§1: Preliminaries:

1.1: The concept of dispersion (Informal)

Example: Schrödinger equation

$$i u_t + \Delta u = 0; \quad \text{here } u : \mathbb{R}^n_x \times \mathbb{R}_t \to \mathbb{C}$$

Let us fix $k \in \mathbb{R}^n$ and let us look for a solution of the form $u(x,t) = e^{i(kx + wt)}$ (for some $w \in \mathbb{R}$). [We drop factors of 2π for now]

We find $i u_t + \Delta u = -(w + |k|^2) e^{i(kx+wt)}$

Hence, we obtain:

$$\underline{w = -|k|^2} \quad ; \quad \text{"Dispersion relation".}$$

Let us fix $\varphi \in C_c^\infty(\mathbb{R}^n)$.

We want to find $r \in \mathbb{R}^n$ such that

$$w_\varepsilon(x,t) := e^{i(kx - |k|^2 t)} \cdot \varphi(\varepsilon(x - rt))$$

satisfies $(i\partial_t + \Delta) w_\varepsilon = \mathcal{O}(\varepsilon^2)$.

"w_ε is obtained by moving the wave packet around. around."

By the Chain Rule, we compute that

$$(i\partial_t + \Delta) w_\varepsilon = i\varepsilon(-r + 2k) \cdot e^{i(kx - |k|^2 t)} \cdot (\nabla \varphi)(\varepsilon(x - rt)) + \mathcal{O}(\varepsilon^2)$$

So, we need to take $r = 2k = -\nabla(-|k|^2) = -\nabla\omega$

"Group velocity".

$$W_\varepsilon(x,t) = e^{i(kx - |k|^2 t)}\,\varphi\big(\varepsilon(x - 2kt)\big)$$

$\rightsquigarrow$ Heuristically, a wave of frequency k moves in space with velocity $\sim 2k$. (up to factors of 2π)

$\rightsquigarrow$ High frequency waves travel faster than low-frequency waves, while keeping the same amplitude, thus broadening the wave packet. $\rightsquigarrow$ "Dispersion".

Moreover, we formally differentiate under the integral sign and find:

$$\frac{d}{dt}\int |u(x,t)|^2\,dx = 2\,\mathrm{Re}\int u_t(x,t)\,\bar{u}(x,t)\,dx$$

$$= 2\,\mathrm{Re}\int i\,\Delta u(x,t)\,\bar{u}(x,t)\,dx$$

$$= -2\,\mathrm{Re}\int i\,|\nabla u(x,t)|^2\,dx = 0$$

<u>Two effects</u>: "Dispersion" of wave packets

+

Conservation of L^2 norm.

$\Rightarrow$ Our goal is to analyze the interplay between these two effects.

We summarize this introduction.

<u>Informal definition of dispersion</u>:

"Wave packets spread out in space with different speeds depending on the frequency while maintaining the same amplitude."

Example: Airy equation

$u_t + u_{xxx} = 0$; here $u: \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{R}$

One finds a solution of the form

$u(x,t) = e^{i(kx + k^3 t)}$

In other words, $\underline{w = k^3}$.

non-Example: Heat equation

$u_t - \Delta u = 0$

$u(x,t) = e^{i(kx + wt)}$

gives

$iw + |k|^2 = 0$

$w = i|k|^2$

so $u(x,t) = e^{ikx} \cdot e^{-|k|^2 t}$

In this case, the amplitude of the wave packet is not conserved.

non-Example: Transport equation

$$u_t + c \cdot \nabla u = 0 \quad \text{for } c \in \mathbb{R}^n$$

$u(x,t) = e^{i k \cdot (x - t \cdot c)}$

The wave packet gets transported with velocity c, independently of k, $\quad \nabla k = (1, 1, \dots, 1)$

Example: Wave equation

$$u_{tt} - \Delta u = 0$$

If we look for solutions of the form $u(x,t) = e^{i(k \cdot x + wt)}$ we obtain $-w^2 + |k|^2 = 0$

$$\Rightarrow w = \pm |k|$$

$$\nabla w = \pm \frac{k}{|k|}$$

- Wave packets move at speed one, but in different directions.

1.2. The Fourier transform on $\mathbb{R}^n$

1.2.1. The Fourier transform on $L^1(\mathbb{R}^n)$

Given $f \in L^1(\mathbb{R}^n)$, we define its Fourier transform as:

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx \quad \text{for } \xi \in \mathbb{R}^n.$$

Basic properties:

① $f \mapsto \widehat{f}$ is a linear map from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ with $\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}$.

② $\widehat{f}$ is continuous.

③ $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$ (Riemann-Lebesgue lemma)

④ If τ_h denotes translation by $h \in \mathbb{R}^n$

$\tau_h f(x) := f(x-h)$, then: for $f \in L^1(\mathbb{R}^n)$

$$(\tau_h f)^\wedge(\xi) = e^{-2\pi i h \cdot \xi} \widehat{f}(\xi).$$

and

$$(e^{-2\pi i h \cdot x} f)^\wedge(\xi) = (\tau_{-h} \widehat{f})(\xi) = \widehat{f}(\xi + h)$$

⑤ Given $\lambda > 0$, we let $f_\lambda(x) := \frac{1}{\lambda^n} f\left(\frac{x}{\lambda}\right)$.

