§2: Interpolation of Operators

- We would like to address the following general question:

Suppose that $\mathcal{A}, \mathcal{B}$ are linear spaces and $T : \mathcal{A} \to \mathcal{B}$ is a linear map.

Moreover, suppose that there exist Banach subspaces $A_0, A_1 \subseteq \mathcal{A}$, $B_0, B_1 \subseteq \mathcal{B}$ such that:

$$T|_{A_0} : A_0 \to B_0$$

$$T|_{A_1} : A_1 \to B_1$$

continuously.

<u>Question:</u> Do there exist "intermediate" Banach subspaces $A \subseteq \mathcal{A}$, $B \subseteq \mathcal{B}$ such that

$$T|_A : A \to B$$

continuously?

<u>Motivation:</u> Let $\mathcal{A} = \mathcal{B} = \mathcal{S}'(\mathbb{R}^n)$.

We know that $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is linear.

$L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n), L^2(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$ are Banach subspaces.

Moreover:

$$\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$$

$$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

continuously.

<u>Goal:</u> Find all possible pairs (p, q) s.t.

$$\mathcal{F} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n).$$

Example: Let $\mathcal{A} = \mathcal{B} = \mathcal{S}'(\mathbb{R}^n)$, and let I = the identity operator.
Suppose that $1 \leq p_0 \leq p_1 \leq \infty$ are given.
Then:

$$I\big|_{L^{p_0}} : L^{p_0} \to L^{p_0}$$

$$I\big|_{L^{p_1}} : L^{p_1} \to L^{p_1}$$

continuously, with operator norm 1.

Let $q \in [p_0, p_1]$ be given.
Then:

$$\frac{1}{q} \in \left[\frac{1}{p_1}, \frac{1}{p_0}\right] \quad \text{(with the convention that } \tfrac{1}{\infty} = 0 \text{)}$$

In particular, there exists $\theta \in [0,1]$ such that:

$$\frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

· Let's assume that $p_1 < \infty$, $\theta \in (0,1)$ for now.
By Hölder's inequality:

$$\|f\|_{L^q} = \big\| |f|^{(1-\theta)} \cdot |f|^{\theta} \big\|_{L^q}$$

$$\leq \big\| |f|^{(1-\theta)} \big\|_{L^{\frac{p_0}{1-\theta}}} \cdot \big\| |f|^{\theta} \big\|_{L^{\frac{p_1}{\theta}}}$$

$$= \|f\|_{L^{p_0}}^{1-\theta} \cdot \|f\|_{L^{p_1}}^{\theta}.$$

· This inequality holds in the case $p_1 = \infty$, and when $\theta \in \{0,1\}$ as well. We leave the verification as an exercise to the reader.
In any case, if we let p_θ be defined by:

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$ it follows that:

$$\|f\|_{L^{p_\theta}} \leq \|f\|_{L^{p_0}}^{1-\theta} \cdot \|f\|_{L^{p_1}}^{\theta}$$

By taking logarithms, it follows that:

$$\log \|f\|_{L^{p_\theta}} \leq (1-\theta) \cdot \log \|f\|_{L^{p_0}} + \theta \cdot \log \|f\|_{L^{p_1}}$$

→ "Logarithmic convexity of L^p norms".

Conclusion: We can control the intermediate norms in terms of the endpoint norms.

Parametrizing the intermediate values of p as

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$ was shown to be useful.

In particular, if $f \in L^{p_0} \cap L^{p_1}$, then $f \in L^{p_\theta}$ for all $\theta \in [0,1]$ and:

$$\|f\|_{L^{p_\theta}} \leq \|f\|_{L^{p_0}}^{1-\theta} \cdot \|f\|_{L^{p_1}}^{\theta}.$$

Observation: If $p_0 \leq p \leq p_1$, then every function $f \in L^p(\mathbb{R}^n)$ can be written as $f = f_0 + f_1$, with $f_0 \in L^{p_0}(\mathbb{R}^n)$, $f_1 \in L^{p_1}(\mathbb{R}^n)$.

Namely, we let:

$$f_0 := f \cdot \chi_{\{|f|>1\}}, \quad f_1 := f \cdot \chi_{\{|f| \leq 1\}}.$$

Then, if $p_1 < \infty$,

$$\int |f_1|^{p_1} dx = \int_{|f|\leq 1} |f|^{p_1} dx \leq \int_{|f|\leq 1} |f|^p dx \leq \int |f|^p dx < \infty$$

$$\int |f_0|^{p_0} dx = \int_{|f|\geq 1} |f|^{p_0} dx \leq \int_{|f|\geq 1} |f|^p dx \leq \int |f|^p dx < \infty.$$

The argument when $p_1 = \infty$ is similar and is left as an exercise.

· In particular, if a linear operator T is defined on $L^{p_0}(\mathbb{R}^n)$ and on $L^{p_1}(\mathbb{R}^n)$ (and if the definitions coincide on $L^{p_0}(\mathbb{R}^n) \cap L^{p_1}(\mathbb{R}^n)$), then the operator T can be extended to $L^{p_\theta}(\mathbb{R}^n)$ by linearity.

Namely, given $f \in L^{p_\theta}(\mathbb{R}^n)$, we can write it as:

$$f = f_0 + f_1 \quad \text{with } f_0 \in L^{p_0}(\mathbb{R}^n),\ f_1 \in L^{p_1}(\mathbb{R}^n).$$

and we can define:

$$Tf := Tf_0 + Tf_1.$$

This is well-defined. Namely, if $\tilde{f}_0 \in L^{p_0}(\mathbb{R}^n)$, $\tilde{f}_1 \in L^{p_1}(\mathbb{R}^n)$ are such that:

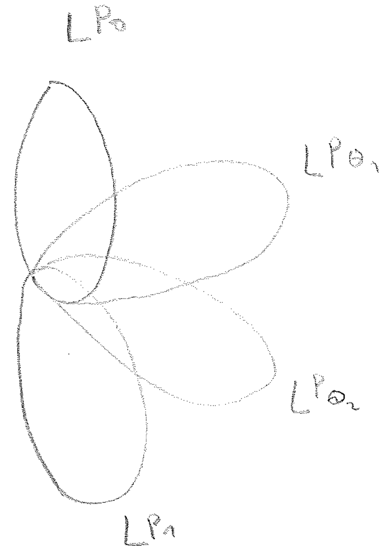
$$f = \tilde{f}_0 + \tilde{f}_1, \text{ then:}$$

$$f_0 + f_1 = \tilde{f}_0 + \tilde{f}_1 \Rightarrow f_0 - \tilde{f}_0 = f_1 - \tilde{f}_1 \in L^{p_0}(\mathbb{R}^n) \cap L^{p_1}(\mathbb{R}^n)$$

so: $T(f_0 - \tilde{f}_0) = T(f_1 - \tilde{f}_1)$

$$\Rightarrow Tf_0 + Tf_1 = T\tilde{f}_0 + T\tilde{f}_1.$$

In this way, we extend T to a linear operator on $L^{p_\theta}(\mathbb{R}^n)$.



§2.1: The Riesz-Thorin Interpolation Theorem; Complex interpolation

- One possible way to answer the question raised earlier is given by the Riesz-Thorin interpolation theorem, which is sometimes also called the M. Riesz convexity theorem.

Theorem 1.1: (Riesz-Thorin interpolation theorem)
Suppose that $T : L^{p_j}(X) \to L^{q_j}(Y)$ is a bounded linear operator with norm k_j, for $j = 0, 1$. Let X, Y be sigma-finite. Then $T : L^{p_\theta}(X) \to L^{q_\theta}(Y)$ is a bounded linear operator with norm $k_\theta \leq k_0^{1-\theta} \cdot k_1^{\theta}$, for all $\theta \in [0,1]$.
Here, p_θ, q_θ are given by:

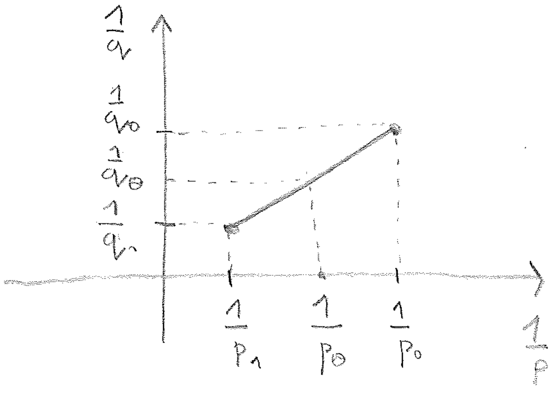
$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Remarks:

① The result holds in general sigma-finite measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$, but in our class we will mostly apply it in the Euclidean setting.

