§3: The nonlinear theory

We now apply the techniques that we have been studying to nonlinear problems.
The two main problems that we will consider are.

① The nonlinear Schrödinger equation (NLS)

$$i u_t + \Delta u = \mu |u|^{p-1} u$$
$$u : \mathbb{R}^n_x \times \mathbb{R}_t \to \mathbb{C} \qquad \text{(Semilinear)}$$
$$\text{or } u : \mathbb{T}^n_x \times \mathbb{R}_t \to \mathbb{C}$$
$$p > 1, \ \mu \in \{1, -1\}$$

② The Korteweg-de Vries equation (KdV)

$$u_t + u_{xxx} - 2\mu u u_x = 0$$
$$u : \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{C} \text{ or } u : \mathbb{T}_x \times \mathbb{R}_t \to \mathbb{C} \qquad \text{(Quasilinear)}$$
$$\mu \in \{1, -1\}.$$

Both equations have the form:

Linear dispersive part + Nonlinearity = 0.

The main idea is to use the regularizing effects of the linear part, (i.e. the Strichartz estimates) to say something about the nonlinear problem.

- Let us focus on the nonlinear Schrödinger equation. We consider the initial value problem

$$(*) \quad \begin{cases} i u_t + \Delta u = \mu |u|^{p-1} u \\ u|_{t=0} = \Phi \end{cases}$$

We usually take $\Phi \in H^s$.

By a formal integration by parts, one can check that the following quantities are conserved.

$$\text{Mass} = \int |u(x,t)|^2 dx$$

$$\text{Energy} = \frac{1}{2} \int |\nabla u(x,t)|^2 dx + \frac{\mu}{p+1} \int |u(x,t)|^{p+1} dx$$

$$\text{Momentum} = \left(- \int \mathrm{Im} (\bar{u}\, \partial_{x_j} u)\, dx \right)_j$$

(a vector in $\mathbb{R}^n$)

For instance,

$$\frac{d}{dt} \int |u(x,t)|^2 dx = 2\,\mathrm{Re} \int u_t(x,t)\, \overline{u(x,t)}\, dx$$

$$= 2\,\mathrm{Re} \int \left(i\Delta u(x,t) - i |u(x,t)|^{p-1} u(x,t) \right) \overline{u(x,t)}\, dx$$

$$= -2\,\mathrm{Im} \int \Delta u(x,t)\, \overline{u(x,t)}\, dx$$
$$+ 2\,\mathrm{Im} \int |u(x,t)|^{p+1} dx$$

$$= 2\,\mathrm{Im} \int |\nabla u(x,t)|^2 dx$$
$$+ 2\,\mathrm{Im} \int |u(x,t)|^{p+1} dx$$

$$= 0$$

The other formal calculations proceed in a similar manner.

- If $\mu = +1$, we call the equation <u>defocusing</u>. In this case, the energy is always non-negative and it gives us control on the $\dot{H}^1$ norm of the solution.

• If $\mu = -1$, the equation is called focusing.
In this case, the energy can be negative.

[For the KdV, the energy is $\int \left(\frac{1}{2}u_x^2 + \frac{\mu}{3}u^3\right)dx$]

Another way to see the conservation laws is to use the pseudo-stress-energy tensor $T_{\alpha,\beta}$: $\alpha,\beta = 0,1,\dots,n$

$T_{00} := |u|^2$ (mass density)

$T_{0j} := T_{j0} := \operatorname{Im}(\bar{u}\,\partial_{x_j} u)$ (momentum density)

For $j,k = 1,\dots,n$

$$T_{jk} := \operatorname{Re}\left(\partial_{x_j} u \overline{\partial_{x_k} u}\right) - \frac{1}{4}\delta_{jk}\Delta(|u|^2) + \mu\frac{p-1}{p+1}\delta_{jk}|u|^{p+1}$$

(stress tensor)

We obtain the following identities

$$\boxed{\begin{aligned} &\partial_t T_{00} + \partial_{x_j} T_{0j} = 0 \rightsquigarrow \text{Conservation of mass} \\ &\partial_t T_{0j} + \partial_{x_k} T_{jk} = 0 \rightsquigarrow \text{Conservation of momentum,} \end{aligned}}$$

for $j,k = 1,\dots,n$

Symmetries of the equation

(We assume that we are working on $\mathbb{R}^n$)

• Translation symmetry
If $u = u(x,t)$ is a solution of NLS, then $\tilde{u}(x,t) = u(x - x_0, t - t_0)$ is also a solution of the NLS.

Modulational symmetry.

If $u = u(x,t)$ is a solution of the NLS, then for all $\vartheta \in \mathbb{R}$, $\tilde{u}(x,t) := e^{i\vartheta} u(x,t)$ is a solution of the NLS.

Spatial rotation symmetry:

If $\mathcal{O} \in O(n)$, then $\tilde{u}(x,t) := u(\mathcal{O}x, t)$ is a solution of NLS.

Galilean symmetry:

For $c \in \mathbb{R}^n$,

$$u_c(x,t) := e^{icx} \cdot e^{-i|c|^2 t} u(x-2ct, t)$$

solves NLS with initial data $\Phi_c(x) = e^{icx} \cdot \Phi(x)$ (we modulate the initial data)

Pseudoconformal symmetry:

If u solves the NLS, we define for $t \neq 0$

$$v(x,t) := \frac{1}{|t|^{n/2}} \overline{u}\left(\frac{x}{t}, \frac{1}{t}\right) e^{\frac{i|x|^2}{4t}}$$

Then, v solves

$$iv_t + \Delta v = t^{\frac{n}{2} \cdot \left(p - \underbrace{\left(1+\frac{4}{n}\right)}_{=: p_{L^2_x}}\right)} \cdot |v|^{p-1} v$$

This gives us a solution of the NLS for $t \neq 0$ when $p = 1 + \frac{4}{n}$.

Scaling symmetry:

If u solves the NLS with initial data Φ, then for $\lambda > 0$

$u_\lambda(x,t) := \lambda^{-\frac{2}{p-1}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$ solves the NLS with initial data $\Phi_\lambda(x) = \lambda^{-\frac{2}{p-1}} \Phi\left(\frac{x}{\lambda}\right)$.

Applications of the symmetries

① Galilean symmetry:

Suppose that we have a solution

$u(x,t) = e^{it\tau} \cdot Q(x)$, $\tau > 0$ $\quad (\Delta)$

of the NLS

(one can show that such solutions exist under appropriate assumptions).

By applying the Galilean symmetry, we obtain a solution

$$u_c(x,t) = e^{icx} \cdot e^{-i|c|^2 t + it\tau} Q(x - 2ct)$$

"a traveling wave".

② Pseudoconformal symmetry

Suppose that $p = 1 + \frac{4}{n}$ and suppose that we know that a solution of the form (Δ) exists. Assume Q is real-valued.

Then,

$$v(x,t) = \frac{1}{|t|^{n/2}} Q\left(\frac{x}{t}\right) e^{\frac{i|x|^2}{4t}} e^{-\frac{i\tau}{t}}$$

solves the NLS for $t \neq 0$.

Observe that:

- v - blows up pointwise at $t = 0$!
- $\|v(t)\|_{L^2_x} = \text{constant}$
- $\|\nabla v(t)\|_{L^2_x} \sim \frac{1}{|t|}$. "Self-similar blow-up rate".

③ Scaling symmetry:

This is the most important symmetry.
For $f_\lambda(x) := \lambda^{-\frac{2}{p-1}} f\left(\frac{x}{\lambda}\right)$, we can compute
$$\|f_\lambda\|_{\dot{H}^s} = \lambda^{-s+s_c} \|f\|_{\dot{H}^s}$$
where $s_c := \frac{n}{2} - \frac{2}{p-1}$.

$\rightsquigarrow$ If u is defined on a time interval of length T, u_λ is defined on a time interval of length $\lambda^2 T$.

We take $\lambda \gg 1$

Consider several cases.

① $s > s_c$: Sub-critical regime

The norm of the initial data is small

$\Updownarrow$

The interval of existence is large.
Heuristically, this regime is good.

② $s = s_c$: Critical regime

The norm is invariant under the scaling, but the scaling changes the interval of existence.
This looks problematic.

③ $s < s_c$: Super-critical regime

If the norm is scaled to be large, the interval becomes large.
Heuristically, scaling works against us in this regime.

An informal explanation of the previous formula

- We want to compare the dispersive part Δu and the nonlinear part $|u|^{p-1}u$.

Suppose u_0 has amplitude A and is localized in frequency to an annulus $|\xi| \sim N$.

One expects most of the support of u_0 to be inside of a ball of radius $\frac{1}{N}$ [Recall that $(f(\cdot N))^\wedge(\xi) = \frac{1}{N}\hat{f}\left(\frac{\xi}{N}\right)$.]

$\rightsquigarrow \|u_0\|_{L^2} \sim A N^{-\frac{n}{2}}$

$\|u_0\|_{\dot{H}^s} \sim A N^{s-\frac{n}{2}}$

We want $A N^{s-\frac{n}{2}} \ll 1$ (small initial data)

$$\Rightarrow A \ll N^{\frac{n}{2}-s}. \qquad (\square)$$

$|\Delta u_0| \sim A N^2$, $|u|^p \sim A^p$.

<u>More linear:</u> $A N^2 \gg A^p$

i.e. $N^2 \gg A^{p-1}$

If $s > \frac{n}{2} - \frac{2}{p-1}$, this follows from $(\square)$.

<u>More nonlinear:</u> $A^{p-1} \gg N^2$ i.e. $A \gg N^{\frac{2}{p-1}}$

From $(\square)$, we deduce $N^{\frac{n}{2}-s} \gg N^{\frac{2}{p-1}}$

so $s < \frac{n}{2} - \frac{2}{p-1} = s_c$.

<u>Remarks:</u>

1) $s_c = 0$: L^2-critical $\Rightarrow p = 1 + \frac{4}{n} = p_{L^2_x}$

$\rightsquigarrow$ recall that the pseudoconformal transformation gives a solution of the NLS for $t \neq 0$.

2) The scaling heuristic doesn't always hold on manifolds.

Burq-Gérard-Tzvetkov

$iu_t + \Delta u = |u|^2 u$; on S^2, $s_c = 0$

ill-posed in $H^s(S^2)$ for $s < \frac{1}{4}$.

Summary:

$$s_c = \frac{n}{2} - \frac{2}{p-1}$$

$p = 1 + \frac{4}{n-2s_c}$ $\rightsquigarrow$ NLS is H^{s_c}- critical

If

$p > 1 + \frac{4}{n-2s_c}$ $\rightsquigarrow$ H^{s_c}- supercritical

$p < 1 + \frac{4}{n-2s_c}$ $\rightsquigarrow$ H^{s_c}- subcritical.

3.1: (Local) well-posedness theory

We are considering the Cauchy problem:

$$(*)\begin{cases} i u_t + \Delta u = \pm |u|^{p-1} u \\ u\big|_{t=0} = \Phi \end{cases}$$

We say that $(*)$ is locally well-posed in H^s if for any ball B in H^s centered at 0, there exists a time $T>0$ and a Banach space of functions $X \subseteq L^\infty([-T,T], H^s)$ such that for every $u_0 \in B$ there exists a unique solution u in the space X to the integral equation:

$$u(x,t) = S(t)\Phi \mp i \int_0^t S(t-\tau)|u|^{p-1}u(\tau)\,d\tau. \quad \left(\begin{array}{c}\text{e.g. in the} \\ \text{sense of} \\ \text{distributions}\end{array}\right)$$

"mild solution"

Furthermore, we require the map

$\Phi \mapsto u$ to be continuous from H^s into $C([-T,T], H^s)$.

"stability".

- If uniqueness is obtained in $L^\infty([-T,T],H^s)$, we say that the local well-posedness is unconditional. (a difficult problem in general!)
- If the local well-posedness condition holds for all $T>0$, we say that the problem $(*)$ is globally well-posed in H^s.

- Let us now work on $\mathbb{R}^n$ with $n \geq 3$.

We recall that, by the Strichartz estimate, we have

$$\|S(t)f\|_{L^q_t L^r_x} \leq C(q,r,n)\|f\|_{L^2_x}$$

whenever $2 \leq q, r \leq \infty$ and

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}.$$

Let us note that the set $\{(\frac{1}{q},\frac{1}{r}) : (q,r)$ is admissible in n dimensions$\}$ is compact.

In particular, the constant $C(q,r,n)$ can be taken to be uniform in q and r.

Let $I \subseteq \mathbb{R}$ be a finite interval.

We define the Strichartz space $S^0(I \times \mathbb{R}^n)$ to be the closure of the Schwartz functions under the following norm.

$$\|u\|_{S^0(I\times\mathbb{R}^n)} := \sup_{\substack{(q,r)\\ \text{admissible}\\ \text{in dimension}\\ n}} \|u\|_{L^q_t L^r_x (I\times\mathbb{R}^n)}.$$

Let us define

$$\|F\|_{N^0(I\times\mathbb{R}^n)} := \inf_{\substack{(q,r)\\ \text{admissible}\\ \text{in dimension}\\ n}} \|F\|_{L^{q'}_t L^{r'}_x (I\times\mathbb{R}^n)}$$

- Let $t_0 \in I$.
- If $iu_t + \Delta u = F$ and if (q,r), $(\tilde{q},\tilde{r})$, are admissible in dimension n, we know that

$$\|u\|_{L^q_t L^r_x(I\times\mathbb{R}^n)} \lesssim_n \|u(t_0)\|_{L^2_x(\mathbb{R}^n)} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x (I\times\mathbb{R}^n)}.$$

Taking suprema over (q,r) and infima over $(\tilde{q},\tilde{r})$, it follows that

$$\|u\|_{S^0(I\times\mathbb{R}^n)} \lesssim_n \|u(t_0)\|_{L^2_x(\mathbb{R}^n)} + \|F\|_{N^0(I\times\mathbb{R}^n)}.$$

Note that this estimate unifies all of the Strichartz estimates in one.

Local well-posedness

Theorem 3.1 (L^2-subcritical local well-posedness)

Let p be an L^2-subcritical exponent, i.e. $1 < p < 1 + \frac{4}{n}$ and let $\mu = \pm 1$.
Then, the NLS

$$\begin{cases} i u_t + \Delta u = \mu |u|^{p-1} u \\ u|_{t=0} = \Phi \end{cases}$$

is locally well-posed in $L^2 = H^0$.