Then $\widehat{f_\lambda}(\xi) = \widehat{f}(\lambda \xi)$.

⑥ If $f, g \in L^1(\mathbb{R}^n)$, then the convolution $(f * g)(x) := \int f(x-y) g(y) dy$ satisfies $(f * g)^\wedge(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi)$.

⑦ If $f, g \in L^1(\mathbb{R}^n)$, then

$$\int \widehat{f}(y)\, g(y)\, dy = \int f(y) \widehat{g}(y)\, dy.$$

Sketch of Proof:

① Linearity follows by definition.
By the triangle inequality:
$|\hat{f}(\xi)| \leq \int |f(x)|\,dx = \|f\|_{L^1}$.

② $\hat{f}(\xi+h)-\hat{f}(\xi) = \int \underbrace{f(x)}_{\in L^1} \cdot \underbrace{\left(e^{-2\pi i x\cdot(\xi+h)} - e^{-2\pi i x\cdot \xi}\right)}_{|\cdot|\leq 1, \text{ and tends to } 0 \text{ as } h\to 0} dx$

Use the Dominated Convergence Theorem.

③ It suffices to show the claim for $\varphi \in C_c^\infty$, by using the density of C_c^∞ in L^1 and $\|(f-\varphi)^\wedge\|_{L^\infty} \leq \|f-\varphi\|_{L^1}$ (i.e. Part ①). Suppose that $n=1$, and $\xi \neq 0$.
Observe that $e^{-2\pi i x \xi} = \frac{-1}{2\pi i \xi} \cdot \left(e^{-2\pi i x \xi}\right)'$, so

$$\hat{\varphi}(\xi) = \int \varphi(x)\, e^{-2\pi i x\xi}\,dx$$
$$= \int \varphi(x) \cdot \frac{-1}{2\pi i \xi} \cdot \left(e^{-2\pi i x\xi}\right)' dx$$
$$= \frac{1}{2\pi i \xi} \cdot \int \varphi'(x)\, e^{-2\pi i x\xi}\,dx \to 0 \quad \text{as } |\xi| \to \infty.$$

For general n, if $\xi \neq 0$, we find $j \in \{1,\dots,n\}$ such that $|\xi_j| \geq \frac{1}{\sqrt{n}}|\xi|$ and repeat this argument by integrating by parts in x_j.

Properties ④–⑦ follow by a direct calculation. This is left as an exercise. It is important to note that, at each step, the function to which we are applying the Fourier transform belongs to L^1.

1.2.2. The Fourier transform on $\mathcal{S}(\mathbb{R}^n)$

$\mathcal{S}(\mathbb{R}^n)$:= class of functions such that for all $\alpha, \beta \in \mathbb{N}^n$; $p_{\alpha,\beta}(f) := \sup_x |x^\alpha \partial_x^\beta f| < \infty$

$x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $\partial_x^\beta := \partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n}$

$\to$ Dense in $L^1(\mathbb{R}^n)$.

We can consider the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$.

Properties:

① If $f \in \mathcal{S}(\mathbb{R}^n)$ then $\frac{\partial f}{\partial x_j}$, $-2\pi i x_j f \in \mathcal{S}(\mathbb{R}^n)$ and

a) $\left(\frac{\partial f}{\partial x_j}\right)^\wedge(\xi) = 2\pi i \xi_j \hat{f}(\xi)$

b) $(-2\pi i x_j f)^\wedge(\xi) = \frac{\partial \hat{f}}{\partial \xi_j}(\xi)$.

② Let $\mathbb{D} := \frac{1}{2\pi i}\partial$, then for any $f \in \mathcal{S}(\mathbb{R}^n)$ and for any polynomial P we have

$$P(\mathbb{D})\hat{f}(\xi) = \left(P(-x)f\right)^\wedge(\xi)$$
$$\left(P(\mathbb{D})f\right)^\wedge(\xi) = P(\xi)\hat{f}(\xi).$$

③ The Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ to itself.

④ $\left(e^{-\pi|x|^2}\right)^\wedge(\xi) = e^{-\pi|\xi|^2}$.

⑤ For $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we have

$$f(x) = \int \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi \quad \text{(Fourier inversion formula).}$$

⑥ For $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\int |\hat{f}(\xi)|^2 d\xi = \int |f(x)|^2 dx \quad \text{(Plancherel's theorem).}$$

Sketch of Proof: [for full details see Harmonic Analysis notes]

(1) a) Use integration by parts.
b) Use the Dominated Convergence Theorem.

(2) Use (1).