② We can graphically represent the result as follows:



→ The operator $T : L^p(X) \to L^q(Y)$ is bounded whenever $\left(\frac{1}{p}, \frac{1}{q}\right)$ lies on the line in the diagram.

(3) This result is called a convexity theorem because the function $\Theta \mapsto \log(k_\Theta)$ is convex.

- Before we give the proof of Theorem 1.1, let us first give the main ingredient:

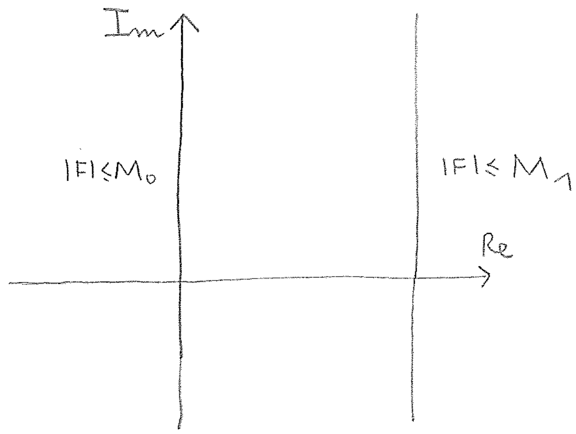
<u>Lemma 1.2:</u> ("Three lines lemma")

Suppose F is a <u>continuous</u> complex-valued function on $S = \{z \in \mathbb{C}; 0 \leq \operatorname{Re} z \leq 1\}$ which is <u>analytic on $0 < \operatorname{Re} z < 1$</u>. Suppose also that F is <u>bounded</u> and that there exist $M_0, M_1 > 0$ such that

$|F(z)| \leq M_0$ whenever $\operatorname{Re} z = 0$ and

$|F(z)| \leq M_1$ whenever $\operatorname{Re} z = 1$.

Then $|F(z)| \leq M_0^{1-\operatorname{Re} z} \cdot M_1^{\operatorname{Re} z}$.



<u>Proof:</u> Let us first note that it suffices to consider the case when $M_0, M_1 > 0$.

Namely, if $M_j = 0$, then we can replace M_j with $\varepsilon > 0$ and then let $\varepsilon \to 0$.

If we replace F by $\dfrac{F}{M_0^{1-z} \cdot M_1^{z}}$, it follows that we can consider the case when $M_0 = M_1 = 1$.

So, we have to prove that:

$$|F(z)| \leq 1 \text{ for all } z \in \mathbb{C} \text{ with } 0 \leq \operatorname{Re} z \leq 1.$$

- If we knew that $\lim\limits_{|\operatorname{Im} z| \to 0} |F(z)| = 0$ for $0 < \operatorname{Re} z < 1$, the claim would follow from the Maximum principle.

- Given $n \in \mathbb{N}$, we let:

$$F_n(z) := F(z) \cdot e^{\frac{z^2-1}{n}}$$

We note that:

F_n is continuous on $0 \leq \operatorname{Re} z \leq 1$ and analytic on $0 < \operatorname{Re} z < 1$.

If $z = x + iy$: $z^2 - 1 = (x^2 - y^2 - 1) + 2ixy$

Hence: $|F_n(z)| = |F(z)| \cdot e^{\frac{x^2-y^2-1}{n}} \leq |F(z)|$ for $0 \leq x \leq 1$,

So, F_n is bounded and $|F_n(z)| \leq 1$ for $\operatorname{Re} z = 0, 1$.

$F_n(z) = F(z) \cdot e^{\frac{z^2-1}{n}} \longrightarrow F(z)$ as $n \to \infty$, pointwise in z.

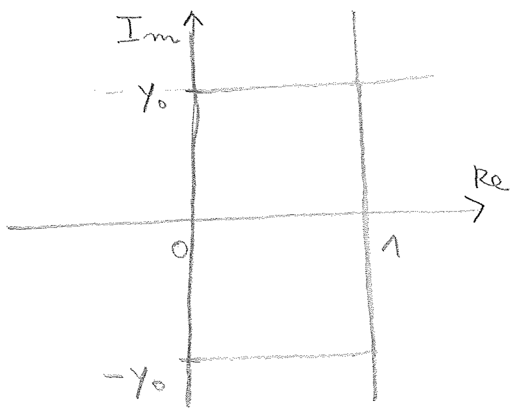
Finally, we also know that:

$$|F_n(z)| \leq |F(z)| \cdot e^{-\frac{y^2}{n}} \leq \|F\|_{L^\infty} \cdot e^{-\frac{y^2}{n}} \to 0 \text{ as } |y| \to \infty.$$

(uniformly in $x = \operatorname{Re} z$)

Thus, given n, we can find $y_0 = y_0(n) > 0$ s.t.

$|F_n(z)| \leq 1$ for $|\operatorname{Im} z| \geq y_0$.



We then apply the Maximum principle on the rectangle $[0,1] \times [-y_0, y_0]$, and deduce that $|F_n| \leq 1$ on this rectangle.

Since $|F_n| \leq 1$ outside of the rectangle, it follows that $|F_n| \leq 1$ on $0 \leq \operatorname{Re} z \leq 1$.

Since $F_n \to F$ pointwise, it follows that:

$|F| \leq 1$ on $0 \leq \operatorname{Re} z \leq 1$. $\square$

<u>Remark:</u> Lemma 2.1 is related to the Phragmen-Lindelöf theorem from Complex analysis.

Proof of Theorem 1.1:

Let us fix $\Theta \in [0,1]$.

Let $\alpha_0 := \frac{1}{p_0}$, $\beta_0 := \frac{1}{q_0}$, $\alpha_1 := \frac{1}{p_1}$, $\beta_1 := \frac{1}{q_1}$, $\alpha := \frac{1}{p_\Theta}$, $\beta := \frac{1}{q_\Theta}$.

For $z \in \mathbb{C}$, $0 \leq \operatorname{Re} z \leq 1$, we define:

$\alpha(z) := (1-z)\cdot\alpha_0 + z\cdot\alpha_1$

$\beta(z) := (1-z)\cdot\beta_0 + z\cdot\beta_1$

Let $p := p_\Theta$, $q := q_\Theta$. We note that $\alpha(\Theta)=\alpha$, $\beta(\Theta)=\beta$.

- Suppose first that f is a simple function with finite measure support. Since X is sigma-finite, such functions are dense in $L^r(X)$ for all $1 \leq r < \infty$.

Claim: $\|Tf\|_{L^q} \leq K_0^{1-\Theta}\cdot K_1^{\Theta}\cdot \|f\|_{L^p}$ $(*)$.

- Let us first observe that:

By scaling (i.e. by replacing f by $\frac{f}{\|f\|_{L^p}}$; we assume $f \not\equiv 0$), it suffices to consider the case when $\|f\|_{L^p} = 1$.

We know that:

$$\|Tf\|_{L^q} = \sup_{\substack{g \text{ simple, with finite measure support} \\ \|g\|_{L^{q'}} \leq 1}} \left\{ \left| \int Tf\cdot g \, dx \right| \right\} \quad \text{for all } 1 \leq q \leq \infty$$

(The fact that this identity holds for $q=\infty$ follows from the assumption that Y is σ-finite.)

So, we suppose that g is a simple function with finite measure support and $\|g\|_{L^{q'}} \leq 1$!

We write:

$$f = \sum_{j=1}^{m} a_j \cdot \chi_{E_j}, \quad g = \sum_{l=1}^{n} b_l \cdot \chi_{F_l}; \quad (E_j), (F_l) \text{ - disjoint}.$$

$a_j = |a_j| e^{i\psi_j}$, $b_l = |b_l| e^{i\Phi_l}$; $j = 1,\ldots,m$; $l = 1,\ldots,n$.

• Let us assume for the moment that $\theta > 0$ and $\beta < 1$.

We define, for $z \in \mathbb{C}$ with $0 \leq \operatorname{Re} z \leq 1$:

$$f_z := \sum_{j=1}^{m} |a_j|^{\frac{\gamma(z)}{\gamma}} \cdot e^{i\psi_j} \cdot \chi_{E_j}$$

$$g_z := \sum_{l=1}^{n} |b_l|^{\frac{1-\beta(z)}{1-\beta}} \cdot e^{i\Phi_l} \cdot \chi_{F_l}$$

(then $f_\theta = f$, $g_\theta = g$)

Let $F(z) := \int Tf_z \cdot g_z$ for $0 \leq \operatorname{Re} z \leq 1$. This equals:

$$\sum_{j=1}^{m} \sum_{l=1}^{n} |a_j|^{\frac{\gamma(z)}{\gamma}} \cdot |b_l|^{\frac{1-\beta(z)}{1-\beta}} e^{i(\psi_j + \Phi_l)} \cdot \underbrace{\int (T\chi_{E_j}) \cdot \chi_{F_l}\, dx}_{=: b_{jl} \text{ (finite!)}}$$

$\left[\chi_{E_j} \in L^{p_0} \Rightarrow T\chi_{E_j} \in L^{q_0},\ \chi_{F_l} \in L^{q_0'} \Rightarrow (T\chi_{E_j}) \cdot \chi_{F_l} \in L^1 \right.$

F is an analytic function (on all of $\mathbb{C}$)

It is bounded on $0 \leq \operatorname{Re} z \leq 1$.