Proof: Let $I := [-T, T]$, for $T > 0$.
$\mathcal{S} := S^0(I \times \mathbb{R}^n)$, $\mathcal{N} := N^0(I \times \mathbb{R}^n)$.
Note that $\mathcal{S}, \mathcal{N}$ are Banach spaces.
Let

$$DF(t) := -i \int_0^t S(t-\tau)\, F(\tau)\, d\tau$$

$$Nu := \mu |u|^{p-1} u.$$

So, we are looking for a fixed point of

$$\mathcal{T}(u) := S(t)\Phi + DNu,$$

which we expect to live in $\mathcal{S}$, and for which we expect that $Nu \in \mathcal{N}$.

Goal: $\| Nu \|_{\mathcal{N}} \lesssim T^{\theta} \| u \|_{\mathcal{S}}^{p}$, for some $\theta > 0$,
which should imply
$\| u \|_{\mathcal{S}} \lesssim \| \Phi \|_{L^2}$, whenever T is chosen to be sufficiently small.

- Let (q,r) be such that $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $\frac{p}{r} = \frac{1}{r'}$ (recall that $p =$ degree of nonlinearity).

$\Rightarrow \quad \frac{p}{r} = 1 - \frac{1}{r}$

$\frac{p+1}{r} = 1$

$\Rightarrow \boxed{r = p+1 > 2}$ (recall that $p>1$)

$$\Rightarrow \frac{2}{q} = \frac{n}{2} - \frac{n}{r} = \frac{n}{2} - \frac{n}{p+1} = \frac{n(p+1) - 2n}{p+1}$$

(since $p < 1 + \frac{4}{n}$) $<$

$$\frac{n(1+n+1) - 2n}{2(p+1)} = \frac{4}{2(p+1)} = \frac{2}{p+1} = \frac{2}{r}$$

$\Rightarrow \quad \boxed{q > r > 2} \quad \Rightarrow$ <u>(q,r) is admissible in dimension n</u>.

$\Rightarrow \quad q' < r' = \underset{\substack{\downarrow \\ \text{by choice of } r}}{\frac{r}{p}} < \frac{q}{p}.$

i.e. $\boxed{\frac{q}{p} > q'}$ (note that this condition relies on $p < 1 + \frac{4}{n}$.)

Hence:

$$\|N u\|_{\mathcal{N}} \leq \|N u\|_{L_t^{q'} L_x^{r'}(I \times \mathbb{R}^n)}$$

by Hölder's ineq. in time

$$\lesssim T^{\gamma} \|N u\|_{L_t^{\frac{q}{p}} L_x^{r'}(I \times \mathbb{R}^n)},$$

where $\gamma := \frac{1}{q'} - \frac{p}{q} > 0.$

Now,

$$\|Nu\|_{L_t^{\frac{q}{p}} L_x^{r'}(I\times\mathbb{R}^n)} = \| |u|^{p-1}u\|_{L_t^{\frac{q}{p}} L_x^{r'}(I\times\mathbb{R}^n)}$$

$$= \| |u|^{p-1}u\|_{L_t^{\frac{q}{p}} L_x^{\frac{r}{p}}(I\times\mathbb{R}^n)}$$

by Hölder's ineq.

$$\lesssim \|u\|^p_{L_t^q L_x^r} \leq \|u\|^p_{\mathcal{S}} .$$

Hence

$$\|Nu\|_{\mathcal{N}} \lesssim T^{\gamma} \|u\|^p_{\mathcal{S}} . \qquad (\mathrm{I})$$

Since

$$\left| |u|^{p-1}u - |v|^{p-1}v \right| \lesssim |u-v| \cdot \left(|u|^{p-1} + |v|^{p-1}\right),$$

we can use analogous arguments and obtain:

$$\|Nu - Nv\|_{\mathcal{N}} \lesssim T^{\gamma} \|u-v\|_{\mathcal{S}} \cdot \left(\|u\|^{p-1}_{\mathcal{S}} + \|v\|^{p-1}_{\mathcal{S}}\right). \,(\mathrm{II})$$

- Let us now denote by $C_1 > 0$ the constant from Strichartz's inequality

$$\|S(t)\Phi\|_{\mathcal{S}} \leq C_1 \|\Phi\|_{L^2_x} .$$

Let $\mathcal{A} := \{u \in \mathcal{S}, \|u\|_{\mathcal{S}} \leq 2C_1 \|\Phi\|_{L^2_x}\}$.

Then (I) implies that, for $u \in \mathcal{A}$ we have

$$\|\mathcal{I}(u)\|_{\mathcal{S}} \leq C_1 \|\Phi\|_{L^2_x} + C(n) \cdot T^{\gamma} \|u\|^p_{\mathcal{S}}$$

$$\leq C_1 \|\Phi\|_{L^2_x} + C(n)\cdot T^{\gamma}\cdot 2^p C_1^{p-1}\|\Phi\|^{p-1}_{L^2_x} C_1 \|\Phi\|_{L^2_x}$$

$$\leq 2C_1\|\Phi\|_{L^2_x} \qquad (\mathrm{III})$$

if $\quad C(n)T^{\gamma}\cdot 2^p C_1^{p-1}\|\Phi\|^{p-1}_{L^2_x} \leq 1$

In particular, for $T \sim \|\Phi\|^{-B}_{L^2_x}$ with appropriately chosen $B>0$, (III) implies that

$\mathcal{T}$ maps $\mathcal{A}$ into itself.

Similarly, we use (II) in order to obtain that, for all $u,v \in \mathcal{A}$ we have:

$$\|\mathcal{T}(u)-\mathcal{T}(v)\|_{\mathcal{S}} = \|DN(u)-DN(v)\|_{\mathcal{S}}$$

$$\leq C(n)\,\|N(u)-N(v)\|_{\mathcal{N}}$$

$$\leq C(n,p)T^{\gamma}\|\Phi\|^{p-1}_{L^2_x}\cdot\|u-v\|_{\mathcal{S}}$$

Hence, choosing $T \sim \|\Phi\|^{-B}_{L^2_x}$ possibly smaller, it follows that

$$\|\mathcal{T}(u)-\mathcal{T}(v)\|_{\mathcal{S}} \leq \tfrac{1}{2}\|u-v\|_{\mathcal{S}} \text{ for all } u,v\in\mathcal{A}. \quad (\mathrm{IV})$$

Note that $\mathcal{A}$ is complete w.r.t. $\|\cdot\|_{\mathcal{S}}$.

By the Banach fixed point theorem, it follows that $\mathcal{T}$ has a unique fixed point in $\mathcal{A}$. We call it u.

- From the formula for u, one can check that
$u \in C([-T,T], L^2_x)$.
We omit the details.

- Let us now check uniqueness in $\mathcal{S}(I \times \mathbb{R}^n) \subseteq L^\infty(I, L^2_x)$.

Namely, let $u, v \in \mathcal{S}(I \times \mathbb{R}^n)$ be solutions of the NLS with initial data Φ.
One then has, for all subintervals $J \subseteq I$:

$$\|u-v\|_{L^q_t L^r_x (J \times \mathbb{R}^n)} = \|DNu - DNv\|_{L^q_t L^r_x (J \times \mathbb{R}^n)}$$

$$\lesssim C(n) \|Nu - Nv\|_{L^{q'}_t L^{r'}_x (J \times \mathbb{R}^n)}$$

$$\leq C(n,q) \cdot |J|^\gamma \cdot \|Nu - Nv\|_{L^{\frac{q}{p}}_t L^{\frac{r}{p}}_x (J \times \mathbb{R}^n)}$$

(since $q' < \frac{q}{p}$, $r' = \frac{r}{p}$). Here $\gamma > 0$.

By Hölder's inequality, this expression is

$$\leq C(n,q,p) \cdot |J|^\gamma \cdot \|u-v\|_{L^q_t L^r_x (J \times \mathbb{R}^n)} \cdot \left(\|u\|^{p-1}_{L^q_t L^r_x (J \times \mathbb{R}^n)} + \|v\|^{p-1}_{L^q_t L^r_x (J \times \mathbb{R}^n)}\right).$$

We partition I as $J_1 \cup J_2 \cup \ldots \cup J_k$ such that

$$C(n,q,p) \cdot |J_\ell|^\gamma \cdot \left(\|u\|^{p-1}_{L^q_t L^r_x (J \times \mathbb{R}^n)} + \|v\|^{p-1}_{L^q_t L^r_x (J \times \mathbb{R}^n)}\right) \leq \frac{1}{2}$$

for all $1 \leq \ell \leq k$.

In particular, it follows that for all $1 \leq \ell \leq k$ we have:

$$\|u-v\|_{L^q_t L^r_x (J_\ell \times \mathbb{R}^n)} \leq \frac{1}{2} \|u-v\|_{L^q_t L^r_x (J_\ell \times \mathbb{R}^n)}$$

$\Rightarrow u = v$ on J_ℓ for all $1 \leq \ell \leq k$.

$\Rightarrow u = v$ on I.

- Finally, we need to check the continuity of $\Phi \mapsto u$ from $L^2(\mathbb{R}^n)$ into $\mathcal{S}(I\times\mathbb{R}^n)$.

Let $\Phi, \Psi \in L^2(\mathbb{R}^n)$ with $\|\Phi\|_{L^2_x}, \|\Psi\|_{L^2_x} \leq R$ be given.

Let u, v be the solutions corresponding to the initial data Φ, Ψ respectively.

In particular, we note: for $(\tilde{q}, \tilde{r})$ admissible, (q, r) as before:

$$\|u-v\|_{L^{\tilde{q}}_t L^{\tilde{r}}_x (I\times\mathbb{R}^n)}$$

$$\leq \|S(t)\Phi - S(t)\Psi\|_{L^{\tilde{q}}_t L^{\tilde{r}}_x (I\times\mathbb{R}^n)}$$

$$+ \|DNu - DNv\|_{L^{\tilde{q}}_t L^{\tilde{r}}_x (I\times\mathbb{R}^d)}$$

$$\leq C_1 \|\Phi - \Psi\|_{L^2_x} + CT^{\gamma} \|u-v\|_{L^q_t L^r_x (I\times\mathbb{R}^n)} \underbrace{\left(\|u\|^{p-1}_{L^q_t L^r_x (I\times\mathbb{R}^n)} + \|v\|^{p-1}_{L^q_t L^r_x (I\times\mathbb{R}^n)}\right)}$$

$= \mathcal{O}(R^{p-1})$ by the construction of u and v.

Take suprema over $(\tilde{q}, \tilde{r})$

$$\Rightarrow \|u-v\|_{\mathcal{S}(I\times\mathbb{R}^n)} \leq C_1 \|\Phi - \Psi\|_{L^2_x} + C_2 T^{\gamma} R^{p-1} \|u-v\|_{\mathcal{S}(I\times\mathbb{R}^n)}$$

If $C_2 T^{\gamma} R^{p-1} \leq \frac{1}{2}$, it follows that

$$\|u-v\|_{\mathcal{S}(I\times\mathbb{R}^n)} \leq 2C_1 \|\Phi - \Psi\|_{L^2_x}.$$

Hence, we obtain the claim if $T \sim R^{-\beta}$ is chosen to be sufficiently small. $\square$

Remarks:

① If we consider initial data in the ball $B = \{ \Phi : \|\Phi\|_{L^2_x} \leq R \}$, the time of local well-posedness / existence is $T \sim R^{-\beta}$ for some $\beta > 0$, determined by p and n.

② This construction is the same for $\mu = +1$ and $\mu = -1$.

③ We do not need to use the full strength of the $\mathcal{S}$ and $\mathcal{N}$ spaces.
It suffices to work with the right pairs (q, r).

Theorem 3.2 (L^2 critical local well-posedness)

Let $p = 1 + \frac{4}{n}$. Then, the NLS

$$\begin{cases} iu_t + \Delta u = \pm |u|^{p-1} u \\ u|_{t=0} = \Phi \in L^2 \end{cases}$$

is locally well-posed in the critical sense.
More precisely, we can find $\varepsilon_0 = \varepsilon_0(n)$ such that whenever

$$\| S(t) \Phi_* \|_{L^{p+1}_{t,x}(I \times \mathbb{R}^n)} \leq \varepsilon_0, \text{ then for all}$$

$$\Phi \in B := \{ \tilde{\Phi} \in L^2, \| \tilde{\Phi} - \Phi_* \|_{L^2} \leq \varepsilon_0 \},$$

there exists a unique solution $u \in S^0(I \times \mathbb{R}^n) = \mathcal{S}$ to the NLS. The map $\Phi \mapsto u$ is Lipschitz from B to $S^0(I \times \mathbb{R}^n) = \mathcal{S}$. (Here, $I = [-T, T]$ for some $T > 0$.)

Proof: Before proceeding with the proof, we note that the pair $(p+1, p+1)$ is admissible in dimension n.
From the Strichartz inequality, we hence deduce that

$$\| S(t) \Phi_* \|_{L^{p+1}_{t,x}([-T,T] \times \mathbb{R}^n)} \leq \varepsilon_0$$

whenever $T > 0$ is chosen to be sufficiently small.
It is important to note that the precise choice of T depends on the function Φ_* and not on any norm of Φ_*.

We take $(q,r) = (p+1, p+1)$.

In this case, we note that

$$q' = \frac{p+1}{p} = \frac{q}{p}.$$

Hence $\gamma = 0$ in the previous notation.

- Let us fix $\Phi \in B$.

One then obtains that:

$$\|S(t)\Phi\|_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)}$$

$$\leq \|S(t)(\Phi - \Phi_*)\|_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)} + \|S(t)\Phi_*\|_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)}$$

$$\leq C(d)\|\Phi - \Phi_*\|_{L^2_x} + \|S(t)\Phi_*\|_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)}$$

$$\leq C_1\varepsilon_0, \text{ for some } C_1 = C_1(n) > 0. \qquad (\Delta)$$

Let $\mathcal{A} := \{u, \|u\|_{L^{p+1}(I\times\mathbb{R}^n)} \leq 2C_1\varepsilon_0\}$.

Let $F(u) := S(t)\Phi + DNu(t)$

We have

$$\|F(u)\|_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)} \leq \|S(t)\Phi\|_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)} + \|DNu\|_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)}$$

$$\leq C_1\varepsilon_0 + C\|Nu\|_{L^{\frac{p+1}{p}}_{t,x}(I\times\mathbb{R}^n)}$$ (use (Δ) and the Strichartz inequality).

$$\leq C_1\varepsilon_0 + \|u\|^p_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)}$$

$$\leq C_1\varepsilon_0 + (2C_1\varepsilon_0)^p \leq 2C_1\varepsilon_0$$ if $\varepsilon_0 > 0$ is sufficiently small.