(3) Use (2) and $\hat{\ }: \mathcal{S}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$
$(\subseteq L^1(\mathbb{R}^n))$

(4) Use (1) to deduce that $e^{-\pi|x|^2}$ and $(e^{-\pi|x|^2})^\wedge$ solve the same ODE initial-value problem when $n=1$

$$\begin{cases} F' + 2\pi x \cdot F = 0 \\ F(0) = 1 \end{cases}$$

Deduce the claim for general n from here.

(5) First reduce to the claim for $x=0$.
Use (4) and the dilation property. (Prop. (5) from earlier).

(6) Use (5) and property (7) from earlier.

1.2.3: The Fourier transform on $L^2(\mathbb{R}^n)$

Plancherel's theorem allows us to extend the Fourier transform to a unitary operator on $L^2(\mathbb{R}^n)$.
It is a bijection.

1.2.4. The Fourier transform on the space of tempered distributions

Definition: Let (φ_j) be a sequence in $\mathcal{S}(\mathbb{R}^n)$. We say that $\varphi_j \to 0$ if $p_{\alpha,\beta}(\varphi_j) \to 0$ as $j \to \infty$ for all $\alpha,\beta \in \mathbb{N}^n$.

Definition: We say that u is a tempered distribution, $u \in \mathcal{S}'(\mathbb{R}^n)$ if
i) $u: \mathcal{S}'(\mathbb{R}^n) \to \mathbb{C}$ is linear.
ii) u is continuous, i.e. $\varphi_j \to 0$ implies $p_{\alpha,\beta}(\varphi_j) \to 0$.

· Examples:

· If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then
$u(\varphi) := \int f \cdot \varphi \, dx$ defines an element of $\mathcal{S}'(\mathbb{R}^n)$.

· $\delta_0(\varphi) := \varphi(0)$. The delta function.

non-Example:

$$u(\varphi) = \int e^{x^2} \cdot \varphi \, dx.$$

We can define the Fourier transform of $u \in \mathcal{S}'(\mathbb{R}^n)$ by duality

$$\hat{u}(\varphi) := u(\hat{\varphi}).$$ [Example: $\hat{\delta}_0 = 1$.]

(This corresponds to

$$\int \hat{f} \varphi = \int f \hat{\varphi} \quad \text{for } f, \varphi \in L^1.)$$

Fact: $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$. This extends our previous definition + $\wedge$ is a bijection.

· We define $x_j u$ and $\frac{\partial u}{\partial x_j}$ by duality:

$$x_j u(\varphi) := u(x_j \varphi)$$

$$\frac{\partial u}{\partial x_j}(\varphi) := -u\left(\frac{\partial \varphi}{\partial x_j}\right)$$

One can check that

$$\left(\frac{\partial u}{\partial x_j}\right)^\wedge = 2\pi i \xi_j \hat{u}$$

$$(-2\pi i x_j u)^\wedge = \frac{\partial \hat{u}}{\partial \xi_j}$$

hold.

Moreover, given $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ one can show that $(u * \varphi)(x) := u(\varphi(x-\cdot))$ is an element of $C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ such that $(u*\varphi)^\wedge = \widehat{u}\cdot\widehat{\varphi}$ where $\widehat{u}\cdot\widehat{\varphi}\,(\psi) := \widehat{u}(\widehat{\varphi}\,\psi)$, for $\psi \in \mathcal{S}(\mathbb{R}^n)$.

Example: We consider the initial-value problem

$$\begin{cases} i u_t + \Delta u = 0 & \text{on } \mathbb{R}^n_x \times \mathbb{R}_t \\ u\big|_{t=0} = f & \text{on } \mathbb{R}^n_x \end{cases}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Suppose that $u \in C^\infty(\mathbb{R}_t)\,\mathcal{S}(\mathbb{R}^n_x)$. We take Fourier transforms in x:

$$\begin{cases} i\,\widehat{u}_t - 4\pi^2|\xi|^2\,\widehat{u} = 0 \\ \widehat{u}\,\big|_{t=0} = \widehat{f} \end{cases}$$

For fixed ξ, we obtain an ODE which has a solution

$$\widehat{u}(\xi,t) = e^{-4\pi^2 i|\xi|^2 t}\cdot\widehat{f}(\xi)$$

We want to show that

$$e^{-4\pi^2 i|\xi|^2 t} = \underbrace{\left(\frac{e^{i|x|^2/4t}}{(4\pi i t)^{n/2}}\right)}_{\in\,\mathcal{S}'(\mathbb{R}^n)}{}^{\wedge}(\xi),\quad t \neq 0 \quad (*)$$

because we then obtain $u(x,t) = \dfrac{e^{i|x|^2/4t}}{(4\pi i t)^{n/2}} * f\,(x)$.

In order to prove $(*)$, it suffices to prove that

$$\left(e^{-a|x|^2}\right)^\wedge = \left(\frac{\pi}{a}\right)^{n/2} e^{-\pi^2|\xi|^2/a}, \text{ for } \operatorname{Re} a \geq 0,\ a \neq 0. \qquad (**)$$

(we take $a = \frac{1}{4it}$).