We know that:

$F(\theta) = \int (Tf) \cdot g \, dx$. This is the quantity that we want to bound.

One now needs to show that F satisfies the assumptions of Lemma 1.2 for $\operatorname{Re} z = 0, 1$.

• $\gamma(iy) = \gamma_0 + iy \cdot (\gamma_1 - \gamma_0)$

$1 - \beta(iy) = (1 - \beta_0) + iy\,(\beta_0 - \beta_1)$

So: $|f_{iy}| = \left| |f|^{\frac{\gamma(iy)}{\gamma}} \right| = |f|^{\frac{\gamma_0}{\gamma}} = |f|^{\frac{p}{p_0}}$

$\Rightarrow |f_{iy}|^{p_0} = |f|^p \Rightarrow \|f_{iy}\|_{L^{p_0}} = (\|f\|_{L^p})^{\frac{p}{p_0}} = 1$

$|g_{iy}| = \left| |g|^{\frac{1-\beta(iy)}{1-\beta}} \right| = |g|^{\frac{1-\beta_0}{1-\beta}} = |g|^{\frac{q'}{q_0'}}$

$\Rightarrow |g_{iy}|^{q_0'} = |g|^{q'} \Rightarrow \|g_{iy}\|_{L^{q_0'}} = (\|g\|_{L^{q'}})^{\frac{q'}{q_0'}} \leq 1.$

Hence, by Hölder's inequality:

$\left| \int (Tf_{iy}) g_{iy} \, dx \right| \leq \|Tf_{iy}\|_{L^{q_0}} \cdot \|g_{iy}\|_{L^{q_0'}} \leq$

$$\leq K_0 \cdot \|f_{iy}\|_{L^{p_0}} \cdot \|g_{iy}\|_{L^{q_0'}} \leq K_0$$

$$\Rightarrow \boxed{F(iy) \leq K_0 \quad \forall y \in \mathbb{R}}$$

- $q(1+iy) = q_1 + iy \cdot (q_1 - q_0)$

$1 - \beta(1+iy) = (1-\beta_1) + iy(\beta_0 - \beta_1)$

So: $|f_{1+iy}| = \left| |f|^{\frac{q(1+iy)}{q}} \right| = |f|^{\frac{q_1}{q}} = |f|^{\frac{p}{p_1}}$

$$|g_{1+iy}| = \left| |g|^{\frac{1-\beta(1+iy)}{1-\beta}} \right| = |g|^{\frac{1-\beta_1}{1-\beta}} = |g|^{\frac{q'}{q_1'}}$$

$$\Rightarrow |f_{1+iy}|^{p_1} = |f|^p \Rightarrow \|f_{1+iy}\|_{L^{p_1}} = \left(\|f\|_{L^p}\right)^{\frac{p}{p_1}} = 1$$

Also:

$$|g_{1+iy}|^{q_1'} = |g|^{q'} \Rightarrow \|g_{1+iy}\|_{L^{q_1'}} = \left(\|g\|_{L^{q'}}\right)^{\frac{q'}{q_1'}} \leq 1$$

Hence, by Hölder's inequality:

$$\left| \int (Tf_{1+iy}) \cdot g_{1+iy} \, dx \right| \leq \|Tf_{1+iy}\|_{L^{q_1}} \cdot \|g_{1+iy}\|_{L^{q_1'}}$$

$$\leq K_1 \cdot \|f_{1+iy}\|_{L^{p_1}} \cdot \|g_{1+iy}\|_{L^{q_1'}}$$

$$\leq K_1$$

$$\Rightarrow \boxed{F(1+iy) \leq K_1 \quad \forall y \in \mathbb{R}}$$

- It follows from the <u>Three-Lines Lemma</u> that:

$$|F(z)| \leq K_0^{1-\mathrm{Re}\, z} \cdot K_1^{\mathrm{Re}\, z} \quad \text{for all } 0 \leq \mathrm{Re}\, z \leq 1.$$

In particular:

$$\left| \int (Tf) \cdot g \, dx \right| = |F(\theta)| \leq K_0^{1-\theta} \cdot K_1^{\theta}.$$

Thus, for all simple functions f with finite measure support, it is the case that:

$$\|Tf\|_{L^{q_\theta}} \leq K_0^{1-\theta} \cdot K_1^{\theta} \cdot \|f\|_{L^{p_\theta}} \quad \forall \theta \in [0,1]$$

· Let us now prove the claim in general.

In order to do this, it suffices to find a sequence (f_n) of simple functions of finite measure support such that

(*) $f_n \longrightarrow f$ in L^p and $Tf_n \to Tf$ almost everywhere.

(we again use the notation $p := p_\theta$, $q := q_\theta$ for some fixed $\theta \in [0,1]$)

· For $q < \infty$, we could then use Fatou's Lemma and the previous result for simple functions of finite measure support in order to deduce that:

$$\|Tf\|_{L^q} \leq \liminf_n \|Tf_n\|_{L^q} \leq \liminf_n K_0^{1-\theta} \cdot K_1^{\theta} \cdot \|f_n\|_{L^p}$$
$$= K_0^{1-\theta} \cdot K_1^{\theta} \cdot \|f\|_{L^p}. \quad (**)$$

· Furthermore, we note that, by linearity of T, it suffices to prove (*) when $f \geq 0$.
Namely, we write $f = \mathrm{Re} f + i \cdot \mathrm{Im} f$ and we decompose $\mathrm{Re} f$ and $\mathrm{Im} f$ into its positive and negative parts, respectively.

· We now prove (*) for $f \in L^p$, with $f \geq 0$.

We can find a sequence g_n of non-negative simple functions with finite measure support such that:

$$g_n \nearrow f.$$

Let us assume WLOG that $p_0 \leq p_1$.

- Given a function h, we define $h^{(0)} := h \cdot \chi_{\{|h|>1\}}$, i.e.

$$h^{(0)}(x) := \begin{cases} h(x), & \text{when } h(x) > 1 \\ 0, & \text{otherwise.} \end{cases}$$

We then define:

$$h^{(1)}(x) := h(x) - h^{(0)}(x).$$

It is then the case that:

$$g_n^{(0)} \nearrow f^{(0)} \text{ and } g_n^{(1)} \nearrow f^{(1)}$$ (as monotone increasing pointwise almost everywhere limits)

Note that it is crucial to take strict inequality in the definition of $h^{(0)}$.

- Since $f \in L^p$, we obtain that $f^{(0)} \in L^{p_0}$, $f^{(1)} \in L^{p_1}$.
- If $\underline{p_1 < \infty}$, it follows by Lebesgue's Monotone Convergence Theorem that $g_n \to f$ in L^p, $g_n^{(0)} \to f_n^{(0)}$ in L^{p_0}, $g_n^{(1)} \to f^{(1)}$ in L^{p_1}.
- If $\underline{p_1 = \infty}$, the above convergence still holds, but we need to modify the choice of g_n slightly.

In particular, since $f^{(1)} \in L^p \cap L^\infty$, it can be written as an L^∞ limit of simple functions with finite measure support (we are using the sigma-finiteness of X here). This limit can be chosen to be pointwise increasing.

We now choose g_n s.t. $g_n^{(1)}$ is the n^{th} term in this sequence.

From $g_n^{(0)} \to f^{(0)}$ in L^{p_0}, $g_n^{(1)} \to f^{(1)}$, we deduce $Tg_n^{(0)} \to Tf^{(0)}$ in L^{q_0}, $Tg_n^{(1)} \to Tf^{(1)}$ in L^{q_1}. (~~***~~)

In particular, up to a subsequence:

$$Tg_n^{(0)} \to Tf^{(0)} \text{ and } Tg_n^{(1)} \to Tf^{(1)}$$

pointwise almost everywhere.

By linearity of T, it follows that

$$Tg_n \to Tf, \text{ almost everywhere (up to a subsequence).}$$

We then take $f_n := n^{th}$ term in the subsequence. The result when $q < \infty$ now follows from $(*)$ and $(**)$.
When $q = \infty$, we note that $q_0 = q_1 = \infty$. Hence, from $(***)$ and linearity, we deduce that $Tg_n \to Tf$ in L^∞ hence pointwise a.e., so we can argue as before. This proves the theorem when $\gamma > 0$ and $\beta < 1$.
We now need to consider the other cases.
It suffices to assume that $\theta \neq 0, 1$ since the claim easily holds if $\theta = 0, 1$.