Likewise, we obtain for $u, v \in \mathcal{A}$

$$\|F(u)-F(v)\|_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)}$$

$$\leq C\|u-v\|_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)}\cdot\left(\|u\|^{p-1}_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)}+\|v\|^{p-1}_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)}\right)$$

$$\leq C\|u-v\|_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)}\cdot 2\cdot(2C_1\varepsilon_0)^{p-1}$$

$$\leq \frac{1}{2}\|u-v\|_{L^{p+1}_{t,x}(I\times\mathbb{R}^n)}$$

for $\varepsilon_0>0$ chosen sufficiently small.

$\Rightarrow$ F has a unique fixed point in $\mathcal{A}$. We call the fixed point u.

We now check that $\underline{u\in\mathcal{S}(I\times\mathbb{R}^n)}$.
This follows immediately from the Strichartz estimates and since $u\in L^{p+1}_{t,x}(I\times\mathbb{R}^n)$.

We now check $\underline{\text{uniqueness in } \mathcal{S}(I\times\mathbb{R}^n)}$.
Suppose that $u,v\in\mathcal{S}(I\times\mathbb{R}^n)$ are solutions of the NLS with initial data Φ.
The same calculation as earlier implies that, for all subintervals $J\subseteq I$ containing 0, we have:

$$\|u-v\|_{L^{p+1}_{t,x}(J\times\mathbb{R}^n)}\leq C(n,p)\cdot\|u-v\|_{L^{p+1}_{t,x}(J\times\mathbb{R}^n)}\cdot\left(\|u\|^{p-1}_{L^{p+1}_{t,x}(J\times\mathbb{R}^n)}+\|v\|^{p-1}_{L^{p+1}_{t,x}(J\times\mathbb{R}^n)}\right)$$

We now partition I as $J_1\cup J_2\cup\ldots\cup J_k$ such that for all $1\leq \ell\leq k$ we have:

$$C(n,p)\cdot\left(\|u\|^{p-1}_{L^{p+1}_{t,x}(J_\ell\times\mathbb{R}^n)}+\|v\|^{p-1}_{L^{p+1}_{t,x}(J_\ell\times\mathbb{R}^n)}\right)\leq\frac{1}{2}.$$

As in the subcritical setting, we deduce that $u = v$ on I.

Finally, we check <u>the continuity of $\Phi \mapsto u$ from $L^2(\mathbb{R}^n)$ into $S(I \times \mathbb{R}^n)$.</u>

Let $\Phi, \Psi \in L^2(\mathbb{R}^n)$ and let $u, v \in S(I \times \mathbb{R}^n)$ be the associated solutions.

For $(\tilde{q}, \tilde{r})$ admissible, we have for all subintervals $J \subseteq I$

$$\|u-v\|_{L_t^{\tilde{q}} L_x^{\tilde{r}} (J \times \mathbb{R}^n)} \leq C \|\Phi - \Psi\|_{L_x^2} + C \|u-v\|_{L_{t,x}^{p+1}(J \times \mathbb{R}^n)} \cdot \left(\|u\|^{p-1}_{L_{t,x}^{p+1}(J \times \mathbb{R}^n)} + \|v\|^{p-1}_{L_{t,x}^{p+1}(J \times \mathbb{R}^n)} \right)$$

In particular, we have

$$\|u-v\|_{S(J \times \mathbb{R}^n)} \leq C \|\Phi - \Psi\|_{L_x^2} + C \|u-v\|_{S(J \times \mathbb{R}^n)} \cdot \left(\|u\|^{p-1}_{S(J \times \mathbb{R}^n)} + \|v\|^{p-1}_{S(J \times \mathbb{R}^n)} \right)$$

We now partition the interval I as $J_1 \cup J_2 \cup \dots \cup J_k$ such that for all $1 \leq l \leq k$ we have:

$$C\left(\|u\|^{p-1}_{S(\mathcal{F}_\ell \times \mathbb{R})} + \|v\|^{p-1}_{S(\mathcal{F}_\ell \times \mathbb{R})}\right) \leq \frac{1}{2}.$$

Hence, for all $1 \leq \ell \leq K$, we obtain:

$$\|u-v\|_{S(\mathcal{F}_\ell \times \mathbb{R}^n)} \leq C \|\Phi - \Psi\|_{L^2_x}.$$

The continuity claim now follows. $\square$

In the H^1-subcritical / H^1-critical regime, one has the following results.

<u>H^1-subcritical regime</u>: $1 < p < 1 + \frac{4}{n-2}$, for $n \geq 3$ and $1 < p < \infty$ for $n = 1, 2$.

$\leadsto$ The NLS is locally well-posed in H^1.

The solution space is $X = S^1(I \times \mathbb{R}^n)$, where

$$\|f\|_{S^1(I \times \mathbb{R}^n)} := \|f\|_{S^0(I \times \mathbb{R}^n)} + \|\nabla f\|_{S^0(I \times \mathbb{R}^n)}.$$

<u>H^1-critical regime</u>: If $p = 1 + \frac{4}{n-2}$ for $n \geq 3$, then the NLS is locally well-posed in the critical sense, i.e. we can find a solution in $S^1([-T,T] \times \mathbb{R}^n)$ if we assume that

$$\|S(t)\Phi\|_{L_t^{\frac{2(n+2)}{n-2}} W_x^{1, \frac{2n(n+2)}{n^2+4}}([-T,T] \times \mathbb{R}^n)}$$

is small enough.

A note on unconditional well-posedness

Suppose that we know that

$$\begin{cases} i u_t + \Delta u = \pm |u|^{p-1} u & \text{on } \mathbb{R}^n_x \times [-T,T] \\ u|_{t=0} = \Phi & \text{on } \mathbb{R}^n_x \end{cases}$$

is locally well-posed in H^s, $s > \frac{n}{2}$. Assume that p is an integer.

Claim: The problem is unconditionally well-posed, i.e. if $u, v \in C([-T,T], H^s)$ solve the initial value problem.

Proof: We know that

$$u(t) - v(t) = \mp i \int_0^t S(t-\tau)\left(|u|^{p-1}u - |v|^{p-1}v\right)(\tau)\,d\tau$$

$\forall t \in [-T,T]$.

In particular, by Minkowski's inequality, we have

$$\|u(t)-v(t)\|_{H^s} \leq \int_0^t \|S(t-\tau)\left(|u|^{p-1}u - |v|^{p-1}v\right)(\tau)\|_{H^s}\,d\tau$$

$$= \int_0^t \|\left(|u|^{p-1}u - |v|^{p-1}v\right)(\tau)\|_{H^s}\,d\tau$$

We know that

$$\left||u|^{p-1}u - |v|^{p-1}v\right| \lesssim |u-v| \cdot \left(|u|^{p-1} + |v|^{p-1}\right).$$

By assumption, p is an integer, so we can use Exercise 2 from Homework assignment 1 and deduce that

$$\|\left(|u|^{p-1}u - |v|^{p-1}v\right)(\tau)\|_{H^s}$$

$$\leq C\|u(\tau)-v(\tau)\|_{H^s} \cdot \left(\|u(\tau)\|_{H^s}^{p-1} + \|v(\tau)\|_{H^s}^{p-1}\right).$$

Hence, we obtain that, for some $M>0$ we have

$$\|u(t)-v(t)\|_{H^s} \leq M \cdot \int_0^t \|u(\tau)-v(\tau)\|_{H^s} \, d\tau \quad \text{for all } t \in [-T,T].$$

By Grönwall's inequality, it follows that

$u = v$ on $[-T,T]$.

⌈ We are using Grönwall's inequality in the following form.

Let $f: [-T,T] \to \mathbb{R}$ be non-negative and continuous and suppose that

$$f(t) \leq A + \int_0^t B(s) f(s) \, ds$$

where $A \geq 0$ and $B: [-T,T] \to \mathbb{R}$ is non-negative and continuous.

Then, we have:

$$f(t) \leq A \exp\Big(\int_0^t B(s) \, ds\Big).$$

<u>Proof:</u> By a limiting argument, it suffices to consider $A>0$.

We can rewrite the assumption as:

$$\frac{d}{dt}\Big(\underbrace{A + \int_0^t B(s) f(s) \, ds}_{>0}\Big) \leq B(t)\Big(\underbrace{A + \int_0^t B(s) f(s) \, ds}_{>0}\Big)$$

$$\Rightarrow \frac{d}{dt} \log\Big(A + \int_0^t B(s) f(s) \, ds\Big) \leq B(t)$$

We integrate in t and obtain

$$\log\Big(A + \int_0^t B(s) f(s) \, ds\Big) \leq \log(A) + \int_0^t B(s) \, ds$$

Exponentiating, we obtain the claim. □ ⌋

Persistence of Regularity

We consider

$$\begin{cases} i u_t + \Delta u = \pm |u|^{p-1} u \\ u|_{t=0} = \Phi \end{cases}$$

with $1 < p < 1 + \frac{4}{n}$ (L^2-subcritical).
We assume that $\Phi \in H^\sigma(\mathbb{R}^n)$ for some $\sigma > 0$.
By the results from last week, we know that
$(*)$ has a solution on the time interval $[-T, T] = I$
for $T \sim \|\Phi\|_{L^2}^{-\beta}$; $\beta > 0$ and the solution belongs
to $\mathcal{S}(I \times \mathbb{R}^n)$.
The natural question to ask is whether the
norm $\|\cdot\|_{\mathcal{S}^\sigma(I \times \mathbb{R}^n)}$ given by

$$\|u\|_{\mathcal{S}^\sigma(I\times\mathbb{R}^n)} := \| |\nabla|^\sigma u \|_{\mathcal{S}(I\times\mathbb{R}^n)} + \|u\|_{\mathcal{S}(I\times\mathbb{R}^n)}$$

is finite.
In other words, does $(*)$ propagate regularity?

The answer is yes.
We will sketch the argument now.
Taking derivatives, it follows that, on $I \times \mathbb{R}^n$
we have:

$$|\nabla|^\sigma u(t) = S(t)|\nabla|^\sigma \Phi \mp i \int_0^t S(t-\tau) |\nabla|^\sigma (|u|^{p-1}u)(\tau) d\tau$$

Working with the same admissible pair
(q, r) (where $r' = \frac{r}{p}$) as before, it follows
that

$$\| |\nabla|^{\sigma} u \|_{S(I\times\mathbb{R}^n)} \lesssim \| |\nabla|^{\sigma} \Phi \|_{L^2(\mathbb{R}^n)} + T^{\vartheta} \| |\nabla|^{\sigma} (|u|^{p-1} u) \|_{L_t^{\frac{q}{p}} L_x^{\frac{r}{p}} (I\times\mathbb{R}^n)}$$

for $\vartheta > 0$.

We can use a Leibniz rule and formally estimate this by

$$\lesssim \|\Phi\|_{\dot{H}^{\sigma}} + T^{\vartheta} \| |\nabla|^{\sigma} u \|_{L_t^q L_x^r (I\times\mathbb{R}^n)} \cdot \|u\|^{p-1}_{L_t^q L_x^r (I\times\mathbb{R}^n)}$$

$$\lesssim \|\Phi\|_{\dot{H}^{\sigma}} + T^{\vartheta} \|u\|_{S^{\sigma}(I\times\mathbb{R}^n)} \cdot \|u\|^{p-1}_{S(I\times\mathbb{R}^n)}$$

$$\lesssim \|\Phi\|_{\dot{H}^{\sigma}} + T^{\vartheta} \|u\|_{S^{\sigma}(I\times\mathbb{R}^n)} \cdot \|\Phi\|^{p-1}_{L^2}$$

We choose T s.t. $T^{\vartheta} \|\Phi\|^{p-1}_{L^2}$ is sufficiently small.

In particular, if $\|u\|_{S^{\sigma}(I\times\mathbb{R}^n)} < \infty$

we obtain

$$\|u\|_{S^{\sigma}(I\times\mathbb{R}^n)} \lesssim \|\Phi\|_{\dot{H}^{\sigma}}.$$

In order to show that $\|u\|_{S^{\sigma}(I\times\mathbb{R}^n)}$ is indeed finite, we would need to redo the fixed-point argument used to establish local well-posedness.

In doing so, one should use a nontrivial fact from harmonic analysis:
A closed ball in $\mathcal{S}^6$ is complete with respect to the metric from $\mathcal{S}$. (c.f. Cazenave, Theorem 1.2.5.)
In other words, for a specific choice of $C>0$, one considers

$$\mathcal{A} := \{ u ; \|u\|_{\mathcal{S}} \leq C \|\Phi\|_{\dot{H}^6}, \|u\|_{\mathcal{S}} \leq C\|\Phi\|_{L^2} \}$$

and one shows that

$$Lv := S(t)\Phi \mp i \int_0^t S(t-\tau) |v|^{p-1} v (\tau) d\tau$$

maps $\mathcal{A}$ to itself and is a contraction wrt. $\|\cdot\|_{\mathcal{S}}$ for T chosen sufficiently small depending on $\|\Phi\|_{L^2}$.
Note that this T does not depend on $\|\Phi\|_{\dot{H}^6}$.

<u>Important:</u>

If we want to estimate
$\| |\nabla|^6 (|u|^{p-1}u - |v|^{p-1}v) \|_{L_t^{\frac{q}{p}} L_x^{\frac{r}{p}}}$, it is only possible to bound this by
$C \|u-v\|_{\mathcal{S}^6} \cdot (\|u\|_{\mathcal{S}^6}^{p-1} + \|v\|_{\mathcal{S}^6}^{p-1})$.
This is why we need to look at contractions wrt. $\|\cdot\|_{\mathcal{S}}$.

• By similar arguments, it can be shown that, in the H^1-subcritical regime, the time of local existence for initial data $\Phi \in H^\sigma$, $\sigma \geq 1$ is given by $T \sim \|\Phi\|_{H^1}^{-\beta}$; $\beta > 0$.

3.2: Global well-posedness theory (on $\mathbb{R}^n$).

• Global well-posedness follows from local well-posedness by putting solutions together, provided that we have appropriate a priori bounds.

1. The L^2-subcritical regime $1 < p < 1 + \frac{4}{n}$.

We obtain global well-posedness in H^s for $s \geq 0$ by using conservation of mass and persistence of regularity.

⇝ Let us explain the conservation of mass in full rigor.

