§2: The linear theory on $\mathbb{R}^n$

2.1: Preliminaries: An informal definition

Given $1 \leq q, r \leq \infty$ and spaces X, Y, we define, for $u: Y \times X \to \mathbb{C}$

$$\|u\|_{L^q_x L^r_y} := \big\| \|u(\cdot,x)\|_{L^r(Y)} \big\|_{L^q(X)}$$

(we assume that this object is well-defined).

In other words, we fix $x \in X$ and we then take the $L^r(Y)$ of $u(\cdot, x)$. We finally take the $L^q(X)$ of the remaining function.

We sometimes also write $\|u\|_{L^q_x L^r_y}$.

How does one make this more precise?

- Let B be a separable Banach space.

A function $F: \mathbb{R}^n \to B$ is <u>weakly measurable</u> if for all $b' \in B^* =$ the dual of B, the map $x \mapsto \langle F(x), b' \rangle$ is measurable.

This also can be shown to imply that

$$x \mapsto \|F(x)\|_B \text{ is measurable.}$$

In this case, we define:

$$\|F\|_{L^p(B)} := \begin{cases} \left(\int_{\mathbb{R}^n} \|F(x)\|_B^p \, dx \right)^{1/p} & \text{if } 1 \leq p < \infty \\ \|F\|_{L^\infty(B)} := \sup \{ \|F(x)\|_B : x \in \mathbb{R}^n \} \end{cases}$$

$L^p(B)$ is a Banach space.

- Let $1 \le p < \infty$ and let B be a <u>reflexive</u>, separable Banach space. Let $F \in L^p(B)$ and $G \in L^{p'}(B^*)$. Then, the function $\langle F, G\rangle(x) := \langle \underset{\in B}{F(x)}, \underset{\in B^*}{G(x)} \rangle$ is integrable and

$$\|G\|_{L^{p'}(B^*)} = \sup \left\{ \left| \int_{\mathbb{R}^n} \langle F(x), G(x)\rangle \, dx \right|, \|F\|_{L^p(B)} \le 1 \right\}$$

- In particular, for $1 < q, r < +\infty$ we have

$$\left(L^q L^r (\mathbb{R}^n \times \mathbb{R}^m)\right)^* = L^{q'} L^{r'} (\mathbb{R}^n \times \mathbb{R}^m).$$

2.2: Strichartz estimates

We consider the Cauchy problem:

$$\begin{cases} i u_t + \Delta u = 0 & \text{on } \mathbb{R}^n_x \times \mathbb{R}_t \\ u|_{t=0} = \Phi & \text{on } \mathbb{R}^n_x \end{cases}$$

We assume that $\Phi \in \mathcal{S}(\mathbb{R}^n)$.

We saw earlier that

$$u(x,t) = \frac{e^{i|x|^2/4t}}{(4\pi i t)^{n/2}} * \Phi .$$

Let us henceforth write $S(t)$ for the solution map.
In other words,

$$u(x,t) = S(t)\Phi .$$

Note that

$$\begin{cases} \|S(t)\Phi\|_{L^\infty_x} \le \dfrac{C}{|t|^{n/2}} \cdot \|\Phi\|_{L^1_x} \\ \|S(t)\Phi\|_{L^2_x} = \|\Phi\|_{L^2_x} \end{cases}$$

Lemma 2.1 (First dispersive estimate)

For all $2 \leq p \leq +\infty$ we have

$$\|S(t)\Phi\|_{L^p_x} \leq \frac{C_p}{|t|^{\frac{n}{2}(1-\frac{2}{p})}} \cdot \|\Phi\|_{L^{p'}_x}.$$

Proof: We use the previous observation and the Riesz-Thorin interpolation theorem.
In particular, if $\Theta \in [0,1]$ is chosen such that

$$\frac{\Theta}{\infty} + \frac{1-\Theta}{2} = \frac{1}{p}$$

(in order to match the integrability exponents on the left-hand side), then $\Theta = 1 - \frac{2}{p}$.

Hence, $\frac{\Theta}{1} + \frac{1-\Theta}{2} = \frac{1+\Theta}{2} = \frac{1}{p'}$. $\square$

We also consider the inhomogeneous Cauchy problem:

$$\begin{cases} iu_t + \Delta u = F \text{ on } \mathbb{R}^n_x \times \mathbb{R}_t \\ u|_{t=0} = \Phi \end{cases}$$

Here, the term on the right-hand side is $F = F(x,t)$.
By Duhamel's principle, we can write the solution as:

$$u(x,t) = S(t)\Phi - i \cdot \int_0^t S(t-\tau)\, F(\cdot,\tau)\, d\tau.$$

- For $1 \leq q, r \leq +\infty$, we work with

$$\|u\|_{L^q_t L^r_x} := \| \|u(\cdot,t)\|_{L^r_x} \|_{L^q_t}.$$

Definition: We say that a pair (q,r) is admissible in dimension n if

$2 \leq q, r \leq +\infty$; $(q,r,n) \neq (2,+\infty,2)$ and

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}.$$

Theorem 2.2 (Strichartz estimates for the Schrödinger equation on $\mathbb{R}^n$)
[Strichartz 1977 for $q, \tilde{q} > 2$; Keel-Tao 1998 for the endpoint]

Let (q,r) be admissible in dimension n.
The following estimates then hold.

i) $$\| S(t)\Phi \|_{L^q_t L^r_x} \lesssim_{q,r,n} \|\Phi\|_{L^2_x}$$

(Homogeneous Strichartz estimate).

ii) If $(\tilde{q}, \tilde{r})$ is also admissible in dimension n, we have

$$\Big\| \int_0^t S(t-\tau) F(\cdot,\tau) \Big\|_{L^q_t L^r_x} \lesssim_{q,r,\tilde{q},\tilde{r},n} \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

(Inhomogeneous Strichartz estimate).

Proof: We will prove only the "non-endpoint case" $q, \tilde{q} > 2$.

i) We fix (q,r) admissible in dimension n.
The claim immediately follows when $(q,r) = (\infty, 2)$.
Indeed $\|S(t)\Phi\|_{L^\infty_t L^2_x} = \|\Phi\|_{L^2_x}$.

