Functional-Type A Posteriori Error Estimates for Mixed Finite Element Methods

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This paper is dedicated to the jubilee of Prof. Yu. A. Kuznetsov

Abstract

The work concerns the a posteriori error estimation for the primal and the dual mixed finite element methods applied to the diffusion problem. The problem is considered in a general setting, with inhomogeneous mixed Dirichlet/Neumann boundary conditions. The new, functional-type a posteriori error estimators are proposed that exhibit the ability both to indicate the local error distribution and to provide guaranteed upper bounds for the discretization errors in the primal and the dual (flux) variables. The latter property is a direct consequence of the absence in the estimators of any mesh-dependent constants; the only constants present in the estimates stem from the Friedrichs and the trace inequalities and, thus, are global and dependent solely on the domain geometry and the bounds of the diffusion matrix. The estimators are computationally cheap and require only the projections of piecewise constant functions onto the spaces of the lowest-order Raviart-Thomas or of the continuous piecewise linear elements. It is shown how these projections can be easily realized with a simple local averaging.

1 Introduction

Mixed finite element methods are frequently used for solving the problems of computational mechanics, especially in the cases when one needs accurate and conserving approximations for the fluxes and stresses. These variables

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are of primary importance in many problems of heat conduction, mass diffusion, electrostatics, elasticity and flow in porous media, which causes the permanently growing interest in the development and application of the mixed methods. Recently, the tight link between the mixed and the finite volume methods have been emphasized in [1], [21], [23]; for distorted polygonal and polyhedral meshes a new variant of mixed finite element approximation has been presented in [13].

The fundamental analysis of the mixed formulations and the *a priori* estimates for the mixed finite element methods can be found in the books [6], [20] and [12]. However, for the practical applications, the *a posteriori* error estimates seem to be even more important, as they should provide both the guidelines for an adaptive improvement of the approximation and the stopping criterion for the computational process. Although this ultimate goal of the a posteriori error estimation was not always achieved, many different estimators were presented for the mixed finite elements. The residual-based estimates are developed in [2], [4], [7], [1], [11] for the diffusion-type equation and extended in [9] and [14] to the equations of linear elasticity. The superconvergence-based (averaging-type) error estimators are proposed in [5] and [8] to control the L_2-error of the flux variable. Further, the estimators based on the solution of local problems are presented in [2], [11] and [14], and the hierarchical estimator can be found in [22]. Finally, a comparison of these four types of error estimators for mixed finite element discretizations by Raviart-Thomas elements is presented in [22].

In this paper, we apply a posteriori error estimator of another type, the so-called *functional-type estimator* (see [16], [17], [18], [19]), to the approximations obtained by the primal and dual mixed finite element methods. While the proposed estimator can be easily used for the indication of local error distribution, its main distinctive feature and the strength is the absence of any discretization-dependent constants; the only constants present in the estimator stem from the Friedrichs and the trace inequalities, they depend solely on the domain geometry and the known bounds of the diffusion matrix and can be readily evaluated just once before the beginning of the computation. That is why the estimator provides a *guaranteed upper bound* for the error and can be utilized as a stopping criterion in the adaptive approximation.

It should be noted that in this work we restrict ourselves to the consideration of a classical diffusion problem. However, the suggested method can be applied to other boundary-value problems where the form of functional-type error estimates is known. The rest of the paper is organized as follows. In Section 2, we introduce the notation and discuss the primal and dual mixed formulations for the diffusion problem with inhomogeneous mixed Dirichlet/Neumann boundary conditions. Although both formulations have been addressed in many different works, a thorough derivation for the case of non-homogeneous mixed boundary conditions does not seem to be a commonplace and is presented here for the sake of completeness. In Section 3, the a posteriori error estimator for the primal mixed method is derived, whilst Section 4 is devoted to the a posteriori estimation for the dual mixed finite element method based on the Raviart-Thomas space.

2 Notation and basic relations

We consider the diffusion problem

$$\operatorname{div} \mathbf{A}\nabla u + f = 0 \quad \text{in } \Omega\,, \tag{1}$$

$$u = u_0 \quad \text{on } \partial_1\Omega\,, \tag{2}$$

$$\mathbf{A}\nabla u \cdot \mathbf{n} = F \quad \text{on } \partial_2\Omega\,, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz continuous boundary $\partial\Omega$ that consists of two non-intersecting parts $\partial_1\Omega$ and $\partial_2\Omega$ and $\mathbf{n}$ is the outward normal to $\partial\Omega$. We assume that $meas\,(\partial_1\Omega) \neq 0$.

