HOMEWORK ASSIGNMENT 3

Exercise 1. *(Blow-up solutions for the NLS)*
Throughout this problem, we consider $p \in [1+\frac{4}{n}, 1+\frac{4}{n-2})$ *and the* ***focusing NLS*** *on* $\mathbb{R}^n$.

$$\begin{cases} \mathrm{i}\partial_t u + \Delta u \; = \; -|u|^{p-1}u \\ u\big|_{t=0} = \Phi \in H^1(\mathbb{R}^n)\,. \end{cases}$$

i) Consider first the mass-critical case $p = 1 + \frac{4}{n}$.

a) *Integrate by parts and show that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int |x|^2 |u(x,t)|^2 \,\mathrm{d}x \; = \; 4\,\mathrm{Im} \int \Big\{ \sum_{j=1}^n x_j \bar{u}\, \partial_{x_j} u \Big\} \,\mathrm{d}x\,. \tag{1}$$

b) *Furthermore, show that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Im} \int \Big\{ \sum_{j=1}^n x_j \bar{u}\, \partial_{x_j} u \Big\} \,\mathrm{d}x \; = \; 4E(\Phi)\,. \tag{2}$$

[HINT: Part b) is a bit involved. One would like to prove the following claim, for general $p \in [1+\frac{4}{n}, 1+\frac{4}{n-2})$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Im} \int \Big\{ \sum_{j} x_j \bar{u}\, \partial_{x_j} u \Big\} \,\mathrm{d}x \; = \; 2\int |\nabla u(x,t)|^2 \,\mathrm{d}x + \Big(\frac{2n}{p+1} - n\Big) \int |u(x,t)|^{p+1} \,\mathrm{d}x\,. \tag{3}$$

A possible way to prove it is to use integration by parts and write

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Im} \int \Big\{ \sum_{j=1}^n x_j \bar{u}\, \partial_{x_j} u \Big\} \,\mathrm{d}x \; = \; \sum_{j=1}^n \mathrm{Re}\Big(\mathrm{i}\int x_j \,(\bar{u}_{x_j}\, u_t - \bar{u}_t\, u_{x_j})\,\mathrm{d}x\Big) + n\,\mathrm{Re}\Big(-\mathrm{i}\int \bar{u}_t\, u \,\mathrm{d}x\Big)\,.$$

One can use the equation to evaluate the expression on the right-hand side. A notational convenience in these calculations is

$$rv_r \; := \; \sum_{j=1}^n x_j\, \partial_{x_j} v\,.$$

]

c) *Use parts a) and b) to deduce that there exists* $T_* \in (0,\infty)$ *such that*

$$\lim_{t\to T_*} \int |x|^2 |u(x,t)|^2 \,\mathrm{d}x \; = \; 0\,, \tag{4}$$

provided that the initial data Φ *is chosen such that*

$$E(\Phi) \; < \; 0\,. \tag{5}$$

d) *Prove the* ***Weyl-Heisenberg inequality****, which states that*

$$\|f\|^2_{L^2(\mathbb{R}^n)} \; \leq \; C(n) \big\| |x| f \big\|_{L^2(\mathbb{R}^n)} \|\nabla f\|_{L^2(\mathbb{R}^n)}\,. \tag{6}$$

[HINT: It suffices to argue by density and assume that $f \in \mathcal{S}(\mathbb{R}^n)$*. Then, one can integrate by parts and write* $\int |f|^2 \,\mathrm{d}x = -\frac{1}{n}\sum_j \int x_j\, \partial_{x_j}(|f|^2)\,\mathrm{d}x$*.]*

e) *Use parts c) and d) to deduce that, for T_* as in part c) we have*

$$\lim_{t \to T_*} \|\nabla u(t)\|_{\dot{H}^1(\mathbb{R}^n)} \; = \; \infty \,. \tag{7}$$

f) *Explain how one can choose $\Phi \in H^1(\mathbb{R}^n)$ in such a way that $E(\Phi) < 0$. [HINT: Use scaling.]*

ii) Explain briefly why the above analysis extends to show existence of solutions satisfying (7) for some $T_* \in (0,\infty)$ whenever $p \in [1+\frac{4}{n}, 1+\frac{4}{n-2})$.
(As was mentioned in class, this construction is due to Robert Glassey in 1977).

Exercise 2. *(Bilinear Strichartz estimates on $\mathbb{R}^n$ for $n \geq 2$)*
Let $N_1 \geq N_2 \in 2^{\mathbb{N}_0}$ be dyadic integers. Suppose that $\psi_1, \psi_2 \in L^2(\mathbb{R}^n)$ are functions such that $\operatorname{supp} \widehat{\psi}_j \subset \{\xi,\, N_j/2 \leq |\xi| \leq 2N_j\}$ if $N_j > 1$ and $\operatorname{supp} \widehat{\psi}_j \subset \{\xi,\, |\xi| \leq 1\}$ if $N_j = 1$. Show that the following ***bilinear Strichartz estimate*** *holds.*

$$\big\| S(t)\psi_1 \, S(t)\psi_2 \big\|_{L^2_{t,x}} \; \lesssim_n \; \frac{N_2^{\frac{n-1}{2}}}{N_1^{\frac{1}{2}}} \, \|\psi_1\|_{L^2_x} \, \|\psi_2\|_{L^2_x} \,. \tag{8}$$

[HINT:

i)First consider the case when the frequencies N_1 and N_2 are comparable. In this case apply Hölder's inequality and estimate the expression as a product of $L^4_{t,x}$ norms. Reduce to the Strichartz inequality by Bernstein's inequality (recall Homework assignment 1).

ii) When N_1 is much bigger than N_2, argue by duality on the Fourier side (in both x and t). Compute

$$\Big(S(t)\psi_1 \, S(t)\psi_2 \Big)^{\sim}(\xi, \tau)$$

and note that it suffices to estimate an expression of the form

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(\xi_1 + \xi_2, -2\pi|\xi_1|^2 - 2\pi|\xi_2|^2) \, \widehat{\psi}_1(\xi_1) \, \widehat{\psi}_2(\xi_2) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \,. \tag{9}$$

Use a change of variables of the form which simplifies the arguments of F.

$$s \; := \; \xi_1 + \xi_2, \quad r \; := \; -2\pi|\xi_1|^2 - 2\pi|\xi_2|^2, \quad \omega \; := \; (\xi_{2,2}, \xi_{2,3}, \ldots, \xi_{2,n}) \,.$$

Explain that this makes sense if $|\xi_{1,1} - \xi_{2,1}|$ is large, i.e. that this is an invertible change of variables. Compute the jacobian J of this change of variables. How large is the jacobian in absolute value in terms of N_1? What bounds does $|\omega|$ satisfy? Rewrite the integral in (9) in the new variables. Apply Cauchy-Schwarz in all of the new variables s, r, ω. Bound $\frac{1}{J^2}$ in the obtained integral in terms of $\frac{1}{J}$ using the previously obtained estimate for J. What factor do we obtain from the ω integration? Undo the change of variables and conclude the claim.]