We assume now that $q<\infty$.
We know that $q\geq 2>1$.
Also $r\geq 2$. Let's assume for now that $r<\infty$.
(although in 1D the pair $(4,\infty)$ is admissible).
In particular, we know that

$$\left(L_t^q L_x^r\right)^* = L_t^{q'} L_x^{r'}$$

Let $G\in L_t^{q'}L_x^{r'}$ be given. We compute:

$$\left|\int_{-\infty}^{+\infty}\langle S(t)\Phi, G\rangle_{L_x^2}\,dt\right| = \left|\left\langle \Phi, \int_{-\infty}^{+\infty} S(-t)G\,dt\right\rangle_{L_x^2}\right|$$

$$\leq \|\Phi\|_{L_x^2}\cdot\left\|\int_{-\infty}^{+\infty}S(-t)G\,dt\right\|_{L_x^2} \qquad (*),$$

- In the first equality, we used $(S(t))^*=S(-t)$ as well as linearity of the integral.

In order to see the former claim, we can use the Fourier transform and see that

$$\langle S(t)f,g\rangle_{L_x^2} = \langle \widehat{S(t)f}, \widehat{g}\rangle_{L_\xi^2} =$$

$$= \int_{\mathbb{R}^n} e^{-4\pi^2 i|\xi|^2 t}\cdot \widehat{f}(\xi)\cdot\overline{\widehat{g}(\xi)}\,d\xi$$

$$= \int_{\mathbb{R}^n} \widehat{f}(\xi)\cdot \overline{e^{4\pi^2 i|\xi|^2 t}\,\widehat{g}(\xi)}\,d\xi$$

$$= \langle \widehat{f}, \widehat{S(-t)g}\rangle_{L_\xi^2} = \langle f, S(-t)g\rangle_{L_x^2}.$$

- In the subsequent line, we used the Cauchy-Schwarz inequality.

We compute

$$\left\| \int_{-\infty}^{+\infty} S(-t)\, G(\cdot,t)\, dt \right\|_{L^2_x}^2$$

$$= \left\langle \int_{-\infty}^{+\infty} S(-t_1)\, G(\cdot,t_1)\, dt_1 , \int_{-\infty}^{+\infty} S(-t_2)\, G(\cdot,t_2)\, dt_2 \right\rangle_{L^2_x}$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle S(t_2-t_1)\, G(\cdot,t_1), G(\cdot,t_2) \rangle_{L^2_x}\, dt_1\, dt_2$$

$$= \int_{-\infty}^{+\infty} \left\langle \int_{-\infty}^{+\infty} S(t_2-t_1)\, G(\cdot,t_1)\, dt_1 , G(\cdot,t_2) \right\rangle_{L^2_x} dt_2$$

$$\leq \left\| \int_{-\infty}^{+\infty} S(t-t_1)\, G(\cdot,t_1)\, dt_1 \right\|_{L^q_t L^r_x}$$

$$\cdot \| G \|_{L^{q'}_t L^{r'}_x} . \qquad (*)_2$$

(Here, we used our earlier duality claim. Alternatively, one can view this as an application of Hölder's inequality in mixed-norm spaces.)

We can write the first factor as

$$\left\| \left\| \int_{-\infty}^{+\infty} S(t-t_1)\, G(\cdot,t_1)\, dt_1 \right\|_{L^r_x} \right\|_{L^q_t}$$

which, by Minkowski's inequality, is

$$\leq \left\| \int_{-\infty}^{+\infty} \| S(t-t_1)\, G(\cdot,t_1) \|_{L^r_x}\, dt_1 \right\|_{L^q_t}$$

We know that $2 \leq r < \infty$.

Hence, by the first dispersive estimate, this term is

$$\leq C_r \cdot \left\| \int_{-\infty}^{+\infty} \frac{1}{|t-t_1|^{\frac{n}{2}\cdot(1-\frac{2}{r})}} \cdot \|G(\cdot,t_1)\|_{L^{r'}_x} \, dt_1 \right\|_{L^q_t} \quad (\Delta)_1$$

<u>Recall:</u> HLS inequality

$$\left\| f * \frac{1}{|x|^\gamma} \right\|_{L^q(\mathbb{R}^d)} \leq C_{p,q,d} \|f\|_{L^p(\mathbb{R}^d)}, \text{ for}$$

$1 < p < q < +\infty$, $0 < \gamma < d$ and $\frac{1}{q} = \frac{1}{p} - 1 + \frac{\gamma}{d}$

- We let $d = 1$ and $\gamma = \frac{n}{2} \cdot (1 - \frac{2}{r})$.

By admissibility, we have

$$\gamma = \frac{n}{2} \cdot \left(\frac{1}{2} - \frac{1}{r}\right) = \frac{2}{q}.$$

Hence $\frac{1}{p} = \frac{1}{q} + 1 - \frac{\gamma}{d} = \frac{1}{q} + 1 - \frac{2}{q} = 1 - \frac{1}{q}$

$\Rightarrow$ $\underline{p = q'}$.

It follows that the expression in $(\Delta)_1$ is

$$\lesssim_{r,q,n} \left\| \|G(\cdot,t)\|_{L^{r'}_x} \right\|_{L^{q'}_t} = \|G\|_{L^{q'}_t L^{r'}_x}. \quad (\Delta)_2$$

Substituting this into $(*)_1$, $(*)_2$, it follows that

$$\left| \int_{-\infty}^{+\infty} \langle S(t)\Phi, G \rangle_{L^2_x} dt \right| \lesssim_{q,r,n} \|\Phi\|_{L^2_x} \cdot \|G\|_{L^{q'}_t L^{r'}_x}$$

- The claim i) for $r < \infty$ now follows by duality.
- The claim i) for $r = \infty$ follows by a limiting argument, which we sketch now. Note that r can equal ∞ only when $\underline{n = 1}$.

