§1: The Fourier transform

1.1: The L^1 theory of the Fourier transform

· Given $n \in \mathbb{N}$, we consider $\mathbb{R}^n$ = n-dimensional Euclidean space
For $1 \leq p < +\infty$, we let $L^p = L^p(\mathbb{R}^n)$ denote the set of all functions for which $\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{\frac{1}{p}}$ is finite.
We let $L^\infty = L^\infty(\mathbb{R}^n)$ denote the set of all functions for which $\|f\|_{L^\infty} := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|$ is finite. The functions are regarded as equivalent if they agree a.e.
Here, the functions that we are considering are complex-valued and (Borel) measurable.

· Let $C_0 = C_0(\mathbb{R}^n)$ denote the space of all continuous functions vanishing at infinity, with respect to $\|\cdot\|_{L^\infty}$.

__Fact__: $\left(L^p(\mathbb{R}^n), \|\cdot\|_{L^p}\right)$ is a Banach space for all $1 \leq p \leq \infty$,
Likewise: $\left(C_0(\mathbb{R}^n), \|\cdot\|_{L^\infty}\right)$ is a Banach space.

__Definition:__ Given $f \in L^1(\mathbb{R}^n)$, we define its __Fourier transform__ $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ as:

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) \cdot e^{-2\pi i x \cdot \xi} \, dx$$

Here, $x \cdot \xi = \sum_{j=1}^{n} x_j \cdot \xi_j$.

Since $\int_{\mathbb{R}^n} |f(x)| \, dx < \infty$, this quantity is well-defined.

__Question:__ Which function space does $\widehat{f}$ belong to?

Theorem 1.1: Let $f \in L^1(\mathbb{R}^n)$. Then, the following properties hold:

i) $\widehat{f} \in L^\infty(\mathbb{R}^n)$. Moreover, $\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$.

ii) $\widehat{f}$ is uniformly continuous.

iii) $\widehat{f} \in C_0$, i.e. $\widehat{f}(x) \to 0$ as $|x| \to \infty$.

Proof:

i) Let $\xi \in \mathbb{R}^n$ be given. Then:

$$|\widehat{f}(\xi)| = \left| \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx \right|$$

By the triangle inequality, this is:

$$\leq \int_{\mathbb{R}^n} |f(x) \cdot e^{-2\pi i x \cdot \xi}| \, dx = \int_{\mathbb{R}^n} |f(x)| \, dx = \|f\|_{L^1}.$$

ii) We will use an approximation argument.

By $C_c^\infty(\mathbb{R}^n)$, we denote the set of all smooth, compactly-supported functions on $\mathbb{R}^n$.

Fact: $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.

Let $\xi, \eta \in \mathbb{R}^n$ and $\varepsilon > 0$ be given.

We can find $g \in C_c^\infty(\mathbb{R}^n)$ such that $\|f - g\|_{L^1} < \frac{\varepsilon}{4}$.

Now,

$$|\widehat{f}(\xi) - \widehat{f}(\eta)| = |(\widehat{f}(\xi) - \widehat{g}(\xi)) + (\widehat{g}(\xi) - \widehat{g}(\eta)) + (\widehat{g}(\eta) - \widehat{f}(\eta))|$$

$$\leq |\widehat{f}(\xi) - \widehat{g}(\xi)| + |\widehat{g}(\xi) - \widehat{g}(\eta)| + |\widehat{g}(\eta) - \widehat{f}(\eta)|$$

$$\leq 2\|(f-g)^\wedge\|_{L^\infty} + |\widehat{g}(\xi) - \widehat{g}(\eta)|$$

$$\leq 2\|f - g\|_{L^1} + |\widehat{g}(\xi) - \widehat{g}(\eta)|$$

$$< \frac{\varepsilon}{2} + |\widehat{g}(\xi) - \widehat{g}(\eta)|$$

Here, we used the fact that: $\hat{f} - \hat{g} = (f-g)^{\wedge} := \widehat{(f-g)}$, and part ii).

Now:

$$|\hat{g}(\xi) - \hat{g}(\eta)| = \left| \int_{\mathbb{R}^n} g(x) e^{-2\pi i x\cdot\eta} \cdot \left(e^{-2\pi i x\cdot(\xi-\eta)} - 1\right) dx \right|$$

$$\leq \int_{\mathbb{R}^n} |g(x)| \cdot |e^{-2\pi i x\cdot(\xi-\eta)} - 1| \, dx$$

$$\leq C \int_{\mathbb{R}^n} |g(x)| \cdot |x| \cdot |\xi - \eta| \, dx$$

$$= \left(C \int_{\mathbb{R}^n} |g(x)| \cdot |x| \, dx \right) \cdot |\xi - \eta| < \frac{\varepsilon}{2}$$

if $|\xi - \eta|$ is sufficiently small. (Here, we use the fact that $|x| \cdot g \in L^1$.)

$\Rightarrow |\hat{f}(\xi) - \hat{f}(\eta)| < \varepsilon$ if $|\xi - \eta|$ is sufficiently small.

iii) Let's first prove the claim for dimension $n=1$.

Let $\varepsilon > 0$ be given.

As in part ii), we find $g \in C_c^\infty(\mathbb{R})$, such that:

$\| f - g \|_{L^1} < \frac{\varepsilon}{2}$.

As in part ii), we need to find M s.t. for $|\xi| > M$, it is the case that:

$$|\hat{g}(\xi)| < \frac{\varepsilon}{2} \text{ for } |\xi| > M.$$

<u>Observe:</u> $e^{-2\pi i x \xi} = -\frac{1}{2\pi i \xi} \cdot \frac{d}{dx}\left(e^{-2\pi i x \xi}\right)$

So:

$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-2\pi i x \xi} dx = -\frac{1}{2\pi i \xi} \int g(x) \cdot \frac{d}{dx}\left(e^{-2\pi i x \xi}\right) dx$$

$$= \frac{1}{2\pi i \xi} \int g'(x) \cdot e^{-2\pi i x \xi} dx$$

Since $g \in C_c^\infty(\mathbb{R})$, we know that $g' \in L^1(\mathbb{R})$

So: $|\hat{g}(\xi)| \leq \frac{1}{2\pi|\xi|} \cdot \|g'\|_{L^1} \longrightarrow 0$ as $|\xi| \to \infty$. (Since $g' \in L^1$!)

In other words, for $|\xi|$ sufficiently large, it is the case that $|\hat{g}(\xi)| < \frac{\varepsilon}{2}$. This proves the claim when $n = 1$.