Namely, suppose that $\Phi \in H^\infty(\mathbb{R}^n) = \bigcap_{s \geq 0} H^s(\mathbb{R}^n)$.

Then, by persistence of regularity, the local-in-time solution u with initial data Φ belongs to $L^\infty([-T,T], H^\infty)$.

For such solutions, the formal calculation showing conservation of mass is rigorous, i.e.

$M(u(t)) = M(\Phi)$ for all $t \in [-T,T]$.

For general $\Phi \in L^2$, we can find (Φ_n) a sequence in $H^\infty(\mathbb{R}^n)$ such that
$\Phi_n \to \Phi$ in L^2.
It follows that
$M(\Phi_n) \to M(\Phi)$. Let $u_n :=$ time evolution of Φ_n.
By local well-posedness, we know that also
$u_n(t) \to u(t)$ in L^2 for $t \in [-T, T]$.
Hence
$$M(u_n(t)) \to M(u(t))$$
Since $M(u_n(t)) = M(\Phi)$, it follows that indeed
$$M(u(t)) = M(\Phi).$$

2. The H^1-subcritical regime

We consider $\Phi \in H^1$; and $\begin{cases} p < 1 + \frac{4}{n-2} & \text{for } n \geq 3 \\ p < \infty & \text{for } n = 1, 2. \end{cases}$

- Here, we need to use conservation of energy.
- In order to do this, we need to study the energy functional more carefully.

<u>Claim:</u> If $s_c = \frac{n}{2} - \frac{2}{p-1} \leq 1$, then the energy functional
$$E(f) = \frac{1}{2}\int |\nabla f|^2 dx \pm \frac{1}{p+1} \int |f|^{p+1} dx$$
is well-defined and continuous on $H^1(\mathbb{R}^n)$.

Proof: One can rewrite the assumption as

$$-\frac{1}{p+1}+\frac{1}{2}\leq\frac{1}{n}.$$

i.e. $\frac{1}{p+1}\geq\frac{1}{2}-\frac{1}{n}$, so $\frac{1}{p+1}=\frac{1}{2}-\frac{s_0}{n}$ for some $s_0\leq 1$.
By Sobolev embedding, we hence obtain

$$H^1(\mathbb{R}^n)\hookrightarrow L^{p+1}(\mathbb{R}^n).$$

The claim now follows.

- Conservation of energy is again proved by an approximation argument.

Note that

$$\frac{n}{2}-\frac{2}{p-1}\leq 1$$

$$\Leftrightarrow\ p\leq 1+\frac{4}{n-2}.$$

In what follows, we will in fact consider the full H^1-subcritical regime with initial data $\Phi\in H^1$.

- The defocusing problem $\Rightarrow$ Global well-posedness is immediate.

· We now consider the <u>focusing</u> problem, $\mu = -1$.
Here, it is no longer clear whether we have uniform bounds on $\|u(t)\|_{H^1}$.

We need to consider several cases.

① $\underline{p < 1 + \frac{4}{n}}$. ($L^2$-subcritical)

In this case, we obtain a global solution in L^2.
By persistence of regularity, if the initial data belongs to H^1, then the solution $u(t)$ will belong to H^1 for (almost) all t.
a priori, iterating the estimate

$$\|u(t+T)\|_{H^1} \leq C\|u(t)\|_{H^1}$$

(for T = time of local existence)
we obtain an exponential bound

$$\|u(t)\|_{H^s} \leq C_1 e^{C_2|t|} \text{ for some } C_1, C_2 > 0.$$

One can improve this bound.

We use the Gagliardo - Nirenberg - Sobolev inequality to obtain

$$\|f\|_{L^{p+1}} \leq C\|\nabla f\|_{L^2}^{\theta} \cdot \|f\|_{L^2}^{1-\theta}$$

for $\theta \in [0,1]$ chosen s.t.

$$\frac{1}{p+1} = \frac{1}{2} - \frac{\theta}{n} \quad ; \text{ observe that}$$

$$\Theta = n \cdot \left(\frac{1}{2} - \frac{1}{p+1}\right) < n \cdot \left(\frac{1}{2} - \frac{1}{2+\frac{4}{n}}\right) = \frac{1}{1+\frac{2}{n}} < 1$$

In particular,

$$(p+1) \cdot \Theta < \frac{2+\frac{4}{n}}{1+\frac{2}{n}} = 2.$$

Hence:

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t)|^2 dx - \frac{1}{p+1}\int |u(t)|^{p+1} dx$$

$$\geq \frac{1}{2}\|\nabla u(t)\|^2_{L^2_x} - C\|u(t)\|^{(p+1)\cdot(1-\Theta)}_{L^2_x} \cdot \|\nabla u(t)\|^{(p+1)\cdot\Theta}_{L^2_x}$$

for some $C > 0$.

Let $\Psi(t) := \|\nabla u(t)\|^2_{L^2_x}$

Using conservation of mass and energy, it follows that

$$\Psi(t) \leq C\tilde{E} + CM^r \cdot \Psi(t)^{\Theta_1},$$

for some $C > 0$.

Here $\tilde{E} := \frac{1}{2}\|\nabla \Phi\|^2_{L^2}$, $M := M(\Phi) \in \mathbb{R}$,

$r := \frac{(p+1)\cdot(1-\Theta)}{2} \in \mathbb{R}^+$

$\Theta_1 := \frac{(p+1)\cdot\Theta}{2} \in (0,1)$.

The above estimate then implies that $\Psi(t)$ is bounded.

Here, we use the <u>general bootstrap method</u>:

If $F:[0,+\infty)\to[0,+\infty)$ is continuous such that

$$F(t)\leq A+B(F(t))^{\Theta}, \text{ with } A,B\geq 0,\ \Theta\in(0,1)$$

and

$$F(0)\leq A+B^{\frac{1}{1-\Theta}},$$

then

$$F(t)\leq C(\Theta)\cdot\left(A+B^{\frac{1}{1-\Theta}}\right).$$

· In order to prove this fact, let $\mathcal{H}_K(t)$ denote the assumption that $F(t)\leq K\cdot(A+B^{\frac{1}{1-\Theta}})$ for some $K>0$.

We note that $\mathcal{H}_1(0)$ holds by assumption.

· The claim follows if we show that there exist $K>L>1$ depending on Θ such that $\mathcal{H}_K(t)\Rightarrow\mathcal{H}_L(t)$.

Then, by continuity $\mathcal{H}_L(t)$ will hold for all $t\geq 0$.

· Suppose now that $\mathcal{H}_K(t)$ holds.

$$F(t)\leq K\cdot(A+B^{\frac{1}{1-\Theta}})$$

Then

$$F(t)\leq A+B(F(t))^{\Theta}$$

$$\leq A+B\left(K\cdot(A+B^{\frac{1}{1-\Theta}})\right)^{\Theta}$$

$$= A + BK^{\theta}\cdot\left(A + B^{\frac{1}{1-\theta}}\right)^{\theta}$$

$$\leq A + BK^{\theta}\cdot\left(A^{\theta} + B^{\frac{\theta}{1-\theta}}\right)$$

$$= A + BK^{\theta}\cdot A^{\theta} + BK^{\theta}\cdot B^{\frac{\theta}{1-\theta}}$$

$$= A + B^{\frac{1}{1-\theta}}\cdot K^{\theta} + \left(B^{\frac{1}{1-\theta}}\right)^{(1-\theta)}(KA)^{\theta}$$

$$\leq A + K^{\theta}\cdot B^{\frac{1}{1-\theta}} + (1-\theta)B^{\frac{1}{1-\theta}} + \theta KA$$

$$\leq (1+\theta K)A + \left(K^{\theta} + 1 - \theta\right)B^{\frac{1}{1-\theta}}$$

So:

$$F(t) \leq (1+\theta K)A + \left(K^{\theta} + (1-\theta)\right)B^{\frac{1}{1-\theta}}.$$

We choose K sufficiently large such that

$$\begin{cases} (1+\theta K) < K \\ \left(K^{\theta} + (1-\theta)\right) < K. \end{cases}$$

Note that K then depends on θ.

We can then let

$$L := \max\left\{C(\theta)\cdot(1+\theta K),\ C(\theta)\cdot\left(K^{\theta} + (1-\theta)\right)\right\}.$$

② $p = 1 + \frac{4}{n}$: L^2-critical regime

A. Weinstein 1983*: Sharp Gagliardo-Nirenberg inequality:

$$\mathcal{J}(v) := \frac{\left(\int|\nabla v|^2\right)\cdot\left(\int|v|^2\right)^{\frac{4}{n}}}{\int|v|^{2+\frac{4}{n}}}$$ is minimized

on the family

*A. Weinstein, "Nonlinear Schrödinger equation and sharp interpolation estimates", CMP no 87, 1983, vol 4

$$\lambda_0^{n/2} Q(\lambda_0 x + x_0) e^{i\gamma_0}; \quad \lambda_0 > 0, \; x_0 \in \mathbb{R}^n, \; \gamma_0 \in \mathbb{R}$$

where Q is the unique positive radial solution of

$$\begin{cases} \Delta Q - Q + Q^{1+\frac{4}{n}} = 0 \\ Q(r) \to 0 \text{ as } r \to +\infty \end{cases}$$

By multiplying the equation for Q with $\frac{n}{2}Q + x\nabla Q$ and integrating by parts, one can deduce that $E(Q)=0$.

In particular, $\frac{1}{2+\frac{4}{n}} \int |Q|^{2+\frac{4}{n}} = \frac{1}{2}\int |\nabla Q|^2$. $(\square)$

By the minimization property, we have:

$$E(v) \geq \frac{1}{2}\left[\int |\nabla v|^2 - \frac{\left(\int |\nabla v|^2\right)\cdot\left(\int |v|^2\right)^{\frac{4}{n}}}{\left(\int |\nabla Q|^2\right)\cdot\left(\int |Q|^2\right)^{\frac{4}{n}}} \cdot \int |Q|^{2+\frac{4}{n}} \cdot \frac{2}{2+\frac{4}{n}}\right]$$

by $(\square)$

$$\geq \frac{1}{2}\int\left[|\nabla v|^2 - \left(\frac{\|v\|_{L^2}^2}{\|Q\|_{L^2}^2}\right)^{\frac{4}{n}}\right].$$

Thus, we can obtain a uniform bound on $\|u(t)\|_{H^1}$ (and hence global well-posedness in H^1) provided that $\|\Phi\|_{L^2} < \|Q\|_{L^2}$.

- Recall that when $\|\Phi\|_{L^2} = \|Q\|_{L^2}$, one can construct blow-up solutions.

③ $1+\frac{4}{n} < p < 1+\frac{4}{n-2}$

The Gagliardo-Nirenberg-Sobolev embedding inequality gives us that

$$\|f\|_{L^{p+1}} \leq C \|\nabla f\|_{L^2}^{\theta} \cdot \|f\|_{L^2}^{1-\theta}$$

where

$$\frac{1}{p+1} = \frac{1}{2} - \frac{\theta}{n}. \quad \text{Hence } \theta = \frac{n(p-1)}{2(p+1)}.$$

Recall the general statement:

If $1 < p < q \leq \infty$, $s > 0$ and $\frac{1}{q} = \frac{1}{p} - \frac{\theta s}{n}$ for some $\theta \in (0,1)$, then

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C(n,p,q,s) \, \|u\|_{L^p(\mathbb{R}^n)}^{1-\theta} \cdot \||\nabla|^s u\|_{L^p(\mathbb{R}^n)}^{\theta}.$$

In particular, it follows that the functional

$$\mathcal{F}(v) := \frac{\left(\int |v|^2\right)^{1-\frac{(n-2)(p-1)}{4}} \cdot \left(\int |\nabla v|^2\right)^{\frac{n(p-1)}{4}}}{\int |v|^{p+1}} \geq C$$

Since $p > 1+\frac{4}{n}$, it follows that

$$\frac{n(p-1)}{4} > 1.$$

Since $p < 1+\frac{4}{n-2}$, it follows that

$$1 - \frac{(n-2)(p-1)}{4} > 0.$$

In particular, it follows that

$$E(v) \geq \frac{1}{2}\int|\nabla v|^2 - C\left(\int|v|^2\right)^{\gamma}\cdot\left(\int|\nabla v|^2\right)^{1+\beta}$$

for some $\gamma, \beta > 0$.

We choose $\varepsilon > 0$ small.

If $\|\Phi\|_{L^2}^{2\gamma} = \varepsilon$, then by conservation of mass and energy, it follows that

$$E(\Phi) = E(u(t)) \geq \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - C\, M(u(t))^{\gamma}\cdot\|\nabla u(t)\|_{L^2}^{2(1+\beta)}$$

$$= \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - C\varepsilon\|\nabla u(t)\|_{L^2}^{2(1+\beta)}$$

Let $y(t) := \frac{1}{2}\|\nabla u(t)\|_{L^2}^2$, $K := \frac{1}{2}\|\nabla\Phi\|_{L^2}^2$.
(a continuous function of t)

It follows that

$$y(t) \leq K + C\varepsilon\big(y(t)\big)^{1+\beta}. \qquad (\Delta)$$

Note that $y(0) = K$.

We show that $y(t) \leq 3K \Rightarrow y(t) \leq 2K$ for $\varepsilon > 0$ small.

Namely, if $y(t) \leq 3K$, then (Δ) implies that

$$y(t) \leq K + C\varepsilon(3K)^{1+\beta}$$

$$= K + \varepsilon\cdot(C\, 3^{1+\beta}K^{\beta})\cdot K$$

- We choose ε such that

$$\varepsilon \cdot (C 3^{1+\beta} K^{\beta}) \leq 1 .$$

If $K = \frac{1}{2} \|\nabla \Phi\|_{L^2}^2 \leq C$, we can choose a uniform ε.

In general, the threshold for ε depends on $\|\nabla \Phi\|_{L^2}$.

$\rightsquigarrow$ If $\|\Phi\|_{H^1}$ is sufficiently small, we obtain a global H^1 solution with uniform bounds on $\|u(t)\|_{H^1}$.

Remark: In the H^1-critical setting $p = 1 + \frac{4}{n-2}$, one has to argue a bit differently.

In order to establish global well-posedness, one needs to assume the a priori spacetime bound

$$\|u\|_{L^{\frac{2(n+2)}{n-2}}_{[-T,T]} L_x^{\frac{2(n+2)}{n-2}}} \leq C$$

for all local solutions.

This spacetime norm is used to bound the S^1 norm.