Further, by letters u, v, w we denote the functions associated with the solution to the problem (the primal variable), by letters $\mathbf{p}, \mathbf{q}, \mathbf{y}, \boldsymbol{\eta}$ the vector-valued functions associated with the solution flux (the dual variable) $\mathbf{A}\nabla u$ (this is, in fact, the anti-flux because of the absence of the minus-sign, but we will still call it "flux" for the sake of brevity). Also, we assume that $\mathbf{A}$ is a symmetric matrix with the components from $L_\infty(\Omega)$ that satisfies the following condition with some positive constants c_1 and c_2:

$$c_1|\xi|^2 \leq \mathbf{A}(x)\xi \cdot \xi \leq c_2|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d\,, \text{ for a.e. } x \in \Omega\,, \tag{4}$$

where $|\xi| = (\xi \cdot \xi)^{1/2}$ and the dot denotes the scalar product of vectors in $\mathbb{R}^d$.

Concerning the problem data we suppose that

$$u_0 \in H^1(\Omega)\,,\ f \in L_2(\Omega)\,,\ F \in L_2(\partial_2\Omega)\,.$$

The following functional spaces will be frequently used throughout the paper:

$$V := H^1(\Omega)\,,\ V_0 := \{v \in V \mid v = 0 \text{ on } \partial_1\Omega\}\,,\ Q := L_2(\Omega; \mathbb{R}^d)\,,$$
$$\widehat{V} := L_2(\Omega)\,,\ \widehat{Q} := H(\Omega; \operatorname{div})\,,\ \widehat{Q}^+ := \{\mathbf{y} \in \widehat{Q} \mid \mathbf{y} \cdot \mathbf{n}\big|_{\partial_2\Omega} \in L_2(\partial_2\Omega)\}\,.$$

Here the letters "V" correspond to the spaces for the primal variable u, "Q" to the spaces for the dual variable $\mathbf{p}$, the spaces without "hats" will be used for the primal mixed method, while the spaces with "hats" for the dual mixed one.

Further, by $\|\cdot\|$ and $\|\cdot\|_{\partial_2\Omega}$ we denote the L_2-norm on Ω and on $\partial_2\Omega$ respectively, by $\|\cdot\|_1$ the H^1-norm on Ω and by $\|\cdot\|_{\mathrm{div}}$ the norm in $H(\Omega; \mathrm{div}\,)$ ($\|\cdot\|_{\mathrm{div}} := (\|\cdot\|^2 + \|\mathrm{div}\,\cdot\|^2)^{1/2}$); we also define $|\!|\!| \,\mathbf{q}\, |\!|\!| := \left(\int_\Omega \mathbf{A}\mathbf{q}\cdot\mathbf{q}\,dx\right)^{1/2}$ and $|\!|\!| \,\mathbf{q}\, |\!|\!|_* := \left(\int_\Omega \mathbf{A}^{-1}\mathbf{q}\cdot\mathbf{q}\,dx\right)^{1/2}$ for all $\mathbf{q}\in Q$.

In the a posteriori estimates we will also use the functional

$$D\left(\nabla v, \mathbf{y}\right) := \int_\Omega \left(\frac{1}{2}\mathbf{A}\nabla v\cdot\nabla v + \frac{1}{2}\mathbf{A}^{-1}\mathbf{y}\cdot\mathbf{y} - \nabla v\cdot\mathbf{y}\right) dx = \frac{1}{2}\, |\!|\!| \,\mathbf{A}\nabla v - \mathbf{y}\, |\!|\!|_*^2 \tag{5}$$

that is defined for all $v\in V$ and $\mathbf{y}\in Q$.

It is well known (see, e.g., [10], [6]) that the generalized solution of problem (1)–(3) can be viewed as a saddle point of the Lagrangian

$$L(v,\mathbf{q}) := \int_\Omega \left(\nabla v\cdot\mathbf{q} - \frac{1}{2}\mathbf{A}^{-1}\mathbf{q}\cdot\mathbf{q}\right) dx - l(v)\,,$$

where

$$l(v) = \int_\Omega f v\,dx + \int_{\partial_2\Omega} F v\,ds\,.$$

The saddle point $(u,\mathbf{p})$ is sought on the pair of sets $(V_0+u_0)\times Q$ and satisfies the conditions

$$\int_\Omega \left(\mathbf{A}^{-1}\mathbf{p} - \nabla u\right)\cdot\mathbf{q}\,dx = 0 \quad \forall \mathbf{q}\in Q\,, \tag{6}$$

$$\int_\Omega \mathbf{p}\cdot\nabla w\,dx - l(w) = 0 \quad \forall w\in V_0\,. \tag{7}$$

In this formulation, the condition $\mathbf{p} = \mathbf{A}\nabla u$ is satisfied in the L_2-sense and (7) shows that the relations $\mathrm{div}\,\mathbf{p} + f = 0$ in Ω and $\mathbf{p}\cdot\mathbf{n} = F$ on $\partial_2\Omega$ are satisfied in a weak sense.