· In general, we note that:

$$e^{-2\pi i x \cdot \xi} = -\frac{1}{2\pi i \xi_j} \cdot \frac{\partial}{\partial x_j}\left(e^{-2\pi i x \cdot \xi}\right) \quad \text{for all } j = 1, 2, \ldots, n.$$

Given $\xi \in \mathbb{R}^n \setminus \{0\}$, we can find $j \in \{1, 2, \ldots, n\}$ such that $|\xi_j| \geq \frac{1}{\sqrt{n}}|\xi|$. In particular, if $g \in C_c^\infty(\mathbb{R}^n)$, it is the case that:

$$|\hat{g}(\xi)| \leq \frac{1}{2\pi|\xi_j|} \cdot \left\|\frac{\partial g}{\partial x_j}\right\|_{L^1} \leq \frac{\sqrt{n}}{2\pi|\xi|} \underbrace{\|\nabla g\|_{L^1}}_{= \left\|\left(\sum_{\ell=1}^{n} \left|\frac{\partial g}{\partial x_\ell}\right|^2\right)^{\frac{1}{2}}\right\|_{L^1}}.$$

The argument now follows as in the case $n = 1$.

$\square$

Remarks:

① Part iii) is called the Riemann–Lebesgue Lemma.

② In parts ii) and iii), it is also possible to perform the approximation argument by using (finite) linear combinations of characteristic functions of rectangles in $\mathbb{R}^n$, instead of elements of $C_c^\infty(\mathbb{R}^n)$.
In this case, one can compute the Fourier transforms directly. This approach is left as an exercise to the reader.

③ It is in general unknown how to characterize all of the functions that one obtains under the image of $L^1(\mathbb{R}^n)$ under the Fourier transform.
We will see that, in the case of $L^2(\mathbb{R}^n)$, the situation is much simpler.

- Let us note that $\mathcal{F}: L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ given by $\mathcal{F}(f) := \hat{f}$ is linear.

Namely, given $f, g \in L^1(\mathbb{R}^n)$, $\alpha, \beta \in \mathbb{C}$, it follows from the definition that

$$\mathcal{F}(\alpha f + \beta g)(\xi) = \int_{\mathbb{R}^n} (\alpha f + \beta g)(x) e^{-2\pi i x \cdot \xi} \, dx$$

$$= \alpha \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx + \beta \int_{\mathbb{R}^n} g(x) e^{-2\pi i x \cdot \xi} \, dx =$$

$$= \alpha \mathcal{F}(f)(\xi) + \beta \mathcal{F}(g)(\xi).$$

§1.2: The Schwartz class

- We saw in the proof of Theorem 1.1 that, when working with the Fourier transform, it is helpful to work with a class of functions which is closed under multiplication by polynomials and under differentiation.

It would be reasonable to expect that a class of functions on which we can give a more precise characterization of the Fourier transform (hopefully as a bijection) should satisfy these properties.

- We note that $L^1(\mathbb{R}^n)$ is not closed under multiplication by polynomials nor under differentiation.

- $C_c^\infty(\mathbb{R}^n)$ is closed under multiplication by polynomials and under differentiation. However, it is "too small".

<u>Fact</u> (which we will not prove here): If $f \in C_c^\infty(\mathbb{R}^n)$ and $\hat{f} \in C_c^\infty(\mathbb{R}^n)$, then $f \equiv 0$.

- <u>Some notation:</u>

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we let:

$x^\alpha := x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \dots \cdot x_n^{\alpha_n}$, $\partial_x^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} \dots \frac{\partial^{\alpha_n}}{\partial x_n}$

Definition: $\mathcal{S}(\mathbb{R}^n)$ is the class of all functions $f: \mathbb{R}^n \to \mathbb{C}$ such that for all $\alpha, \beta \in \mathbb{N}^n$,

$$p_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)| < \infty .$$

- Let us observe that:

$$C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$$

The fact that $\mathcal{S}(\mathbb{R}^n)$ follows from the fact that:

$$\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^{n+1} f(x)\right| < \infty$$

so $$|f(x)| \leq \frac{c}{(1+|x|)^{n+1}} \Rightarrow f \in L^1(\mathbb{R}^n).$$

The inclusion is strict:

$$e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n) \setminus C_c^\infty(\mathbb{R}^n)$$

$$\frac{1}{(1+|x|)^{n+1}} \in L^1(\mathbb{R}^n) \setminus \mathcal{S}(\mathbb{R}^n).$$

- We observe that:

$$f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow x_j f, \frac{\partial f}{\partial x_j} \in \mathcal{S}(\mathbb{R}^n)$$

- Let us prove this claim when $n=1$, for simplicity of notation. The claim for general n holds similarly.

If $f \in \mathcal{S}(\mathbb{R})$, $\alpha, \beta \in \mathbb{N}$:

$$|x^\alpha \partial_x^\beta (xf)| = |x^{\alpha+1} \partial_x^\beta f + \beta x^\alpha \partial_x^{\beta-1} f|$$
$$\leq |x^{\alpha+1}\partial_x^\beta f| + |\beta| \cdot |x^\alpha \partial_x^{\beta-1} f| \leq C_1(\alpha,\beta)$$
$$|x^\alpha \partial_x^\beta (f')| = |x^\alpha \partial_x^{\beta+1} f| \leq C_2(\alpha,\beta)$$

Remark: We say that a sequence $(\varphi_k)_k$ in $\mathcal{S}(\mathbb{R}^n)$ converges to 0 if $p_{\alpha,\beta}(\varphi_k) \to 0$ as $k \to \infty$ for all $\alpha, \beta \in \mathbb{N}^n$.

Properties of the Fourier transform:

Proposition 2.1 Let $f \in \mathcal{S}(\mathbb{R}^n)$, and let $j \in \{1, 2, \dots, n\}$. Then:

i) $$\left(\frac{\partial f}{\partial x_j}\right)^\wedge(\xi) = 2\pi i \xi_j \widehat{f}(\xi)$$

ii) $$\left(-2\pi i x_j f\right)^\wedge(\xi) = \frac{\partial \widehat{f}}{\partial \xi_j}(\xi)$$

Proof: We note that $\frac{\partial f}{\partial x_j}, x_j f \in \mathcal{S}(\mathbb{R}^n)$.

i) $$\left(\frac{\partial f}{\partial x_j}\right)^\wedge(\xi) = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) \cdot e^{-2\pi i x \cdot \xi_j}\, dx$$

by integration by parts, this equals:

$$-\int_{\mathbb{R}^n} f(x) \frac{\partial}{\partial x_j}\left(e^{-2\pi i x \cdot \xi}\right) dx$$

(the boundary terms at infinity vanish)

$$= 2\pi i \xi_j \cdot \widehat{f}(\xi)$$

ii) $$\frac{\widehat{f}(\xi + h e_j) - \widehat{f}(\xi)}{h} = \int_{\mathbb{R}^n} f(x) \cdot \frac{e^{-2\pi i x \cdot (\xi + h e_j)} - e^{-2\pi i x \cdot \xi}}{h}\, dx$$

$$\left| f(x) \cdot \frac{e^{-2\pi i x \cdot (\xi + h e_j)} - e^{-2\pi i x \cdot \xi}}{h} \right| \leq C |f(x)| \cdot |x_j| \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$$