Remark: Blow-up solutions

R. Glassey 1977.

For $p \in [1 + \frac{4}{n}, 1 + \frac{4}{n-2})$, one can construct solutions to the focusing NLS blowing up in H^1

provided that $E(\Phi) < 0$.

This is done by showing that

$$\frac{d^2}{dt^2} \int |x|^2 |u(x,t)|^2 dx = 16 E(\Phi) < 0$$

("Virial identity")

and using

$$\|f\|_{L^2}^2 \leq \frac{2}{n} \|x f\|_{L^2} \cdot \|\nabla f\|_{L^2}$$

(Weyl - Heisenberg inequality).

The precise details of this construction will be outlined on the homework assignment.

3.3: Low regularity well-posedness

- We consider the following Cauchy problem.

$$(*) \begin{cases} iu_t + \Delta u = |u|^2 u & \text{on } \mathbb{R}^2_x \times \mathbb{R}_t \\ u|_{t=0} = \Phi \in H^s(\mathbb{R}^2) \end{cases}$$

We note that $p = 3 = 1 + \frac{4}{2} \rightsquigarrow$ The problem is <u>L^2-critical.</u>

We obtain global well-posedness in H^1 (if $s \geq 1$).
Also, we obtain local well-posedness in L^2 in the critical sense.

<u>Question:</u> What happens when $s \in (0,1)$?

We outline two methods based on
"<u>almost conservation laws</u>"

① <u>Bourgain's Fourier Multiplier Method</u>

- due to <u>Bourgain (1998)</u>.

② <u>I - method</u>

- due to the "<u>I - team</u>": <u>Colliander - Keel - Staffilani - Takaoka - Tao. (early 2000s)</u>.

I) Bourgain's Fourier Multiplier Method

[we follow J. Bourgain: "Refinements of Strichartz's inequality and applications to 2D-NLS with critical nonlinearity", IMRN 5 (1998), 253 – 283.]

· We work with $s \in (0,1)$. Take initial data $\Phi \in H^s(\mathbb{R}^2)$.
· Let us take $N > 1$ to be a parameter (frequency threshold).
We write

$$\Phi = \varphi_0 + \Psi_0 \text{, where}$$

$$\underbrace{\widehat{\varphi_0} = \widehat{\Phi} \cdot \mathbb{1}_{\{|\xi| \leq N\}}}_{\text{Low frequency}}, \quad \underbrace{\widehat{\Psi_0} = \widehat{\Phi} \cdot \mathbb{1}_{\{|\xi| > N\}}}_{\text{High frequency}}$$

We obtain

$$\|\varphi_0\|_{H^1} \lesssim N^{1-s} \cdot \|\varphi_0\|_{H^s} \leq N^{1-s} \|\Phi\|_{H^s} \lesssim N^{1-s} \quad (3.1)$$

Namely

$$\|\varphi_0\|_{H^1} \sim \left(\int |\widehat{\varphi_0}(\xi)|^2 \cdot (1+|\xi|)^2 d\xi \right)^{1/2}$$

$$= \left(\int_{|\xi| \leq N} |\widehat{\Phi}(\xi)|^2 \cdot (1+|\xi|)^2 d\xi \right)^{1/2}$$

$$\lesssim N^{1-s} \left(\int_{|\xi| \leq N} |\widehat{\Phi}(\xi)|^2 \cdot (1+|\xi|)^{2s} d\xi \right)^{1/2}$$

$$\leq N^{1-s} \|\Phi\|_{H^s}.$$

By the Gagliardo - Nirenberg - Sobolev inequality, we have

$$\|\varphi_0\|_{L^4}^4 \lesssim \|\varphi_0\|_{L^2}^2 \cdot \|\varphi_0\|_{\dot{H}^1}^2.$$

In particular,

$$\|\varphi_0\|_{L^4}^4 \lesssim \|\Phi\|_{L^2}^2 \cdot \|\varphi_0\|_{H^1}^2 \overset{\text{by (3.1)}}{\lesssim} N^{2(1-s)} \|\Phi\|_{H^s}^4 \lesssim N^{2(1-s)} \quad (3.2)$$

(3.1) and (3.2) imply that

$$E(\varphi_0) \lesssim N^{2(1-s)}. \quad (3.3)$$

By H^1 global-wellposedness we can find a global H^1 solution of the following Cauchy problem.

$$(**) \begin{cases} i\partial_t u_0 + \Delta u_0 = |u_0|^2 u_0 \\ u_0|_{t=0} = \varphi_0 \end{cases}$$

$$E(u_0(t)) = E(\varphi_0).$$

(Low frequency evolution).

We can choose $\delta \lesssim \|\varphi_0\|_{H^1}^{-2}$ small enough such that

$\|u_0\|_{L^4_{[0,\delta]} L^4_x} = \varepsilon$ is sufficiently small (to be determined later).

(Recall that $\|\varphi_0\|_{H^1}^{-2} \gtrsim N^{-2(1-s)}$.)

Throughout what follows, we will consider

$$\underline{\underline{\delta \sim N^{-2(1-s)-}}}.$$

· We write $u_0 = u + v$. Then v has to solve the Cauchy problem of the difference equation.

$$(***)\begin{cases} i v_t + \Delta v = |u_0+v|^2(u_0+v) - |u_0|^2 u_0 \\ v|_{t=0} = \Psi_0 \end{cases}$$

$$M(v(t)) = M(\Psi_0)$$

(High frequency evolution).

Goal: $v = S(t)\Psi_0 + w$, where for $t \in [0,\delta]$ we have $w(t) \in H^1$. "Smoothing occurs in the Duhamel term."

More precisely, choosing $\varepsilon > 0$ small and $N > 1$ large we have that the map

$$Lv := S(t)\Psi_0 - i\int_0^t S(t-\tau)\left(|u_0+v|^2(u_0+v) - |u_0|^2 u_0\right)(\tau)\,d\tau$$

is a contraction on $\mathcal{A} := \{ v \in L^4_I L^4_x : \|v\|_{L^4_I L^4_x} \le 2C_0 N^{-s} \}$

Here $I = [0,\delta]$ and $C_0 > 0$ is such that

$$\|S(t)\Psi_0\|_{L^4_I L^4_x} \lesssim C_0 N^{-s}. \quad \text{(by Strichartz's estimate).}$$

By the inhomogeneous Strichartz estimate, we have

$$\Big\| \int_0^t S(t-\tau)\big(\underbrace{|u_0+v|^2(u_0+v)^2 - |u_0|^2 u}_{2|u_0|^2 v + u_0^2\bar{v} + 2u_0|v|^2 + \bar{u}_0 v_2 + |v|^2 v}\big)(\tau)\,d\tau \Big\|_{L^4_I L^4_x}$$

$$\lesssim \|u_0^2 v\|_{L_I^{4/3} L_x^{4/3}} + \|u_0 v^2\|_{L_I^{4/3} L_x^{4/3}} + \|v^3\|_{L_I^{4/3} L_x^{4/3}}$$

$$\leq \|u_0\|^2_{L_I^4 L_x^4} \cdot \|v\|_{L_I^4 L_x^4} + \|u_0\|_{L_I^4 L_x^4} \cdot \|v\|^2_{L_I^4 L_x^4} + \|v\|^3_{L_I^4 L_x^4}$$

$$\leq C\varepsilon \cdot (\|v\|_{L_I^4 L_x^4} + \|v\|^2_{L_I^4 L_x^4}) + \|v\|^3_{L_I^4 L_x^4}.$$

Hence

$$\|Lv\|_{L_I^4 L_x^4} \leq C_0 N^{-s} + C\varepsilon \cdot (\|v\|_{L_I^4 L_x^4} + \|v\|^2_{L_I^4 L_x^4}) + \|v\|^3_{L_I^4 L_x^4}$$

also

$$\|Lv_1 - Lv_2\|_{L_I^4 L_x^4}$$

$$\leq C\varepsilon \cdot (\|v_1 - v_2\|_{L_I^4 L_x^4} + (\|v_1\|_{L_I^4 L_x^4} + \|v_2\|_{L_I^4 L_x^4}) \cdot \|v_1 - v_2\|_{L_I^4 L_x^4})$$

$$+ C \cdot (\|v_1\|^2_{L_I^4 L_x^4} + \|v_2\|^2_{L_I^4 L_x^4}) \cdot \|v_1 - v_2\|_{L_I^4 L_x^4}.$$

Thus, we obtain a fixed point v for L satisfying $\|v\|_{L_I^4 L_x^4} \leq 2C_0 N^{-s}$ if $\varepsilon > 0$ is chosen sufficiently small and if $N > 1$ is chosen sufficiently large.

Let $w := v - S(t)\Psi_0$.

We show that w has the following properties.

$$\left.\begin{aligned} &i)\ \|w(t)\|_{L^2} \leq CN^{-s}. \\ &ii)\ \|w(t)\|_{H^1} \leq CN^{(1-2s)+} \end{aligned}\right\} \quad (\star)$$

Let us assume $(\star)$ for the moment.

For $t \in [0, \delta]$, we can write

$$u(t) = u_0(t) + v(t) = \underbrace{u_0(t) + w(t)}_{\in H^1} + \underbrace{S(t)\Psi_0}_{\substack{\text{linear}\\ \text{evolution}}}$$

For $t_1 = \delta$, we write

$u(t_1) = \varphi_1 + \Psi_1$ where

$$\begin{cases} \varphi_1 := u_0(t_1) + w(t_1) & \in H^1 \\ \Psi_1 := S(t)\Psi_0 & \text{supported on } |\xi| > N. \end{cases}$$

"Redistribution of the initial data".

We write

$$E(\varphi_1) = E(\varphi_0) + [E(\underbrace{u_0(t_1) + w(t_1)}_{\varphi_1}) - \underbrace{E(u_0(t_1))}_{= E(\varphi_0)}]$$

Now,

$$\begin{aligned} &|E(u_0(t_1) + w(t_1)) - E(u_0(t_1))| \\ &\lesssim (\|u(t_1)\|_{\dot{H}^1} + \|w(t_1)\|_{\dot{H}^1}) \cdot \|w(t)\|_{\dot{H}^1} \\ &+ \|(|u_0(t_1)| + |w(t_1)|)^3 \cdot |w(t_1)|\|_{L^1}. \qquad (3.4) \end{aligned}$$

Let us estimate

$$\| |u_0(t_1)|^3 \cdot |w(t_1)| \|_{L^1} \leq \|u_0(t_1)\|^3_{L^4} \cdot \|w(t_1)\|_{L^4}$$

$$\overset{GNS}{\lesssim} \|u_0(t_1)\|^{\frac{3}{2}}_{L^2} \cdot \|u_0(t_1)\|^{\frac{3}{2}}_{H^1} \cdot \|w(t_1)\|^{\frac{1}{2}}_{L^2} \cdot \|w(t_1)\|^{\frac{1}{2}}_{H^1}$$

$$\lesssim \underbrace{1^{\frac{3}{2}}}_{\substack{\text{by conservation}\\ \text{of mass}}} \cdot \underbrace{N^{\frac{3}{2}(1-s)}}_{\substack{\text{by (3.3)}\\ \text{+ conservation}\\ \text{of energy}}} \cdot \underbrace{N^{-\frac{s}{2}}}_{\text{by } (\star)(i)} \cdot \underbrace{N^{\frac{1}{2}(1-2s)+}}_{\text{by } (\star)(ii)} \lesssim N^{(2-3s)+}$$

The other terms in (3,4) are estimated analogously.

We obtain

$$\underline{E(\varphi_1) \leq E(\varphi_0) + C N^{(2-3s)+}}. \qquad (3.5)$$

- The energy is "almost conserved", i.e. it cannot grow too fast.

Recalling the construction of δ, it follows that (3.5) can be iterated as long as $E(\varphi_j) \lesssim N^{2(1-s)}$ (this guarantees $\|\varphi_j\|_{H^1} \lesssim N^{1-s}$).

- In order to reach time $T>0$, we need to take

$$\sim \frac{T}{\delta} \sim T \cdot N^{2(1-s)+} \text{ timesteps.}$$

(recall that $\delta \sim N^{-2(1-s)-}$).

We hence want

$$T N^{2(1-s)+} \cdot N^{(2-3s)+} \lesssim N^{2(1-s)}$$

$$\Rightarrow T \lesssim N^{(3s-2)-}.$$

Hence, for fixed $T > 0$ (large), we can take the frequency threshold

$$N \sim T^{\frac{1}{3s-2}+}, \text{ as long as } \underline{s > \tfrac{2}{3}}.$$

Hence, we obtain global well-posedness of $(*)$ in H^s provided that $s > \frac{2}{3}$.

Moreover, $u - S(t)\Phi \in H^1$. ("Smoothing in the nonlinearity.")

- Let us now outline the proof of $(\star)$.

We first prove i)

By the inhomogeneous Strichartz estimate we can deduce

$$\|w\|_{L^\infty_I L^2_x} \lesssim \|u_0^2 v\|_{L^{\frac{4}{3}}_I L^{\frac{4}{3}}_x} + \|u_0 v^2\|_{L^{\frac{4}{3}}_I L^{\frac{4}{3}}_x}$$

$$+ \|v^3\|_{L^{\frac{4}{3}}_I L^{\frac{4}{3}}_x}$$

$$\leq \|u_0\|^2_{L^4_I L^4_x} \cdot \|v\|_{L^4_I L^4_x} + \|u_0\|_{L^4_I L^4_x} \cdot \|v\|^2_{L^4_I L^4_x}$$

$$+ \|v\|^3_{L^4_I L^4_x} \lesssim (N^{-s}. \text{ (by construction of } u_0 \text{ and } v).$$

The proof of ii) is more involved.
It relies on the __Improved Bilinear Strichartz Estimate__
(Bourgain 1998)

Let $\Psi_1, \Psi_2 \in L^2(\mathbb{R}^2)$ be such that
$\operatorname{supp} \widehat{\Psi_j} \subseteq \{\xi,\ |\xi| \sim N_j\}$, where $\left[\begin{array}{l} N_1, N_2 \in 2^{\mathbb{N}} \\ N_1 \leq N_2 \end{array}\right.$ are dyadic integers.