The Lagrangian L generates two functionals

$$J(v) := \sup_{\mathbf{q}\in Q} L(v,\mathbf{q}) = \frac{1}{2}\, |\!|\!| \,\nabla v\, |\!|\!|^2 - l(v)$$

and

$$I^*(\mathbf{q}) := -\frac{1}{2}\, |\!|\!| \,\mathbf{q}\, |\!|\!|_*^2 - l(u_0) + \int_\Omega \nabla u_0\cdot\mathbf{q}\,dx\,.$$

It is well known (see [10]) that

$$\inf_{v\in V_0+u_0} J(v) =: \inf \mathcal{P} = \sup \mathcal{P}^* := \sup_{\mathbf{q}\in Q_l} I^*(\mathbf{q})\,. \tag{8}$$

Here

$$Q_l := \{\mathbf{q}\in Q \;\mid\; \int_\Omega \mathbf{q}\cdot\nabla w\,dx = l(w)\;\;\forall w\in V_0\}\,.$$

By standard arguments, one can prove that the unique saddle point $(u,\mathbf{p})$ exists and satisfies the relation

$$L(u,\mathbf{p}) = \inf \mathcal{P} = \sup \mathcal{P}^*\,. \tag{9}$$

If $Q_h \subset Q$ and $V_{0h}\subset V_0$ are certain subspaces constructed by finite element approximation, then a discrete analog of (6)–(7) is the *primal mixed finite element method* (see [20]): Find $(u_h,\mathbf{p}_h)\in (V_{0h}+u_0)\times Q_h$ such that

$$\int_\Omega \left(\mathbf{A}^{-1}\mathbf{p}_h - \nabla u_h\right)\cdot\mathbf{q}_h\,dx = 0 \quad \forall \mathbf{q}_h\in Q_h\,, \tag{10}$$

$$\int_\Omega \mathbf{p}_h\cdot\nabla w_h\,dx - l(w_h) = 0 \quad \forall w_h\in V_{0h}\,. \tag{11}$$

Another mixed formulation arises if we rewrite the Lagrangian L in a somewhat different form. First, we introduce the functional $g\,:\,(V_0+u_0)\times\widehat{Q}\to\mathbb{R}$ by the relation

$$g(v,\mathbf{q}) := \int_\Omega (\nabla v\cdot\mathbf{q} + v(\operatorname{div}\mathbf{q}))\,dx\,. \tag{12}$$

We have

$$L(v,\mathbf{q}) = -\frac{1}{2}\,|\!|\!|\,\mathbf{q}\,|\!|\!|_*^2 - \int_\Omega v(\operatorname{div}\mathbf{q})\,dx - l(v) + g(v,\mathbf{q})\,.$$

Introduce the set

$$\widehat{Q}_F := \{\mathbf{q}\in\widehat{Q}\;\mid\; g(w,\mathbf{q}) = \int_{\partial_2\Omega} Fw\,ds\;\;\forall w\in V_0\}\,.$$

Note that for $\mathbf{q}\in\widehat{Q}_F$ we have

$$g(v,\mathbf{q}) = g(w+u_0,\mathbf{q}) = g(w,\mathbf{q}) + g(u_0,\mathbf{q}) = \int_{\partial_2\Omega} Fw\,ds + g(u_0,\mathbf{q})\;\forall w\in V_0\,.$$

Therefore, if the variable $\mathbf{q}$ is taken not from Q but from the narrower set $\widehat{Q}_F$, then the Lagrangian can be written as

$$\widehat{L}(v,\mathbf{q}) := -\frac{1}{2}\,|\!|\!|\,\mathbf{q}\,|\!|\!|_*^2 - \int_\Omega v(\operatorname{div}\mathbf{q})\,dx - \int_\Omega fv\,dx - \int_{\partial_2\Omega} Fu_0\,ds + g(u_0,\mathbf{q})\,.$$

We observe that the Lagrangian $\widehat{L}$ is defined on a wider set of primal functions: here $v \in \widehat{V}$.

