HOMEWORK ASSIGNMENT 1: FOURIER TRANSFORM AND SOBOLEV SPACES

You need to solve only one of the exercises. A tentative due date is October 14.

Exercise 1. *(Bernstein's Inequality)*
Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be given. For $j \in \mathbb{Z}$, we define $\psi_j(\xi) := \psi\big(\frac{\xi}{2^j}\big)$. Let $1 \leq q \leq p \leq \infty$ be given.

a) *Show that, for all $j \in \mathbb{Z}$ we have*

$$\|\psi_j(D)f\|_{L^p(\mathbb{R}^n)} \leq C\, 2^{nj\,(\frac{1}{q}-\frac{1}{p})}\, \|f\|_{L^q(\mathbb{R}^n)}$$

for some constant $C > 0$ which depends on ψ, n, p, q, but which is independent of j.
[HINT: Write $\psi_j(D)$ as a convolution.]

b) *Suppose that $f \in L^p(\mathbb{R}^n)$ has the property that $\widehat{f}$ is supported on the set $B(0, c\,2^j)$ for some $c > 0$. Show that*

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C\, 2^{nj\,(\frac{1}{q}-\frac{1}{p})}\, \|f\|_{L^q(\mathbb{R}^n)}\,,$$

for some constant $C > 0$ independent of j.
*The results in parts a) and b) are called **Bernstein's Inequality**.*

c) *Suppose that $f \in H^{\frac{n}{2}}(\mathbb{R}^n)$ has the property that $\widehat{f}$ is supported in an annulus*

$$\mathcal{A}_j = \big\{\xi \in \mathbb{R}^n : 2^{j-1} < |\xi| \leq 2^j\big\}$$

for some $j \in \mathbb{N} = \{1, 2, 3, \ldots\}$ or in the ball

$$\mathcal{A}_0 = \big\{\xi \in \mathbb{R}^n : |\xi| \leq 1\big\}\,.$$

Show that

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C\, \|f\|_{H^{\frac{n}{2}}(\mathbb{R}^n)}\,,$$

for some constant $C > 0$ independent of j.

d) *Compare the result of part c) with Proposition 1.4.3. from class.*

Exercise 2. *(Algebra property of $H^s(\mathbb{R}^n)$ for $s > n/2$.)*
Let $s > n/2$ be given. Show that for $f, g \in H^s(\mathbb{R}^n)$ one has

$$\|fg\|_{H^s(\mathbb{R}^n)} \;\leq\; C\, \|f\|_{H^s(\mathbb{R}^n)}\, \|g\|_{H^s(\mathbb{R}^n)}\,,$$

for some constant $C > 0$ depending on s and n. In particular, $H^s(\mathbb{R}^n)$ is closed under multiplication.
[HINT: Work on the Fourier side. Recall Young's inequality.]