Then we have

$$\| S(t)\Psi_1 \cdot S(t)\Psi_2 \|_{L^2_{t,x}} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \cdot \|\Psi_1\|_{L^2_x} \cdot \|\Psi_2\|_{L^2_x}$$

This is an "improved" estimate since, in general, we have for $f_1, f_2 \in L^2(\mathbb{R}^2)$

$$\| S(t)f_1 \cdot S(t)f_2 \|_{L^2_{t,x}}$$

$$\leq \| S(t)f_1 \|_{L^4_{t,x}} \cdot \| S(t)f_2 \|_{L^4_{t,x}}$$

$$\leq C \|f_1\|_{L^2_x} \cdot \|f_2\|_{L^2_x} .$$

__Observation:__ For $f_1, f_2 \in L^2(\mathbb{R}^2)$, $s \geq 0$, we can write

$$\| D^s (S(t)f_1 \cdot S(t)f_2) \|_{L^2_{t,x}}$$

$$= \Big\| \sum_{N_1, N_2} D^s (S(t)P_{N_1}f_1 \cdot S(t)P_{N_2}f_2) \Big\|_{L^2_{t,x}}$$

$$\lesssim \Big\| \sum_{\substack{N_1, N_2 \\ N_1 \sim N_2}} D^s \big(S(t) P_{N_1} f_1 \cdot S(t) P_{N_2} f_2 \big) \Big\|_{L^2_{t,x}}$$

$$+ \Big\| \sum_{\substack{N_1, N_2 \\ N_1 \gg N_2}} D^s \big(S(t) P_{N_1} f_1 \cdot S(t) P_{N_2} f_2 \big) \Big\|_{L^2_{t,x}}$$

$$+ \Big\| \sum_{\substack{N_1, N_2 \\ N_1 \ll N_2}} D^s \big(S(t) P_{N_1} f_1 \cdot S(t) P_{N_2} f_2 \big) \Big\|_{L^2_{t,x}}$$

$$=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad \text{"Paraproduct"}$$

We estimate $\mathrm{I}, \mathrm{II}, \mathrm{III}$ separately.

$$\mathrm{I} \leq \sum_{N_1 \sim N_2} \big\| D^s \big(S(t) P_{N_1} f_1 \cdot S(t) P_{N_2} f_2 \big) \big\|_{L^2_{t,x}}$$

$$\lesssim \sum_{N_1 \sim N_2} N_1^s \cdot \| P_{N_1} f_1 \|_{L^2_x} \cdot \| P_{N_2} f_2 \|_{L^2_x},$$

which by the Cauchy-Schwarz inequality in $N_1 \sim N_2$ is

$$\lesssim \Big(\sum_N N^{2s} \cdot \| P_N f_1 \|^2_{L^2_x} \Big) \cdot \Big(\sum_N \| P_N f_2 \|^2_{L^2_x} \Big)$$

$$\lesssim \| f_1 \|^2_{H^s_x} \cdot \| f_2 \|^2_{L^2_x}$$

[We use $\Big(\sum_N N^{2s} \| P_N f \|^2_{L^2_x} \Big)^{1/2} \lesssim \| f \|_{H^s}$.]

- If $N_1 \gg N_2$, then $\left(S(t)P_{N_1}f_1\, S(t)P_{N_2}f_2\right)^{\wedge}$ is supported on $|\xi| \sim N_1$.

Hence, we have (by L^2 orthogonality in ξ-space)

$$\| D^s \sum_{N_1 \gg N_2} S(t)P_{N_1}f_1 \cdot S(t)P_{N_2}f_2 \|^2_{L^2_{t,x}}$$

$$\sim \sum_{N_1} N_1^{2s} \cdot \| \sum_{N_2 ; N_2 \ll N_1} S(t)P_{N_1}f_1\, S(t)P_{N_2}f_2 \|^2_{L^2_{t,x}}$$

$$\leq \sum_{N_1} N_1^{2s} \cdot \left(\sum_{N_2 \ll N_1} \| S(t)P_{N_1}f_1\, S(t)P_{N_2}f_2 \|_{L^2_{t,x}} \right)^2$$

$$\lesssim \sum_{N_1} N_1^{2s} \cdot \left(\sum_{N_2 \ll N_1} \frac{N_2^{1/2}}{N_1^{1/2}} \cdot \|P_{N_1}f_1\|_{L^2_x} \cdot \|P_{N_2}f_2\|_{L^2_x} \right)^2$$

$$= \sum_{N_1} N_1^{2s} \cdot \|P_{N_1}f_1\|^2_{L^2_x} \cdot \left(\sum_{N_2 \ll N_1} \frac{N_2^{1/2}}{N_1^{1/2}} \cdot \|P_{N_2}f_2\|_{L^2_x} \right)^2$$

By applying the Cauchy-Schwarz inequality in N_2, this quantity is

$$\lesssim \left(\sum_{N_1} N_1^{2s} \cdot \|P_{N_1}f_1\|^2_{L^2_x} \right) \cdot \left(\sum_{N_2} \|P_{N_2}f_2\|^2_{L^2_x} \right)$$

$$\lesssim \|f_1\|^2_{H^s_x} \cdot \|f_2\|^2_{L^2_x}.$$

· If $N_2 \gg N_1$, then $\left(S(t)P_{N_1}f_1 \, S(t)P_{N_2}f_2\right)^\wedge$ is supported on $|\xi| \sim N_2$. Hence

$$\mathrm{III} \lesssim \sum_{N_2} N_2^{2s} \cdot \Big\| \sum_{N_1 \ll N_2} S(t)P_{N_1}f_1 \cdot S(t)P_{N_2}f_2 \Big\|^2_{L^2_{t,x}}$$

$$\leq \sum_{N_2} N_2^{2s} \cdot \Big(\sum_{N_1 \ll N_2} \frac{N_1^{1/2}}{N_2^{1/2}} \cdot \|P_{N_1}f_1\|_{L^2_x} \cdot \|P_{N_2}f_2\|_{L^2_x} \Big)^2$$

$$= \sum_{N_2} N_2^{2s} \cdot \|P_{N_2}f_2\|^2_{L^2_x} \cdot \Big(\sum_{N_1 \ll N_2} \frac{N_1^{1/2}}{N_2^{1/2}} \cdot \|P_{N_1}f_1\|_{L^2_x} \Big)^2$$

<u>Case 1:</u> $0 \leq s < \frac{1}{2}$.

In this case, we write

$$\mathrm{III} \lesssim \sum_{N_2} \|P_{N_2}f_2\|^2_{L^2_x} \cdot \Big(\sum_{N_1 \ll N_2} \frac{N_1^{\frac{1}{2}-s}}{N_2^{\frac{1}{2}-s}} \cdot N_1^s \|P_{N_1}f_1\|_{L^2_x} \Big)^2$$

We apply Cauchy-Schwarz in N_1 and use $\frac{1}{2} - s > 0$ to note that

$$\mathrm{III} \lesssim \Big(\sum_{N_2} \|P_{N_2}f_2\|^2_{L^2_x} \Big) \cdot \Big(\sum_{N_1} N_1^{2s} \cdot \|P_{N_1}f_1\|^2_{L^2_x} \Big)$$

$$\lesssim \|f_1\|^2_{H^s} \cdot \|f_2\|^2_{L^2_x}$$

Case 2: If $s \geq \frac{1}{2}$, we need to argue differently

$$\mathrm{III} \lesssim \Big(\sum_{N_2} N_2^{2s-1} \|P_{N_2} f_2\|_{L^2_x}^2\Big) \cdot \Big(\sum_{N_1} N_1^{1/2} \cdot \|P_{N_1} f_1\|_{L^2_x}\Big)^2$$

$$= \Big(\sum_{N_2} N_2^{2s-1} \|P_{N_2} f_2\|_{L^2_x}^2\Big) \cdot \Big(\sum_{N_1} N_1^{-\varepsilon} \cdot N_1^{1/2+\varepsilon} \cdot \|P_{N_1} f_1\|_{L^2_x}\Big)^2$$

which by Cauchy-Schwarz in N_1 is

$$\lesssim \Big(\sum_{N_2} N_2^{2s-1} \|P_{N_2} f_2\|_{L^2_x}^2\Big) \cdot \Big(\sum_{N_1} N_1^{1+2\varepsilon} \|P_{N_1} f_1\|_{L^2_x}^2\Big)$$

$$= \|f_1\|^2_{H_x^{\frac{1}{2}+\varepsilon}} \cdot \|f_2\|^2_{H_x^{s-\frac{1}{2}}} .$$

Conclusion:

$$\|D^s (S(t)f_1\, S(t) f_2)\|_{L^2_{t,x}} \lesssim \begin{cases} \|f_1\|_{H^s_x} \cdot \|f_2\|_{L^2_x} & \text{if } 0 \leq s < \frac{1}{2} \\ \|f_1\|_{H^s_x} \cdot \|f_2\|_{L^2_x} + \|f_1\|_{H_x^{\frac{1}{2}+\varepsilon}} \cdot \|f_2\|_{H_x^{s-\frac{1}{2}}} & \text{if } s \geq \frac{1}{2}. \end{cases}$$

$X^{s,b}$ spaces

- In the application of the bilinear Strichartz estimate, one needs to consider functions which are not necessarily products of free Schrödinger evolutions.

In order to overcome this difficulty, one works in the following spaces.

- Given $s, b \in \mathbb{R}$, we let $X^{s,b}$ denote the space of $u = u(x,t)$ given by the norm

$$\|u\|_{X^{s,b}} := \|(1+|\xi|)^s (1+|\tau+2\pi|\xi|^2|)^b \cdot \tilde{u}(\xi,\tau)\|_{L^2_{\xi,\tau}},$$

where $\tilde{\ }$ denotes the spacetime Fourier transform.

We note that

$$\|u\|_{X^{s,b}} = \|S(-t)u\|_{H^s_x H^b_t}$$ "Dispersive Sobolev Space"

Recall: $(S(t)f)^{\sim}(\xi) = \delta(\tau + 2\pi|\xi|^2) \cdot \hat{f}(\xi)$ for $f = f(x)$.

Fundamental properties: The $X^{s,b}$ spaces are well-adapted to the study of the Schrödinger equation.

1 (Free evolutions localized in time belong to $X^{s,b}$).

$\eta \in \mathcal{S}(\mathbb{R})$, $f \in H^s$

$$\|\eta(t) S(t) f\|_{X^{s,b}} \lesssim_{\eta, b} \|f\|_{H^s}.$$

② (Relationship to Strichartz estimates)

For admissible pairs (q,r) we have

$$\|u\|_{L^q_t L^r_x} \lesssim \|u\|_{X^{0,\frac{1}{2}+}} .$$

"Transference principle"

We use the Fourier inversion formula to write

$$u(x,t) = \iint e^{2\pi i t\tau + 2\pi i x\xi}\, \tilde{u}(\xi,\tau)\, d\xi\, d\tau$$

$$= \iint e^{2\pi i t(\tau - 2\pi|\xi|^2) + 2\pi i x\xi}\, \tilde{u}(\xi, \tau - 2\pi|\xi|^2)\, d\tau\, d\xi$$

$$= \int e^{2\pi i t\tau} \left(\int e^{-4\pi^2 i t|\xi|^2 + 2\pi i x\xi}\, \tilde{u}(\xi, \tau - 2\pi|\xi|^2)\, d\xi \right) d\tau$$

$$= \int e^{2\pi i t\tau}\, S(t) f_\tau\, d\tau^{*}, \text{ where}$$

$$f_\tau(x) := \int \tilde{u}(\xi, \tau - 2\pi|\xi|^2)\, e^{2\pi i x\xi}\, d\xi$$

* We wrote u as a superposition of modulated free evolutions.

Hence, for (q,r) an admissible pair, we have by Minkowski's inequality that

$$\|u\|_{L^q_t L^r_x} \leq \int_{-\infty}^{+\infty} \| e^{2\pi i t\tau} S(t) f_\tau \|_{L^q_t L^r_x}\, d\tau$$

$$\lesssim \int_{-\infty}^{+\infty} \|f_\tau\|_{L^2_x}\, d\tau = \int_{-\infty}^{+\infty} (1+|\tau|)^{-\frac{1}{2}-\varepsilon} \cdot (1+|\tau|)^{\frac{1}{2}+\varepsilon} \cdot \|f_\tau\|_{L^2_x}\, d\tau$$

$$\lesssim \|(1+|\tau|)^{\frac{1}{2}+\varepsilon} f_\tau\|_{L^2_x L^2_\tau}$$

$$= \|(1+|\tau|)^{\frac{1}{2}+\varepsilon} \widehat{f_\tau}\|_{L^2_\xi L^2_\tau}$$

$$= \|u\|_{X^{0,\frac{1}{2}+}}$$

- By using a bilinear version of this construction (and estimating $\|FG\|_{L^2_{t,x}} \leq \|F\|_{L^4_{t,x}} \cdot \|G\|_{L^4_{t,x}}$), one can show that, for $N_1 \geq N_2$ - dyadic integers

$$\| P_{N_1} u_1 \, P_{N_2} u_2 \|_{L^2_{t,x}} \lesssim \frac{N_2^{1/2}}{N_1^{1/2}} \cdot \|u_1\|_{X^{0,\frac{1}{2}+}} \cdot \|u_2\|_{X^{0,\frac{1}{2}+}}.$$

In fact,

$$\| P_{N_1} u_1 \, P_{N_2} u_2 \|_{L^2_{t,x}} \lesssim \frac{N_2^{1/2}}{N_1^{1/2}} \cdot \|P_{N_1} u_1\|_{X^{0,\frac{1}{2}+}} \cdot \|P_{N_2} u_2\|_{X^{0,\frac{1}{2}+}}$$

(we are now working on $\mathbb{R}^2$).

Consequently, one can argue as before and show that

$$\|u_1 u_2\|_{L^2_{t,x}} \lesssim \begin{cases} \|u_1\|_{X^{s,\frac{1}{2}+}} \cdot \|u_2\|_{X^{0,\frac{1}{2}+}} & \text{if } 0 \leq s < \frac{1}{2}. \\ \|u_1\|_{X^{s,\frac{1}{2}+}} \cdot \|u_2\|_{X^{0,\frac{1}{2}+}} \\ + \|u_1\|_{X^{\frac{1}{2}+,\frac{1}{2}+}} \cdot \|u_2\|_{X^{s-\frac{1}{2},\frac{1}{2}+}} & \text{if } s \geq \frac{1}{2}. \end{cases}$$

This is the type of multilinear estimate used in the proof of ($\star$) (ii).