By the Dominated Convergence Theorem, we can let $h \to 0$ and deduce that:

$$\left(-2\pi i x_j f\right)^\wedge(\xi) = \frac{\partial \widehat{f}}{\partial \xi_j}(\xi). \qquad \square$$

Let us now note some additional properties of the Fourier transform. The properties below hold for general L^1 functions:

<u>Proposition 2.2:</u> Let $f \in L^1(\mathbb{R}^n)$

i) Let $g \in L^1(\mathbb{R}^n)$. Then:

$$(f * g)^\wedge = \hat{f} \cdot \hat{g}$$

ii) Let $h \in \mathbb{R}^n$; $\tau_h f := f(\cdot - h)$

$$(\tau_h f)^\wedge(\xi) = e^{-2\pi i h \cdot \xi} \cdot \hat{f}(\xi)$$

$$(e^{2\pi i h \cdot x} f)^\wedge(\xi) = \hat{f}(\xi - h) = \tau_h \hat{f}(\xi)$$

iii) Let $\lambda > 0$, $f_\lambda(x) := \frac{1}{\lambda^n} \cdot f\left(\frac{x}{\lambda}\right)$.

Then:

$$\widehat{f_\lambda}(\xi) = \hat{f}(\lambda \xi).$$

<u>Proof:</u>

i. If $f, g \in L^1(\mathbb{R}^n)$, then:

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y) \cdot g(y)\,dy = \int_{\mathbb{R}^n} f(y) \cdot g(x-y)\,dy$$

($*$ is called the <u>convolution operator</u>)

If $f, g \in L^1(\mathbb{R}^n)$, then $f * g \in L^1(\mathbb{R}^n)$.

We see this as follows:

$$\int_{\mathbb{R}^n} |(f*g)(x)|\,dx = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y) \cdot g(y)\,dy \right| dx$$

$$\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)| \cdot |g(y)|\,dy\,dx$$

By Fubini's theorem, this equals:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)| \cdot |g(y)| \, dx \, dy = \|f\|_{L^1(\mathbb{R}^n)} \cdot \|g\|_{L^1(\mathbb{R}^n)} < \infty.$$

So, $(f*g)^\wedge$ is well-defined.

$$(f*g)^\wedge(\xi) = \int_{\mathbb{R}^n} (f*g)(x) \cdot e^{-2\pi i x\cdot\xi} dx =$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y) \cdot g(y) dy \right) \cdot e^{-2\pi i x\cdot\xi} dx =$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) \cdot g(y) \cdot e^{-2\pi i x\cdot\xi} \, dx \, dy =$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) \cdot g(y) \cdot e^{-2\pi i (x-y)\cdot\xi} \cdot e^{-2\pi i y\cdot\xi} \, dx dy$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y) e^{-2\pi i (x-y)\xi} dx \right) \cdot g(y) \cdot e^{-2\pi i y\cdot\xi} dy$$

$$= \hat{f}(\xi) \cdot \int_{\mathbb{R}^n} g(y) e^{-2\pi i y\cdot\xi} dy =$$

$$= \hat{f}(\xi) \cdot \hat{g}(\xi).$$

ii) $$(\tau_h f)^\wedge(\xi) = \int_{\mathbb{R}^n} (\tau_h f)(x) \cdot e^{-2\pi i x\cdot\xi} dx =$$

$$= \int_{\mathbb{R}^n} f(x-h) \cdot e^{-2\pi i x\cdot\xi} dx =$$

$$= e^{-2\pi i h\cdot\xi} \cdot \int_{\mathbb{R}^n} f(x-h) \cdot e^{-2\pi i (x-h)\cdot\xi} dx$$

$$= e^{-2\pi i h\cdot\xi} \cdot \hat{f}(\xi)$$

$$\left(e^{2\pi i h\cdot x} f\right)^{\wedge}(\xi) =$$

$$= \int_{\mathbb{R}^n} e^{2\pi i h\cdot x} f(x)\cdot e^{-2\pi i x\cdot\xi}\, dx =$$

$$= \int_{\mathbb{R}^n} f(x)\cdot e^{-2\pi i x\cdot(\xi - h)}\, dx =$$

$$= \widehat{f}(\xi - h).$$

iii) $$\widehat{f_\lambda}(\xi) = \int_{\mathbb{R}^n} f_\lambda(x)\cdot e^{-2\pi i x\cdot\xi}\, dx =$$

$$= \frac{1}{\lambda^n} \int_{\mathbb{R}^n} f\left(\frac{x}{\lambda}\right)\cdot e^{-2\pi i x\cdot\xi}\, dx =$$

$$= \left\{ \begin{array}{l} y = \frac{x}{\lambda} \;;\; x = \lambda\cdot y \\ dy = \frac{1}{\lambda^n} dx \end{array} \right\}$$

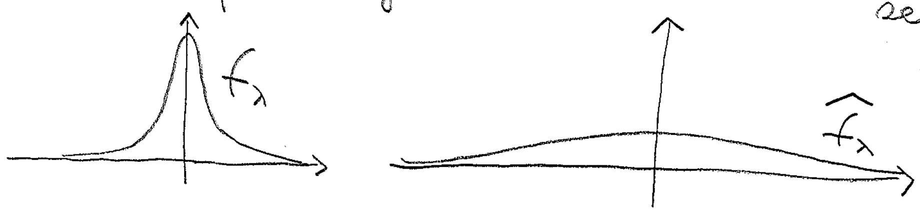
$$= \int_{\mathbb{R}^n} f(y)\cdot e^{-2\pi i (\lambda y)\cdot\xi}\, dy =$$

$$= \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i y\cdot(\lambda\xi)}\, dy =$$

$$= \widehat{f}(\lambda\xi). \quad \square$$

<u>Remark:</u> Property iii) is related to the "<u>Uncertainty Principle</u>". In other words, it is not possible for both f and $\widehat{f}$ to be localized near the origin.

For example, if λ is small: $\longrightarrow$ f_λ is concentrated on a set of diameter $\sim \lambda$



$\widehat{f_\lambda}$ is spread out over a set of size $\frac{1}{\lambda}$.

Let's return to the mapping properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$.

Proposition 2.3: Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$.

Proof: Let $\alpha, \beta \in \mathbb{N}^n$

$$\xi^\alpha \partial_\xi^\beta \widehat{f}(\xi) \sim (\partial_x^\alpha (x^\beta f))^\wedge(\xi)$$

by Proposition 2.1.

We observe that $\partial_x^\alpha (x^\beta f) \in \mathcal{S}(\mathbb{R}^n)$.

In particular $\partial_x^\alpha (x^\beta f) \in L^1(\mathbb{R}^n)$, so its Fourier transform belongs to $L^\infty(\mathbb{R}^n)$.