We note that it suffices to show

$$\| S(t)\Phi \|_{L^4_t L^\infty_x (\mathcal{K}_x \times \mathcal{A}_t)} \leq C \| \Phi \|_{L^2_x}$$

for all $\Phi \in \mathcal{S}(\mathbb{R})$, $\mathcal{K}, \mathcal{A} \subseteq \mathbb{R}$ compact, provided that the constant $C > 0$ is independent of $\Phi, \mathcal{K}, \mathcal{A}$.

We know that we can find $(q_\varepsilon, r_\varepsilon)$ admissible in dimension 1 such that $q_\varepsilon \to 4$, $r_\varepsilon \to \infty$ as $\varepsilon \to 0$ and a constant $C > 0$ independent of ε such that

$$\| S(t)\Phi \|_{L^{q_\varepsilon}_t L^{r_\varepsilon}_x (\mathbb{R}_x \times \mathbb{R}_t)} \leq C \| \Phi \|_{L^2_x}$$

for all $\Phi \in \mathcal{S}(\mathbb{R})$.

(the independence of C on ε requires a slightly more detailed analysis of the Hardy-Littlewood-Sobolev inequality and we will not go into this here).

Now, fixing $\mathcal{K}, \mathcal{A} \subseteq \mathbb{R}$ compact, we obtain for $\Phi \in \mathcal{S}(\mathbb{R})$ that

$$\| S(t)\Phi \|_{L^4_t L^\infty_x (\mathcal{K}_x \times \mathcal{A}_t)} =$$

$$= \lim_{\varepsilon \to 0} \| S(t)\Phi \|_{L^{q_\varepsilon}_t L^{r_\varepsilon}_x (\mathcal{K}_x \times \mathcal{A}_t)},$$

and so the claim follows.

ii) For the inhomogeneous estimate, let us first prove the estimate without the truncation in time.

$$\left\| \int_{-\infty}^{+\infty} S(t-\tau) F(\cdot,\tau)\, d\tau \right\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} . \qquad (INH_1)$$

This claim holds when $(q,r) = (\tilde{q},\tilde{r})$ (c.f. $(\Delta)_1$ and $(\Delta)_2$).

We can also deduce from the proof of i) that:

$$\left\| \int_{-\infty}^{+\infty} S(t-\tau) F(\cdot,\tau)\, d\tau \right\|_{L^\infty_t L^2_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} .$$

Namely, we know that, for fixed t we have

$$\left\| \int_{-\infty}^{+\infty} S(t-\tau) F(\cdot,\tau)\, d\tau \right\|^2_{L^2_x} \lesssim \|F\|^2_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

$$\|$$

$$\left\| \int_{-\infty}^{+\infty} S(-\tau) F(\cdot,\tau)\, d\tau \right\|^2_{L^2_x}$$

(see the calculations starting from $(*)_2$).

By interpolation in mixed norm spaces, the claim (INH_1) follows for $q \leq \tilde{q}$. By an additional duality step, we obtain (INH_1) for the full range of (q,r).

<u>Interpolation in mixed norm spaces</u>:

Let $T: L^{q_0}_t L^{r_0}_x \to L^{\tilde{q}_0}_t L^{\tilde{r}_0}_x$

$\quad T: L^{q_1}_t L^{r_1}_x \to L^{\tilde{q}_1}_t L^{\tilde{r}_1}_x$

be linear and bounded with norms K_0 and K_1 respectively. Then $T: L^{q_\theta}_t L^{r_\theta}_t \to L^{\tilde{q}_\theta}_t L^{\tilde{r}_\theta}_x$ is bounded with norm $K_\theta \leq K_0^{1-\theta} K_1^{\theta}$; for $\theta \in [0,1]$, where $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, $\frac{1}{\tilde{q}_\theta} = \frac{1-\theta}{\tilde{q}_0} + \frac{\theta}{\tilde{q}_1}$, $\frac{1}{\tilde{r}_\theta} = \frac{1-\theta}{\tilde{r}_0} + \frac{\theta}{\tilde{r}_1}$.

· More precisely, we suppose that $q \leq \tilde{q}$ and we take $G \in L^{q'}_t L^{r'}_x$ with $\|G\|_{L^{q'}_t L^{r'}_x} \leq 1$.

We compute:

$$\int_{-\infty}^{+\infty} \Big\langle \int_{-\infty}^{+\infty} S(t-\tau) F(\cdot,\tau)\, d\tau,\ G(\cdot,t) \Big\rangle_{L^2_x} dt$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle F(\cdot,\tau), S(\tau-t) G(\cdot,t) \rangle_{L^2_x}\, dt\, d\tau$$

$$= \int_{-\infty}^{+\infty} \Big\langle F(\cdot,\tau), \int_{-\infty}^{+\infty} S(\tau-t) G(\cdot,t)\, dt \Big\rangle_{L^2_x} d\tau$$

which is in absolute value

$$\leq \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \cdot \Big\| \int S(\tau-t) G(\cdot,t)\, dt \Big\|_{L^{\tilde{q}}_t L^{\tilde{r}}_x}$$

which, by the earlier result is

$$\lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \cdot \|G\|_{L^{q'}_t L^{r'}_x}$$

$$\leq \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} .$$

This proves (INH_1).

In order to deduce the claim of ii) from (INH_1), we can use the following kernel truncation result.

Christ - Kiselev Lemma:

Let X, Y be Banach spaces and let $I \subseteq \mathbb{R}$ be an interval (possibly all of $\mathbb{R}$, or one of the intervals $(0,+\infty)$, $(-\infty,0)$).

