§3: Maximal functions

3.1: The Hardy-Littlewood maximal function

We denote by B_r the Euclidean ball $B(0,r)$ centered at the origin with radius $r>0$.
Given $f \in L^1_{loc}(\mathbb{R}^n)$, we define its <u>Hardy-Littlewood maximal function</u> Mf by:

$$Mf(x) := \sup_{r>0} \frac{1}{|B_r|} \cdot \int_{B_r} |f(x-y)|\,dy \,.$$ Note that M is sublinear.

We can write this as:

$$Mf(x) = \sup_{r>0} \fint_{B_r(x)} |f(y)|\,dy \,.$$

Here $\fint_Q (\dots)\,dy$ denotes the average over Q.

· Some related notions are given as follows:

$$M'f(x) := \sup_{r>0} \frac{1}{(2r)^n} \int_{Q_r} |f(x-y)|\,dy$$

where $Q_r = [-r,r]^n$ is a cube.
We note that when $n=1$, M and M' coincide.
More generally, one can consider:

$$M''f(x) := \sup_{\substack{Q \ni x \\ Q \text{ is a cube}}} \frac{1}{|Q|} \int_Q |f(y)|\,dy$$

It can be shown that $Mf(x) \sim M'f(x) \sim M''f(x)$.

· A related concept can be associated to a family of linear operators.
Namely, let $\{T_t\}$ be a family of linear operators on $L^p(X,\mu)$ and let $T^*f(x) := \sup_t |T_t f(x)|$.
Then, T^* is called the <u>maximal operator associated with the family $\{T_t\}$</u>. Note that T^* is sublinear.

· These notions can be related as follows:

Let $\varphi := \frac{1}{|B_1|} \cdot \chi_{B_1}$

$$\varphi_r := \frac{1}{r^n}\varphi\left(\frac{\cdot}{r}\right) = \frac{1}{|B_r|}\cdot \chi_{B_r} \Rightarrow \frac{1}{|B_r|}\int_{B_r} |f(x-y)|\,dy = \varphi_r * |f|\,(x)$$

Then $Mf(x) = \sup_{r>0} \varphi_r * |f|$

Hence, for $f \geq 0$, $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator coincides with the maximal operator associated to the approximation of the identity corresponding to φ.

A fundamental result is the following:

<u>Theorem 3.1:</u> The Hardy-Littlewood maximal operator is weak $(1,1)$ and strong (p,p) for $1 < p \leq \infty$.

<u>Proof:</u> By construction, it follows that

$$\|Mf\|_{L^\infty} \leq \|f\|_{L^\infty}.$$

The operator M is sublinear.

Hence, the claim will follow from the Marcinkiewicz Interpolation theorem if we show that M is weak $(1,1)$.

In order to prove this claim, we can use the following result:

<u>Lemma 3.2:</u> (Wiener's Vitali-type Covering Lemma)

Let $\{B_1, \dots, B_N\}$ be a collection of balls in $\mathbb{R}^n$.

Then, there exists a subcollection $\{B_{j_1}, \dots, B_{j_m}\}$ of disjoint balls such that:

$$\mu\left(\bigcup_{k=1}^{m} B_{j_k}\right) \geq 3^{-n} \cdot \mu\left(\bigcup_{k=1}^{N} B_k\right)$$

<u>Proof of Lemma:</u>

We use a greedy algorithm.

· Let B_{j_1} be the ball of largest radius. (If there are more, choose one.)

We then choose B_{j_2} to be the ball disjoint from B_{j_1} which has the largest radius. In general, given $B_{j_1}, \dots, B_{j_k}$, we let $B_{j_{k+1}}$ be the ball in the collection which is disjoint from $B_{j_1} \dots B_{j_k}$ and has maximal radius.

• This procedure terminates after finitely many steps, when we obtain $\{B_{j_1}, \dots, B_{j_m}\}$.
Since the total number of balls is finite, this procedure terminates after a finite number of steps. By construction, $\{B_{j_1}, \dots, B_{j_m}\}$ is a disjoint collection.

• If $l \in \{1, \dots, N\}$, then by construction B_l intersects B_{j_k} for some $k \in \{1, 2, \dots, m\}$. Otherwise, we could continue the procedure.
We choose k such that B_{j_k} has maximal radius, i.e. we choose k to be minimal.

