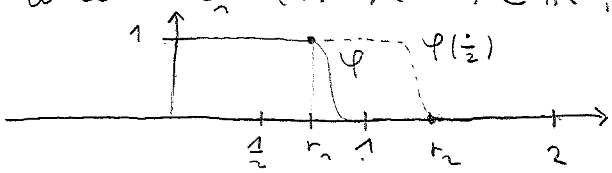
4.5: Littlewood - Paley Theory

- We know from Plancherel's Theorem that $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$.

In particular, if we change the sign of the Fourier transform of an L^2 function on some set, the L^2 norm will not change (in particular, the resulting function will belong to L^2).
The question that we will address is what happens in L^p.

- Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a non-negative, radial, (radially) decreasing function such that $\varphi(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$, and $\varphi(\xi) = 1$ for $|\xi| \geq 1$
We can construct φ such that $\Psi(\xi) := \sqrt{\varphi(\frac{\xi}{2}) - \varphi(\xi)}$ is a smooth function. Namely, we know that $\varphi(\frac{\xi}{2}) - \varphi(\xi) \geq 0$ for all ξ. Moreover, there exist $r_1, r_2 > 0$ such that $\varphi(\frac{\xi}{2}) - \varphi(\xi)$ equals zero for $|\xi| = r_1$, $|\xi| = r_2$ and it is positive for $r_1 < |\xi| < r_2$. We want to construct φ such that $\varphi(\frac{\xi}{2}) - \varphi(\xi)$ vanishes to sufficiently high order for $|\xi| = r_1, r_2$. Let us note that $r_1 \in [\frac{1}{2}, 1]$, $r_2 \in [1, 2]$. In particular $\varphi(\frac{\xi}{2}) - \varphi(\xi) = 1 - \varphi(r_1 \cdot e_1)$ for $|\xi| = r_1$, $\varphi(\frac{\xi}{2}) - \varphi(\xi) = \varphi(\frac{r_2}{2} \cdot e_1)$ for $|\xi| = r_2$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Thus, such a φ can be constructed.



- By construction $\Psi \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp} \Psi \subseteq \{1/2 \leq |\xi| \leq 2\}$.
Also: $\sum_{j \in \mathbb{Z}} \left(\Psi\left(\frac{\xi}{2^j}\right)\right)^2 = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ $\quad (\Delta)$

Theorem 4.8 (Littlewood - Paley Inequality)

Let $\Psi_j(\xi) := \Psi\left(\frac{\xi}{2^j}\right)$ for $j \in \mathbb{Z}$ and $\Psi \in C_c^\infty(\mathbb{R}^n)$ as above.
Then, for all $1 < p < \infty$ and $f \in L^p$, it is the case that:

$$\left\| \sqrt{\sum_j |\Psi_j(D) f|^2} \right\|_{L^p} \sim_{n,p} \|f\|_{L^p}.$$

Remarks: ① $\sqrt{\sum_j |\Psi_j(D) f|^2}$ is called the Littlewood-Paley square function.

② If $n = 1$, it can be shown that one can take rough truncations (see Duoandikoetxea, Chapter 8.2) This uses some properties of the Hilbert transform.

· One approach to prove Theorem 4.8 is based on Vector-valued Calderón - Zygmund theory.
We will not take this approach here (for a detailed explanation, see Duoandikoetxea Chapter 5).
Instead, we will take an approach based on some ideas from Probability Theory.
The starting point is:

<u>Theorem 4.9 (Khinchine's Inequality)</u>

Let $N \in \mathbb{N}$. Let $x_1, \dots, x_N \in \mathbb{C}$ be given.
Suppose that $\varepsilon_1, \dots, \varepsilon_N \in \{-1, 1\}$ are independent random signs chosen each with probability $\frac{1}{2}$.
Then, for all $1 \leq p < \infty$:

$$\left(\mathbb{E}\left(\left|\sum_{j=1}^{N} \varepsilon_j x_j\right|^p\right)\right)^{\frac{1}{p}} \sim_p \sqrt{\sum_{j=1}^{N} |x_j|^2}$$

In particular:

$$\left(\mathbb{E}\left(\left|\sum_{j=1}^{\infty} \varepsilon_j x_j\right|^p\right)\right)^{\frac{1}{p}} \sim_p \sqrt{\sum_{j=1}^{\infty} |x_j|^2}.$$

· Theorem 4.9 will be proved on Homework Assignment 4.