The problem of finding $(\widehat{u}, \widehat{\mathbf{p}}) \in \widehat{V} \times \widehat{Q}_F$ such that

$$\widehat{L}(\widehat{u}, \widehat{\mathbf{q}}) \leq \widehat{L}(\widehat{u}, \widehat{\mathbf{p}}) \leq \widehat{L}(\widehat{v}, \widehat{\mathbf{p}}) \quad \forall \widehat{\mathbf{q}} \in \widehat{Q}_F\,, \ \forall \widehat{v} \in \widehat{V} \tag{13}$$

is known as the dual mixed formulation of problem (1)–(3) (see, e.g., [6]).

From (13) the following necessary conditions can be derived:

$$\int_\Omega \left(\mathbf{A}^{-1}\widehat{\mathbf{p}} \cdot \widehat{\mathbf{q}} + (\operatorname{div} \widehat{\mathbf{q}})\widehat{u}\right) dx = g(u_0, \widehat{\mathbf{q}}) \quad \forall \widehat{\mathbf{q}} \in \widehat{Q}_0\,, \tag{14}$$

$$\int_\Omega (\operatorname{div} \widehat{\mathbf{p}} + f)\widehat{v}\, dx = 0 \quad \forall \widehat{v} \in \widehat{V}\,. \tag{15}$$

Note that the condition $\operatorname{div} \widehat{\mathbf{p}} + f = 0$ is satisfied in a "strong" (i.e. in L_2) sense, the Neumann type boundary condition (3) is viewed as the essential boundary condition, and the relation $\widehat{\mathbf{p}} = \mathbf{A}\nabla\widehat{u}$ and the Dirichlet type boundary condition (2) are satisfied in a weak sense.

The Lagrangian $\widehat{L}$ also generates two functionals

$$\widehat{J}(\widehat{v}) := \sup_{\widehat{\mathbf{q}} \in \widehat{Q}_F} \widehat{L}(\widehat{v}, \widehat{\mathbf{q}}) \quad \text{and} \quad \widehat{I}^*(\widehat{\mathbf{q}}) := \inf_{\widehat{v} \in \widehat{V}} \widehat{L}(\widehat{v}, \widehat{\mathbf{q}})\,.$$

The two corresponding variational problems are $\inf\limits_{\widehat{v} \in \widehat{V}} \widehat{J}(\widehat{v})$ and $\sup\limits_{\widehat{\mathbf{q}} \in \widehat{Q}_F} \widehat{I}^*(\widehat{\mathbf{q}})$. They are called Problems $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{P}}^*$, respectively. Note that the functional $\widehat{J}$ (unlike J) has no simple explicit form. However, we can prove the solvability of Problem $\widehat{\mathcal{P}}$ by the following Lemma.

Lemma 2.1 *For any $\widehat{v} \in \widehat{V}$ and $F \in L_2(\partial_2\Omega)$ there exists $\mathbf{p}^v \in \widehat{Q}_F$ such that*

$$\operatorname{div} \mathbf{p}^v + \widehat{v} = 0 \ \text{ in } \Omega\,, \tag{16}$$

$$\| \mathbf{p}^v \|_* \leq C_\Omega \left(\|\widehat{v}\| + \|F\|_{\partial_2\Omega}\right). \tag{17}$$

Proof. We know that the boundary-value problem

$$\operatorname{div} \mathbf{A}\nabla u^v + \widehat{v} = 0 \quad \text{in } \Omega\,, \tag{18}$$

$$u^v = 0 \quad \text{on } \partial_1\Omega\,, \tag{19}$$

$$\mathbf{A}\nabla u^v \cdot \mathbf{n} = F \quad \text{on } \partial_2\Omega \tag{20}$$

possesses the unique solution $u^v \in V_0$, and the energy estimate

$$\| \nabla u^v \| \leq C_\Omega \, (\|\widehat{v}\| + \|F\|_{\partial_2\Omega})$$

holds. Let $\mathbf{p}^v := \mathbf{A}\nabla u^v$. Obviously, $\mathbf{p}^v \in \widehat{Q}_F$ and, since

$$\| \mathbf{p}^v \|_*^2 = \int_\Omega \mathbf{A}^{-1}(\mathbf{A}\nabla u^v) \cdot (\mathbf{A}\nabla u^v)\, dx = \| \nabla u^v \|^2 \,,$$

we find that (17) holds. By (18) we also observe the validity of (16).