• By construction, we get that $r(B_l) \leq r(B_{j_k})$, for $l > k$.
Namely, if $r(B_l) > r(B_{j_k})$, then we would have had to choose B_l as an element of the subcollection at some previous step of the procedure.

• From the triangle inequality and the fact that $B_l \cap B_{j_k} \neq \emptyset$, it follows that:

$$B_l \subseteq 3 B_{j_k}$$

$$\Rightarrow \bigcup_{l=1}^{N} B_l \subseteq \bigcup_{k=1}^{N} 3 B_{j_k}$$

$$\Rightarrow \mu\Big(\bigcup_{k=1}^{N} B_k\Big) \leq \mu\Big(\bigcup_{k=1}^{m} 3B_{j_k}\Big) \leq \sum_{k=1}^{m} \mu(3B_{j_k})$$

$$= \sum_{k=1}^{m} 3^n \cdot \mu(B_{j_k}) = 3^n \cdot \mu\Big(\bigcup_{k=1}^{m} B_{j_k}\Big)$$

$$\Rightarrow \mu\Big(\bigcup_{k=1}^{m} B_{j_k}\Big) \geq 3^{-n} \cdot \mu\Big(\bigcup_{k=1}^{N} B_k\Big). \quad \square$$

We can now finish the proof of Theorem 3.1.
Namely, let $\lambda > 0$ be given.
Let $\mathcal{A} := \{x \in \mathbb{R}^n, \ Mf(x) > \lambda\}$.
We know that, $\forall x \in \mathcal{A}$, $\exists\, B_x = B_x(r_x)$ s.t. $\displaystyle ⨍_{B_x} |f| > \lambda$.

Let $K \subseteq \mathcal{A}$ be a compact set.
Then, by compactness $\exists\, x_1, \dots, x_N$ s.t. $K \subseteq B_{x_1} \cup \dots \cup B_{x_N}$.

Let $B_i := B_{x_i}$ for $i = 1, 2, \dots, N$.

Hence, $K \subseteq B_1 \cup \dots \cup B_N$ and $\fint_{B_i} |f| \, dy > \lambda$ for all $i = 1, 2, \dots, N$.

By Wiener's Lemma, we can find a disjoint subcollection $\{B_{j_1}, \dots, B_{j_m}\}$ such that:

$$\mu\left(\bigcup_{k=1}^{m} B_{j_k}\right) \geq 3^{-n} \cdot \mu\left(\bigcup_{k=1}^{N} B_k\right) \geq 3^{-n} \cdot \mu(K)$$

$$\Rightarrow \mu(K) \leq 3^n \cdot \mu\left(\bigcup_{k=1}^{m} B_{j_k}\right) = 3^n \cdot \sum_{k=1}^{m} \mu(B_{j_k})$$

$$< 3^n \cdot \sum_{k=1}^{m} \frac{1}{\lambda} \cdot \int_{B_{j_k}} |f| \, dy \leq \frac{3^n}{\lambda} \cdot \int_{\mathbb{R}^n} |f| \, dy = \frac{3^n \|f\|_{L^1}}{\lambda}.$$

We take suprema over $K \subseteq \mathcal{a}$ and we obtain that:

$$\mu(\mathcal{a}) = \mu\left(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}\right) \leq \frac{3^n \|f\|_{L^1}}{\lambda}.$$

$\Rightarrow M$ is weak $(1,1)$.

Theorem 3.1 now follows. $\square$

<u>Proposition 3.2:</u> Suppose that $f \in L^1(\mathbb{R}^n)$, $f \not\equiv 0$. Then $Mf \notin L^1(\mathbb{R}^n)$.

In particular, M is not strong $(1,1)$.

<u>Proof:</u> We let f be as above.

Then, there exists $R > 0$ s.t. $\int_{B_R(0)} |f| \, dy := \varepsilon > 0$

If $|x| > R$, then $B_R(0) \subseteq B_{2|x|}(x)$ by the triangle inequality.