- If $\gamma = 0$ and $\beta = 1$, then $(p_0, q_0) = (p_1, q_1) = (p_\theta, q_\theta) = (\infty, 1)$ and the claim holds.

- If $\gamma = 0$ and $\beta < 1$, then $p_0 = p_1 = \infty$.
The previous proof is then modified by taking $f_z := f$ for all $z \in \mathbb{C}$ with $0 \leq \operatorname{Re} z \leq 1$.

- If $\gamma > 0$ and $\beta = 1$, then $q_0 = q_1 = \infty$.
The previous proof is then modified by taking $g_z := g$ for all $z \in \mathbb{C}$ with $0 \leq \operatorname{Re} z \leq 1$.

The Riesz-Thorin theorem now follows. $\square$

Remark: Let us note that the second step in the proof was necessary. Namely, the fact that $\|Tf\|_{L^{q_\theta}} \leq K_0^{1-\theta} \cdot K_1^{\theta} \cdot \|f\|_{L^{p_\theta}}$ for all simple functions f of finite measure support does not immediately imply the claim in general. Namely, since T is defined on all of L^{p_0} and on all of L^{p_1}, it follows that T is defined on all of L^{p_θ}.
It is true that, by the inequality from step 1, it is the case that T extends uniquely to a continuous linear operator $\overline{T}$ on L^{p_θ}.
However, we do not automatically know that $\overline{T} = T$ (for this, we would need to know that T is continuous on L^{p_θ}, which we are trying to prove.)

Examples and applications for the Riesz-Thorin theorem

Example 1: (Motivating example)

Let $1 \leq p_0 \leq p_1 \leq \infty$ be given.

Let $f \in L^{p_0}(X) \cap L^{p_1}(X)$.

We define the operator T to be multiplication by f.

In other words:

$$T(g) := f \cdot g. \quad T \text{ is a linear operator.}$$

- Let us recall the following notation:

Given $1 \leq p \leq \infty$, we define the number $1 \leq p' \leq \infty$ to satisfy:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

p' is then called the Hölder conjugate of p.

Hölder's inequality states that:

$$\|f_1 \cdot f_2\|_{L^1(X)} \leq \|f_1\|_{L^p(X)} \cdot \|f_2\|_{L^{p'}(X)}.$$

Moreover:

$$\|f_1\|_{L^p(X)} = \sup \left\{ \int_X f_1 \cdot f_2 \, dx \, ; \, \|f_2\|_{L^{p'}(X)} = 1 \right\}$$

(by sigma-finiteness of X)

- In particular, we observe that:

$$T : L^{p_0'}(X) \to L^1(X)$$

$$T : L^{p_1'}(X) \to L^1(X).$$

Let us compute the norm of $T\big|_{L^{p_0'}(X)}$.

It is:

$$\sup_{\|g\|_{L^{p_0'}(X)} = 1} T(g) = \sup \left\{ \int_X f \cdot g \, dx \, ; \, \|g\|_{L^{p_0'}(X)} = 1 \right\} = \|f\|_{L^{p_0}(X)}$$

We write this as: $\|T\|_{L^{p_0'}(X) \to L^1(X)} = \|f\|_{L^{p_0}(X)}$.

Analogously:

$$\|T\|_{L^{p_1'}(X)\to L^1(X)} = \|f\|_{L^{p_1}(X)}$$

- Given $\Theta \in [0,1]$, we define $1 \le p_\Theta \le \infty$ as:

$$\frac{1}{p_\Theta} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$$ (observe that $p_0 \le p_\Theta \le p_1$).

Then, as before, we know that:

$$T : L^{p_\Theta'}(X) \to L^1(X)$$

$$\|T\|_{L^{p_\Theta'}(X)\to L^1(X)} = \|f\|_{L^{p_\Theta}(X)} < \infty$$

- Let us note that:

$$\frac{1}{p_\Theta'} = \frac{1-\Theta}{p_0'} + \frac{\Theta}{p_1'}, \qquad \frac{1}{1} = \frac{1-\Theta}{1} + \frac{\Theta}{1}$$

By the logarithmic convexity of L^p norms, we know that:

$$\|f\|_{L^{p_\Theta}(X)} \le \|f\|_{L^{p_0}(X)}^{1-\Theta} \cdot \|f\|_{L^{p_1}(X)}^{\Theta}$$

Putting everything together, it follows that:

$$\|T\|_{L^{p_\Theta'}(X)\to L^1(X)} \le \|T\|_{L^{p_0'}(X)\to L^1(X)}^{1-\Theta} \cdot \|T\|_{L^{p_1'}(X)\to L^1(X)}^{\Theta}$$

- This is consistent with the Riesz-Thorin theorem.

Applications:

① Generalized Young's Inequality:

$1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \cdot \|g\|_{L^q(\mathbb{R}^n)}$$

Proof: Let $f \in L^p(\mathbb{R}^n)$ be fixed.
Let $T := f * (\cdot)$. T is linear.
From the ordinary Young's inequality, T is bounded from $L^1(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with norm $\leq \|f\|_{L^p(\mathbb{R}^n)}$.
From Hölder's inequality:

$$\|f*g\|_{L^\infty(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y) \cdot g(y)\, dy \right|$$

$$\leq \sup_{x \in \mathbb{R}^n} \|f(x-\cdot)\|_{L^p(\mathbb{R}^n)} \cdot \|g\|_{L^{p'}(\mathbb{R}^n)}$$

$$= \|f\|_{L^p(\mathbb{R}^n)} \cdot \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

In other words,
T is bounded from $L^{p'}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ with operator norm $\leq \|f\|_{L^p(\mathbb{R}^n)}$.
The Riesz-Thorin theorem then implies that
T is bounded from $L^{q_\theta}(\mathbb{R}^n)$ to $L^{r_\theta}(\mathbb{R}^n)$ with operator norm $\leq \|f\|_{L^p(\mathbb{R}^n)}$ where, for $\theta \in [0,1]$:

$$\frac{1}{q_\theta} = \frac{1-\theta}{1} + \frac{\theta}{p'}, \quad \frac{1}{r_\theta} = \frac{1-\theta}{p} + \frac{\theta}{\infty}$$

So, if $q_\theta = q$, then:

$$\frac{1-\theta}{1} + \frac{\theta}{p'} = \frac{1}{q}$$

$$1 - \theta + \theta - \frac{\theta}{p} = \frac{1}{q}$$

$$\frac{\theta}{p} = 1 - \frac{1}{q}$$

$$\theta = p \cdot \left(1 - \frac{1}{q}\right) = p \cdot \left(\frac{1}{p} - \frac{1}{r}\right) = 1 - \frac{p}{r} \in [0,1], \text{ since } r \geq p.$$

$$\Rightarrow \frac{1}{r_\theta} = \frac{1-\theta}{p} = \frac{1-\left(1-\frac{p}{r}\right)}{p} = \frac{\frac{p}{r}}{p} = \frac{1}{r}.$$

$\Rightarrow$ T is bounded from $L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ with operator norm $\leq \|f\|_{L^p(\mathbb{R}^n)}$.

In other words:

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \cdot \|g\|_{L^q(\mathbb{R}^n)}. \quad \square$$

---

(2) Hausdorff-Young Inequality:

$$\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } 1 \leq p \leq 2.$$

Proof: Let $T := \hat{\cdot}$. T is linear.

We know that T is bounded from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ with operator norm ≤ 1, by Theorem 1.1 in §1.

By Plancherel's theorem (Theorem 3.1 in §1), we know that T is an isometry on $L^2(\mathbb{R}^n)$. In particular, it is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with operator norm equal to 1.

By the Riesz-Thorin theorem, it follows that, for all $\theta \in [0,1]$, T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where:

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{1}, \quad \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{\infty}$$

From the first equation, it follows that:

$$\theta = 2 \cdot \left(1 - \frac{1}{p}\right)$$

So:

$$\frac{1}{q} = \frac{\theta}{2} = 1 - \frac{1}{p} = \frac{1}{p'}$$

$$\Rightarrow q = p'$$

Hence, for all $1 \leq p \leq 2$, T is bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ with operator norm 1.

In other words:

$$\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 \leq p \leq 2$. $\square$

Remark: It is possible to show that, if (p,q) are s.t.

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } f \in L^p(\mathbb{R}^n),$$

then:

i) $q = p'$

ii) $p \in [1,2]$.

§2.2: The Marcinkiewicz Interpolation Theorem: Real interpolation

Weak-type inequalities:

Let (X,μ) and (Y,ν) be measure spaces and let $1 \le p, q \le \infty$ be given.