Let $K \in C^0(I \times I \to B(X \times Y))$ be an integral kernel and $1 \leq p < q \leq +\infty$ s.t.

$$\left\| \int_I K(t,s) f(s)\, ds \right\|_{L^q_t(I \to Y)} \leq A \|f\|_{L^p_t(I \to X)} .$$

Then one has

$$\left\| \int_{\substack{s \in I: \\ s<t}} K(t,s) f(s)\, ds \right\|_{L^q_t(I \to Y)} \lesssim_{p,q} A \|f\|_{L^p_t(I \to X)} .$$

$\leadsto$ We take $Y = L^r_x$, $X = L^{\tilde{r}'}_x$ and we note that $\infty > q > 2 > \tilde{q}' > 1$. $\square$

[More precisely, a minor modification of the earlier arguments shows that: for (q,r), $(\tilde{q},\tilde{r})$ admissible

$$\left\| \int_0^{+\infty} S(t-\tau) F(\cdot,\tau)\, d\tau \right\|_{L^q_{(0,+\infty)} L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_{(0,+\infty)} L^{\tilde{r}'}_x}$$

we take $I = (0,+\infty)$ and apply the Christ-Kiselev lemma to deduce:

$$\left\| \int_0^t S(t-\tau) F(\cdot,\tau)\, d\tau \right\|_{L^q_{(0,+\infty)} L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_{(0,+\infty)} L^{\tilde{r}'}_x}$$

$$\leq \|F\|_{L^{\tilde{q}'}_{(-\infty,+\infty)} L^{\tilde{r}'}_x}$$

The estimate

$$\left\| \int_0^t S(t-\tau) F(\cdot,\tau)\, d\tau \right\|_{L^q_{(-\infty,0)} L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_{(-\infty,+\infty)} L^{\tilde{r}'}_x}$$ follows analogously.]

Digression (from last time)

- Let B be a Banach space

A function $F: \mathbb{R}^n \to B$ is weakly measurable if for all $b' \in B^*$, the map $x \mapsto \langle F(x), b' \rangle$ is measurable.

It is Bochner (strongly) measurable if it can be written as a limit $F_n \to F$ in norm, where each F_n is of the form

$$\sum_{j=1}^{N} b_j \chi_{E_j}$$

$E_j \subseteq \mathbb{R}^n$ measurable, $b_j \in B$.

- If B is separable, these two notions coincide.

Theorem (Pettis) $F: \mathbb{R}^n \to B$ is Bochner measurable if and only if it is weakly measurable and if there exists $E \subseteq \mathbb{R}^n$ of measure zero such that $F(\mathbb{R}^n \setminus E) \subseteq B$ is separable.

2.3: Strichartz estimates for the Airy equation

We consider the Airy Cauchy problem:

$$\begin{cases} (\partial_t + \partial_x^3) u = 0 \quad \text{on } \mathbb{R}_x \times \mathbb{R}_t \\ u|_{t=0} = \Phi \in \mathcal{S}(\mathbb{R}) \end{cases}$$

A similar calculation as before implies that:

$$\hat{u}(\xi, t) = e^{8\pi^3 i t \xi^3} \cdot \hat{\Phi}(\xi)$$

So

$$u(x,t) = K_t * \Phi(x)$$

where

$$K_t(x) := \int_{-\infty}^{+\infty} e^{2\pi i x \xi + 8\pi^3 i t \xi^3} \, d\xi$$

$$= \int_{-\infty}^{+\infty} e^{i(2\pi x \xi + 8\pi^3 \xi^3 t)} \, d\xi \qquad = \begin{cases} \eta^3 = 8\pi^3 \xi^3 t \\ \eta = 2\pi \xi t^{1/3} \\ d\xi = \frac{1}{2\pi t^{1/3}} d\eta \end{cases}$$

$$= \frac{1}{2\pi t^{1/3}} \int_{-\infty}^{+\infty} e^{i\left(\frac{x}{t^{1/3}} \eta + \eta^3\right)} \, d\eta$$

$$= \frac{1}{2\pi t^{1/3}} A\left(\frac{x}{t^{1/3}}\right), \text{ where}$$

$$A(x) := \int_{-\infty}^{+\infty} e^{i x \xi + i \xi^3} \, d\xi$$

If we show that $A \in L^\infty(\mathbb{R})$, we will be able to deduce the dispersive estimate:

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{|t|^{1/3}} \cdot \|\Phi\|_{L^1(\mathbb{R})}. \qquad (3.1)$$

By interpolation and unitarity of the flow map, (3.1) implies:

$$\|u(t)\|_{L^p(\mathbb{R})} \leq \frac{C}{|t|^{\frac{1}{3}\cdot(1-\frac{2}{p})}} \cdot \|\Phi\|_{L^{p'}(\mathbb{R})} \quad \text{for all } 2 \leq p \leq +\infty.$$

(3.2).

Note that the analogue of (3.1) for the Schrödinger equation was proved by using an explicit formula for the kernel (which could be deduced from the formula for the inverse Fourier transform of a Gaussian).

In the proof of (3.1), we need to argue in a more indirect way.

<u>3.3.2</u>: Tools from harmonic analysis; Oscillatory integrals.

Let $\varphi: \mathbb{R} \to \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{C}$ be smooth.

For $a, b \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, we consider:

$$I(\lambda) := \int_a^b e^{i\lambda\varphi(x)} \cdot f(x)\,dx.$$

We want to see how $I(\lambda)$ behaves when $|\lambda| \to +\infty$.

<u>Proposition 3.1</u>: If $f \in C_c^\infty([a,b])$ and $\varphi' \neq 0$ for all $x \in [a,b]$, then:

$$I(\lambda) = \mathcal{O}_k(\lambda^{-k}) \quad \text{as } |\lambda| \to +\infty \text{ for all } k \in \mathbb{N}.$$

<u>Proof</u>: Let us define the operator $\mathcal{L}$ by:

$\mathcal{L}(g) := \frac{1}{i\lambda\varphi'} \cdot \frac{dg}{dx}$. Then, $\mathcal{L}^t g = -\frac{d}{dx}\left(\frac{g}{i\lambda\varphi'}\right)$, and for all k $\mathcal{L}^k(e^{i\lambda\varphi}) = e^{i\lambda\varphi}$. We deduce the claim by integration by parts. $\square$