In particular:

$$Mf(x) \geq \frac{1}{c_n |x|^n} \cdot \int_{B_{2|x|}(x)} |f| \, dy \geq \frac{1}{c_n |x|^n} \cdot \int_{B_R(0)} |f| \, dy = \frac{\varepsilon}{c_n |x|^n}$$

So, $Mf \notin L^1(\mathbb{R}^n)$. $\square$

• The following fact concerning the maximal operator associated to a family $\{T_t\}$ is useful:

<u>Proposition 3.3:</u> Let $\{T_t\}$ be a family of linear operators on $L^p(X,\mu)$.
Let $T^* := \sup |T_t|$ (we suppose that T^* is well-defined)
Suppose that T^* is weak (p,q), for some $1 \leq p,q < \infty$.
Then, $\mathcal{A} := \{ f \in L^p(X,\mu),\ \lim_{t\to t_0} T_t f = f \text{ a.e.}\}$ is closed in $L^p(X,\mu)$, for all t_0.

<u>Proof:</u> Suppose that (f_n) is a sequence in $\mathcal{A}$ such that

$$f_n \xrightarrow{L^p} f.$$

Let $\lambda > 0$ be given. Then, since $\lim_{t\to t_0} T_t f_n = f_n$ a.e., it follows that:

$$\mu\left(\{x : \limsup_{t\to t_0} |T_t f - f| > \lambda\}\right)$$

$$= \mu\left(\{x : \limsup_{t\to t_0} |T_t(f-f_n) - (f-f_n)| > \lambda\}\right)$$

We note that we are using the linearity of T_t at this step.
This expression is:

$$\leq \mu\left(\{x : \limsup_{t\to t_0} |T_t(f-f_n)| > \tfrac{\lambda}{2}\}\right) + \mu\left(\{x : |f-f_n| > \tfrac{\lambda}{2}\}\right)$$

$$\leq \mu\left(\{x : |T^*(f-f_n)| > \tfrac{\lambda}{2}\}\right) + \mu\left(\{x : |f-f_n| > \tfrac{\lambda}{2}\}\right)$$

$$\leq \left(\frac{2C\|f-f_n\|_{L^p}}{\lambda}\right)^q + \left(\frac{2\|f-f_n\|_{L^p}}{\lambda}\right)^p \to 0 \text{ as } n\to\infty$$

since $f_n \to f$ in L^p.

It follows that:

$$\mu\left(\{x : \limsup_{t\to t_0} |T_t f - f| > 0\}\right) \leq \sum_{n=1}^{\infty} \mu\left(\{x : \limsup_{t\to t_0} |T_t f - f| > \tfrac{1}{n}\}\right),$$

which equals to zero

Hence $\lim_{t \to t_0} T_t f = f$ a.e.

Thus, $f \in \mathcal{A}$. So, $\mathcal{A}$ is closed. $\square$

3.2: The dyadic maximal function

Given $k \in \mathbb{Z}$, let $\mathcal{Q}_k$ denote the set of all cubes of the form

$$[2^{-k} \cdot m_1, 2^{-k} \cdot (m_1+1)) \times [2^{-k} \cdot m_2, 2^{-k} \cdot (m_2+1)) \times \dots \times [2^{-k} \cdot m_n, 2^{-k} \cdot (m_n+1))$$

for $m_1, m_2, \dots, m_n \in \mathbb{Z}$.

These are cubes of sidelength 2^{-k} in $\mathbb{R}^n$.

Given $f \in L^1_{loc}(\mathbb{R}^n)$, we define:

$$E_k f := \sum_{Q \in \mathcal{Q}_k} \left(\fint_Q f \right) \cdot \chi_Q$$

This can be interpreted as the conditional expectation of f with respect to the σ-algebra generated by $\mathcal{Q}_k$.

The dyadic maximal function M_d is defined by:

$$M_d f := \sup_k |E_k f|.$$

• We will now study the properties of M_d.

Given a cube $Q \subseteq \mathbb{R}^n$, we say that it is a dyadic cube if $Q \in \mathcal{Q}_k$ for some $k \in \mathbb{Z}$.

Theorem 3.4:

1) M_d is weak $(1,1)$ and strong (p,p) for all $1 < p \leq \infty$.
2) $E_k f \xrightarrow[k \to \infty]{} f$ a.e. for all $f \in L^1_{loc}(\mathbb{R}^n)$.

Proof: 1) By construction $\|M_d f\|_{L^\infty} \leq \|f\|_{L^\infty}$.

Hence, by the Marcinkiewicz interpolation theorem, it suffices to show that M_d is weak $(1,1)$.

By decomposing f into its real and imaginary part and then by decomposing each of these into its non-negative and negative part and by using the sublinearity of M_d, it suffices to show the claim when $f \in L^1(\mathbb{R}^n)$ and $f \geq 0$.