Furthermore, let T be an operator from $L^p(X,\mu)$ to the space of all ν-measurable functions from Y to $\mathbb{C}$.

We say that T is weak (p,q) if there exists $C>0$ s.t:

① $\nu(\{y \in Y : |Tf(y)| > \lambda\}) \le \left(\frac{C\|f\|_{L^p}}{\lambda}\right)^q$

for all $\lambda > 0$ if $q < \infty$.

② $\|Tf\|_{L^\infty} \le C\|f\|_{L^p}$ if $q = \infty$.

- From now on, if $T: L^p(X,\mu) \to L^q(Y,\nu)$ is bounded (in the sense that there exists $C \ge 0$ s.t. $\|Tf\|_{L^q} \le C\|f\|_{L^p}$, for all $f \in L^p$), then we will say that T is strong (p,q).

Observation: If T is strong (p,q), then it is weak (p,q).

- It suffices to show this when $q < \infty$.

Then: for all $f \in L^p(X,\mu)$, $\lambda > 0$:

$$\nu(\{y \in Y : |Tf(y)| > \lambda\}) = \int_{\{y\in Y: |Tf(y)|>\lambda\}} d\nu \le \int_{\{y\in Y: |Tf(y)|>\lambda\}} \frac{|Tf(x)|^q}{\lambda^q} d\nu$$

$$\le \frac{\|Tf\|_{L^q}^q}{\lambda^q} \le \left(\frac{C\|f\|_{L^p}}{\lambda}\right)^q.$$

- When $(X,\mu) = (Y,\nu)$, $p = q < \infty$ and $T = Id$, the weak (p,q) inequality is the Chebyshev Inequality:

$\nu(\{y \in Y : |f(y)| > \lambda\}) \le \frac{\|f\|_{L^p}^p}{\lambda^p}$. (also known as Markov's Inequality.)

- Let (X,μ) be a measure space and let $f: X \to \mathbb{C}$ be a measurable function. The function $a_f : (0,+\infty) \to [0,+\infty]$ given by:

$$a_f(\lambda) := \mu(\{x \in X : |f(x)| > \lambda\})$$

is called the distribution function of f (associated with μ).

- We observe that this quantity appears in the definition of weak boundedness.

The following identity holds for distribution functions:

Proposition 2.1: Let $\varphi : [0,+\infty) \to [0,+\infty)$ be differentiable, non-decreasing and such that $\varphi(0) = 0$. Then:

$$\int_X \varphi(|f(x)|)\, dx = \int_0^{+\infty} \varphi'(\lambda) \cdot a_f(\lambda)\, d\lambda$$

In particular, if $\varphi(\lambda) = \lambda^p$, then:

$$\int_X |f(x)|^p\, dx = p \cdot \int_0^{+\infty} \lambda^{p-1} \cdot a_f(\lambda)\, d\lambda$$

$$\text{i.e. } \|f\|_{L^p}^p = p \cdot \int_0^{+\infty} \lambda^{p-1} \cdot a_f(\lambda)\, d\lambda .$$

Proof: Let us note that the left-hand side equals:

$$\int_X \left(\int_0^{|f(x)|} \varphi'(\lambda)\, d\lambda \right) d\mu$$

By Tonelli's theorem, we can interchange the order of integration.
($\varphi' \geq 0$ by assumption)

In particular, this expression equals:

$$\int_0^{+\infty} \int_{\{x : |f(x)| > \lambda\}} \varphi'(\lambda)\, d\mu\, d\lambda =$$

$$= \int_0^{+\infty} \varphi'(\lambda) \cdot \mu(\{x : |f(x)| > \lambda\})\, d\lambda$$

$$= \int_0^{+\infty} \varphi'(\lambda) \cdot a_f(\lambda)\, d\lambda. \qquad \square$$

· This leads us to the following:

Definition: We say that a measurable function on (X,μ) belongs to weak L^p, which we write as $f \in L^{p,\infty}(X)$ if:

- $$\|f\|_{L^{p,\infty}(X)} := \sup_{\lambda>0} \lambda \cdot \left[\mu(\{x\in X : |f(x)|>\lambda\})\right]^{\frac{1}{p}} < \infty \;, \text{ for } p\in[1,\infty)$$

- For $p=\infty$, we set $L^{\infty,\infty}(X)$ and $\|\cdot\|_{L^{\infty,\infty}(X)} := \|\cdot\|_{L^\infty(X)}$

Hence, the condition that an operator T on $L^p(X,\mu)$ to the space of all ν-measurable functions on Y is of weak-type (p,q) can be written more concisely as:

$$\|Tf\|_{L^{q,\infty}(Y,\nu)} \leq C\|f\|_{L^p(X,\mu)}$$

· Chebyshev's inequality allows us to deduce that

$$\|f\|_{L^{p,\infty}(X)} \leq \|f\|_{L^p(X)}, \qquad (\text{for } p\in[1,+\infty)).$$

Namely: for all $\lambda>0$

$$\lambda \cdot \left[\mu(\{x\in X : |f(x)|>\lambda\})\right]^{\frac{1}{p}} \leq \lambda \cdot \left(\frac{\|f\|^p_{L^p(X)}}{\lambda^p}\right)^{\frac{1}{p}} = \|f\|_{L^p(X)}$$

In particular, $L^p(X) \hookrightarrow L^{p,\infty}(X)$.

$L^{p,\infty}(X)$ is not a normed space. It is a quasi-normed space in the sense that the quasi-triangle inequality holds:

$$\|f+g\|_{L^{p,\infty}(X)} \leq C\cdot\left(\|f\|_{L^{p,\infty}(X)} + \|g\|_{L^{p,\infty}(X)}\right)$$

for some constant $C\geq 1$. (this will be elaborated on the homework assignment.)

· More generally, one can study Lorentz spaces: for $0<p<\infty$, $0<q\leq\infty$

$$\|f\|_{L^{p,q}(X)} := \left\| \lambda\cdot\left(\mu(|f|>\lambda)\right)^{\frac{1}{p}} \right\|_{L^q\left((0,+\infty),\frac{d\lambda}{\lambda}\right)}$$

Hereafter, we will write $\mu(\{x : |f(x)|>\lambda\})$ as $\mu(|f|>\lambda)$.

If $q < \infty$, this is:

$$\|f\|_{L^{p,q}(X)} = \left(\int_0^{+\infty} \lambda^{q-1} \cdot \left(\mu(|f| \geq \lambda)\right)^{\frac{q}{p}} d\lambda \right)^{\frac{1}{q}}.$$

In this notation:

$$L^{p,p}(X) = L^p(X) \text{ for } 1 \leq p < \infty.$$

We have now come to the main result in this section.

We say that an operator T is <u>sublinear</u> if

$$|T(f_1+f_2)| \leq |Tf_1| + |Tf_2| \quad , \quad |T(cf)| = |c| \cdot |Tf|$$

In particular, if S is a linear operator, then $T := |S|$ is quasilinear.

<u>Theorem 2.2:</u> (Marcinkiewicz Interpolation Theorem)

Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ be such that <u>$p_0 \leq q_0$, $p_1 \leq q_1$, $q_0 \neq q_1$.</u>

Suppose that T is an operator defined on $L^{p_0}(X,\mu) + L^{p_1}(X,\mu)$ which takes values in the class of ν-measurable functions on Y such that:

i, T is sublinear.

ii, T is weak (p_j, q_j) for $j = 0, 1$.

Then T is <u>strong (p_θ, q_θ)</u> for all <u>$\theta \in (0,1)$</u> where p_θ and q_θ are defined as:

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

$$\frac{1}{q_\theta} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

<u>Proof:</u> Let us first consider the case when <u>$p_0 \neq p_1$; $q_0, q_1, p_0, p_1 < \infty$.</u>

For $n \in \mathbb{Z}$, we let:

$$m_n(f) := \mu\left(2^{n-1} < |f| \leq 2^n\right).$$

Then, for $r \in [1, \infty)$, it is the case that:

1) $\|f\|_{L^r}^r \leq \sum_n 2^{nr} \cdot m_n(f) \leq 2^r \|f\|_{L^r}^r$

1) follows from the fact that:

$$|f| \leq \sum_n 2^n \cdot \chi\left(2^{n-1} < |f| \leq 2^n\right) \leq 2 \cdot |f|$$

and from the construction of the Lebesgue integral.