■

With the help of Lemma 2.1 we can easily prove the coercivity of $\widehat{J}$ on $\widehat{V}$. Indeed,

$$\widehat{J}(\widehat{v}) \geq \widehat{L}(\widehat{v}, \alpha\mathbf{p}^v) = -\frac{1}{2}\alpha^2 \| \mathbf{p}^v \|_*^2 + \alpha\|\widehat{v}\|^2 - \|f\|\|\widehat{v}\| + g(u_0, \alpha\mathbf{p}^v) - \int_{\partial_2\Omega} F u_0 \, ds \,.$$

Here $|g(u_0, \alpha\mathbf{p}^v)| \leq \alpha\|\mathbf{p}^v\|_{\mathrm{div}}\,\|u_0\|_1$ and

$$\|\mathbf{p}^v\|_{\mathrm{div}}^2 = \|\mathbf{p}^v\|^2 + \|\mathrm{div}\,\mathbf{p}^v\|^2 \leq \frac{1}{c_1} \| \mathbf{p}^v \|_*^2 + \|\widehat{v}\|^2 \leq \frac{1}{c_1} C_\Omega^2 \, (\|\widehat{v}\| + \|F\|_{\partial_2\Omega})^2 + \|\widehat{v}\|^2 \,.$$

Therefore

$$\widehat{J}(\widehat{v}) \geq -\frac{1}{2}\alpha^2 C_\Omega^2 \|\widehat{v}\|^2 + \alpha\|\widehat{v}\|^2 + \Theta(\|\widehat{v}\|) + \Theta_0 \,,$$

where $\Theta(\|\widehat{v}\|)$ contains the terms linear with respect to $\|\widehat{v}\|$ and Θ_0 does not depend on $\widehat{v}$. Take $\alpha = 1/C_\Omega^2$. Then

$$\widehat{J}(\widehat{v}) \geq \frac{1}{2C_\Omega^2}\|\widehat{v}\|^2 + \Theta(\|\widehat{v}\|) + \Theta_0 \;\longrightarrow +\infty \;\text{ as } \|\widehat{v}\| \to \infty \,.$$

It is not difficult to prove that the functional $\widehat{J}$ is convex and lower semicontinuous. Therefore, Problem $\widehat{\mathcal{P}}$ has a solution $\widehat{u}$.

Remark. Lemma 2.1 implies the *inf-sup* condition

$$\inf_{\substack{\phi \in L_2(\Omega) \\ \psi \in L_2(\partial_2\Omega)}} \sup_{\mathbf{q} \in \widehat{Q}_F} \frac{\int_\Omega \phi \,\mathrm{div}\, \mathbf{q} \, dx + \int_{\partial_2\Omega} \psi \mathbf{q} \cdot \mathbf{n} \, ds}{\|\mathbf{q}\|_{\mathrm{div}} \, (\|\phi\|^2 + \|\psi\|_{\partial_2\Omega}^2)^{1/2}} \geq C_0 > 0 \,. \tag{21}$$

Consider now the functional $\widehat{I}^*$. $\widehat{I}^*(\widehat{\mathbf{q}}) = -\frac{1}{2} \parallel\!| \widehat{\mathbf{q}} \parallel\!|_*^2 + g(u_0, \widehat{\mathbf{q}}) - \int_{\partial_2\Omega} F u_0 \, ds$, if $\operatorname{div} \widehat{\mathbf{q}} + f = 0$ in $L_2(\Omega)$, and $\widehat{I}^*(\widehat{\mathbf{q}}) = -\infty$ in all other cases. Since $\widehat{\mathbf{q}} \in \widehat{Q}_F$, we have

$$\int_\Omega \nabla w \cdot \widehat{\mathbf{q}} \, dx = -\int_\Omega (\operatorname{div} \widehat{\mathbf{q}}) w \, dx + \int_{\partial_2\Omega} F w \, ds \quad \forall w \in V_0 .$$

Recalling that $\operatorname{div} \widehat{\mathbf{q}} = -f$ in $L_2(\Omega)$, we find that the dual functional for such a case has the form

$$\begin{aligned} -\frac{1}{2} \parallel\!| \widehat{\mathbf{q}} \parallel\!|_*^2 + \int_\Omega (\nabla u_0 \cdot \widehat{\mathbf{q}} - f u_0) \, dx - \int_{\partial_2\Omega} F u_0 \, ds \\ = \int_\Omega \nabla u_0 \cdot \widehat{\mathbf{q}} \, dx - \frac{1}{2} \parallel\!| \widehat{\mathbf{q}} \parallel\!|_*^2 - l(u_0) , \end{aligned}$$

and here $\widehat{\mathbf{q}}$ must satisfy the relation

$$\int_\Omega \nabla w \cdot \widehat{\mathbf{q}} \, dx = l(w) \quad \forall w \in V_0 .$$