Here, $\sqrt{\cdot}$ is the branch of the square root on $\operatorname{Re} > 0$.

· If $a > 0$, $(**)$ immediately follows from dilation properties of the Fourier transform.

Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we let

$$F(a) := \left(e^{-a|x|^2}\right)^\wedge(\varphi) = \left(e^{-a|x|^2}\right)(\hat{\varphi}) = \int e^{-a|x|^2}\hat{\varphi}(x)\,dx$$

$$G(a) := \left(\frac{\pi}{a}\right)^{n/2} \cdot e^{-\pi^2|\xi|^2/a}(\varphi) = \left(\frac{\pi}{a}\right)^{n/2} \cdot \int e^{-\pi^2|\xi|^2/a}\varphi(\xi)\,d\xi$$

Then F, G are analytic on $\operatorname{Re} > 0$.

By analytic continuation, we obtain that $F = G$ on $\operatorname{Re} > 0$.

Moreover, by Dominated Convergence we know that

$$F\left(\frac{1}{4it}\right) = \lim_{\varepsilon \to 0+} F\left(\varepsilon + \frac{1}{4it}\right)$$

$$G\left(\frac{1}{4it}\right) = \lim_{\varepsilon \to 0+} G\left(\varepsilon + \frac{1}{4it}\right).$$

Hence, we deduce $(**)$.

In conclusion, we obtain:

$$\boxed{u(x,t) = \frac{e^{i|x|^2/4t}}{(4\pi i t)^{n/2}} * f}$$

$$\Rightarrow \quad |u(x,t)| \leq \frac{1}{(4\pi|t|)^{n/2}} \cdot \|f\|_{L^1}\ ; \quad \|u(x,t)\|_{L^2_x} = \|f\|_{L^2_x}.$$

Example: Consider the first-order equation

$$\partial_t u(x,t) = Lu(x,t) \qquad (*)$$

where $L = \sum_{|\alpha| \le m} c_\alpha \partial_x^\alpha$.

We take Fourier transforms in x and argue as earlier to obtain

$$\partial_t \widehat{u}(\xi,t) = i \cdot \sum_{|\alpha| \le m} i^{|\alpha|-1} \cdot (2\pi)^{|\alpha|} c_\alpha \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \widehat{u}(\xi,t)$$

$$= i \cdot h(\xi) \cdot \widehat{u}(\xi,t)$$

In order for $(*)$ to be a dispersive PDE, we want h to be real and ∇h to be non-constant.

When $(*)$ is the Schrödinger equation, we have $h(\xi) \sim |\xi|^2$.

1.3 The Fourier transform on $\mathbb{T}^n = [0,1]^n$

Given $f \in L^1(\mathbb{T}^n)$, we define its Fourier coefficients by

$$\hat{f}(k) = \int_{\mathbb{T}^n} f(x) e^{-2\pi i k\cdot x}\,dx\,;\ k \in \mathbb{Z}^n.$$

This definition makes sense for Borel measures $\mu \in \mathcal{B}(\mathbb{T}^n)$:

$$\hat{\mu}(k) = \int_{\mathbb{T}^n} e^{-2\pi i k\cdot x}\,d\mu(x)$$

Fourier series: $\quad f(x) \sim \sum_{k\in\mathbb{Z}^n} \hat{f}(k) e^{2\pi i k\cdot x}$

(compare to Fourier inversion formula).

Plancherel's theorem:

If $f \in L^2(\mathbb{T}^n)$, then

$$\Big\| f - \sum_{|k|\leq N} \hat{f}(k) e^{2\pi i k\cdot x} \Big\|_{L^2(\mathbb{T}^n)} \to 0 \text{ as } N \to \infty.$$

In other words

$$f(x) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k) e^{2\pi i k\cdot x} \quad \text{in the } L^2 \text{ sense.}$$

Moreover,

$$\|f\|_{L^2}^2 = \sum_k |\hat{f}(k)|^2.$$

Important feature: Scaling arguments do not in general apply in the periodic setting.

Example / Heuristic: Consider the linear Schrödinger equation on $\mathbb{T}^n$:

$$\begin{cases} iu_t + \Delta u = 0 & \text{on } \mathbb{T}^n_x \times \mathbb{R}_t \\ u|_{t=0} = f & \in L^1(\mathbb{T}^n) \end{cases}$$

On $\mathbb{R}^n$, we could obtain decay in L^∞.
If we could prove L^∞ decay in this case, we could deduce that

$$\|u(x,t)\|_{L^2_x} \lesssim \|u(x,t)\|_{L^\infty_x} \to 0 \quad \text{as } |t| \to \infty.$$

This is a contradiction since $\|u(x,t)\|_{L^2_x}$ is constant in t.