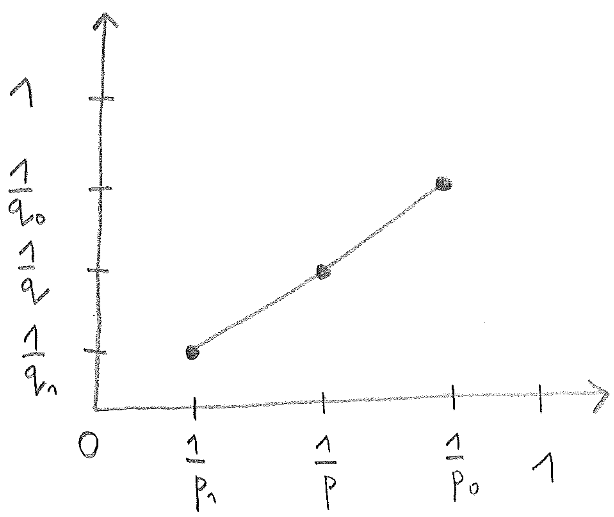
· In what follows, we will also use the notation:

$$m_n(Tf) := \nu\left(2^{n-1} < |Tf| \leq 2^n\right).$$

In other words, we will use the same notation for the measures μ and ν.

Let $\Theta \in (0,1)$ be fixed.
We let $p := p_\Theta$, $q := q_\Theta$.
Throughout, we will assume WLOG that $\underline{p_0 < p_1}$.

We know that $\left(\frac{1}{p_0}, \frac{1}{q_0}\right)$, $\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$ and $\left(\frac{1}{p}, \frac{1}{q}\right)$ lie on the same line.



The slope of this line is:

$$\sigma := \frac{\frac{1}{q} - \frac{1}{q_1}}{\frac{1}{p} - \frac{1}{p_1}} = \frac{\frac{1}{q} - \frac{1}{q_0}}{\frac{1}{p} - \frac{1}{p_0}}, \text{ i.e. } \sigma = \frac{q_1 - q}{p_1 - p} \cdot \frac{p p_1}{q q_1} = \frac{q - q_0}{p - p_0} \cdot \frac{p_0 p}{q_0 q}.$$

Let $\eta := 6 \cdot \frac{q}{p}$.

Then $\eta \cdot (p_0 - p) \cdot \frac{q_0}{p_0} = 6 \cdot (p_0 - p) \cdot \frac{q_0}{p_0} \cdot \frac{q}{p} = \frac{q - q_0}{p - p_0} \cdot \frac{p_0 p}{q_0 q} \cdot (p_0 - p) \cdot \frac{q_0 q}{p_0 p}$

$= q_0 - q.$

$$\eta \cdot (p_1 - p) \cdot \frac{q_1}{p_1} = 6 \cdot (p_1 - p) \cdot \frac{q_1}{p_1} \cdot \frac{q}{p} = \frac{q_1 - q}{p_1 - p} \cdot \frac{p p_1}{q q_1} \cdot (p_1 - p) \cdot \frac{q_1 q}{p_1 p} =$$

$= q_1 - q.$

By scaling, we can assume WLOG that $\|f\|_{L^p} < 1$ is sufficiently small.

Hence, we need to show that $\|Tf\|_{L^q} \le C$. (Note: Here, we use the assumption that T is sublinear.)

If $\|f\|_{L^p} \le \frac{1}{2}$, then we deduce:

$$\sum_n m_n(f) \cdot 2^{np} \le 1 \qquad (\mathrm{I})$$

(here, we used 1))

<u>Goal:</u> Estimate $m_n(Tf)$.

Let us write $f = f_{0,n} + f_{1,n}$, for:

$$\left.\begin{array}{l} f_{0,n}(x) := f(x) \cdot \mathbb{1}(|f| > 2^{\eta n}) \\ f_{1,n}(x) := f(x) \cdot \mathbb{1}(|f| \le 2^{\eta n}). \end{array}\right\} \text{Here, we use the parameter } \eta.$$

Since $f \in L^p$, it follows that:

$$f_{0,n} \in L^{p_0}, \quad f_{1,n} \in L^{p_1}.$$

Since $\{x : |Tf(x)| > \lambda\} \subseteq \{x : |Tf_{0,n}(x)| > \frac{\lambda}{2}\} \cup \{x : |Tf_{1,n}(x)| > \frac{\lambda}{2}\}$

by the assumption that <u>T is sublinear</u>, it follows that:

$m_n(Tf) \le \nu(|Tf| > 2^{n-1})$

$\le \nu(|Tf_{0,n}| > 2^{n-2}) + \nu(|Tf_{1,n}| > 2^{n-2})$

$$\le \frac{\|Tf_{0,n}\|_{L^{q_0,\infty}}^{q_0}}{(2^{n-2})^{q_0}} + \frac{\|Tf_{1,n}\|_{L^{q_1,\infty}}^{q_1}}{(2^{n-2})^{q_1}}, \quad \text{(by definition of } \|\cdot\|_{L^{q,\infty}}\text{)}$$

$$= \frac{4^{q_0} \cdot \|Tf_{0,n}\|_{L^{q_0,\infty}}^{q_0}}{2^{nq_0}} + \frac{4^{q_1} \cdot \|Tf_{1,n}\|_{L^{q_1,\infty}}^{q_1}}{2^{nq_1}}$$

by the definition of $\|\cdot\|_{L^{q_j,\infty}}$.

By the boundedness assumptions, this is

$$\leq \frac{C \cdot \|f_{0,n}\|_{L^{p_0}}^{q_0}}{2^{nq_0}} + \frac{C \cdot \|f_{1,n}\|_{L^{p_1}}^{q_1}}{2^{nq_1}}$$

(Here $C>0$ depends on q_0, q_1.)

which by 1) is

$$\leq C \cdot 2^{-nq_0} \cdot \Big(\sum_{k > \gamma n} m_k(f) \cdot 2^{kp_0} \Big)^{\frac{q_0}{p_0}}$$

$$+ \; C \cdot 2^{-nq_1} \cdot \Big(\sum_{k \leq \gamma n+1} m_k(f) \cdot 2^{kp_1} \Big)^{\frac{q_1}{p_1}} \qquad (\mathrm{II})$$

The first term in (II) is:

$$\lesssim 2^{-nq_0} \cdot \Big(\sum_{k > \gamma n} m_k(f) \cdot 2^{kp_0} \Big)^{\frac{q_0}{p_0}}$$

$$= 2^{-nq} \cdot \Big(\sum_{k > \gamma n} m_k(f) \cdot 2^{kp + k(p_0 - p) - n(q_0 - q) \cdot \frac{p_0}{q_0}} \Big)^{\frac{q_0}{p_0}}$$

$$= 2^{-nq} \cdot \Big(\sum_{k > \gamma n} m_k(f) \cdot 2^{kp + (k - \gamma n)\cdot(p_0 - p)} \Big)^{\frac{q_0}{p_0}}$$

since $\gamma \cdot (p_0 - p) = (q_0 - q) \cdot \frac{p_0}{q_0}$

Now, in the above expression, the sum is ≤ 1.
Namely, $2^{(k-\gamma n)\cdot(p_0-p)} \leq 1$ and $\sum_{k \geq \gamma n} m_k(f) \cdot 2^{kp} \leq 1$

Hence, the first term in (II) is:

$$\lesssim 2^{-nq} \cdot \sum_{k > \eta n} m_k(f) \cdot 2^{kp + (k-\eta n)\cdot(p_0 - p)}$$

Here, it was crucial to use the fact that $\frac{q_0}{p_0} \geq 1$.

By similar arguments, the second term in (II) is:

$$\lesssim 2^{-nq_1} \cdot \Big(\sum_{k \leq \eta n + 1} m_k(f) \cdot 2^{kp_1} \Big)^{\frac{q_1}{p_1}} =$$

$$= 2^{-nq} \cdot \Big(\sum_{k \leq \eta n+1} m_k(f) \cdot 2^{kp + k(p_1 - p) - n(q_1 - q)\cdot \frac{p_1}{q_1}} \Big)^{\frac{q_1}{p_1}}$$

$$= 2^{-nq} \cdot \Big(\sum_{k \leq \eta n + 1} m_k(f) \cdot 2^{kp + (\eta n - k)\cdot(p - p_1)} \Big)^{\frac{q_1}{p_1}}$$

$$\leq 2^{-nq} \cdot \sum_{k \leq \eta n + 1} m_k(f) \cdot 2^{kp + (\eta n - k)\cdot(p - p_1)} =$$

$$= 2^{-nq} \cdot \sum_{k \leq \eta n} m_k(f) \cdot 2^{kp + (\eta n - k)\cdot(p - p_1)}$$

$$+ 2^{-nq} \cdot m_{k_n}(f) \cdot 2^{k_n p + (p_1 - p)},$$

where $k_n \in \mathbb{Z}$ is the unique integer such that $0 < k_n - \eta n \leq 1$.

This is the extra term.

