HOMEWORK ASSIGNMENT 2

Exercise 1. *(Bound on the Airy function A)*
Recall from class that we are interested in bounding the expression

$$I(x) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}x\xi + \mathrm{i}\xi^3}\, \phi_0(\xi)\, \mathrm{d}\xi \tag{1}$$

uniformly in $x \in \mathbb{R}$. *Here* $\phi_0 \in C^\infty(\mathbb{R})$ *is chosen such that*

$$\phi_0(\xi) = \begin{cases} 1 & \text{for } |\xi| > 4 \\ 0 & \text{for } |\xi| < 3\,. \end{cases} \tag{2}$$

Note that we can rewrite (1) *as*

$$I(x) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Psi_x(\xi)}\, \phi_0(\xi)\, \mathrm{d}\xi \tag{3}$$

where, for fixed $x \in \mathbb{R}$ *we define*

$$\Psi_x(\xi) := x\xi + \xi^3\,. \tag{4}$$

In other words, in the phase function defined in (4), x *is treated as a parameter.*

a) *If* $x > -3$, *show that we have*

$$|x + 3\xi^2| \geq |x| + |\xi|^2 \tag{5}$$

for all ξ *in the support of* ϕ_0. *Explain how we can deduce the claim for* $x > -3$ *from* (5) *and integration by parts.*

b) *If* $x \leq -3$ *define the following sets.*

$$A := \left\{ \xi \in \mathbb{R},\, |x + 3\xi^2| \leq \frac{|x|}{2} \right\}. \tag{6}$$

$$B := \left\{ \xi \in \mathbb{R},\, |x + 3\xi^2| < \frac{|x|}{3} \right\}. \tag{7}$$

We know that there exist nonnegative functions $\phi_1 \in C_c^\infty(\mathbb{R}), \phi_2 \in C^\infty(\mathbb{R})$ *such that*

$$\begin{cases} \phi_1 + \phi_2 = 1 \\ \operatorname{supp} \phi_1 \subset A \\ \phi_2 = 0 \quad \text{on } B\,. \end{cases} \tag{8}$$

(This fact can be used without proof). For $j = 1, 2$, *we define the following quantities.*

$$I_j(x) := \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}x\xi + \mathrm{i}\xi^3} \phi_0(\xi)\, \phi_j(\xi)\, \mathrm{d}\xi\,. \tag{9}$$

In particular, we can write

$$I = I_1 + I_2\,. \tag{10}$$

In what follows, we show that I_1 *and* I_2 *are both bounded which by* (10) *implies the claim.*

b1) *Show that for $\xi \in A$ we have*

$$|\Psi_x''(\xi)| \sim |x|^{\frac{1}{2}}\,. \tag{11}$$

In the proof of (11)*, it is helpful to note that for $\xi \in A$ we have*

$$|\xi|^2 \sim |x|\,. \tag{12}$$

Explain how one can combine (11) *and the Van der Corput lemma and deduce that I_1 is bounded.*
Note: *In order to do this properly, one should choose ϕ_1 in the correct way. Observe that by* (8) *and* (12) *we have that ϕ_1 can be chosen to be a function adapted to an annulus of size $\sim |x|^{1/2}$. In other words, it can be chosen as a rescaling of a function F independent of x supported on an annulus of size ~ 1 in such a way that the result is supported on an annulus of size $\sim |x|^{1/2}$.*

b2) *For $\xi \notin B$, show that we have*

$$|x + 3\xi^2| \geq C(|x| + |\xi|^2) \tag{13}$$

for some $C > 0$. Conclude that I_2 is bounded.