$\Rightarrow |\xi^\alpha \partial_\xi^\beta \widehat{f}| \leq C(\alpha, \beta)$

$\Rightarrow \widehat{f} \in \mathcal{S}(\mathbb{R}^n)$. $\square$

Proposition 2.4: Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then:

$$\int_{\mathbb{R}^n} f \widehat{g} \, dx = \int_{\mathbb{R}^n} \widehat{f} g \, dx$$

Proof: $\int_{\mathbb{R}^n} f \widehat{g} \, dx = \int_{\mathbb{R}^n} f(x) \cdot \left(\int_{\mathbb{R}^n} g(\xi) \cdot e^{-2\pi i x \cdot \xi} d\xi \right) dx =$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) \cdot g(\xi) \cdot e^{-2\pi i x \cdot \xi} d\xi dx =$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) \cdot g(\xi) \cdot e^{-2\pi i \xi \cdot x} dx d\xi =$$

$$= \int_{\mathbb{R}^n} \widehat{f} g \, dx. \quad \square$$

Remark: We can also work with $f, g \in L^1(\mathbb{R}^n)$ here.

Lemma 2.5: If $f(x) = e^{-\pi|x|^2}$, then $\widehat{f}(\xi) = e^{-\pi|\xi|^2}$.

Proof: We note that:

$$\widehat{f}(\xi) = \prod_{j=1}^{n} \left(\int_{\mathbb{R}} e^{-\pi x_j^2} \cdot e^{-2\pi i x_j \cdot \xi_j} \, dx_j \right),$$

so it suffices to prove the claim when $n=1$.

When $n=1$, we note that $f(x) = e^{-\pi x^2}$ solves the ODE:

$$\begin{cases} f' + 2\pi x f = 0 \\ f(0) = 1 \end{cases}$$

We take Fourier transforms and see that $\widehat{f}$ solves:

$$2\pi i \xi \cdot \widehat{f}(\xi) - \frac{1}{i} \frac{\partial \widehat{f}}{\partial \xi} = 0$$

i.e. $$\frac{\partial \widehat{f}}{\partial \xi} + 2\pi \xi \widehat{f} = 0$$

$$\widehat{f}(0) = \int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} e^{-\pi x^2} dx = 1$$

So:

$$\begin{cases} \widehat{f}\,' + 2\pi \xi \widehat{f} = 0 \\ \widehat{f}(0) = 1. \end{cases}$$

$\Rightarrow \widehat{f} = f.$ $\square$

Alternatively, we can show Lemma 2.5 by using a contour integral.

As before, we know that it suffices to show that the formula holds when $n=1$.

That is, given $\xi \in \mathbb{R}$, we need to compute:

$$\int_{-\infty}^{+\infty} e^{-\pi x^2} \cdot e^{-2\pi i x \xi} \, dx = \int_{-\infty}^{+\infty} e^{-\pi x^2 - 2\pi i x \xi} \, dx$$

We note that:

$$-\pi x^2 - 2\pi i x \xi = -(\sqrt{\pi} \cdot x)^2 - 2\sqrt{\pi} \cdot x \cdot (i \xi \sqrt{\pi}) - (i \xi \sqrt{\pi})^2 + (i \xi \sqrt{\pi})^2 =$$

$$= -(\sqrt{\pi} \cdot (x - i\xi))^2 - \pi \xi^2$$

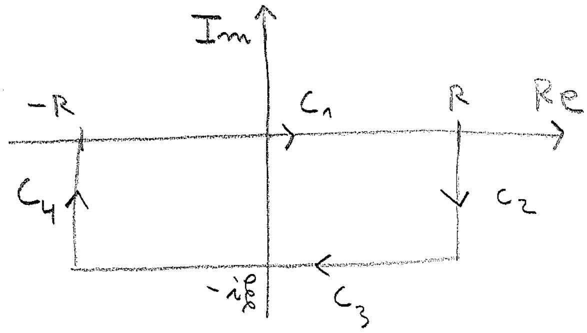
So, the above integral equals:

$$\int_{-\infty}^{+\infty} e^{-(\sqrt{\pi} \cdot (x - i\xi))^2 - \pi \xi^2} dx = \left(\int_{-\infty}^{+\infty} e^{-(\sqrt{\pi} \cdot (x - i\xi))^2} dx \right) \cdot e^{-\pi \xi^2}$$

By a contour deformation:

$$\int_{-\infty}^{+\infty} e^{-(\sqrt{\pi} \cdot (x - i\xi))^2} dx = \int_{-\infty}^{+\infty} e^{-\pi x^2} dx = 1$$

In order to justify the last step, we let $R>0$ be given, and we look at the contour C in $\mathbb{C}$ given by: (we assume WLOG $\xi \geq 0$)



By Cauchy's theorem:

$$\oint_C e^{-\pi z^2} dz = 0$$

Now: $$\oint_C e^{-\pi z^2} dz = \sum_{j=1}^{4} \int_{C_j} e^{-\pi z^2} dz$$

$$\int_{C_1} e^{-\pi z^2} dz = \int_{-R}^{R} e^{-\pi x^2} dx, \quad \int_{C_3} e^{-\pi z^2} dz = -\int_{-R}^{R} e^{-\pi (x - i\xi)^2} dx$$

So: $\int_{C_1} e^{-\pi z^2} dz \to \int_{-\infty}^{+\infty} e^{-\pi x^2} dx$

$\int_{C_3} e^{-\pi z^2} dz \to -\int_{-\infty}^{+\infty} e^{-\pi (x-i\xi)^2} dx$, as $R \to +\infty$

Thus, we need to show that:

$$\int_{C_2} e^{-\pi z^2} dz, \int_{C_4} e^{-\pi z^2} dz \to 0, \text{ as } R \to +\infty.$$

$$\left| \int_{C_2} e^{-\pi z^2} dz \right| = \left| \int_0^{\xi} e^{-(R-is)^2} \cdot (-is) ds \right| \leq \xi^2 e^{-R^2+\xi^2} \to 0 \text{ as } R \to +\infty$$

Similarly:

$$\left| \int_{C_4} e^{-\pi z^2} dz \right| \to 0, \text{ as } R \to +\infty.$$

Hence, it follows that: $\left(e^{-\pi x^2}\right)^{\wedge}(\xi) = e^{-\pi \xi^2}$.

The formula for general n follows as before. $\square$

Theorem 2.6: (Fourier inversion formula)

Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then:

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) \cdot e^{2\pi i x \cdot \xi} d\xi \text{ for all } x \in \mathbb{R}^n.$$

Proof: · Let us first observe that it suffices to prove the claim when $x=0$. Namely, if $f(0) = \int_{\mathbb{R}^n} \widehat{f}(\xi) d\xi$ were to hold, we could apply this formula with f replaced by $\tau_{-x} f = f(\cdot + x) \in \mathcal{S}(\mathbb{R}^n)$.