Let $\lambda > 0$.

Since $f \in L^1(\mathbb{R}^n)$, we know that $\lim_{k \to -\infty} E_k f = 0$ a.e.

Now, we deduce that:

$$\{x \in \mathbb{R}^n, M_d f(x) > \lambda\} = \bigcup_{k \in \mathbb{Z}} \Omega_k$$

where:

$$\Omega_k := \{x \in \mathbb{R}^n : E_k f(x) > \lambda, E_j f(x) \leq \lambda \text{ for all } j < k\}$$

In other words, we find the largest dyadic cube over which the average is $> \lambda$.

The Ω_k are then disjoint sets which are unions of elements of $\mathcal{Q}_k$.

Hence: (throughout this discussion, μ denotes the Lebesgue measure on $\mathbb{R}^n$)

$$\mu(|M_d f| > \lambda) = \mu(M_d f > \lambda) = \sum_k \mu(\Omega_k)$$

We note that, for each k:

$$\mu(\Omega_k) < \frac{1}{\lambda} \int_{\Omega_k} E_k f = \frac{1}{\lambda} \int_{\Omega_k} f \text{, by construction.}$$

So:

$$\mu(|M_d f| > \lambda) < \sum_k \frac{1}{\lambda} \int_{\Omega_k} f \leq \frac{1}{\lambda} \|f\|_{L^1}.$$

2) By replacing f with $f \cdot \chi_Q$ for Q a cube of sidelength 2^M centered at 0 it suffices to consider the case $f \in L^1(\mathbb{R}^n)$.

We know that $E_k f \underset{k\to\infty}{\longrightarrow} f$ pointwise if $f \in C_0(\mathbb{R}^n)$.

We know from part a) that $M_d = \sup |E_k|$ is weak $(1,1)$.

Hence, by Proposition 3.3, it follows that

$$\mathcal{A} := \{ f \in L^1(\mathbb{R}^n),\ E_k f \underset{k\to\infty}{\to} f \text{ a.e.} \} \text{ is closed in } L^1(\mathbb{R}^n).$$

Since $C_0(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, it follows that $\mathcal{A} = L^1(\mathbb{R}^n)$. The claim now follows. $\square$

The following result can be deduced by using the methods of the above proof:

<u>Theorem 3.5.</u> (Calderón - Zygmund Decomposition)

Let $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$ and let $\lambda > 0$ be given. Then, there exists a countable collection (Q_j) of disjoint dyadic cubes such that:

1) $f \leq \lambda$ a.e. on $\left(\bigcup\limits_j Q_j\right)^c$

2) $\mu\left(\bigcup\limits_j Q_j\right) \leq \frac{1}{\lambda} \|f\|_{L^1}$

3) $\lambda < \fint_{Q_j} f \leq 2^n \lambda \quad \forall j$

→ Such a decomposition of $\mathbb{R}^n$ is called a <u>Calderón - Zygmund Decomposition of f at height λ</u>.

<u>Proof:</u> We use the proof of Theorem 3.4.

The Q_j are obtained from decomposing each Ω_k as a disjoint union of dyadic cubes, (and taking the union over k)

- By construction, we know that $E_k f \leq \lambda$ a.e. on $\left(\bigcup\limits_j Q_j\right)^c$.

Hence, from Part 2) of Theorem 3.4, it follows that

$f \leq \lambda$ a.e. on $(\bigcup_j Q_j)^c$, which proves 1).

- Part 2) follows from the construction of the family (Q_j).
- Let us take an element Q_j of the family.

Then, we know that:

$$\lambda < \fint_{Q_j} f = \frac{1}{\mu(Q_j)} \cdot \int_{Q_j} f$$

- Let $\widetilde{Q}_j$ be the dyadic cube containing Q_j of twice the sidelength.

