^{jn} = |x|^{-n} \cdot (2^j|x|)^n$.

- In order to obtain the second bound, we note that: for $x \neq 0$,

$$\left(\frac{-ix}{2\pi |x|^2} \cdot \nabla_\xi \right) e^{2\pi i x \cdot \xi} = e^{2\pi i x \cdot \xi}$$

- In particular, we can apply this formula iteratively to deduce that, for all $0 \leq k \leq n+2$:

$$K_j(x) = \int m_j(\xi) e^{2\pi i x \cdot \xi} d\xi = \int m_j(\xi) \left(\frac{-ix}{2\pi |x|^2} \cdot \nabla_\xi \right)^k e^{2\pi i x \cdot \xi} d\xi$$

$$= \int \left(\frac{ix}{2\pi |x|^2} \cdot \nabla_\xi \right)^k m_j(\xi) \cdot e^{2\pi i x \cdot \xi} d\xi$$

$$= O_n \Big(\underbrace{\frac{1}{|x|^k}}_{\text{decay factor from } x} \cdot \underbrace{2^{-jk}}_{\text{bound for } |\nabla_\xi^k m_j| \text{ (see Remark)}} \cdot \underbrace{2^{jn}}_{\text{bound for the measure of supp } m_j} \Big)$$

["$O_n(\dots)$" means that the bound depends on n.]

Remark: In the bound for $|\nabla_\xi^k m_j|$, we use the assumptions on m, as well as the Leibniz rule and the bounds on $|\nabla_\xi^\ell \psi_j| \lesssim 2^{-j\ell}$

In particular, for $k = n+2$:

$$|K_j(x)| \lesssim \frac{1}{|x|^{n+2}} \cdot 2^{-j(n+2)} \cdot 2^{jn} = \frac{1}{|x|^{n+2}} \cdot 2^{-2j} =$$

$$= \frac{1}{|x|^n} \cdot \left(2^j |x| \right)^{-2}$$

$$\Rightarrow |K_j(x)| \lesssim_n |x|^{-n} \cdot \min \left\{ (2^j|x|)^n, (2^j|x|)^{-2} \right\}.$$

- Similarly, we get an estimate for $|\nabla K_j(x)|$

We know: $\nabla K_j(x) = \int m_j(\xi) \cdot 2\pi i \xi \, e^{2\pi i x \cdot \xi} d\xi$.

The "ξ" gives us an extra factor of 2^j in the estimates we had before. The claim now follows.

- From the claim, we deduce that for all $x \neq 0$:

$$\sum_j |K_j(x)| \lesssim_n |x|^{-n} \cdot \sum_j \min \left\{ (2^j|x|)^n, (2^j|x|)^{-2} \right\}$$

$$= |x|^{-n} \cdot \left(\sum_{j,\, 2^j|x| < 1} (2^j|x|)^n + \sum_{j,\, 2^j|x| \geq 1} (2^j|x|)^{-2} \right)$$

$\lesssim_n |x|^{-n}$.

Here, we use the fact that the sum of a geometric series is comparable to its largest term.

Similarly,

$$\sum_j |\nabla K_j(x)| \lesssim_n |x|^{-n-1}.$$

- It follows that that $\sum_j K_j$ converges in $C^1_{loc}(\mathbb{R}^n \setminus \{0\})$ (by using the Weierstrass M-test locally).

We call the limit K. By construction $K \in C^1_{loc}(\mathbb{R}^n \setminus \{0\})$ and $|K(x)| \lesssim \frac{1}{|x|^n}$, $|\nabla K(x)| \lesssim \frac{1}{|x|^{n+1}}$ for $x \neq 0$. $(*)$

<u>Claim:</u> For all $f \in L^2(\mathbb{R}^n)$ with compact support, it is the case that:

$$(m(D)f)(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\, dy, \text{ whenever } x \notin \operatorname{supp}(f).$$

- Let us note that the claim above then implies the Theorem.

Namely, since $m \in L^\infty(\mathbb{R}^n)$, it follows that $m(D)$ is strong $(2,2)$. Combining the claim and the gradient bound in $(*)$, the proof of Theorem 4.6 implies that $m(D)$ is weak $(1,1)$ and strong (p,p) for all $1 < p < \infty$.
Duality then shows that $m(D)$ is strong (p,p) for $2 < p < \infty$.

<u>Proof of the claim:</u>

- Let $f \in L^2(\mathbb{R}^n)$ have compact support.

Suppose that $g \in L^2(\mathbb{R}^n)$ has compact support disjoint from the support of f. Hence $f, g \in L^1(\mathbb{R}^n)$.

- Let us first fix $j \in \mathbb{Z}$. Then:

$K_j = \check{m}_j \in L^\infty$ and we compute:

$$\int \Big(\underbrace{\int K_j(x-y) f(y)\, dy}_{\in L^\infty \text{ by Young's inequality}} \Big) \cdot \underbrace{\overline{g(x)}}_{\in L^1}\, dx \quad \text{(Everything is well-defined)}$$

$$= \iiint m_j(\xi) e^{2\pi i (x-y)\cdot \xi} f(y) \overline{g(x)}\, d\xi\, dy\, dx = \left\{ \begin{array}{l} \text{by Fubini's} \\ \text{Theorem} \end{array} \right\}$$

$$= \int \Big(\int \Big(\int f(y) e^{-2\pi i y\cdot \xi} dy \Big) \cdot m_j(\xi) \cdot e^{2\pi i x \cdot \xi} d\xi \Big) \overline{g(x)}\, dx =$$

$$= \int \Big(\int \widehat{f}(\xi) m_j(\xi) e^{2\pi i x\cdot \xi}\, d\xi \Big) \cdot \overline{g(x)}\, dx =$$

$$= \int (m_j(D) f)(x) \cdot \overline{g(x)}\, dx =$$

$$= \langle m_j(D) f, g \rangle \qquad (**)$$

- We now sum in j:

Let us first note that there exists $\delta > 0$ such that for all $x \in \text{supp}(f)$, $y \in \text{supp}(g)$, it is the case that $|x-y| \geq \delta$.
Hence, $|K(x-y)| \lesssim \delta^{-n}$ by $(*)$. In other words $\sum_j |K_j(x-y)| \lesssim \delta^{-n}$ by the proof of $(*)$. We observe that $f(y) \cdot \overline{g(x)} \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.
By the Dominated Convergence Theorem, it follows that:

$$\sum_j \iint K_j(x-y) f(y) \overline{g(x)}\, dy\, dx =$$

$$= \iint \sum_j K_j(x-y) f(y) \overline{g(x)}\, dy\, dx =$$

$$= \iint K(x-y) f(y) \overline{g(x)}\, dy\, dx.$$

Moreover,

$$\sum_j \iint K_j(x-y) f(y) \overline{g(x)}\, dy\, dx = \{\text{by } (**)\}$$

$$= \sum_j \langle m_j(D) f, g \rangle = \langle m(D) f, g \rangle.$$

· Here, we used the fact that $\sum_j m_j(D)f \to m(D)f$ in L^2 for all $f \in L^2$ and the Cauchy-Schwarz inequality.

So: $\langle m(D)f, g\rangle = \int \left(\int K(x-y) f(y)\, dy \right) \cdot \overline{g(x)}\, dx$

for all $g \in L^2(\mathbb{R}^d)$ with compact support which is disjoint from that of f.

$\Rightarrow m(D)f(x) = \int K(x-y) f(y)\, dy$, for all $x \notin \operatorname{supp}(f)$.

The claim now follows. $\square$