Proposition 3.2: Let $k \in \mathbb{N}$ and $|\varphi^{(k)}(x)| \geq 1$ on $[a,b]$.
If $k=1$, we also assume that φ' is monotonic if $k=1$.
Then:

$$\left|\int_a^b e^{i\lambda\varphi(x)}\,dx\right| \leq C_k \lambda^{-1/k} \quad ; \; C_k>0 \text{ is independent of } a,b.$$

Proof: • For $k=1$, we have:

$$\int_a^b e^{i\lambda\varphi(x)}\,dx = \int_a^b \frac{1}{i\lambda\varphi'(x)} \cdot \frac{d}{dx}\left(e^{i\lambda\varphi(x)}\right) dx$$

$$= \frac{e^{i\lambda\varphi(x)}}{i\lambda\varphi'(x)}\Big|_a^b - \int_a^b \frac{e^{i\lambda\varphi(x)}}{i\lambda} \frac{d}{dx}\left(\frac{1}{\varphi'}\right) dx$$

$$\Rightarrow \left|\int_a^b e^{i\lambda\varphi(x)}\,dx\right| \leq \frac{2}{\lambda} + \frac{1}{\lambda}\int_a^b \left|\frac{d}{dx}\left(\frac{1}{\varphi'}\right)\right| dx$$

$= \left\{ \begin{array}{l} \text{by the monotonicity} \\ \text{of } \varphi' \end{array} \right\}$

$$= \frac{2}{\lambda} + \frac{1}{\lambda}\left|\frac{1}{\varphi'(b)} - \frac{1}{\varphi'(a)}\right| \leq \frac{4}{\lambda}.$$

• We now argue by induction.
Suppose that the claim holds for $k \geq 1$.

Let $x_0 \in [a,b]$ be such that:

$$|\varphi^{(k)}(x_0)| = \min_{a \leq x \leq b} |\varphi^{(k)}(x)|.$$

Since $|\varphi^{(k+1)}| \geq 1$, it follows that $|\varphi^{(k)}| \geq \delta$ outside of $(x_0-\delta, x_0+\delta)$.

Hence:

$$|I(\lambda)| \leq \left|\int_a^{x_0-\delta} e^{i\lambda\varphi(x)}dx\right| + \left|\int_{x_0-\delta}^{x_0+\delta} e^{i\lambda\varphi(x)}dx\right|$$
$$+ \left|\int_{x_0+\delta}^{b} e^{i\lambda\varphi(x)}dx\right|$$

$\leq$ {replace λ by $\lambda\delta$ and φ by $\frac{\varphi}{\delta}$ and use the inductive assumption}

$$\leq c_k (\lambda\delta)^{-\frac{1}{k}} + 2\delta + c_k (\lambda\delta)^{-\frac{1}{k}}$$

$$= 2c_k (\lambda\delta)^{-\frac{1}{k}} + 2\delta$$

We choose $\delta > 0$ s.t.

$$(\lambda\delta)^{-\frac{1}{k}} = \delta$$

$$\underline{\delta = \lambda^{-\frac{1}{k+1}}}$$

$$\Rightarrow |I(\lambda)| \leq (2c_k + 2)\lambda^{-\frac{1}{k+1}}$$

so we can choose $c_{k+1} := 2c_k + 2$. $\square$

<u>Proposition 3.3</u> (Van der Corput Lemma)
Under the hypotheses of Proposition 3.2, we have for $f \in C^\infty(\mathbb{R})$

$$\left|\int_a^b e^{i\lambda\varphi(x)} \cdot f(x)\,dx\right| \leq C_k \lambda^{-\frac{1}{k}} \left(\|f\|_{L^\infty([a,b])} + \|f'\|_{L^1([a,b])}\right)$$

<u>Proof:</u>
Let $G(x) := \int_a^x e^{i\lambda\varphi(y)}\,dy$.
By Proposition 3.2, we know that

$$|G(x)| \leq C_k \lambda^{-\frac{1}{k}}, \text{ for } \underline{c_k > 0 \text{ independent of } x}.$$

Integration by parts implies:

$$\left|\int_a^b e^{i\lambda\varphi} \cdot f\,dx\right| = \left|\int_a^b G' f\,dx\right| = \left|(Gf)\Big|_a^b - \int_a^b G f'\,dx\right|$$

$$\leq c_k \lambda^{-1/k} \cdot \left(\|f\|_{L^\infty([a,b])} + \|f'\|_{L^1([a,b])}\right). \quad \square$$

Sketch of proof of the fact that $A \in L^\infty(\mathbb{R})$

We take $\varphi_0 \in C^\infty(\mathbb{R})$ s.t.

$$\varphi_0(\xi) = \begin{cases} 1 & \text{for } |\xi| > 4 \\ 0 & \text{for } |\xi| < 3 \end{cases}$$

We now write

$$A(x) = \int_{-\infty}^{+\infty} e^{ix\xi + i\xi^3} \varphi_0(\xi)\, d\xi$$

$$+ \int_{-\infty}^{+\infty} e^{ix\xi + i\xi^3} \cdot (1 - \varphi_0(\xi))\, d\xi .$$

Note that $1 - \varphi_0 \in L^1$.

Hence, the second term is bounded uniformly in x.

We need to consider the first term.

$$I(x) := \int_{-\infty}^{+\infty} e^{ix\xi + i\xi^3} \varphi_0(\xi)\, d\xi$$

$\psi_x(\xi) := x\xi + \xi^3$ is the phase function

$\psi_x'(\xi) = x + 3\xi^2$.

One then deduces the result by an appropriate use of Van der Corput's Lemma.

(This will be outlined on the homework).

A more detailed analysis (based on the same techniques) can be used to show

$$|\partial_x|^\beta A \in L^\infty(\mathbb{R}) \text{ for all } \beta \in [0, \tfrac{1}{2}].$$

Also,

$$|A(x)| \lesssim \frac{1}{(1+x_-)^{1/4}} e^{-c x_+^{3/2}}$$

The Strichartz estimates for the Airy equation exhibit a derivative gain.