Then, by construction:

$$\fint_{\widetilde{Q}_j} f = \frac{1}{|\widetilde{Q}_j|} \cdot \int_{\widetilde{Q}_j} f \leq \lambda$$

So: $$\frac{1}{\mu(Q_j)} \cdot \int_{Q_j} f = \underbrace{\frac{\mu(\widetilde{Q}_j)}{\mu(Q_j)}}_{=2^n} \cdot \frac{1}{\mu(\widetilde{Q}_j)} \cdot \int_{Q_j} f \leq 2^n \cdot \frac{1}{\mu(\widetilde{Q}_j)} \cdot \int_{\widetilde{Q}_j} f \leq 2^n \lambda.$$

$\square$

<u>Note:</u> We know by Hölder's inequality that for $1 \leq p < \infty$, for $Q \subseteq \mathbb{R}^n$ a bounded set that:

$$\fint_Q |f| \leq \left(\fint_Q |f|^p \right)^{\frac{1}{p}}$$

So, if $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ and $f \geq 0$, by applying the previous construction, we get a decomposition (Q_j) of $\mathbb{R}^n$ that satisfies 1) and 3). Namely, we know that $f \leq \lambda$ on $(\bigcup Q_j)^c$ because M_d is strong (p,p).

· We can use Theorem 3.4 to give an alternative proof of Theorem 3.1. This is done by relating M' and M_d:

<u>Proposition 3.5:</u> Let $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$ and $\lambda > 0$ be given. Then:

$$\mu\left(\{x \in \mathbb{R}^n : M'f(x) > 4^n \lambda\}\right) \leq 2^n \mu\left(\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}\right)$$

<u>Proof:</u> We use the same notation as in the previous arguments to write:

$$\{x \in \mathbb{R}^n, M_d f(x) > \lambda\} = \bigcup_j Q_j,$$ a disjoint union of dyadic cubes (with Q_j satisfying the assumptions given earlier.)

We will now show that:

$$\{x \in \mathbb{R}^n, M'f(x) > 4^n \lambda\} \subseteq \bigcup_j \widehat{Q}_j \qquad (\Delta)$$

Here, each $\widehat{Q}_j$ is the cube with same center as Q_j of twice the sidelength.

Let us note that (Δ) indeed implies that:

$$\mu\left(\{x \in \mathbb{R}^n, M'f(x) > 4^n \lambda\}\right) \leq \sum_j \mu(\widehat{Q}_j) =$$

$$= \sum_j 2^n \mu(Q_j) = 2^n \cdot \mu\left(\bigcup_j Q_j\right) = 2^n \cdot \mu\left(\{x \in \mathbb{R}^n, M'f(x) > \lambda\}\right)$$

(This step is reminiscent of the Vitali covering method used earlier)

· Let us now show (Δ).

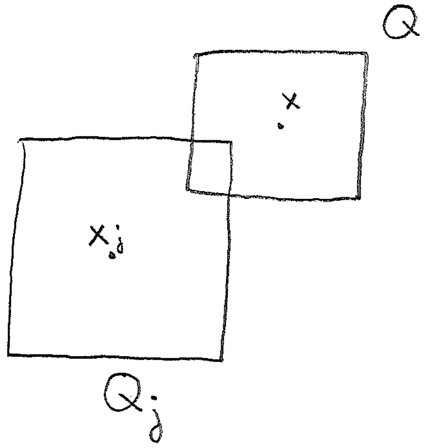
Suppose that $x \notin \underline{\bigcup_j \widehat{Q}_j}$.

Let Q be a cube centered at x. We find $k \in \mathbb{Z}$ such that $2^{-k-1} \leq \ell(Q) < 2^{-k}$, where $\ell(Q) :=$ sidelength of Q.

The cube Q intersects cubes $R_1, \dots, R_m$ that belong to $\mathcal{Q}_k$. Since $\ell(Q) < 2^{-k}$, it follows that $m \leq 2^n$. Let us fix one such R_i.

• Let us observe that R_i cannot be contained in any of the Q_j.
Namely, if this were the case, then one would obtain (in the decomposition) that $x \in \widehat{Q_j}$, which is a contradiction.

More precisely: If $R_i \subseteq Q_j$, then $\ell(Q_j) \geq 2^{-k}$ and $Q \cap Q_j \neq \emptyset$



We observe that each coordinate of $x - x_j$ is bounded in absolute value by $\frac{1}{2}(\ell(Q_j) + \ell(Q)) \leq \ell(Q_j)$

$\Rightarrow x \in \widehat{Q_j}$, which is the desired contradiction.

• In particular, since $R_i \in \mathcal{Q}_k$, it follows that $\fint_{R_i} f \leq \lambda$.