1.4: Sobolev spaces

Definition: Given $s \in \mathbb{R}$, we define the Sobolev space of order s on $\mathbb{R}^n$, $H^s(\mathbb{R}^n)$ by:

$$H^s(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n),\ (D^s f)^\wedge(\xi) := (1+|\xi|^2)^{\frac{s}{2}} \hat{f}(\xi) \in L^2(\mathbb{R}^n) \right\}$$

Here D^s is the operator given by:

$$(D^s f)^\wedge(\xi) = (1+|\xi|^2)^{\frac{s}{2}} \hat{f}(\xi).$$

We sometimes also write $\langle \xi \rangle := (1+|\xi|^2)^{\frac{1}{2}}$.

This is called the Japanese bracket.

Given $f \in H^s(\mathbb{R}^n)$, we write

$$\|f\|_{H^s(\mathbb{R}^n)} := \| \langle \xi \rangle^s \hat{f} \|_{L^2(\mathbb{R}^n)}.$$

(We sometimes omit the argument $\mathbb{R}^n$ if it is clear that this is the domain.)

Motivation:

Suppose that $s = k \in \mathbb{N}$.

Then, for all $\alpha \in \mathbb{N}^n$ with

$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq k$, we know that

$$(\partial^\alpha f)^\wedge(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi) \quad \text{and}$$

$$|\xi^\alpha| \leq |\xi|^k \leq \langle \xi \rangle^k \leq C \cdot \sum_{|\alpha| \leq k} |\xi^\alpha|$$

Hence, the norms

$$\|f\|_{H^k} \quad \text{and} \quad \sum_{|\alpha| \leq k} \|\partial_x^\alpha f\|_{L^2}$$

are equivalent. (1.4.1)

- This definition applies on $\mathbb{T}^n$; we define

$$H^s(\mathbb{T}^n) := \left\{ \mu \in B(\mathbb{T}^n),\ (D^s \mu)^\wedge(k) = (1+|k|^2)^{\frac{s}{2}} \hat{\mu}(k) \in \ell^2(\mathbb{Z}^n) \right\}$$

In this case, we write $\|\mu\|_{H^s(\mathbb{T}^n)} := \|\langle k\rangle^s \hat{\mu}\|_{\ell^2(\mathbb{Z}^n)}$.

Moreover, (1.4.1) holds by the same proof.

<u>Remark 1:</u>

We note that, by Plancherel's theorem we have $H^0 = L^2$.

Furthermore, $L^2 \subseteq H^s$ for $s > 0$.

If $s < 0$, $H^s(\mathbb{R}^n)$ consists of tempered distributions $\notin L^2(\mathbb{R}^n)$ and $H^s(\mathbb{T}^n)$ consists of Borel measures $\notin L^2(\mathbb{T}^n)$

In other words,

$$H^s(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$$

$$H^s(\mathbb{T}^n) \subseteq \mathcal{B}(\mathbb{T}^n) \setminus L^2(\mathbb{T}^n).$$

<u>Example:</u> Let $u = \delta_0 \in \mathcal{S}'(\mathbb{R}^n)$.

Then $\hat{u} = 1$. Hence $\hat{u} \cdot \langle \xi \rangle^{-\frac{n}{2}-\varepsilon} \in L^2$ for all $\varepsilon > 0$

$\Rightarrow u \in H^{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n)$

(Analogously, if $u = \delta_0 \in \mathcal{B}(\mathbb{T}^n)$, i.e. $u(f) = f(0)$ for $f \in C(\mathbb{T}^n)$ we know that $\hat{u}(k) = 1$ for all $k \in \mathbb{Z}^n$.

Hence $\langle k \rangle^{-\frac{n}{2}-\varepsilon} \cdot \hat{u}(k) \in \ell^2(\mathbb{Z}^n)$

$\Rightarrow u \in H^{-\frac{n}{2}-\varepsilon}(\mathbb{T}^n)$.)

<u>Example:</u> Let $n = 1$, and let $f(x) = \chi_{[-1,1]}(x)$.

We can compute $\hat{f}(\xi) = \dfrac{\sin(2\pi\xi)}{\pi\xi}$.

This implies that $f \in H^s(\mathbb{R})$ for $s < \frac{1}{2}$.

From now on, we mostly work with $H^s(\mathbb{R}^n)$, unless it is specifically mentioned otherwise

Proposition 1.4.1:

1) If $s < s'$, then $H^{s'}(\mathbb{R}^n) \subsetneq H^s(\mathbb{R}^n)$.

2) $H^s(\mathbb{R}^n)$ is a Hilbert space with respect to the inner product $\langle \cdot , \cdot \rangle_s$, which is defined by

$$\langle f, g \rangle_{H^s} := \int_{\mathbb{R}^n} D^s f \cdot \overline{D^s g}\, dx = \int_{\mathbb{R}^n} \left(\langle \xi \rangle^s \hat{f}(\xi)\right) \cdot \overline{\left(\langle \xi \rangle^s \hat{g}(\xi)\right)} d\xi .$$

3) If $s_0 \leq s \leq s_1$ and $s = (1-\theta)s_0 + \theta s_1$, then

$$\|f\|_{H^s} \leq \|f\|_{H^{s_0}}^{1-\theta} \cdot \|f\|_{H^{s_1}}^{\theta} . \qquad \text{"Interpolation"}$$

Proof: Points 1) and 2) follow from the definition.
We now prove part 3).