Let $S(t)$ be the flowmap for the Airy equation.

$$S(t)\Phi = e^{-t\partial_x^3}\Phi .$$

For (q,r) with $4 \leq q \leq +\infty$, $2 \leq r \leq +\infty$ s.t. $\frac{4}{q}+\frac{2}{r}=1$, one has:

i) $\| |\partial_x|^{\frac{1}{q}} u \|_{L^q_t L^r_x} \lesssim \|\Phi\|_{L^2_x}$

ii) $\| |\partial_x|^{\frac{2}{q}} \int_0^t S(t-\tau)F(\cdot,\tau)\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{q'}_t L^{r'}_x}$

2.4: Strichartz estimates in the periodic setting

Let us consider the domain $\mathbb{T}^1 = [0,1]$ and the Schrödinger equation on this domain.

Given $f \in L^2(\mathbb{T}^1)$, we write its Fourier series expansion

$$f(x) = \sum_k a_k e^{2\pi i k x}$$

The free Schrödinger evolution is given by

$$S(t)f = \sum_k a_k e^{-4\pi^2 i k^2 t + 2\pi i k x}$$; 1-periodic in x, $\frac{1}{2\pi}$-periodic in t

By Plancherel's theorem, we know that

$$\|S(t)f\|_{L^2_x} = \|f\|_{L^2_x}.$$

As we saw earlier, it is hence impossible to prove an estimate of the type

$$\|S(t)f\|_{L^p_x} \leq \frac{C(f)}{|t|^q}$$ for some $p>2, q>0$.

Therefore, the previous proofs used on $\mathbb{R}^n$ do not carry over to the periodic setting.

$\rightsquigarrow$ Dispersion is weaker in the periodic setting.

Theorem (Bourgain 1993)

Suppose $f = \sum_k a_k e^{2\pi i k x} \in L^2(\mathbb{T}^1)$. Then, one has:

i) $\|S(t)f\|_{L^4_t L^4_x([0,1]\times[0,\frac{1}{2\pi}])} =: \|S(t)f\|_{L^4_{t,x}(loc)}$

ii) If $a_k = 0$ for $|k| > N$, i.e. $\operatorname{supp} \hat{f}_N \subseteq \{k; |k| \leq N\}$ then, for all $\varepsilon > 0$, one has:

$$\|S(t)f_N\|_{L^6_{t,x}(loc)} \leq C_\varepsilon N^\varepsilon \|f_N\|_{L^2_x}.$$

Recall that $(6,6)$ is admissible in 1D $\left(\frac{2}{6} + \frac{1}{6} = \frac{1}{2}\right)$.

$\rightsquigarrow$ One "loses ε derivatives".

iii) Under the same assumptions as in ii), one has:

$$\|S(t)f_N\|_{L^q_{t,x}(loc)} \leq C_\varepsilon N^{\frac{1}{2}-\frac{3}{q}+\varepsilon} \cdot \|f_N\|_{L^2_x}$$

when $q \in (6, +\infty]$.

<u>Proof:</u>

i) $$\|S(t)f\|_{L^4_{t,x}(loc)} = \| |S(t)f|^2 \|^{1/2}_{L^2_{t,x}(loc)}$$

Now,

$$|S(t)f|^2 = (S(t)f) \cdot \overline{(S(t)f)} =$$

$$= \left(\sum_{k_1} a_{k_1} e^{2\pi i k_1 x - 4\pi^2 i |k_1|^2 t}\right) \cdot \left(\sum_{k_2} \overline{a}_{k_2} e^{-2\pi i k_2 x + 4\pi^2 i |k_2|^2 t}\right)$$

$$= \sum_k |a_k|^2 + \sum_{k_1 \neq k_2} a_{k_1} \overline{a}_{k_2} e^{2\pi i (k_1 - k_2)x - 4\pi^2 i (k_1^2 - k_2^2)t}$$

[We estimate the $L^2_{t,x}(loc)$ norm of this quantity.]

Observe that if $k_1 \neq k_2$ are given and the quantities

$$\begin{cases} k_1^2 - k_2^2 = (k_1 - k_2)\cdot(k_1 + k_2) \\ k_1 - k_2 \end{cases}$$

are specified, then k_1 and k_2 are uniquely determined. In particular, by the triangle inequality, it follows that

$$\| |S(t)f|^2 \|^2_{L^2_{t,x}(loc)}$$

$$\leq C\left(\sum_k |a_k|^2 + \left(\sum_{k_1 \neq k_2} |a_{k_1}|^2 \cdot |a_{k_2}|^2\right)^{1/2}\right)$$

$$\leq C\Big(\sum_k |a_k|^2\Big)^{1/2} = C\|f\|_{L^2_x}.$$

ii) $\|S(t)f_N\|_{L^6_{t,x}(loc)} = \|(S(t)f_N)^3\|^{1/3}_{L^2_{t,x}(loc)}$

$$= \Big\| \sum_{\substack{k_1,k_2,k_3 \\ |k_j|\leq N}} a_{k_1} a_{k_2} a_{k_3} e^{2\pi i(k_1+k_2+k_3)x - 4\pi^2 i(k_1^2+k_2^2+k_3^2)t} \Big\|^{1/3}_{L^2_{t,x}(loc)}$$

We compute:

$$\Big\| \sum_{\substack{k_1,k_2,k_3 \\ |k_j|\leq N}} a_{k_1} a_{k_2} a_{k_3} e^{2\pi i(k_1+k_2+k_3)x - 4\pi^2 i(k_1^2+k_2^2+k_3^2)t} \Big\|^2_{L^2_{t,x}(loc)}$$

$$= C \cdot \sum_{\substack{k, j \\ |k|\leq 3N \\ |j|\leq 3N^2}} \Big| \sum_{\substack{k_1,k_2 \\ k_1^2+k_2^2+(k-k_1-k_2)^2=j}} a_{k_1} a_{k_2} a_{k-k_1-k_2} \Big|^2 = (*)$$

(We look at k_1,k_2,k_3 such that $k_1+k_2+k_3=k$ and $k_1^2+k_2^2+k_3^2=j$).