Then we could deduce that:

$$f(x) = (\tau_{-x} f)(0) = \int_{\mathbb{R}^n} (\tau_{-x} f)^{\wedge}(\xi) \, d\xi ,$$

which by Proposition 2.2 ii) equals:

$$\int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

Hence, we need to prove:

$$f(0) = \int_{\mathbb{R}^n} \widehat{f}(\xi) d\xi , \text{ for all } f \in \mathcal{S}(\mathbb{R}^n) \quad (*).$$

We recall from Proposition 2.4 that, for all $f, g \in \mathcal{S}(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^n} \hat{f}(\xi)g(\xi)\,d\xi$$

Let $\lambda > 0$, and we replace g by $g_\lambda = \frac{1}{\lambda^n} g(\frac{\cdot}{\lambda})$.

Then, by Proposition 2.2 iii), we know that:

$$\widehat{g_\lambda}(\xi) = \hat{g}(\lambda\xi)$$

So: $$\int_{\mathbb{R}^n} f(x)\hat{g}(\lambda x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\cdot\left(\frac{1}{\lambda^n} g\left(\frac{\xi}{\lambda}\right)\right)d\xi$$

$$\Rightarrow \quad \lambda^n \int_{\mathbb{R}^n} f(x)\hat{g}(\lambda x)\,dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\,g\left(\frac{\xi}{\lambda}\right)d\xi$$

Let $y = \lambda x$; then $dy = \lambda^n dx$.

$$\Rightarrow \quad \int_{\mathbb{R}^n} f\left(\frac{y}{\lambda}\right)\hat{g}(y)\,dy = \int_{\mathbb{R}^n} \hat{f}(\xi)\,g\left(\frac{\xi}{\lambda}\right)d\xi$$

Let $\lambda \to \infty$.

It follows that:

$$f(0)\cdot\int_{\mathbb{R}^n} \hat{g}(\xi)\,d\xi = g(0)\cdot\int_{\mathbb{R}^n} \hat{f}(\xi)\,d\xi$$

Let $g(x) := e^{-\pi|x|^2}$.

Then $g(0) = 1$,

$\hat{g}(\xi) = e^{-\pi|\xi|^2}$, so $\int_{\mathbb{R}^n} g(\xi)\,d\xi = \int_{\mathbb{R}^n} e^{-\pi|\xi|^2}d\xi = 1.$

Hence:

$$f(0) = \int_{\mathbb{R}^n} \hat{f}(\xi)\,d\xi.$$

The claim now follows. $\square$

Corollary 2.7: $\hat{\hat{\hat{f}}}(x) = f(-x)$, for all $f \in \mathcal{S}(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$.

In particular $\mathcal{F}^{(4)}(f) = f$, for all $f \in \mathcal{S}(\mathbb{R}^n)$, (i.e. the Fourier transform has period 4.)

Proof: We use the Fourier inversion formula and write:

$$\int_{\mathbb{R}^n} \hat{f}(\xi) e^{-2\pi i x \xi} d\xi = f(-x) \qquad \square$$

$$\shortparallel$$

$$\hat{\hat{f}}(x)$$

Remark: Let $f \in \mathcal{S}(\mathbb{R}^n)$. Let $\Phi := f + \hat{f} + \hat{\hat{f}} + \hat{\hat{\hat{f}}} = f + \mathcal{F}(f) + \mathcal{F}^{(2)}(f) + \mathcal{F}^{(3)}(f)$.

Then $\Phi \in \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F}(\Phi) = \mathcal{F}(f) + \mathcal{F}^{(2)}(f) + \mathcal{F}^{(3)}(f) + f = \Phi$.

Theorem 2.8: $\hat{\ } : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a bijection.

Proof: • We know that $\hat{\ }$ maps $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ by Proposition 2.3.

• *It is injective:* Let $f \in \mathcal{S}(\mathbb{R}^n)$ be such that $\hat{f} \equiv 0$.
Then $f = \mathcal{F}^{(4)}(f) = \mathcal{F}^{(3)}(\hat{f}) \equiv 0$.

• *It is surjective:* Let $f \in \mathcal{S}(\mathbb{R}^n)$ be given. Then,
$g := \mathcal{F}^{(3)}(f) \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{g} = f$. $\square$

[**Remark:** The proof of Proposition 2.3 shows that $\hat{\ } : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous with respect to the seminorms $p_{\alpha,\beta}$, which generate the topology of $\mathcal{S}(\mathbb{R}^n)$.

By linearity, it suffices to check that:

$$p_{\alpha,\beta}(f_n) \to 0 \quad \text{for all } \alpha, \beta \in \mathbb{N}_n$$

$$\Rightarrow p_{\alpha,\beta}(\hat{f}_n) \to 0 \quad \text{for all } \alpha, \beta \in \mathbb{N}_n.$$

We leave the details of this proof as an exercise.]

1.3 : The L^2 theory of the Fourier transform

The Fourier Transform satisfies the following unitarity property on L^2:

Theorem 3.1: (Plancherel's Theorem)

For all $f \in \mathcal{S}(\mathbb{R}^n)$, it is the case that:

$$\|\hat{f}\|_{L^2(\mathbb{R}^n_\xi)} = \|f\|_{L^2(\mathbb{R}^n_x)}.$$

Proof: We recall from Proposition 2.4 that:

$$\int_{\mathbb{R}^n} \hat{f} g \, dx = \int_{\mathbb{R}^n} f \hat{g} \, dx, \text{ for all } f, g \in \mathcal{S}(\mathbb{R}^n).$$

Let us take $g := \overline{\hat{f}}$. Then, the left-hand side of the above identity equals:

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi.$$

In order to prove the claim, it suffices to show that:

$$\hat{g} = \overline{f}, \text{ i.e. } \widehat{\overline{\hat{f}}} = \overline{f}.$$

Let us note that, for $h \in \mathcal{S}(\mathbb{R}^n)$, it is the case that:

$$\widehat{\overline{h}}(\xi) = \int_{\mathbb{R}^n} \overline{h}(x) e^{-2\pi i x \cdot \xi} dx = \overline{\int_{\mathbb{R}^n} h(x) e^{2\pi i x \cdot \xi} dx} = \overline{\hat{h}(-\xi)}$$

So, if we let $h := \hat{f}$, then:

$$\widehat{\overline{\hat{f}}}(\xi) = \overline{\hat{\hat{f}}(-\xi)}$$

By Corollary 2.7, this equals:

$\overline{f(\xi)}$. Hence $\widehat{\overline{\hat{f}}} = \overline{f}$ indeed and the claim follows. $\square$

- Theorem 3.1 allows us to extend the Fourier transform to all of $L^2(\mathbb{R}^n)$.