We write

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^n} \langle \xi \rangle^{2(1-\theta)s_0} |\hat{f}(\xi)|^{2(1-\theta)} \cdot \langle \xi \rangle^{2\theta s_1} |\hat{f}(\xi)|^{2\theta} d\xi$$

$$= \int_{\mathbb{R}^n} \left(\langle \xi \rangle^{2s_0} \cdot |\hat{f}(\xi)|^2\right)^{1-\theta} \cdot \left(\langle \xi \rangle^{2s_1} |\hat{f}(\xi)|^2\right)^{\theta} d\xi ,$$

which by Hölder's inequality is

$$\leq \left(\int \langle \xi \rangle^{2s_0} |\hat{f}(\xi)|^2 d\xi\right)^{1-\theta} \cdot \left(\int \langle \xi \rangle^{2s_1} |\hat{f}(\xi)|^2 d\xi\right)^{\theta}$$

$$= \|f\|_{H^{s_0}}^{2(1-\theta)} \cdot \|f\|_{H^{s_1}}^{2\theta} . \qquad \square$$

Sobolev Embedding

Proposition 1.4.2:

Let $k \in \mathbb{N}$ and $s > \frac{n}{2} + k$.

Then, $H^s(\mathbb{R}^n)$ embeds continuously into

$C^k(\mathbb{R}^n)$ = space of functions with k continuous derivatives

$\|f\|_{C^k} \leq C_{s,n} \cdot \|f\|_{H^s}$. (here $\|f\|_{C^k} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty}$.)

Proof: By a density argument, it suffices to prove the bound we claim for $f \in \mathcal{S}(\mathbb{R}^n)$.

Let's first prove the bound when $\underline{k=0}$.

By Fourier inversion, we know

$$\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1} = \int_{\mathbb{R}^n} |\hat{f}(\xi)| \, d\xi$$

$$= \int_{\mathbb{R}^n} |\hat{f}(\xi)| \cdot \langle \xi \rangle^s \cdot \langle \xi \rangle^{-s} \, d\xi,$$

which, by the Cauchy-Schwarz inequality is

$$\leq \|f\|_{H^s} \cdot \left(\int_{\mathbb{R}^n} \frac{d\xi}{\langle \xi \rangle^{2s}} \right)^{1/2} \leq C_{s,n} \cdot \|f\|_{H^s},$$

since $s > \frac{n}{2}$.

Suppose now that $k \geq 1$.

Let us fix α with $|\alpha| \leq k$. We note that

$$\|\partial^\alpha f\|_{L^\infty} \leq \|\widehat{\partial^\alpha f}\|_{L^1} = \|(2\pi i \xi)^\alpha \hat{f}\|_{L^1}$$

$$\leq C_{\varepsilon,n} \cdot \|(2\pi i \xi)^\alpha f\|_{H^{\frac{n}{2}+\varepsilon}} \quad \forall \varepsilon > 0$$

by applying the previous argument.
The claim now follows. $\square$

<u>Proposition 1.4.3:</u> Let $s \in (0, \frac{n}{2})$ be given.
Then, for p such that $\frac{1}{p} = \frac{1}{2} - \frac{s}{n}$, we have

$$\|f\|_{L^p} \leq C_{n,s} \|f\|_{H^s}.$$

<u>Remark:</u>
In fact, a stronger inequality holds

$$\|f\|_{L^p} \leq C_{n,s} \|f\|_{\dot{H}^s}, \qquad (1.4.3')$$

where $\dot{H}^s(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n),\ |\xi|^s \hat{f}(\xi) \in L^2(\mathbb{R}^n) \}$
and we set $\|f\|_{\dot{H}^s(\mathbb{R}^n)} := \| |\xi|^s \hat{f} \|_{L^2(\mathbb{R}^n)} = \| \dot{D}^s f \|_{L^2(\mathbb{R}^n)}$
where $(\dot{D}^s f)^\wedge(\xi) := |\xi|^s \hat{f}(\xi)$.
This is the <u>homogenous Sobolev space</u>.
In particular, $\|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)}$ so
$H^s(\mathbb{R}^n) \subseteq \dot{H}^s(\mathbb{R}^n)$. This inclusion is strict (Exercise).

Before we prove Proposition 1.4.3, we summarize a few facts.

i) $\left(\frac{1}{|x|^\gamma}\right)^\wedge(\xi) = c_{n,\gamma} \cdot \frac{1}{|\xi|^{n-\gamma}}$ for $\gamma \in (0, n)$

ii) The <u>Hardy-Littlewood-Sobolev inequality.</u>

Let $p, q \in (1, +\infty)$ with $q < p$ and $\gamma \in (0, n)$ be given.
Suppose that $\frac{1}{p} = \frac{1}{q} - \frac{\gamma}{n}$. Then we have

$$\left\| f * \frac{1}{|x|^{n-\gamma}} \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^q(\mathbb{R}^n)}$$

for some $C = C(p, q, n) > 0$.