- For fixed k,j with $|k|\leq 3N$, $|j|\leq 3N^2$, we let

$$r_{k,j} := \{(k_1,k_2)\in \mathbb{Z} \text{ s.t. } |k_1|\leq N, |k_2|\leq N, k_1^2+k_2^2+(k-k_1-k_2)^2=j\}.$$

We use the Cauchy-Schwarz inequality in (k_1,k_2) to deduce that:

$$(*) \leq C \cdot \sum_{\substack{k,j \\ |k|\leq 3N^2 \\ |j|\leq 3N^2}} \Big[r_{k,j} \cdot \sum_{k_1,k_2} |a_{k_1}|^2 \cdot |a_{k_2}|^2 \cdot |a_{k-k_1-k_2}|^2 \Big]$$

$$\leq C \underbrace{\sup_{\substack{|k|\leq 3N \\ |j|\leq 3N^2}} r_{k,j}}_{=:K_N} \cdot \sum_{\substack{k,j \\ |k|\leq 3N \\ |j|\leq 3N}} \sum_{\substack{k_1,k_2 \\ |k|\leq 3N \\ |j|\leq 3N^2}} |a_{k_1}|^2 \cdot |a_{k_2}|^2 \cdot |a_{k-k_1-k_2}|^2$$

$$\leq K_N \cdot \sum_{k,k_1,k_2} |a_{k_1}|^2 \cdot |a_{k_2}|^2 \cdot |a_{k-k_1-k_2}|^2$$

$$= K_N \cdot \Big(\sum_k |a_k|^2\Big)^3$$

$$\Rightarrow \|S(t)f\|_{L^6_{t,x}(\mathrm{loc})} \leq K_N^{\frac{1}{6}} \|f\|_{L^2_x}.$$

We need to show that $K_N = \mathcal{O}(N^{\varepsilon})$.

- Let us fix $|k| \leq N$, $|j| \leq 3N^2$.

Suppose that (k_1,k_2) are s.t. $|k_1|, |k_2| \leq N$ and

$k^2 + k_1^2 + (k-k_1-k_2)^2 = j$

By elementary manipulations, one can show:

$(3m_1 - 2k)^2 + 3m_2^2 = 6j - 2k^2$

where $m_1 := k_1 + k_2$, $m_2 := k_1 - k_2$.

$\ulcorner$

$$k_1^2 + k_2^2 - kk_1 - kk_2 + k_1k_2 = \frac{j-k^2}{2}$$

$$\frac{3}{4}(k_1+k_2)^2 + \frac{1}{4}(k_1-k_2)^2 - k(k_1+k_2) = \frac{j-k^2}{2}$$

$$\frac{3}{4}m_1^2 + \frac{1}{4}m_2^2 - km_1 = \frac{j-k^2}{2} \quad / \cdot 12$$

$$9m_1^2 - 12km_1 + 3m_2^2 = 6j - 6k^2$$

$$9m_1^2 - 12km_1 + 4k^2 + 3m_2^2 = 6j - 2k^2$$

$$(3m_1 - 2k)^2 + 3m_2^2 = 6j - 2n^2$$

$$X^2 + 3Y^2 = Z, \quad |Z| \lesssim N^2.$$

<u>Fact:</u> $\mathbb{Z} + i\sqrt{3}\mathbb{Z}$ is a <u>Euclidean division domain</u>.

Recall that an Euclidean division domain is an integral domain R (commutative ring s.t. $ab=0 \Rightarrow a=0$ or $b=0$) with a Euclidean function $e: R \to \mathbb{R}\setminus\{0\}$ s.t. for all $a, b \in R$, $b \neq 0$, there exist $q, r \in R$ with $a = bq + r$ and $r = 0$ or $f(r) < f(b)$.

$\rightsquigarrow$ The number of divisors of $A \neq 0$ in R is at most

$$\exp\left(c \, \frac{\log A}{\log A \log A}\right) = \mathcal{O}(A^{\varepsilon}).$$

We rewrite

$$X^2 + 3Y^2 = Z \text{ as}$$

$$(X + i\sqrt{3}\,Y)(X - i\sqrt{3}\,Y) = Z \; ; \; X \pm i\sqrt{3}Y \in \mathbb{Z} + i\sqrt{3}\mathbb{Z}$$

$$\Rightarrow \mathcal{O}(Z^{\varepsilon/2}) = \mathcal{O}(N^{\varepsilon}) \text{ solutions.}$$

$$\Rightarrow \underline{K_N = \mathcal{O}(N^{\varepsilon})}.$$

We hence obtain ii).

iii) Let us note that for $g_N \in L^2$ with $\mathrm{supp}\, \widehat{g}_N \subseteq \{|k| \leq N\}$ we have

$$\|g_N\|_{L^\infty} \leq C N^{\frac{1}{2}} \|g\|_{L^2_x}.$$

Namely, for all $x \in \mathbb{T}^1$, we have

$$|g_N(x)| = \Big|\sum_{|k| \leq N} \widehat{g}_N(k) e^{2\pi i k x}\Big| \leq C N^{1/2} \Big(\sum_{|k| \leq N} |\widehat{g}_N(k)|^2\Big)^{1/2}$$

$$= CN^{1/2} \|g_N\|_{L^2_x}$$

by Cauchy-Schwarz and Plancherel's theorem.

We apply this to deduce:

$$\|S(t)f_N\|_{L^\infty_{t,x}(loc)} = \big\| \|S(t)f_N\|_{L^\infty_x} \big\|_{L^\infty_t(loc)}$$

$$\le CN^{1/2} \big\| \|S(t)f_N\|_{L^2_x} \big\|_{L^\infty_t(loc)} = CN^{1/2} \|f_N\|_{L^2_x}$$

From ii), we know

$$\|S(t)f_N\|_{L^6_{t,x}(loc)} \le CN^{\varepsilon} \|f_N\|_{L^2_x}.$$

The result of iii) now follows by interpolation. $\square$

<u>Remarks:</u>

① In the work of Bourgain, it is shown that for $\varphi_N = \frac{1}{\sqrt{N}} \sum_{k=0}^{N} e^{2\pi i k x}$ one has

$$\|S(t)\varphi_N\|_{L^6_{t,x}(loc)} \to \infty \text{ as } N \to \infty.$$

② The higher-dimensional setting is more involved.