Namely, we know that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ with respect to $\|\cdot\|_{L^2}$ and $C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$.
Hence, $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

- Given $f \in L^2(\mathbb{R}^n)$, we can find a sequence (φ_k) in $\mathcal{S}(\mathbb{R}^n)$ s.t. $\varphi_k \to f$ in L^2 as $k \to \infty$.

By Theorem 3.1, we know that $(\widehat{\varphi_k})$ is Cauchy in L^2, hence it converges to a limit in $L^2(\mathbb{R}^n)$, which we define to be $\mathcal{F}_{L^2}(f)$. ("The L^2-Fourier transform of f").

- <u>$\mathcal{F}_{L^2}(f)$ is well-defined.</u>

If (ψ_k) is a sequence in $\mathcal{S}(\mathbb{R}^n)$ s.t. $\psi_k \to f$ in L^2, then, by Theorem 3.1 $\widehat{\psi_k} \xrightarrow[L^2]{} \mathcal{F}_{L^2}(f)$ since $\|\widehat{\psi_k} - \widehat{\varphi_k}\|_{L^2} = \|\psi_k - \varphi_k\|_{L^2} \to 0$ as $k \to \infty$.

- <u>$\mathcal{F}_{L^2}(f) = \widehat{f}$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.</u>

Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$ and $\mathcal{F}_{L^2}(f)$ are well-defined.
We want to show that they are equal (almost everywhere).

- Let us first find (φ_k), a sequence in $\mathcal{S}(\mathbb{R}^n)$ such that $\varphi_k \to f$ in L^2.

Let us then take $\chi \in C_c^\infty(\mathbb{R}^n)$ s.t. $0 \leq \chi \leq 1$, $\chi \equiv 1$ for $|x| \leq 1$, and $\chi \equiv 0$ for $|x| > 2$.
Given $M \in \mathbb{N}$, we let:

$$f_M(x) := \chi\left(\frac{x}{M}\right) \cdot f(x)$$
$$\varphi_{k,M}(x) := \chi\left(\frac{x}{M}\right) \cdot \varphi_k(x)$$

Then, we know that $\varphi_{k,M} \in C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$

Claim: There exists a sequence (k_j, M_j) s.t.

$$\varphi_{k_j, M_j} \to f \text{ in } L^1 \quad \text{as } j \to \infty$$
$$\varphi_{k_j, M_j} \to f \text{ in } L^2, \quad \text{as } j \to \infty$$

Proof: We observe that:

$$\| \varphi_{k,M} - f \|_{L^1(\mathbb{R}^n)} \leq \| \varphi_{k,M} - f_M \|_{L^1(\mathbb{R}^n)} + \| f_M - f \|_{L^1(\mathbb{R}^n)}$$

Now, $\| \varphi_{k,M} - f_M \|_{L^1(\mathbb{R}^n)} = \| \varphi_k \cdot \chi(\frac{\cdot}{M}) - f \cdot \chi(\frac{\cdot}{M}) \|_{L^1(\mathbb{R}^n)}$,

which by the Cauchy-Schwarz inequality is:

$$\leq \| \varphi_k - f \|_{L^2(\mathbb{R}^n)} \cdot \| \chi(\tfrac{\cdot}{M}) \|_{L^2(\mathbb{R}^n)} = C M^{\frac{n}{2}} \cdot \| \varphi_k - f \|_{L^2(\mathbb{R}^n)}$$

So:

$$\| \varphi_{k,M} - f \|_{L^1(\mathbb{R}^n)} \leq C M^{\frac{n}{2}} \cdot \| \varphi_k - f \|_{L^2(\mathbb{R}^n)} + \| f_M - f \|_{L^1(\mathbb{R}^n)} \quad (1)$$

Also:

$$\| \varphi_{k,M} - f \|_{L^2(\mathbb{R}^n)} \leq \| \overbrace{\varphi_{k,M} - f_M}^{= (\varphi_k - f) \cdot \chi(\frac{\cdot}{M})} \|_{L^2(\mathbb{R}^n)} + \| f_M - f \|_{L^2(\mathbb{R}^n)}$$

$$\leq \| \varphi_k - f \|_{L^2(\mathbb{R}^n)} + \| f_M - f \|_{L^2(\mathbb{R}^n)} \quad (2)$$

Here, we used the fact that: $|\varphi_{k,M} - f_M| \leq |\varphi_k - f|$.

We know that $f_M \to f$ in L^1 and $f_M \to f$ in L^2 as $M \to \infty$.

Consequently, given $\varepsilon_j = \frac{1}{j}$ we can find $M_j \in \mathbb{N}$ s.t.

$$\| f_{M_j} - f \|_{L^1(\mathbb{R}^n)}, \| f_{M_j} - f \|_{L^2(\mathbb{R}^n)} < \frac{\varepsilon_j}{2}.$$

Moreover, we can find $k_j \in \mathbb{N}$ s.t.

$$C M_j^{\frac{n}{2}} \cdot \| \varphi_{k_j} - f \|_{L^2(\mathbb{R}^n)}, \ \| \varphi_{k_j} - f \|_{L^1(\mathbb{R}^n)} < \frac{\varepsilon}{2}.$$

This is the wanted sequence (k_j, M_j).

(1),(2) then imply that:

$\varphi_{K_j, M_j} \to f$ in L^1 and in L^2 as $j \to \infty$.

Now $\psi_j := \varphi_{K_j, M_j} \in \mathcal{S}(\mathbb{R}^n)$.

In particular $\mathcal{F}_{L^2}(\psi_j) = \widehat{\psi_j}$.

- By Theorem 1.1, we know that:

$$\widehat{\psi_j} \longrightarrow \widehat{f} \text{ in } L^\infty$$

By Theorem 3.1, we know that:

$$\mathcal{F}_{L^2}(\psi_j) = \widehat{\psi_j} \longrightarrow \mathcal{F}_{L^2}(f) \text{ in } L^2.$$

Hence, it follows that $\mathcal{F}_{L^2}(f) = \widehat{f}$, almost everywhere.

We can describe the image of $L^2(\mathbb{R}^n)$ under the Fourier transform $\mathcal{F}$ (we henceforth omit the subscript L^2).

Theorem 3.2: The mapping $\mathcal{F}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is surjective.

Proof 1: By construction, $\mathcal{F}$ is an isometry defined on the Hilbert space $L^2(\mathbb{R}^n)$. Hence, its image $\mathcal{F}(L^2(\mathbb{R}^n)) \subseteq L^2(\mathbb{R}^n)$ is closed.