Compare to the Generalized Young's Inequality:

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)} \cdot \|g\|_{L^r(\mathbb{R}^n)}$$

for $\frac{1}{q}+\frac{1}{r}=1+\frac{1}{p}$

We note that, in the assumptions of the Hardy-Littlewood-Sobolev inequality, we have

$$1+\frac{1}{p}=\frac{1}{q}+\frac{1}{\left(\frac{n}{n-\gamma}\right)}$$

Now, we know that $\left\|\frac{1}{|x|^{n-\gamma}}\right\|_{L^{\frac{n}{n-\gamma}}}=+\infty$.

Nevertheless, $\left\|\frac{1}{|x|^{n-\gamma}}\right\|_{L^{\frac{n}{n-\gamma},\infty}}<+\infty$, where

$$\|f\|_{L^{p,\infty}}:=\begin{cases}\sup\limits_{\lambda>0}\lambda\cdot\left[\mu\left(\{x\in\mathbb{R}^n:|f(x)|>\lambda\}\right)^{1/p}\right] & \text{if } p<\infty\\ \|f\|_{L^\infty} & \text{if } p=\infty\end{cases}$$

This is the <u>weak L^p space</u>. $\|\cdot\|_{L^{p,\infty}}$ is a quasinorm.

One has the following result:

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)} \cdot \|g\|_{L^{r,\infty}(\mathbb{R}^n)}.$$

for $\frac{1}{q}+\frac{1}{r}=1+\frac{1}{p}$.

This yields the Hardy-Littlewood-Sobolev inequality.

<u>Proof of Proposition 1.43:</u> We prove the stronger claim (1.4.3').

Let $g:=\dot{D}^s f$.

Then $\widehat{g}(\xi)=|\xi|^s\widehat{f}(\xi)$

$$\Rightarrow \widehat{f}(\xi)=|\xi|^{-s}\cdot\widehat{g}(\xi)$$

$$\Rightarrow f=\left(|\xi|^{-s}\cdot\widehat{g}(\xi)\right)^{\vee}$$, where $\vee$ denotes the inverse Fourier transform.

So, $f = (|\xi|^{-s})^{\vee} * g$

By i) with $\sigma = s$, we get

$$f = c g * \frac{1}{|x|^{n-s}}$$

By ii) with $q=2$, $\sigma = s$ we obtain

$$\|f\|_{L^p} \leq C\|g\|_{L^2} = C\|\dot{D}^s f\|_{L^2} = C\|f\|_{\dot{H}^s}. \qquad \square$$

<u>Remark:</u> We cannot set $p=\infty$ and $s=\frac{n}{2}$ in Proposition 1.4.3.

A counterexample is given as follows.

Let f be such that

$$\hat{f}(\xi) = \frac{1}{(1+|\xi|)^n \log(2+|\xi|)}, \text{ for } \xi \in \mathbb{R}^n$$

One then has $f \in H^{\frac{n}{2}}(\mathbb{R}^n)$

However $\hat{f} \geq 0$ and $f(0) = \int \hat{f}(\xi)\, d\xi = +\infty$.

$\Rightarrow f \notin L^\infty(\mathbb{R}^n)$, at least formally.

More rigorously, one considers

$$\operatorname{Re} f(x) = \int \frac{\cos(2\pi x \cdot \xi)}{(1+|\xi|)^n \log(2+|\xi|)}\, d\xi$$

and one sees that this function is unbounded near 0. (Exercise)

Sobolev spaces in the periodic setting

We define for $\mu \in \mathcal{B}(\mathbb{T}^n)$, and $s \in \mathbb{R}$

$$\|\mu\|_{H^s(\mathbb{T}^n)} := \| \langle k \rangle^s \hat{\mu}(k) \|_{\ell^2(\mathbb{Z}^n)}$$

$$\|\mu\|_{\dot{H}^s(\mathbb{T}^n)} := \| |k|^s \hat{\mu}(k) \|_{\ell^2(\mathbb{Z}^n)}$$

Note that, if $\hat{\mu}(0) = \int d\mu = 0$, we have

$$\|\mu\|_{H^s(\mathbb{T}^n)} \sim \|\mu\|_{\dot{H}^s(\mathbb{T}^n)}.$$

The proof of Proposition 1.4.2 carries over to the periodic setting and yields the analogous result on $\mathbb{T}^n$.

The proof of Proposition 1.4.3 does not immediately carry over due to the use of the Hardy-Littlewood-Sobolev inequality.

The result does hold by modifying the HLS inequality step by means of the Poisson summation formula

- If $f, \hat{f} \in L^1(\mathbb{R}^n)$ satisfy $|f(x)| + |\hat{f}(x)| \leq C\langle x \rangle^{-n-\delta}$ for some $C, \delta > 0$ then $f, \hat{f}$ are continuous functions and for all $x \in \mathbb{R}^n$ we have

$$\sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i k \cdot x} = \sum_{k \in \mathbb{Z}^n} f(x+k).$$

"The two ways to make a periodic function starting from f give the same result."