The result that one can prove here is the following.

For $\operatorname{supp} \widehat{f_N} \subseteq \{|k| \le N\}$, one has

$$\|S(t)f_N\|_{L^p_{t,x}} \le C^{\frac{n}{2} - \frac{n+2}{p}} \|f_N\|_{L^2_x}.$$

for $p > \frac{2(n+2)}{n}$.

This question was resolved in the work of <u>Bourgain–Demeter</u> (2014) and <u>Killip–Vişan</u> (20[illegible].

2.5: The geometric interpretation of Strichartz estimates

- Let us consider the Schrödinger equation on $\mathbb{R}^n$

$$\begin{cases} iu_t + \Delta u = 0 \\ u|_{t=0} = \Phi \end{cases}$$

We saw earlier that

$$u(x,t) = S(t)\Phi(x) = \int e^{2\pi i x\xi - 4\pi^2 i |\xi|^2 t}\, \widehat{\Phi}(\xi)\, d\xi$$

i.e.

$$\widehat{u}(\xi,t) = e^{-4\pi^2 i |\xi|^2 t}\, \widehat{\Phi}(\xi)$$

$$\Rightarrow \widetilde{u}(\xi,\tau) = \delta(\tau + 2\pi|\xi|^2)\, \widehat{\Phi}(\xi)$$

where

$$\widetilde{F}(\xi,\tau) := \int F(x,t)\, e^{-2\pi i x\xi - 2\pi i t\tau}\, dx\, dt$$

denotes the spacetime Fourier transform.

In particular, $\widetilde{u}$ is supported on the paraboloid $\underline{\tau = -2\pi|\xi|^2}$.

. How does one set this up rigorously?

Let $S \subseteq \mathbb{R}^{n+1}$ be an n-dimensional hypersurface. Suppose that, up to a change of variables S is locally given as a graph near zero of a smooth function φ s.t.

$$\varphi(0)=0, \nabla\varphi(0)=0, \quad \det\left(\frac{\partial^2\varphi}{\partial x_i \partial x_j}(0)\right) \neq 0.$$

(for example, we can take $\varphi(\xi) = -2\pi|\xi|^2$).

Let $d\sigma = \sqrt{1+|\nabla\varphi|^2}\, dx_1 \dots dx_n$ denote the surface measure on S induced by Lebesgue measure on $\mathbb{R}^n$.

→ One wants to make sense of $\hat{f}|_S$. (Here, $\hat{\ }$ = the Fourier transform on $\mathbb{R}^{n+1}$.)

Theorem (Stein-Tomas)

Let $S \subseteq \mathbb{R}^{n+1}$ be a hypersurface with nonvanishing Gaussian curvature. Then one has

$$\|\hat{f}|_S\|_{L^2(\sigma)} := \left(\int_S |\hat{f}(\xi)|^2 d\sigma(\xi)\right)^{1/2} \leq C \cdot \|f\|_{L^p(\mathbb{R}^{n+1})}$$

for all $p \leq \frac{2n+4}{n+4}$.

Setup: For $\eta \in S \subseteq \mathbb{R}^{n+1}$, we write

$$Rf(\eta) := \int_{\mathbb{R}^{n+1}} e^{-2\pi i y\cdot\eta} f(y)\, dy, = \hat{f}|_S(\eta).$$

$\{ y = (x,t), \eta = (\xi, \tau) \in \mathbb{R}^{n+1} \}$

"restrict $\hat{f}$ to S". So, $\|Rf\|_{L^2(\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^{n+1})}$

Its formal adjoint is given by

$$R^* f(y) = \int_S e^{2\pi i y\cdot\eta} f(\eta)\, d\sigma(\eta), \text{ for } y \in \mathbb{R}^{n+1}. *$$

[* Then, we have $\langle Rf, g\rangle_{L^2(\sigma)} = \langle f, R^* g\rangle_{L^2(\mathbb{R}^{n+1})}$

$$\| \widehat{f}|_S \|^2_{L^2(\sigma)} = \langle Rf, Rf \rangle_{L^2(\sigma)} = \langle R^*Rf, f \rangle_{L^2(\mathbb{R}^{n+1})}$$

Hence, by Hölder's inequality, it is equivalent to prove

$$\| R^* R f \|_{L^{p'}(\mathbb{R}^{n+1})} \leq C \| f \|_{L^p(\mathbb{R}^{n+1})}.$$

One can compute:

$$(R^*Rf)(y) = \int_{\mathbb{R}^{n+1}} \int_S e^{2\pi i \eta \cdot (y-z)} \, d\sigma(\eta) \, f(z) \, dz$$

$$= (K * f)(y)$$

where $K(z) = \int_S e^{2\pi i \eta \cdot z} \, d\sigma(\eta) = (d\sigma)^\wedge(-z)$.

$\rightsquigarrow$ certain regimes can be treated by the Hardy-Littlewood-Sobolev inequality, provided that one knows good estimates on $(d\sigma)^\wedge$.

- a dual version of the Stein-Tomas theorem reads

$$\| (\varphi\sigma)^\wedge \|_{L^q(\mathbb{R}^{n+1})} \leq C \| \varphi \|_{L^2(\sigma)}$$

for all $q \geq \frac{2n+4}{n}$.

- We lift $\widehat{\Phi}$ to the paraboloid and obtain a function φ. We let $\varphi(\xi, -2\pi|\xi|^2) := \widehat{\Phi}(\xi)$.

Then $S(t)\Phi = (\varphi\sigma)^\vee$

$$\rightsquigarrow \| S(t)\Phi \|_{L^{2+\frac{4}{n}}_{t,x}(\mathbb{R}^{n+1})} \leq C \| \widehat{\Phi} \|_{L^2_\xi} = C \| \Phi \|_{L^2_x}.$$