- If it were the case that $\mathcal{F}(L^2(\mathbb{R}^n)) \subsetneq L^2(\mathbb{R}^n)$, then we could find $g \neq 0$ in $L^2(\mathbb{R}^n)$, which is orthogonal to $\mathcal{F}(L^2(\mathbb{R}^n))$ with respect to $(\cdot,\cdot)_{L^2(\mathbb{R}^n)}$. In other words:

$$\int_{\mathbb{R}^n} \widehat{f} \cdot \overline{g}\, dx = 0 \text{ for all } f \in L^2(\mathbb{R}^n). \qquad (\Delta)$$

Let us find (φ_k), (ψ_k) sequences in $\mathcal{S}(\mathbb{R}^n)$ such that

$\varphi_k \to f$, $\psi_k \to \bar{g}$ in L^2.

Then, by Proposition 2.4, we know that, for all k:

$$\int_{\mathbb{R}^n} \widehat{\varphi_k} \cdot \psi_k \, dx = \int_{\mathbb{R}^n} \varphi_k \cdot \widehat{\psi_k} \, dx$$

Moreover:

$$\int_{\mathbb{R}^n} \widehat{\varphi_k} \cdot \psi_k \, dx \longrightarrow \int_{\mathbb{R}^n} \widehat{f} \cdot \bar{g} \, dx \ , \text{ as } k \to \infty \quad (\mathrm{I})$$

$$\int_{\mathbb{R}^n} \varphi_k \cdot \widehat{\psi_k} \, dx \longrightarrow \int_{\mathbb{R}^n} f \cdot \widehat{\bar{g}} \, dx \ , \text{ as } k \to \infty \quad (\mathrm{II})$$

Namely:

$$\left| \int_{\mathbb{R}^n} \widehat{\varphi_k} \cdot \psi_k \, dx - \int_{\mathbb{R}^n} \widehat{f} \cdot \bar{g} \, dx \right| = \left| \int_{\mathbb{R}^n} \left[\widehat{\varphi_k} \cdot (\psi_k - \bar{g}) + (\widehat{\varphi_k} - \widehat{f}) \cdot \bar{g} \right] dx \right|$$

$$\leq \int_{\mathbb{R}^n} |\widehat{\varphi_k}| \cdot |\psi_k - \bar{g}| \, dx + \int_{\mathbb{R}^n} |\widehat{\varphi_k} - \widehat{f}| \cdot |\bar{g}| \, dx$$

which by the Cauchy-Schwarz inequality and Theorem 3.1 is:

$$\leq \|\varphi_k\|_{L^2} \cdot \|\psi_k - \bar{g}\|_{L^2} + \|\varphi_k - f\|_{L^2} \cdot \|\bar{g}\|_{L^2} \to 0 \text{ as } k \to \infty.$$

This proves (I).

(II) is proved analogously.

In particular, it follows from (Δ) that:

$$\int_{\mathbb{R}^n} f \cdot \widehat{\bar{g}} \, dx = 0 \text{ for all } f \in L^2(\mathbb{R}^n)$$

$\Rightarrow \widehat{\overline{g}} = 0$

We will obtain a contradiction, and hence finish the proof if we can deduce that $\overline{g} = 0$ and hence $g = 0$.

We need to use:

<u>Lemma 3.3:</u> Let $F \in L^2(\mathbb{R}^n)$ be such that $\widehat{F} = 0$. Then $F = 0$.

<u>Proof 1:</u> We know by construction that

$$\|\widehat{F}\|_{L^2(\mathbb{R}^n)} = \|F\|_{L^2(\mathbb{R}^n)} = 0$$

so $F = 0$, as was claimed.

<u>Proof 2:</u> We find a sequence (φ_k) in $\mathcal{S}(\mathbb{R}^n)$ such that $\varphi_k \to F$ in L^2. Then $\widehat{\varphi_k} \to \widehat{F} = 0$ in L^2. In particular $\|\widehat{\varphi_k}\|_{L^2} \to 0$ as $k \to \infty$, so $\|\varphi_k\|_{L^2} \to 0$ as $k \to \infty$.

$\Rightarrow \varphi_k \to 0$ in L^2 as $k \to \infty \Rightarrow F = 0$. $\square$

The proof of Theorem 3.2 now follows. $\square$

<u>Proof 2 of Theorem 3.2:</u>

Let $f \in L^2(\mathbb{R}^n)$ be given. We can find a sequence (ψ_k) in $\mathcal{S}(\mathbb{R}^n)$ s.t. $\psi_k \to f$ in L^2.

By Theorem 2.8, we can find for each k $\varphi_k \in \mathcal{S}(\mathbb{R}^n)$ s.t. $\mathcal{F}(\varphi_k) = \psi_k$.

By Theorem 3.1, (φ_k) is Cauchy in L^2, so there exists $g \in L^2(\mathbb{R}^n)$ s.t. $\varphi_k \to g$ in L^2.

By continuity of $\mathcal{F}$ on L^2, it follows that $\mathcal{F}(g) = f$. $\square$

Remark: It is possible to take Fourier transforms of more general objects (which contain L^2 functions).

$$\mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$$

$\mathcal{S}'(\mathbb{R}^n)$ is the class of all Schwartz distributions. (or tempered distributions)
This is the space of all linear functionals on $\mathcal{S}(\mathbb{R}^n)$, which are continuous with respect to the topology on $\mathcal{S}(\mathbb{R}^n)$ defined earlier.

If u belongs to some L^p space,

$$u(\varphi) := \int_{\mathbb{R}} u \cdot \varphi \, dx \quad , \text{ for } \varphi \in \mathcal{S}(\mathbb{R}^n)$$

Given $u \in \mathcal{S}'(\mathbb{R}^n)$, we define $\hat{u}$ to be the element of $\mathcal{S}'(\mathbb{R}^n)$ given by:

$$\hat{u}(\varphi) := u(\hat{\varphi}) \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^n)$$

If u belongs to L^2, this reads:

$$\int_{\mathbb{R}^n} \hat{u} \cdot \varphi \, dx = \int_{\mathbb{R}^n} u \cdot \hat{\varphi} \, dx \,, \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^n)$$

which is consistent with Proposition 2.4 (applied to L^2 functions).

Fact: The Dirac delta function $\delta_0 \in \mathcal{S}'(\mathbb{R}^n)$

What is $\widehat{\delta_0}$?

For all $\varphi \in \mathcal{S}(\mathbb{R}^n)$:

$$\widehat{\delta_0}(\varphi) = \delta_0(\hat{\varphi}) = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x)\, dx = \int_{\mathbb{R}^n} 1 \cdot \varphi(x) dx = 1(\varphi)$$

$$\Rightarrow \widehat{\delta_0} = 1.$$

- The theory of the space $\mathcal{S}'(\mathbb{R}^n)$ and of the Fourier transform is very rich and has a lot of applications.

We will not study this in further detail in our course.