Let $\varepsilon := \min(p - p_0, p_1 - p) > 0$. Then, we have shown that:

$$m_n(Tf) \leq C \cdot 2^{-nq} \cdot \sum_k m_k(f) \cdot 2^{kp - |\eta n - k| \cdot \varepsilon}$$
$$+ C \cdot 2^{-nq} \cdot m_{k_n}(f) \cdot 2^{k_n p + (p_1 - p)}$$

So, by 1) it follows that:

$$\|Tf\|_{L^q}^q \leq \sum_n 2^{nq} \cdot m_n(Tf)$$

$$\lesssim \sum_n \sum_k m_k(f) \cdot 2^{kp - |\eta n - k| \cdot \varepsilon} + \sum_n m_{k_n}(f) \cdot 2^{k_n p} \cdot 2^{p_1 - p}$$

$$= \sum_k \Big(m_k(f) \cdot 2^{kp} \cdot \sum_n 2^{-|\eta n - k| \cdot \varepsilon} \Big) + \sum_n m_{k_n}(f) \cdot 2^{k_n p} \cdot 2^{p_1 - p}$$

$$\lesssim \sum_k m_k(f)\cdot 2^{kp} + \sum_n m_{k_n}(f)\cdot 2^{k_n p} \lesssim \sum_k m_k(f) 2^{kp} \lesssim \|f\|_{L^p}^p$$

In the last step, we used 1) and the fact that, for each fixed k, there are $\leq \frac{C}{|\gamma|}$ possible values of n such that $k_n = k$.

- We need to consider the case when some of the exponents are infinite.
- If $q_0 < \infty$ and $q_1 < \infty$, we argue as follows:

The quantities σ and γ are defined as before.

① If $p_1 < \infty$, then:

$$\|f_{1,n}\|_{L^{p_1}}^{p_1} = \int_{|f|\leq 2^{\gamma n}} |f|^{p_1}\,dx \leq \int_{|f|\leq 2^{\gamma n}} |f|^p \cdot 2^{\gamma n(p_1-p)}\,dx$$

So:

$$\|f_{1,n}\|_{L^{p_1}} \leq \|f\|_{L^p}^{\frac{p}{p_1}} \cdot 2^{\gamma n\cdot\frac{p_1-p}{p_1}} = \|f\|_{L^p}^{\frac{p}{p_1}}\cdot 2^n \text{ since } \gamma\cdot\frac{p_1-p}{p} = 1$$

If we choose $\|f\|_{L^p} << 1$ sufficiently small, then this is $\leq \frac{2^{n-2}}{C_1}$, where $C_1 > 0$ is the constant in $\|Tg\|_{L^\infty} \leq C_1 \|g\|_{L^{p_1}}$

Hence:

$$\|Tf_{1,n}\|_{L^\infty} \leq C_1\cdot\|f_{1,n}\|_{L^{p_1}} \leq C_1\cdot(2^{n-2}/C_1) = 2^{n-2}$$

In other words, $\nu(|Tf_{1,n}| > 2^{n-2}) = 0$

The estimate is proved analogously as before. We only need to bound the terms coming from $f_{0,n}$.

② If $p_1 = \infty$, then we have to modify the argument a little bit.

Let us first note that $\gamma = 1$.

We define $f_{1,n}(x) := f(x)\cdot \mathbb{1}(|f| \leq \frac{1}{4C_1}\cdot 2^n)$

$$f_{0,n}(x) := f(x) - f_{1,n}(x).$$

$$\|Tf_{1,n}\|_{L^\infty} \leq C_1\|f_{1,n}\|_{L^\infty} \leq 2^{n-2}.$$

Hence $\nu(|Tf_{1,n}| > 2^{n-2}) = 0$. The proof follows as in the case when $p_1 < \infty$.

- If $q_0 = \infty$ and $q_1 < \infty$, the argument is analogous.

- If $p_0 = \infty$ or $p_1 = \infty$, then since $q_0 \geq p_0$ and $q_1 \geq p_1$, it follows that $q_0 = \infty$ or $p_0 = \infty$.

Thus, we are reduced to the previous cases.

- Finally, the case that we need to consider is:

$1 \leq p_0 = p_1 < \infty$, $1 \leq q_0, q_1 \leq \infty$, $q_0 \geq p_0$, $q_1 \geq p_1$, $q_0 \neq q_1$

Let $p := p_0 = p_1$. We assume WLOG that $q_0 < q_1$.
We fix $f \in L^p(X, \mu)$. By scaling, we can assume WLOG that $\|f\|_{L^p} = 1$. For now, we assume that $q_1 < \infty$.
We know by assumption that, for some $C_0, C_1 > 0$ independent of f:

$$\|Tf\|_{L^{q_0,\infty}} \leq C_0$$

$$\|Tf\|_{L^{q_1,\infty}} \leq C_1 .$$

Hence:

$$\nu(|Tf| > \lambda) \leq \frac{C_0^{q_0}}{\lambda^{q_0}}$$

$$\nu(|Tf| > \lambda) \leq \frac{C_1^{q_1}}{\lambda^{q_1}}$$

If we let $\hat{q}_\gamma := (1-\gamma) \cdot q_0 + \gamma \cdot q_1$, it follows that:
(for $\gamma \in [0,1]$)

$$\nu(|Tf| > \lambda) \leq \frac{C_\gamma^{\hat{q}_\gamma}}{\lambda^{\hat{q}_\gamma}} \qquad (\Delta)$$

for $C_\gamma := \left(C_0^{(1-\gamma) q_0} \cdot C_1^{\gamma q_1}\right)^{\frac{1}{\hat{q}_\gamma}}$

- The point is that (Δ) can be improved to incorporate some decay in λ.
- Namely, there exists a unique $\lambda_0 > 0$ such that:

$$\frac{C^{q_0}}{\lambda_0^{q_0}} = \frac{C^{q_1}}{\lambda_0^{q_1}}, \text{ for } C := \max(C_0, C_1).$$

i.e. for which the upper bounds "balance out".

<u>Claim:</u> For $\vartheta \in (0,1)$, $\nu(|Tf| > \lambda) \leq M \cdot \frac{C^{\hat{q}_\vartheta}}{\lambda^{\hat{q}_\vartheta}} \cdot \min\left(\frac{\lambda}{\lambda_0}, \frac{\lambda_0}{\lambda}\right)^{\varepsilon}$ for some $M, \varepsilon > 0$, depending on $q_0, q_1, \vartheta, C_0, C_1$.

→ Here $C := \max(C_0, C_1)$ as defined earlier.

<u>Proof of Claim:</u>

- Let us consider first $\underline{\lambda \geq \lambda_0}$. Then $\underbrace{\frac{C^{q_1}}{\lambda^{q_1}}}_{\text{better bound}} \leq \frac{C^{q_0}}{\lambda^{q_0}}$.

We want to find $M, \varepsilon > 0$ s.t.

$$\frac{C^{q_1}}{\lambda^{q_1}} \leq M \cdot \frac{C^{(1-\vartheta)q_0 + \vartheta q_1}}{\lambda^{(1-\vartheta)q_0 + \vartheta q_1}} \cdot \left(\frac{\lambda_0}{\lambda}\right)^{\varepsilon}$$

(we recall that $\nu(|Tf| > \lambda) \leq \frac{C^{q_1}}{\lambda^{q_1}}$)

The above inequality is equivalent to:

$$\frac{C^{(1-\vartheta)q_1}}{\lambda^{(1-\vartheta)q_1}} \leq M \cdot \frac{C^{(1-\vartheta)q_0}}{\lambda^{(1-\vartheta)q_0}} \cdot \left(\frac{\lambda_0}{\lambda}\right)^{\varepsilon} \quad \Big/ \,{}^{\frac{1}{1-\vartheta}}$$

$$\Leftrightarrow \frac{C^{q_1}}{\lambda^{q_1}} \leq \frac{C^{q_0}}{\lambda^{q_0}} \cdot \left(\frac{\lambda_0}{\lambda}\right)^{\frac{\varepsilon}{1-\vartheta}} \cdot M^{\frac{1}{1-\vartheta}}$$

$$\Leftrightarrow \frac{1}{\lambda^{q_1 - q_0}} \leq M^{\frac{1}{1-\vartheta}} \cdot C^{q_0 - q_1} \cdot \left(\frac{\lambda_0}{\lambda}\right)^{\frac{\varepsilon}{1-\vartheta}}, \text{ for } \lambda \geq \lambda_0.$$

We now choose $\varepsilon > 0$ s.t. $\frac{\varepsilon}{1-\vartheta} \leq q_1 - q_0$.

Then, we choose $M > 0$ large enough.