So:

$$\fint_Q f = \frac{1}{\mu(Q)} \cdot \sum_{i=1}^{m} \int_{Q \cap R_i} f \leq \sum_{i=1}^{m} \underbrace{\frac{\mu(R_i)}{\mu(Q)}}_{\leq 2^n} \cdot \underbrace{\left(\frac{1}{\mu(R_i)} \cdot \int_{R_i} f \right)}_{\leq \lambda}$$

$$\leq 2^n m \lambda \leq 4^n \lambda.$$

Hence, $M'f(x) \leq 4^n \lambda$ for $x \notin \bigcup_j \widehat{Q_j}$, i.e.

$$\{x : M'f(x) > 4^n \lambda\} \subseteq \bigcup_j \widehat{Q_j}.$$

This is (Δ). The claim now follows. $\square$

• It is now possible to give an alternative proof of Theorem 3.1 from Theorem 3.4, Proposition 3.5 and the fact that $Mf \sim M'f$.

3.3: Examples

- We can use maximal functions to study approximations of the identity.

Proposition 3.6: Let $\varphi \in L^1(\mathbb{R}^n)$ be a non-negative, radial, decreasing (wrt. radius) function. Then:

$$\sup_{t>0} |\varphi_t * f| \leq \|\varphi\|_{L^1} \cdot Mf, \text{ for all } f \in L^p(\mathbb{R}^n)$$
$$1 \leq p \leq \infty.$$

Proof: By rescaling, it suffices to prove $|\varphi * f| \leq \|\varphi\|_{L^1} \cdot Mf$ (i.e. we use the fact that $\|\varphi_t\|_{L^1} = \|\varphi\|_{L^1}$).

Since $|\varphi * f| \leq \varphi * |f|$ and $M|f| = Mf$, we may just consider the case when $f \geq 0$.

So, we must show that, for $f \geq 0$:

$$\varphi * f \leq \|\varphi\|_{L^1} \cdot Mf.$$

- If φ is a simple function, we can write:

$$\varphi = \sum_{i=1}^{k} a_i \chi_{B_i},$$ where $a_i > 0$ and B_i are balls centered at the origin.

(note that these are all the simple functions of this form).

Then $$\varphi * f = \sum_{i=1}^{k} (a_i \chi_{B_i}) * f = \sum_{i=1}^{k} a_i \mu(B_i) \cdot \underbrace{\left(\frac{1}{\mu(B_i)} \cdot (\chi_{B_i} * f)\right)}_{\leq Mf}$$

$$\leq \sum_{i=1}^{k} a_i \mu(B_i) \cdot Mf = \|\varphi\|_{L^1} \cdot Mf.$$

We observe that every non-negative, radial, decreasing (wrt. radius) integrable function φ can be written as a monotone increasing limit of simple functions $\varphi^{(k)}$ as above. Note that $\varphi^{(k)} \to \varphi$ in L^1.

Therefore, by Young's inequality $\varphi^{(k)} * f \to \varphi * f$ in L^p.

$\Rightarrow$ Up to a subsequence (which we also denote by $\varphi^{(k)}$), we then obtain that $\varphi * f = \lim_k \varphi^{(k)} * f$, almost everywhere.

$$\Rightarrow \varphi * f = \lim_k \varphi^{(k)} * f \leq \lim_k \|\varphi^{(k)}\|_{L^1} \cdot Mf = \|\varphi\|_{L^1} \cdot Mf$$

So: $\sup_{t>0} |\varphi_t * f| \leq \|\varphi\|_{L^1} \cdot Mf$. $\square$

- We can apply this result to study pointwise convergence of approximations of the identity:

<u>Proposition 3.7:</u> Let φ satisfy the assumptions as above. Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^n)$ be given. Then:

$$\varphi_t * f \xrightarrow[t \to 0]{} \left(\int_{\mathbb{R}^n} \varphi\right) \cdot f \quad \text{a.e.}$$

<u>Proof:</u> From Proposition 3.6 and Theorem 3.1, it follows that $T^* f := \sup_{t>0} |\varphi_t * f|$ is a weak (p,p) operator.

If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\varphi_t * f \xrightarrow[t \to 0]{} f$ pointwise a.e.

The claim follows for general $f \in L^p(\mathbb{R}^n)$ by Proposition 3.3. $\square$ (Note that here, we consider $p < \infty$)

<u>Remark:</u> The same conclusions hold if $|\varphi| \leq \Psi$ where Ψ is a non-negative, radial, decreasing (wrt. radius) and integrable function. This is because $|\varphi * f| \leq |\varphi| * |f| \leq \Psi * |f|$ and we then argue as before.