For details, see Á. Bényi, T. Oh, "The Sobolev inequality on the torus revisited", Publ. Math. Debrecen, 83(2013), no. 3, 359-374.

For the Poisson summation formula, see VII.2 of E. Stein, G. Weiss, "Introduction to Fourier analysis on Euclidean Spaces".

Remark: Let (M,g) be a compact Riemannian manifold with Laplace-Beltrami operator Δ. We write the eigenvalues of $-\Delta$ as:

$$-\Delta \varphi_k = \lambda_k^2 \varphi_k \,, \quad \lambda_k \geq 0$$

$$\|\varphi_k\|_{L^2} = 1 \quad ; \; k \in \mathbb{N}_0$$

(φ_k) - orthonormal basis of eigenvectors.

$$\|f\|_{H^s(M)} := \left(\sum_k (1+\lambda_k)^{2s} \left| \langle f, \varphi_k \rangle_{L^2(M)} \right|^2 \right)^{\frac{1}{2}}.$$

This generalizes the definition of $H^s(\mathbb{T}^n)$.

1.5: Interpolation Theory

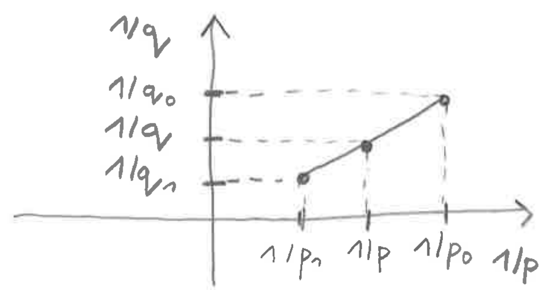
Riesz-Thorin Interpolation theorem:

Let (X,μ), (Y,ν) be sigma-finite measure spaces. Suppose that $1 \le p_0, p_1, q_0, q_1 \le \infty$ and suppose that $T: L^{p_j}(X,\mu) \longrightarrow L^{q_j}(Y,\nu)$ is a <u>bounded linear</u> operator with norm K_j for $j = 0,1$.
Then, $T: L^{p_\theta}(X,\mu) \longrightarrow L^{q_\theta}(Y,\nu)$ is a bounded linear operator with norm $K_\theta \le K_0^{1-\theta} K_1^{\theta}$, for all $\theta \in [0,1]$, where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

We graphically represent the result as



<u>Example 1</u>: (Generalized Young inequality)

$$\|f * g\|_{L^p} \le \|f\|_{L^q} \cdot \|g\|_{L^r}$$

for $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$; $1 \le p, q, r \le \infty$

$\rightsquigarrow$ deduce general result from $p=q$, $r=1$ and $p=\infty$, $r=q'$.

<u>Example 2</u>: (Hausdorff-Young inequality)

For $1 \le p \le 2$ we have:

$$\|\hat{f}\|_{L^{p'}} \le \|f\|_{L^p}.$$

We interpolate between $\|Tf\|_{L^2} = \|f\|_{L^2}$ and $\|Tf\|_{L^\infty} \le \|f\|_{L^1}$ for $T = \hat{\ }$.

1.6: Fourier multiplier operators

Given $m = m(\xi)$ a measurable function, we define the operator $m(D)$ by

$$(m(D)f)^\wedge(\xi) := m(\xi)\hat{f}(\xi), \text{ for } f \in \mathcal{S}(\mathbb{R}^n).$$

$m(D)$ is called a Fourier multiplier operator.

Example: $m(\xi) = \langle \xi \rangle^s \rightsquigarrow m(D) = D^s$.

Example: Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be a radial function s.t. $\psi = 1$ for $\frac{1}{2} \leq |\xi| \leq 2$ and $\psi = 0$ for $|\xi| < \frac{1}{4}$ or $|\xi| > 4$.
For $j \in \mathbb{N}$, we let $\psi_j(\xi) := \psi\left(\frac{\xi}{2^j}\right)$.
$\psi_j(D)$ – Littlewood-Paley projection.

Hörmander - Mikhlin theorem:
Let $m : \mathbb{R}^n \to \mathbb{C}$ be such that $|\partial_\xi^\alpha m| \lesssim |\xi|^{-k}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ whenever $0 \leq k \leq n+2$ and $|\alpha| = k$.
Then $m(D) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded for all $1 < p < \infty$.

The transference principle
(following Stein-Weiss VII.3)

Suppose that $T = m(D) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded, $1 \leq p \leq \infty$, and suppose that m is continuous at every $k \in \mathbb{Z}^n$.
Then the operator $\tilde{T}$ given by

$$\tilde{T}f(x) := \sum_{k \in \mathbb{Z}^n} m(k)\hat{f}(k)e^{2\pi i k \cdot x}$$

is bounded from $L^p(\mathbb{T}^n)$ to $L^p(\mathbb{T}^n)$ with operator norm $\leq \|T\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}$.