③ (Inhomogeneous Estimate)

For $\eta \in \mathcal{S}(\mathbb{R})$, $s \in \mathbb{R}$ and $b > \frac{1}{2}$, one has

$$\| \eta(t) \int_0^t S(t-\tau) F(\cdot,\tau) \, d\tau \|_{X^{s,b}} \lesssim_{\eta,b} \|F\|_{X^{s,b-1}}.$$

A way to heuristically see why the latter estimate should be true is to note that, if $G(x,t) := \int_0^t S(t-\tau)F(\cdot,\tau)\,d\tau$, then

$$(i\partial_t + \Delta)G = F$$

$$\Rightarrow \quad \widetilde{G}(\xi,\tau) = -\frac{\widetilde{F}(\xi,\tau)}{2\pi(\tau+2\pi|\xi|^2)}$$

So, we expect to obtain a bound of the type

$$\text{"}\ \|G\|_{X^{s,b}} \lesssim \|F\|_{X^{s,b-1}}\ \text{"}.$$

> Goal: Solve (~~***~~) as a fixed-point equation in $X^{0,\frac{1}{2}+}_{(loc)}$

The I-method

- As before, we fix $N > 1$ to be a frequency threshold.

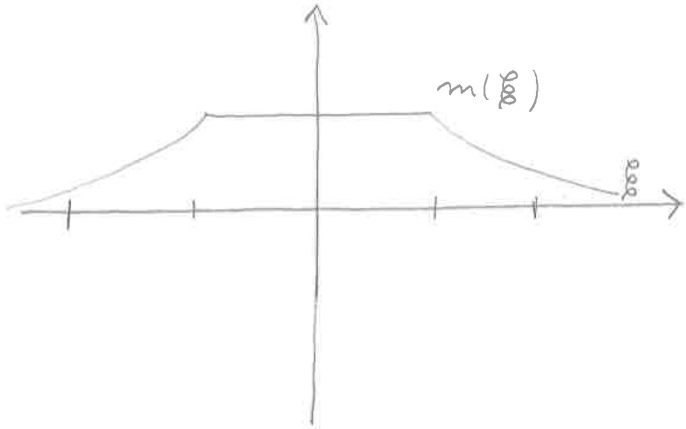
One now considers a smooth projection in frequency.

- Given $s < 1$, $N > 1$, define $I \equiv I_N$ on H^s_x by

$$(If)^\wedge(\xi) := m(\xi)\hat{f}(\xi),$$

where

$$m(\xi) := \begin{cases} 1 & , |\xi| \le N \quad \text{"Identity"} \\ \left(\frac{N}{|\xi|}\right)^{1-s} & , |\xi| > 2N \quad \text{"Integration" (of order } 1-s) \end{cases}$$



Note that $I : H^s_x \to H^1_x$ and

$$\|\varphi\|_{H^s_x} \lesssim \|I\varphi\|_{H^1_x} \lesssim N^{1-s}\|\varphi\|_{H^s_x}.$$

Goal: Show that $E(I\varphi)$ is almost conserved.

Idea: Use symmetry (→ cancelation) and smoothness of m.

→ Work on the Fourier side.

Model calculation:

Conservation of mass by the Fourier transform.

$$\frac{d}{dt}\int |u(x,t)|^2 dx \sim \operatorname{Im}\Big(\int_{\xi_1+\xi_2=0} \xi_1^2 \hat{u}(\xi_1,t)\hat{\bar{u}}(\xi_2,t)\,d\xi_j\Big)$$

$$+\underbrace{\operatorname{Im}\Big(\int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \hat{u}(\xi_1,t)\hat{\bar{u}}(\xi_2,t)\hat{u}(\xi_3,t)\hat{\bar{u}}(\xi_4,t)\,d\xi_j\Big)}_{=0}$$

$$= \frac{1}{2}\operatorname{Im}\Big(\int_{\xi_1+\xi_2=0} (\xi_1^2-\xi_2^2)\hat{u}(\xi_1,t)\hat{\bar{u}}(\xi_2,t)\,d\xi_j\Big) = 0.$$

3.4: The NLS in the periodic setting

Theorem (Bourgain 1993)

Consider the Cauchy problem on $\mathbb{T}^1$

$$(*) \quad \begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1} u \\ u|_{t=0} = \Phi \in H^s(\mathbb{T}^1) \end{cases}$$

where $3 \leq p \leq 5$.

1) If $p=3$, then $(*)$ is locally well-posed in H^s if $s \geq 0$. In particular, by conservation of mass (and persistence of regularity), it is globally well-posed in H^s.

2) If $3 < p \leq 5$, then $(*)$ is locally well-posed in H^s for $s > 0$.
The time of local existence is $T \sim \|\Phi\|_{H^s}^{-\gamma}$ for some $\gamma \equiv \gamma(p) > 0$.

Idea: Recall that:

i) $\|S(t)\Phi\|_{L^4([0,1]\times\mathbb{T}^1)} \leq C\|\Phi\|_{L^2_x}$.

ii) $\|S(t)P_N\Phi\|_{L^4([0,1]\times\mathbb{T}^1)} \leq CN^{\varepsilon}\|P_N\Phi\|_{L^2_x}$.

- One wants to use i) to show part 1), i.e. one wants to construct a fixed point of the map
$Lv := e^{it\Delta} \mp i \int_0^t S(t-\tau)\, |u|^2 u\,(\tau)\, d\tau$
in $L^4([0,T]\times\mathbb{T}^1)$ for $T>0$ small.
A key ingredient here is to note that
$\|f\|_{L^4_{t,x}} \lesssim \|f\|_{X^{0,\frac{3}{8}}}$ (proved in Bourgain 1998)
(for a proof, see also Proposition 2.13 in the textbook by Terence Tao).

Then, the fact that $\frac{3}{8} < \frac{1}{2}$ allows one to obtain an estimate of the type

$$" \quad \|Lv\|_{L^4([0,T]\times\mathbb{T}^1)} \leq C\|\Phi\|_{L^2_x} + CT^{\delta}\|v\|_{L^4([0,T]\times\mathbb{T}^1)} \quad "$$

T^{δ} comes from comparing $X^{0,\frac{3}{8}}$ and $X^{0,\frac{1}{2}+}$.

- The loss of derivative in ii) requires one to consider $s>0$ in 2).

3.5. Gibbs measures for the NLS

Finite dimensional example:

Work on $\mathbb{R}^{2n}$. Denote its elements by $(p_1, \dots, p_n, q_1, \dots, q_n)$.
Consider the Hamiltonian

$H(p_1, \dots, p_n, q_1, \dots, q_n)$. This defines a Hamiltonian flow

$$(H) \quad \begin{cases} \dot{p}_i = \dfrac{\partial H}{\partial q_i} \\ \dot{q}_i = -\dfrac{\partial H}{\partial p_i} \end{cases}$$

$$\operatorname{div}\left(\frac{\partial H}{\partial q_1}, \dots, \frac{\partial H}{\partial q_n}, -\frac{\partial H}{\partial p_1}, \dots, -\frac{\partial H}{\partial p_n} \right) = 0$$

$\Rightarrow$ The Lebesgue measure $\prod_{i=1}^{n} dp_i dq_i =: \mu_{Leb}$ is invariant under the flow of (H).

Since H is conserved under the flow of (H), it follows that the Gibbs measure (unnormalized)

$$\mu_{Gibbs} := e^{-\beta H} \prod_{i=1}^{n} dp_i dq_i = e^{-\beta H} \mu_{Leb} \qquad (\beta \in \mathbb{R} \text{ is a parameter})$$

is invariant under the flow of (H). More precisely, we have

$$\int_S d\mu_{Gibbs} = \int_{\Phi_t(S)} d\mu_{Gibbs} \qquad \forall\, S \text{ - measurable wrt. } \mu_{Gibbs}.$$

i.e. $\mu_{Gibbs}(S) = \mu_{Gibbs}(\Phi_t(S))$.

Formally, we write

$$\int_{\Phi_t(S)} d\mu_{Gibbs} = \int_{\Phi_t(S)} e^{-\beta H(p,q)} d\mu_{Leb}(p,q) =$$

$$= \begin{cases} (\tilde{p}, \tilde{q}) = \Phi_t^{-1}(p,q) \\ d\mu_{Leb}(p,q) = d\mu_{Leb}(\tilde{p}, \tilde{q}) \\ H(p,q) = H(\tilde{p}, \tilde{q}). \end{cases}$$

$$= \int_S e^{-\beta H(\tilde{p},\tilde{q})} \, d\mu_{Leb}(\tilde{p},\tilde{q}) = \int_S d\mu_{Gibbs} \,. \quad \lrcorner$$

(H) can also be written as

$$\nabla_w H = \left(\frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q_1}, \dots, -\frac{\partial H}{\partial q_n} \right)$$

where ∇_w is the symplectic gradient associated to the symplectic form $w = \sum_{j=1}^{n} dp_j \wedge dq_j$ on $\mathbb{R}^{2n}$.

• If one considers the symplectic form $w(f,g) = -\mathrm{Im} \int f \bar{g} \, dx$ on $L^2(\mathbb{R}^n)$, then one can write the NLS (defocusing problem)

$$i \partial_t u + \Delta u = + |u|^{p-1} u \,;$$

as

$$\partial_t u = (\nabla_w H)(u) \,,$$

for $H(u) = \frac{1}{2} \int |\nabla u|^2 dx + \frac{1}{p+1} \int |u|^{p+1} dx.$

→ In particular, one can write the NLS as in (H) with

$$\begin{cases} p_k = \mathrm{Re}\, \hat{u}(k) \\ q_k = \mathrm{Im}\, \hat{u}(k) \end{cases}$$

for $k \in \mathbb{Z}^n$

$$(H') \begin{cases} \dot{p}_k = \dfrac{\partial H}{\partial q_k} \\ \dot{q}_k = -\dfrac{\partial H}{\partial p_k} \end{cases}$$

(We write $H(u)$ as a function of (p_j, q_j) using Fourier series.)

⌜More precisely, we note that:

$$(dH_u)(v) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} H(u+\varepsilon v)$$

$$= 2\mathrm{Re}\left(-\frac{1}{2}\int \Delta u\cdot\bar{v}\,dx\right) + \mathrm{Re}\left(\int |u|^{p-1}u\,\bar{v}\,dx\right)$$

$$= \mathrm{Re}\left(-\int \Delta u\cdot\bar{v}\,dx + \int |u|^{p-1}u\bar{v}\,dx\right)$$

$$= \mathrm{Im}\left(\int (-i\Delta u + |u|^{p-1}u)\bar{v}\,dx\right) = \omega\left(-i\Delta u + |u|^{p-1}u, v\right)$$

So, the Hamiltonian flow of H with the symplectic form ω is $\partial_t u = \nabla_\omega H$, where the above expression $= \omega(\nabla_\omega H, v)$, so

$$\nabla_\omega H = -i\Delta u + |u|^{p-1}u$$

$$\rightsquigarrow \partial_t u = -i\Delta u + |u|^{p-1}u.$$ ⌟

In this infinite dimensional setting, i.e. on $L^2(\mathbb{T}^n)$, the Lebesgue measure is formally given as

$$du := \prod_{k\in\mathbb{Z}^n} dp_k\, dq_k$$

This measure should be invariant under the flow of (H').

- Let us consider the formally defined "<u>unnormalized Gibbs measure</u>"

$$d\tilde{\mu} := e^{-H(u)}\,du$$

$$= e^{-\frac{2}{p+1}\int |u|^{p+1}dx}\cdot \underbrace{e^{-\int |\nabla u|^2 dx}\,du}_{dv}$$

$dv := e^{-\int|\nabla u|^2 dx}\cdot du$ is the "<u>unnormalized Wiener measure</u>"

We consider the "<u>normalized Wiener measure</u>"

$$d\rho := \frac{d\nu}{\int d\nu} \rightsquigarrow \text{"a probability measure"}$$

In order to see on which set $d\rho$ is supported, we need to rewrite it in a different way.

Let us write:

$$a_k := \hat{u}(k).$$

Then, we also write:

$$d^2 a_k := d\,\mathrm{Re}\, a_k \, d\,\mathrm{Im}\, a_k = dp_k \, dq_k \, .$$

By Plancherel's theorem,

$$\int |\nabla u|^2 dx = c \sum_k |k|^2 |a_k|^2 .$$

Hence, we can rewrite:

$$d\rho = \frac{e^{-c\sum_k |k|^2 |a_k|^2} \cdot \prod_{k\in\mathbb{Z}^n} da_k}{\int e^{-c\sum_k |k|^2 |a_k|^2} \cdot \prod_{k\in\mathbb{Z}^n} da_k}$$

$$= \prod_{k\in\mathbb{Z}^n} \frac{e^{-c|k|^2|a_k|^2} da_k}{\int e^{-c|k|^2|a_k|^2} da_k} \, .$$

$\rightsquigarrow$ This gives us a Gaussian distribution for $|k|\, a_k = |k|\, \hat{u}(k)$.

We think of each Fourier coefficient as a random variable.

In particular, the measure $d\rho$ yields for $\varphi \in L^2(\mathbb{T}^n)$ the distribution of a random Fourier series:

$$\varphi \equiv \varphi^\omega = \sum_k \frac{g_k(\omega)}{|k|} e^{2\pi i k\cdot x} \quad , \text{ "Gaussian free field"}$$

(g_k) – i.i.d complex Gaussians.

⇝ If we denote by $T : (\Omega, p) \to$ "space of functions" (or "Fourier series"), $T(\omega) := \varphi^\omega$, then we identify
$d\rho = T_* dp$. (or just $T_* p$).

<u>Note:</u> There is an issue with the zero mode $k=0$. This is dealt with by replacing Δ with $\Delta - C$ for some $C>0$, in which case the random Fourier series becomes

$$\varphi^\omega = \sum_k \frac{g_k(\omega)}{\sqrt{|k|^2+C}} e^{2\pi i k\cdot x}.$$

Note that if u solves $i\partial_t u + \Delta u = |u|^{p-1}u$, then $V := e^{-iCt} u$ solves

$$i\partial_t v + (\Delta - C)v = |v|^{p-1} v.$$

We henceforth ignore this issue.

- Observe that,

$$\| \langle D \rangle^s \varphi^\omega \|^2_{L^2_x} \sim \sum_k \langle k \rangle^{2s} \cdot \frac{|g_k(\omega)|^2}{|k|^2}$$

$$\Rightarrow \| \langle D \rangle^s \varphi^\omega \|^2_{L^2(\Omega\times\mathbb{T}^n)} \sim \sum_k (1+|k|)^{2-2s} < \infty \text{ if } s < 1-\tfrac{n}{2}.$$

$$\Rightarrow \varphi^\omega \in H_x^{(1-\frac{n}{2})-}, \text{ almost surely.}$$

(Moreover, the infinite sum by which we define φ^ω converges almost surely in this topology).