Exercise 2. *(An alternative proof of the inhomogeneous Strichartz estimate) We work on $\mathbb{R}^n$. We assume the homogeneous Strichartz estimate for all admissible pairs (q,r) in dimension n. Throughout the exercise, assume that $q > 2$, i.e. that we are in the non-endpoint case. Define the operator Ψ by the following rule.*

$$\Psi : f \mapsto \Psi_f(x,t) := \int_0^t S(t-s)\, f(\cdot,s)\, \mathrm{d}s\,.$$

Here $S(\cdot)$ denotes the free Schrödinger evolution and we evaluate the expression on the right-hand side at x. We want to prove that

$$\|\Psi_f\|_{L_t^q L_x^r} \lesssim_{q,r,\tilde{q},\tilde{r},n} \|f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{14}$$

for all $(q,r), (\tilde{q},\tilde{r})$ admissible in dimension n. In this exercise, we give a proof of (14) *using duality. In order to do this, we define the following operators.*

$$\Xi_f(x,t) := \int_t^\infty S(t-s)\, f(\cdot,s)\, \mathrm{d}s\,. \tag{15}$$

Furthermore, given $t_0 \in \mathbb{R}$ we define

$$\Psi_{f,t_0}(x,t) := \int_0^{t_0} S(t-s)\, f(\cdot,s)\, \mathrm{d}s\,. \tag{16}$$

a) *Show that for all (q,r) admissible in dimension n we have the following estimates.*

$$\|\Psi_f\|_{L_t^q L_x^r} \lesssim_{q,r,n} \|f\|_{L_t^{q'} L_x^{r'}} \tag{17}$$

$$\|\Xi_f\|_{L_t^q L_x^r} \lesssim_{q,r,n} \|f\|_{L_t^{q'} L_x^{r'}} \tag{18}$$

$$\|\Psi_{f,t_0}\|_{L_t^q L_x^r} \lesssim_{q,r,n} \|f\|_{L_t^{q'} L_x^{r'}}\,. \tag{19}$$

In particular, note that the implied constant in (19) *is uniform in t_0.*

b) *Show that we have*

$$\|\Psi_f\|_{L_t^\infty L_x^2} \lesssim_{q,r,n} \|f\|_{L_t^{q'} L_x^{r'}} \tag{20}$$

for all (q, r) *admissible in dimension* n.

Here, it is helpful to write the expression

$$\left\langle \Psi_f(t), \Psi_f(t) \right\rangle_{L_x^2}$$

in terms of the operator $\Psi_{f,\star}$ *defined in* (16) *and apply the result of a).*

Note also that

$$\|\Xi_f\|_{L_t^\infty L_x^2} \lesssim_{q,r,n} \|f\|_{L_t^{q'} L_x^{r'}} . \tag{21}$$

(This follows from the same proof.)

c) *Show that we have*

$$\|\Psi_f\|_{L_t^q L_x^r} \lesssim_{q,r,n} \|f\|_{L_t^1 L_x^2} \tag{22}$$

for all $(q, r) \neq (\infty, 2)$ *admissible in dimension* n.

Here, it is helpful to use duality and for $g \in L_t^{q'} L_x^{r'}$ *write the pairing*

$$\int_{-\infty}^{\infty} \left\langle \Psi_f(t), g \right\rangle_{L_x^2} \mathrm{d}t$$

in terms of the operator Ξ *defined in* (15) *and apply* (21).

d) *Check directly that*

$$\|\Phi_f(t)\|_{L_x^2} \leq \|f\|_{L_t^1 L_x^2} . \tag{23}$$

e) *Note that parts a)-d) imply that*

$$\begin{cases} \Psi : L_t^{q'} L_x^{r'} \to L_t^q L_x^r \\ \Psi : L_t^{q'} L_x^{r'} \to L_t^\infty L_x^2 \\ \Psi : L_t^1 L_x^2 \to L_t^q L_x^r \end{cases} \tag{24}$$

as bounded operators for all (q, r) *admissible in dimension* n.

Conclude (14) *from* (24) *and interpolation in mixed norm spaces. Here, one should consider two cases depending on the relative sizes of* q *and* $\tilde{q}$.