We can hence deduce the following result:

<u>Proposition 4.10:</u> (Khinchine's Inequality for functions)
Let $1 \leq p < \infty$ be fixed. For $\varepsilon_1, \dots, \varepsilon_N$ as in Theorem 4.9, it is the case that:

$$\left(\mathbb{E}\left(\left\|\sum_{j=1}^{N} \varepsilon_j f_j\right\|^p_{L^p(\mathbb{R}^n)}\right)\right)^{\frac{1}{p}} \sim_p \left\|\sqrt{\sum_{j=1}^{N} |f_j|^2}\right\|_{L^p(\mathbb{R}^n)}$$

whenever $f_1, \dots, f_N \in L^p(\mathbb{R}^n)$.

<u>Proof:</u> We know from Theorem 4.9 that, for all $x \in \mathbb{R}^n$:

$$\left(\mathbb{E}\left(\left|\sum_{j=1}^{N} \varepsilon_j f_j(x)\right|^p\right)\right)^{\frac{1}{p}} \sim_p \sqrt{\sum_{j=1}^{N} |f_j(x)|^2}$$

- The claim now follows after taking p^{th} powers, integrating over $\mathbb{R}^n$ and taking p^{th} roots. $\square$

- We can now prove the Littlewood-Paley Inequality.

Proof of Theorem 4.8:

- We observe that, for fixed $j \in \mathbb{Z}$, it is the case that:

$$\nabla\left[\Psi_j(\xi)\right] = \nabla\left[\Psi\left(\frac{\xi}{2^j}\right)\right] = \frac{1}{2^j}\cdot\nabla\Psi\left(\frac{\xi}{2^j}\right) = \mathcal{O}\left(\frac{1}{2^j}\right)$$

By the support properties of Ψ_j, we deduce that:

$$|\nabla\Psi_j| \lesssim |\xi|^{-1} \text{ on } \operatorname{supp}\Psi_j.$$

In general, we obtain:

$$|\nabla^k\Psi_j| \lesssim 2^{-jk} \left(\sim |\xi|^{-k} \text{ on } \operatorname{supp}\Psi_j\right)$$

This holds for all $k \in \{0,1,2,\dots\}$.

- In particular, given $N \in \mathbb{N}$ and any choice of signs $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N \in \{-1,1\}$, we note that the operator $\sum_{j=1}^{N} \varepsilon_j \cdot \Psi_j(D)$ satisfies the assumptions of Theorem 4.7.

In particular, for all $1<p<\infty$, and for all $f \in L^p(\mathbb{R}^n)$, we deduce that:

$$\left\|\sum_{j=1}^{N} \varepsilon_j \cdot \Psi_j(D) f\right\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

· We hence obtain, by taking p^{th} powers and expectations:

$$\left(\mathbb{E}\left(\left\|\sum_{j=1}^{N} \varepsilon_j \Psi_j(D)f\right\|^p_{L^p(\mathbb{R}^n)}\right)\right)^{\frac{1}{p}} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

By Khinchine's Inequality for functions, it follows that:

$$\left\|\sqrt{\sum_{j=1}^{N} |\Psi_j(D)f|^2}\right\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}.$$

Analogously, we can consider negative j and deduce that

$$\left\|\sqrt{\sum_{j=-N}^{N} |\Psi_j(D)f|^2}\right\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}.$$

Letting $N \to \infty$ and using the Monotone Convergence Theorem, it follows that:

$$\left\|\sqrt{\sum_{j} |\Psi_j(D)f|^2}\right\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}.$$

This is the "Upper Littlewood-Paley Inequality".