In particular, we obtain that
$d\rho$ is a probability measure on $H_x^{1-\frac{n}{2}-}$.
However, one can also show that
$d\rho(H_x^s) = 0$ for all $s \geq 1 - \frac{n}{2}$.

- Let us henceforth fix $\underline{n=1}$. In this case, $\varphi^\omega \in H_x^{\frac{1}{2}-}(\mathbb{T}) \subseteq L^2(\mathbb{T})$ almost surely, i.e. $d\rho$ indeed defines a probability measure on $\bigcap_{s<\frac{1}{2}} H_x^s(\mathbb{T})$ (and on $L^2(\mathbb{T})$).
Moreover, by Sobolev embedding, we have

$$\int |\varphi^\omega|^{p+1} dx < \infty, \text{ almost surely.}$$

It hence follows that
$d\tilde{\mu} = e^{-\frac{1}{p+1}\int |u|^{p+1} dx} \cdot d\gamma$
is absolutely continuous wrt. $d\gamma$
Also $d\tilde{\mu} \neq 0$.
So $d\tilde{\mu}$ is well-defined and supported on $\bigcap_{s<\frac{1}{2}} H_x^s(\mathbb{T})$
Also $\int d\tilde{\mu} \in (0,1)$.
Hence, we can define the <u>Gibbs measure for the NLS</u> by

$$d\mu := \frac{d\tilde{\mu}}{\int d\tilde{\mu}}$$

$\rightarrow$ a probability measure on $\bigcap_{s<\frac{1}{2}} H_x^s(\mathbb{T})$.

- Formally, $d\mu$ is invariant under the flow of the NLS. (by construction).

$\rightsquigarrow$ " a conservation law at regularity $\frac{1}{2}-$".

Theorem (Bourgain 1994)

1) The Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1} u \\ u|_{t=0} = \Phi \end{cases}$$

is globally well-posed for μ-almost all $\Phi \in H_x^{1/2-}$. Here $3 < p \leq 5$.

2) The Gibbs measure μ is invariant under the flow of the NLS.

Difficulty: "• One uses Global existence to show invariance of μ.
One uses invariance of μ to show global existence."

$\rightsquigarrow$ The difficulty is resolved by considering finite dimensional approximations ("Galerkin approximations").

• Given $N \in \mathbb{N}$ a frequency threshold, we define

$$P_N f := \sum_{|k| \leq N} \widehat{f}(k) e^{2\pi i k \cdot x}.$$

We consider the following truncated problem:

$$(5.1) \quad \begin{cases} i\partial_t u^N + \Delta u^N = P_N(|u^N|^{p-1} u^N) \\ u^N|_{t=0} = P_N \varphi =: \varphi_N \end{cases}$$

Here $\varphi \in H_x^{\frac{1}{2}-}$.
Note that (5.1) is globally well-posed since $P_N \varphi \in H_x^1$.

We consider $d\rho_N$, $d\mu_N$ as the truncations of the Wiener and Gibbs measures,

$$d\rho_N := \prod_{|k|\le N} \frac{e^{-c|k|^2|a_k|^2}\,da_k}{\int e^{-c|k|^2|a_k|^2}\,da_k}$$

$$d\mu_N := \frac{e^{-\frac{1}{p+1}|P_N u|^{p+1}}\,d\gamma_N}{\int e^{-\frac{1}{p+1}|P_N u|^{p+1}}\,d\gamma_N} := \frac{e^{-\frac{1}{p+1}|P_N u|^{p+1}}\prod_{|k|\le N} e^{-c|k|^2|a_k|^2}\,da_k}{\int e^{-\frac{1}{p+1}|P_N u|^{p+1}}\prod_{|k|\le N} e^{-c|k|^2|a_k|^2}\,da_k}$$

<u>Fact:</u> Let $\mathcal{U}$ be an open set in $H_x^{\frac{1}{2}-}(\mathbb{T})$. Then, one has

$$(5.2)\begin{cases} \mu(\mathcal{U}) = \lim_N \mu_N(\mathcal{U}\cap E_N) \\ \rho(\mathcal{U}) = \lim_N \rho_N(\mathcal{U}\cap E_N) \end{cases}$$

where $E_N :=$ span of $\{e^{2\pi i k\cdot x}, |k|\le N\}$.

- Note that, in addition to being globally well-posed in H^1, the invariance of μ_N under the flow of (5.1) is immediate. Namely, in this case, we are considering dynamics on a finite dimensional phase space $\mathbb{C}^{2N+1}$ and the formal arguments for invariance are then rigorous.

One can relate the Cauchy problem

$$(5.3)\quad \begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u \\ u|_{t=0} = \varphi \end{cases}$$

as follows.

<u>Approximation Lemma:</u> Suppose that $\varphi\in H_x^s$, for some $s>0$ and $s_1\in(0,s)$. Furthermore, suppose that solutions (in H_x^s) of (5.1), (5.3) exist on a time interval $[-T,T]$.
Then, we have

$$\|u-u^N\|_{L^\infty_{[-T,T]}H_x^{s_1}} \lesssim N^{s_1-s} \xrightarrow{\text{as } N\to\infty} 0.$$

Remark: In fact, if a solution of (5.1) exists on $[-T,T]$, then we can deduce that a solution of (5.3) exists on $[-T,T]$, provided that

$$\|u^N\|_{L^\infty_{[-T,T]} H^{s_1}_x} \text{ is bounded uniformly in } N. \quad (5.4)$$

The obtained solution u of (5.1) is of regularity s_1. In order to see this, one uses the local well-posedness theory of (5.1). (Recall the theorem in Section 3.4). Moreover, one uses the approximation lemma (for possibly a smaller time) to deduce that $\|u(t)\|_{H^{s_1}}$ is bounded if we iterate the local solution of (5.3) on $[-T,T]$.
Thus, in the approximation lemma, the assumption that a solution of (5.1) exists on $[-T,T]$ is sufficient provided that we know (5.4).

Lemma (almost-sure global well-posedness of (5.1))

1) Let $0 < s < \frac{1}{2}$, $T < \infty$, $\delta > 0$ be given.
There exists a set $\Omega_\delta \subseteq \bigcap_{s<\frac{1}{2}} H^s_x(\mathbb{T})$ with $\mu(\Omega_\delta^c) = \mu\left(\left(\bigcap_{s<\frac{1}{2}} H^s_x(\mathbb{T})\right) \setminus \Omega_\delta\right) < \delta$ such that for all $\varphi \in \Omega_\delta$, there exists a solution u^N of (5.1) on $[-T,T]$ satisfying

$$\|u^N(t)\|_{L^\infty_{[-T,T]} H^s_x} \leq C\left(\log \frac{T}{\delta}\right)^{1/2}.$$

2) With $0 < s < \frac{1}{2}$, and $\delta > 0$ as in part 1), there exists a set $\widehat{\Omega}_\delta \subseteq \bigcap_{s<\frac{1}{2}} H^s(\mathbb{T})$ with $\mu(\widehat{\Omega}_\delta^c) < \delta$ such that for all $\varphi \in \widehat{\Omega}_\delta$, there exists a

global in time solution of (5.1) which satisfies

$$\|u(t)\|_{H^s} \leq C \left(\log \frac{1+|t|}{\delta}\right)^{1/2}.$$

<u>Proof / Sketch:</u>

We note that 1) $\Rightarrow$ 2) if we take, $T_j := 2^j$

$\delta_j := \frac{\delta}{2^{j+1}}$, $\Omega_{\delta_j}^{(T_j)} :=$ set from part 1) with

$$T = T_j, \ \delta = \delta_j$$

$\hat{\Omega}_\delta := \left(\bigcup_j \left(\Omega_{\delta_j}^{(T_j)}\right)^c \right)^c = \bigcap_j \Omega_{\delta_j}^{(T_j)}$, satisfies the claim.

<u>We now prove 1)</u>

For $K > 1$, we consider

$$\Omega_{s,K} := \{ \psi \in E_N, \|\psi\|_{H^s(\mathbb{T})} \leq K \} \quad \underline{(5.5)}$$

With our earlier identifications, we have

$$p_N(\Omega_{s,K}^c) = p\left(\Big\| \sum_{|k| \leq N} \frac{g_k(\omega)}{|n|} e^{2\pi i k x} \Big\|_{H^s_x} > K \right)$$

It can be shown that this quantity is

$\lesssim e^{-cK^2}$. $\underline{(5.6)}$ (<u>Recall:</u> $p =$ Probability measure on $\Omega = \{\omega\}$).

<u>" a large deviation estimate "</u>

The precise statement that we are using here is <u>Bernstein's Inequality</u>. See Proposition 5.16 of the lecture notes " Introduction to the non-asymptotic analysis of random matrices " arXiv 1011.3027.

· Note that we are estimating

$$p\left(\sum_{|k| \leq N} \frac{|g_k|^2}{(1+|n|)^{2s+2}} \geq K \right).$$

Then, we apply the proposition in the aforementioned lecture notes with $X_i = |g_i(\omega)|^2$, $a_i = \frac{1}{(1+|i|)^{2-2s}}$ and we get

$$P\Big(\sum_{|k|\le N} \frac{|g_k(\omega)|^2}{(1+|k|)^{2-2s}} \ge K^2\Big) \lesssim e^{-cK^2}.$$ This implies (5.6). $\lrcorner$

Let S_N denote the flow map of (5.1).

We let $S_N^\delta := S_N(\delta)$.

We fix $K>0$ which we will specify later.

Note that, for all $\Psi \in \Omega_{s,K}$, we have

$$\|S_N^\delta \Psi\|_{H^s} \le 2K \text{ for } |t| \le \delta \sim K^{-c} \qquad (5.7)$$

by recalling (5.5) and using the H^s_x local well-posedness theory of (5.1).

$\ulcorner$ Note: The frequency truncations are irrelevant in the construction of the local solution so the local well-posedness theory for (5.1) is the same as that for (5.3). $\lrcorner$

We now define the set:

$$\Omega_\delta := \Omega_{s,K} \cap (S_N^\delta)^{-1}(\Omega_{s,K}) \cap (S_N^\delta)^{-2}(\Omega_{s,K}) \cap \cdots \cap (S_N^\delta)^{-\lfloor \frac{T}{\delta} \rfloor}(\Omega_{s,K})$$

We observe that,

$$\mu_N(\Omega_\delta^c) \le \mu_N\Big(\bigcup_{j=0}^{\lfloor \frac{T}{\delta}\rfloor} \big[(S_N^\delta)^{-j}(\Omega_{s,K})\big]^c\Big)$$

$$\le \sum_{j=0}^{\lfloor \frac{T}{\delta}\rfloor} \mu_N\big((S_N^\delta)^{-j}(\Omega_{s,K}^c)\big),$$ which by the invariance of μ_N under the flow of (5.1) is

$$= \sum_{j=0}^{\lfloor \frac{T}{\delta} \rfloor} \mu_N(\Omega_{s,K}^c) = \left\lfloor \frac{T}{\delta} \right\rfloor \cdot \mu_N(\Omega_{s,K}^c),$$

which by (5.6) and (5.7) is

$$\lesssim \frac{T}{\delta} \cdot e^{-K^2} \sim T K^c e^{-K^2}.$$

Hence, for $K \sim \left(\log \frac{T}{\delta}\right)^{1/2}$, we have

$$\mu_N(\Omega_\delta^c) < \delta \qquad (5.8)$$

By construction, we know that for all $j = 0, 1, \dots, \lfloor \frac{T}{\delta} \rfloor$ we have

$$\| S_N(j\delta)\Psi \|_{H^s} \leq K$$

for all $\Psi \in \Omega_\delta$.

By local well-posedness theory, we hence deduce that, for all $\Psi \in \Omega_\delta$ we have

$$\| S_N(t)\Psi \|_{H^s} \leq 2K \sim \left(\log \frac{T}{\delta}\right)^{1/2} \qquad (5.9)$$

for all $0 \leq t \leq T$.

The negative times are treated analogously.

The claim 1) now follows from (5.8) and (5.9). $\square$

- Now that we have a solution u of (5.3) for almost every initial data φ, we can address the issue of invariance of the Gibbs measure.

- Let us fix a time t. We denote by $\mathcal{S}_N := \mathcal{S}_N(t)$ and let $\mathcal{S} := \mathcal{S}(t)$ denote the flow map of (5.3) for time t.

Let K be a compact set in $\bigcap_{s<\frac{1}{2}} H^s$.

We want to show that

$$\mu(K) = \mu(\mathcal{S}(K)). \qquad (5.10)$$

Let us fix $0 < s_1 < \frac{1}{2}$. Denote by B_ε the ball of radius $\varepsilon > 0$ centered at 0.

$$(5.2) \Rightarrow \mu(\mathcal{S}(K) + B_\varepsilon) = \lim_N \mu_N((\mathcal{S}(K) + B_\varepsilon) \cap E_N) \qquad (5.2')$$

The approximation Lemma implies that, for N sufficiently large $(N > N_0)$ we have

$$\mathcal{S}_N(P_N K) \subseteq \mathcal{S}_N(K) + B_{\varepsilon/2}. \qquad (5.11)$$

By continuity of $\mathcal{S}_N$ on H^{s_1}, it follows that for sufficiently small $\varepsilon_1 > 0$ (independent of N) we have
↳ depending on ε

$$\mathcal{S}_N((K + B_{\varepsilon_1}) \cap E_N) \subseteq \mathcal{S}_N(P_N K) + B_{\varepsilon/2}$$

which, by (5.11) is

$$\subseteq \mathcal{S}(K) + B_\varepsilon.$$

It follows that:

$$\mu_N\big(\mathcal{S}_N\big((K+B_{\varepsilon_1})\cap E_N\big)\big) \le \mu_N\big((\mathcal{S}(K)+B_\varepsilon)\cap E_N\big)$$

$$\parallel$$

$$\mu_N\big((K+B_{\varepsilon_1})\cap E_N\big) \text{ by invariance of } \mu_N$$

Setting $N \to \infty$, we obtain, using (5.2) and (5.2')

$$\mu\,(K+B_{\varepsilon_1}) \le \mu\,(\mathcal{S}(K)+B_\varepsilon)$$

$$\Rightarrow \mu(K) \le \mu(\mathcal{S}(K)+B_\varepsilon).$$

Setting $\varepsilon \to 0$, we obtain

$$\mu(K) \le \mu(\mathcal{S}(K)).$$

By time-reversibility of the flow, we deduce (5.10). $\square$