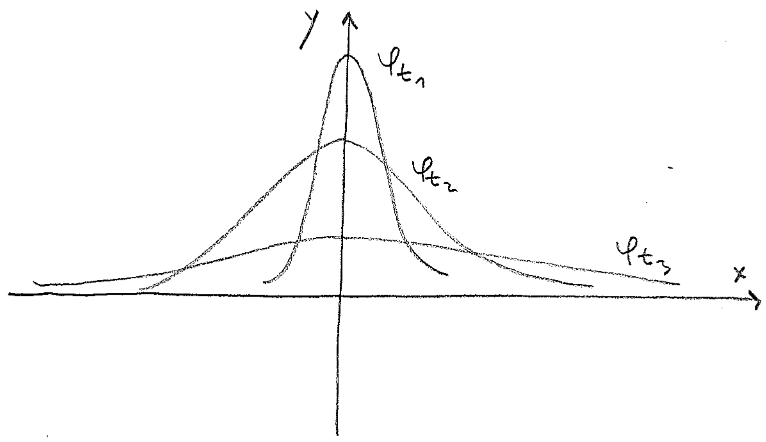
For a precise treatment of this topic, a good reference is E. Stein - G. Weiss: "Introduction to Fourier analysis on Euclidean spaces", Chapter 1, Section 3.

1.4: A brief review of approximations of the identity

- Let $\varphi \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$ be given. For $t>0$, we define:

$$\varphi_t(x) := \frac{1}{t^n}\varphi\left(\frac{x}{t}\right).$$

We call the set $\{\varphi_t : t>0\}$ an <u>approximation of the identity</u>.



$0<t_1<t_2<t_3$

We note that, for all $t>0$:

$$\int_{\mathbb{R}^n} \varphi_t(x)\,dx = \int_{\mathbb{R}^n} \frac{1}{t^n}\varphi\left(\frac{x}{t}\right)dx = \left\{\begin{matrix} y=\frac{x}{t} \\ dx = t^n dy\end{matrix}\right\} = \int_{\mathbb{R}^n}\varphi(y)\,dy = 1.$$

The following result then holds:

<u>Theorem 4.1:</u> Let $\{\varphi_t : t>0\}$ be an approximation of the identity. Then:

$$\lim_{t\to 0}\|\varphi_t * f - f\|_{L^p} = 0 \text{ if } f\in L^p(\mathbb{R}^n), \text{ for } 1\le p<\infty.$$

Furthermore, if $f\in C_0(\mathbb{R}^n)$, then $\lim_{t\to 0}\|\varphi_t * f - f\|_{L^\infty} = 0$.

<u>Proof:</u> • Let us first consider the case when $f\in L^p$ for $1\le p<\infty$.

Then:

$$(\varphi_t * f)(x) - f(x) = \int_{\mathbb{R}^n}\varphi_t(y)\cdot f(x-y)\,dy - f(x)\cdot \underbrace{\int_{\mathbb{R}^n}\varphi_t(y)\,dy}_{=1}$$

$$= \int_{\mathbb{R}^n} \varphi_t(y) \cdot (f(x-y) - f(x))\, dy =$$

$$= \int_{\mathbb{R}^n} \frac{1}{t^n} \varphi\left(\frac{y}{t}\right) \cdot (f(x-y) - f(x))\, dy =$$

$$= \int_{\mathbb{R}^n} \varphi(y) \cdot (f(x-ty) - f(x))\, dy$$

Hence, by Minkowski's inequality:

$$\| \varphi_t * f - f \|_{L^p_x} \leq \int_{\mathbb{R}^n} |\varphi(y)| \cdot \| f(\cdot - ty) - f \|_{L^p_x}\, dy$$

$$= \int_{|y| \leq \frac{\delta}{t}} |\varphi(y)| \cdot \| f(\cdot - ty) - f \|_{L^p_x}\, dy$$

$$+ \int_{|y| > \frac{\delta}{t}} |\varphi(y)| \cdot \| f(\cdot - ty) - f \|_{L^p_x}\, dy$$

We know that:

- $\lim_{h \to 0} \| f(\cdot - h) - f \|_{L^p_x} = 0$
- $\| f(\cdot - ty) - f \|_{L^p_x} \leq 2 \| f \|_{L^p_x}$

Hence:

$$\| \varphi_t * f - f \|_{L^p_x} \leq \| \varphi \|_{L^1_x} \cdot \sup_{|h| \leq \delta} \| f(\cdot - h) - f \|_{L^p_x}$$
$$+ 2 \| f \|_{L^p_x} \cdot \int_{|y| > \frac{\delta}{t}} |\varphi(y)|\, dy$$

- Given $\varepsilon > 0$, we first choose $\delta > 0$ such that the first term is $\leq \frac{\varepsilon}{2}$.

· Then, we choose $t>0$ small enough such that the second term is $\leq \frac{\varepsilon}{2}$. This is possible since $\lim_{R\to\infty} \int_{|y|>R} |\varphi(y)|\,dy = 0$.

· The same proof works when $p=\infty$ provided that
$\lim_{h\to 0} \|f(\cdot - h) - f\|_{L^\infty_x} = 0$, which is the case when $f \in C_0(\mathbb{R}^n)$. $\square$

· In the above proof, we implicitly used the fact that, for $1 \leq p \leq \infty$, $f \in L^p$ and for $\varphi \in L^1$, it is the case that $\varphi * f \in L^p$.
This is called <u>Young's inequality</u> and it is proved as follows: · for $1 \leq p < \infty$:

$$\|\varphi * f\|_{L^p_x} = \left(\int \left| \int \varphi(x-y) \cdot f(y)\, dy \right|^p dx \right)^{\frac{1}{p}}$$

which, by <u>Minkowski's Inequality in L^p</u> is

$$\leq \int \left(\int |\varphi(x-y) \cdot f(y)|^p dx \right)^{\frac{1}{p}} dy =$$

$$= \int \left(\int |\varphi(x-y)|^p dx \right)^{\frac{1}{p}} \cdot |f(y)|\, dy =$$

$$= \|\varphi\|_{L^p} \cdot \|f\|_{L^1}.$$

· for $p=\infty$

$$\|\varphi * f\|_{L^\infty} = \operatorname*{ess\,sup}_x \left| \int \varphi(x-y) \cdot f(y)\, dy \right|$$

$$\leq \operatorname*{ess\,sup}_x \int |\varphi(x-y)| \cdot |f(y)|\, dy$$

$$\leq \left(\operatorname*{ess\,sup}_x \int |\varphi(x-y)|\, dy \right) \cdot \|f\|_{L^\infty}$$

$$= \|\varphi\|_{L^1} \cdot \|f\|_{L^\infty}.$$

Conclusion: (Young's Inequality)

$$\|\varphi * f\|_{L^p} \leq \|\varphi\|_{L^1} \cdot \|f\|_{L^p}, \text{ for all } 1 \leq p \leq \infty.$$

Remark: The form of Minkowski's inequality that we are using is:

$$\left\| \int f \right\|_X \leq \int \|f\|_X$$

for X = a Banach space of functions.