- Similarly, by choosing M possibly larger and ε possibly smaller, we can arrange so that:

$$\frac{C^{q_0}}{\lambda^{q_0}} \leq M \cdot \frac{C^{(1-\vartheta)q_0 + \vartheta q_1}}{\lambda^{(1-\vartheta)q_0 + \vartheta q_1}} \cdot \left(\frac{\lambda}{\lambda_0}\right)^{\varepsilon} \text{ for } \lambda \leq \lambda_0.$$

- By construction, there exists $\theta \in (0,1)$ (which is uniquely determined) such that $\hat{q}_\theta = q$.
We henceforth fix such an θ.
We then compute:

$$\int |Tf|^q \, dx = q \int_0^{+\infty} \lambda^{q-1} \, \nu(|Tf|>\lambda) \, d\nu$$

which, by the claim is:

$$\leq q M C^q \cdot \Big[\int_0^{\lambda_0} \lambda^{-1+\varepsilon} \cdot (\lambda_0^{-\varepsilon}) \, d\lambda + \int_{\lambda_0}^{+\infty} \lambda^{-1-\varepsilon} \cdot (\lambda_0^{\varepsilon}) \, d\lambda \Big]$$

$$< \infty.$$

Here, we used that $\lambda^{-1+\varepsilon}$ is integrable near the origin and that $\lambda^{-1-\varepsilon}$ is integrable near $+\infty$.
Since we assumed that $\|f\|_{L^p} = 1$, the strong (p,q) boundedness of T now follows.

- This argument needs to be modified slightly when $q_1 = \infty$.
In this case, we still have $\nu(|Tf|>\lambda) \leq \frac{C_0^{q_0}}{\lambda^{q_0}}$ for all $\lambda > 0$. Also $q_0 < q < \infty$.
We can apply the argument for $q_1 < \infty$ provided that we know $\nu(|Tf|>\lambda) \leq \frac{C_1^{\tilde{q}_1}}{\lambda^{\tilde{q}_1}}$ for all $\lambda>0$ and for some $\tilde{q}_1 > q$, which is finite.
Note that here $\tilde{q}_1$ is otherwise arbitrary.
Let us explain why this latter condition is indeed satisfied.
By assumption, we know that: T is strong (p,∞)
$\Rightarrow \|Tf\|_{L^\infty} \leq C_1 \|f\|_{L^p}$

Hence, if $\|f\|_{L^p} = 1$, then $\nu(|Tf| > \lambda) = 0$ for λ sufficiently large.

In particular, this is stronger than the bound

$\nu(|Tf| > \lambda) \leq \frac{C_1^{\tilde{q}_1}}{\lambda^{\tilde{q}_1}}$ that was needed before (for finite $\tilde{q}_1 > q$)

For smaller λ, we just need to choose $C_1 > 0$ sufficiently large so that the estimate $\nu(|Tf| > \lambda) \leq \frac{C_1^{\tilde{q}_1}}{\lambda^{\tilde{q}_1}}$ holds.

At this stage, the proof is concluded as when $q_1 < \infty$.

The Marcinkiewicz Interpolation Theorem now follows. $\square$

<u>Remark:</u> The assumption that $q_0 \geq p_0$ and $q_1 \geq p_1$ in general cannot be removed.

We will see an explicit counterexample when this assumption does not hold on the homework.

The counterexample is given as follows:

Let $\gamma \in \mathbb{R}$. Then, μ_γ is defined to be the weighted counting measure $\mu_\gamma := \sum_{n \in \mathbb{N}} 2^{\gamma n} \cdot \delta_n$ on $\mathbb{N}$.

Let $X_\gamma := (\mathbb{N}, \mu_\gamma)$.

Suppose that $0 < \beta < \gamma$ and $1 \leq p, q < \infty$ be such that $\frac{\gamma}{p} = \frac{\beta}{q}$. Hence $p > q$.

Then, the identity map $I : X_\gamma \to X_\beta$ is weak (p,q) but not strong (p,q).

From here, one can construct a counterexample.

<u>Remark:</u> It can be shown that, for $1 < p < \infty$:

$$\|f\|_{L^{p,q}} \sim_{p,q} \sup_g \left\{ \left|\int f \cdot g\right|, \|g\|_{L^{p',q'}} \leq 1 \right\}, \text{ when } 1 \leq q < \infty$$

$$\|f\|_{L^{p,\infty}} \sim_p \sup_E \left\{ \left(\frac{1}{\mu(E)}\right)^{\frac{1}{p'}} \cdot \left|\int f \cdot \chi_E\right| ; \mu(E) < \infty \right\}, \text{ when } q = \infty.$$

Hence, for $1 < p < \infty$, $1 \leq q \leq \infty$, $\|\cdot\|_{L^{p,q}}$ is equivalent to a norm.

When $p = 1$, this is no longer the case when $q > 1$.

We shall see an explicit counterexample on the homework.

<u>Example 1:</u> (Hardy's Inequality)

If $1 < p \leq \infty$ and if $f \in L^1_{loc}(\mathbb{R}^+)$, then:

$$\left\| \frac{1}{x} \int_0^x f(t)\,dt \right\|_{L^p(\mathbb{R}^+)} \lesssim_p \|f\|_{L^p(\mathbb{R}^+)}.$$

<u>Proof:</u> Let $(Tf)(x) := \frac{1}{x} \int_0^x f(t)\,dt$.

Then, $T : L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+) \to$ Space of Lebesgue measurable functions on $\mathbb{R}^+$

is linear.

- By construction, $\|Tf\|_{L^\infty(\mathbb{R}^+)} \leq \|f\|_{L^\infty(\mathbb{R}^+)}$.

- Suppose that $f \in L^1(\mathbb{R}^+)$. Then Tf is continuous (by the Dominated Convergence Theorem). {This holds even if $f \in L^1_{loc}(\mathbb{R}^+)$.}

Let us suppose first that <u>$f \geq 0$.</u>

Then, for $\lambda > 0$, we want to estimate the measure of

$E(\lambda) := \{x \in \mathbb{R}^+ ; |Tf(x)| > \lambda\} = \{x \in \mathbb{R}^+ ; Tf(x) > \lambda\}$

Since Tf is continuous, it follows that $E(\lambda)$ is open.

Since $f \in L^1(\mathbb{R}^+)$, it follows that $\lim_{x \to \infty} Tf(x) = 0$.

Consequently, $E(\lambda)$ is bounded.

$\Rightarrow$ $E(\lambda)$ is a countable union of disjoint open intervals in $\mathbb{R}^+$:

$$E(\lambda) = \bigoplus_{j=1}^{\infty} (a_j, b_j). \quad \text{Each } b_j \text{ is finite.}$$

By continuity of Tf, we know that:

$$Tf(a_j) = Tf(b_j) = \lambda, \quad \text{for all } j.$$

$$\Rightarrow \frac{1}{a_j} \int_0^{a_j} f(t)\,dt = \frac{1}{b_j} \int_0^{b_j} f(t)\,dt = \lambda$$

$$\Rightarrow \int_{a_j}^{b_j} f(t)\,dt = \int_0^{b_j} f(t)\,dt - \int_0^{a_j} f(t)\,dt = (b_j - a_j) \cdot \lambda$$

We can then sum over j to deduce that:

$$\int_{E(\lambda)} f(t)\,dt = \lambda \mu(E(\lambda)) \Rightarrow \mu(E(\lambda)) = \frac{1}{\lambda} \cdot \int_{E(\lambda)} f(t)\,dt.$$

In other words:

$$\mu(|Tf|>\lambda) \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^+)} \qquad (*)$$

whenever $f \geq 0$.

Given a general $f \in L^1(\mathbb{R}^+)$, we can write it as:

$f = (f_1 - f_2) + i(f_3 - f_4)$, with $f_j \geq 0$, $\|f_j\|_{L^1(\mathbb{R}^+)} \leq \|f\|_{L^1(\mathbb{R}^+)}$.

Then:

$$\mu(|Tf|>\lambda) \leq \sum_{j=1}^{4} \mu\left(|Tf_j|>\frac{\lambda}{4}\right)$$

$$\leq \sum_{j=1}^{4} \frac{4}{\lambda}\|f_j\|_{L^1(\mathbb{R}^+)} = \frac{16}{\lambda}\cdot\|f\|_{L^1(\mathbb{R}^+)}.$$

Here, we applied $(*)$ for f_j: $j=1,2,3,4$.

In particular, it follows that T is weak $(1,1)$.

The claim now follows from the Marcinkiewicz interpolation theorem. $\square$

<u>Remark:</u> T is <u>not</u> strong $(1,1)$.

Choose $f \in L^1(\mathbb{R}^+)$, $f>0$. Suppose $\int_0^M f(t)\,dt = \varepsilon > 0$ for some $M>0$.

Then, for $x \geq M$, $Tf(x) \geq \frac{\varepsilon}{x}$ so $Tf \notin L^1(\mathbb{R}^+)$. $\Rightarrow$ T is not strong $(1,1)$.