· We will now prove the "Lower Littlewood-Paley Inequality":

$$\left\|\sqrt{\sum_{j} |\Psi_j(D)f|^2}\right\|_{L^p(\mathbb{R}^n)} \gtrsim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}.$$

• We first prove:

$$\Big\| \sum_j \psi_j(D) f_j \Big\|_{L^p} \lesssim_{n,p} \Big\| \sqrt{\sum_j |f_j|^2} \Big\|_{L^p}. \qquad (*)$$

for $f_j \in L^p$.

In order to prove $(*)$, we use duality:

Let us fix $g \in L^{p'}$.

Then, for all j:

$$\int \psi_j(D) f_j \cdot g\, dx = \int \psi_j(\xi) \cdot \widehat{f_j}(\xi) \cdot \widehat{g}(\xi)\, d\xi = \int f_j \cdot \psi_j(D) g\, dx$$

So: $\Big| \int \Big(\sum_j \psi_j(D) f_j \Big) \cdot g\, dx \Big|$

$$= \Big| \sum_j \int f_j \cdot \psi_j(D) g\, dx \Big| \leq \int \sum_j |f_j| \cdot |\psi_j(D) g|\, dx$$

$$\leq \int \Big(\sum_j |f_j|^2 \Big)^{\frac{1}{2}} \cdot \Big(\sum_j |\psi_j(D) g|^2 \Big)^{\frac{1}{2}} dx$$

by the Cauchy-Schwarz inequality

$$\leq \Big\| \sqrt{\sum_j |f_j|^2} \Big\|_{L^p} \cdot \Big\| \sqrt{\sum_j |\psi_j(D) g|^2} \Big\|_{L^{p'}}$$

by Hölder's inequality

$$\lesssim_{n,p} \Big\| \sqrt{\sum_j |f_j|^2} \Big\|_{L^p} \cdot \| g \|_{L^{p'}}$$

by the Upper Littlewood-Paley inequality applied to g.

• Hence, we deduce $(*)$.

· Let us note that, so far, we have only been using the fact that $\psi_j(\xi) = \psi\left(\frac{\xi}{2^j}\right)$ for some $\psi \in C_c^\infty(\mathbb{R}^n)$ which is supported on the set where $|\xi| \sim 1$.
In other words, we have not yet used the condition (Δ), i.e. that $\sum_{j \in \mathbb{Z}} \left(\psi\left(\frac{\xi}{2^j}\right)\right)^2 = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

· We will use (Δ) in order to prove the lower Littlewood-Paley inequality from $(*)$. In other words, we want to prove that:

$$\left\| \sqrt{\sum_j |\psi_j(D) f|^2} \right\|_{L^p(\mathbb{R}^n)} \gtrsim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in L^p(\mathbb{R}^n)$.

By density, it suffices to prove the claim for $f \in \mathcal{S}(\mathbb{R}^n)$, since by the first part of the Theorem, we know that the expression on the left-hand side is continuous in L^p.

By (Δ), it follows that:

$$f = \sum_j \left(\psi_j(D)\right)^2 f.$$

The convergence is pointwise, uniformly in x.

Namely:

$$\sum_j \left(\psi_j(D) f\right)^2(x) = \int \sum_j \left(\psi_j(\xi)\right)^2 \cdot \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi$$

$$= \int \widehat{f}(\xi) \cdot e^{2\pi i x \cdot \xi} \, d\xi$$

The summation is rigorously justified by using the fact that $f \in \mathcal{S}(\mathbb{R}^n)$ and by the Dominated Convergence Theorem.

In particular:

$$f = \sum_j \Psi_j(D)\left(\Psi_j(D)f\right)$$

So, by $(*)$

$$\Rightarrow \|f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \left\| \sqrt{\sum_j |\Psi_j(D)f|^2} \right\|_{L^p(\mathbb{R}^n)} . \qquad \square$$