In other cases, the functional $\widehat{I}^*(\widehat{\mathbf{q}}) = -\infty$. This, in fact, means that Problems $\mathcal{P}^*$ and $\widehat{\mathcal{P}}^*$ coincide and are reduced to maximization of $\widehat{I}^*$ on the set Q_l (i.e. $\sup \mathcal{P}^* = \sup \widehat{\mathcal{P}}^*$).

Again, we have $\widehat{L}(\widehat{u}, \widehat{\mathbf{p}}) = \inf \widehat{\mathcal{P}} = \sup \widehat{\mathcal{P}}^*$, but $\sup \widehat{\mathcal{P}}^* = \sup \mathcal{P}^* = \inf \mathcal{P}$ (see (9)). Thus, we infer that $\inf \widehat{\mathcal{P}} = \inf \mathcal{P}$.

The latter equality implies that the minimizer $u \in V_0 + u_0$ of Problem $\mathcal{P}$ is also the minimizer of the functional $\widehat{J}$ on $\widehat{V}$. Analogously, if $\mathbf{p} \in Q_l$ is the maximizer of Problem $\mathcal{P}^*$, then

$$\int_\Omega \nabla w \cdot \mathbf{p} \, dx = \int_\Omega f w \, dx + \int_{\partial_2\Omega} F w \, ds \quad \forall w \in V_0 .$$

From here we immediately see that $\operatorname{div} \mathbf{p} + f = 0$ a.e. in Ω and, hence,

$$\int_\Omega (\nabla w \cdot \mathbf{p} + (\operatorname{div} \mathbf{p}) w) \, dx = \int_{\partial_2\Omega} F w \, ds \quad \forall w \in V_0 ,$$

that is $\mathbf{p} \in \widehat{Q}_F$. Thus, we conclude that $\mathbf{p}$ is also the maximizer of Problem $\widehat{\mathcal{P}}^*$.

The reverse statement that the solutions of $\widehat{\mathcal{P}}$, $\widehat{\mathcal{P}}^*$ are also the solutions of $\mathcal{P}$, $\mathcal{P}^*$ is not difficult to prove as well. Hence, both mixed formulations have the same solution $(u, \mathbf{p})$ that is the generalized solution of problem (1)–(3).

If $\widehat{V}_h \subset \widehat{V}$, $\widehat{Q}_{0h} \subset \widehat{Q}_0$ and $\widehat{Q}_{Fh} \subset \widehat{Q}_F$ are certain finite element subspaces, then a discrete analog of (14)–(15) is the *dual mixed finite element method* (see, e.g., [6]): Find $(\widehat{u}_h, \widehat{\mathbf{p}}_h) \in \widehat{V}_h \times \widehat{Q}_{Fh}$ such that

$$\int_\Omega \left(\mathbf{A}^{-1}\widehat{\mathbf{p}}_h \cdot \widehat{\mathbf{q}}_h + (\operatorname{div}\widehat{\mathbf{q}}_h)\widehat{u}_h\right)\, dx = g(u_0, \widehat{\mathbf{q}}_h) \quad \forall \widehat{\mathbf{q}}_h \in \widehat{Q}_{0h}\,, \tag{22}$$

$$\int_\Omega (\operatorname{div}\widehat{\mathbf{p}}_h + f)\widehat{v}_h \, dx = 0 \quad \forall \widehat{v}_h \in \widehat{V}_h\,. \tag{23}$$

Now, our aim is to obtain computable upper bounds for the quantities

$$|\!|\!| \nabla(u - u_h) |\!|\!| \,,\ |\!|\!| \mathbf{p} - \mathbf{p}_h |\!|\!|_* \,,\ \|\mathbf{p} - \widehat{\mathbf{p}}_h\|_{\operatorname{div}} \,.$$

3 A posteriori estimates for the primal mixed formulation

First, we recall the relation (see, e.g., [17])

$$|\!|\!| \mathbf{p} - \mathbf{q} |\!|\!|_*^2 + |\!|\!| \nabla(u - v) |\!|\!|^2 = 2(J(v) - I^*(\mathbf{q}))\,, \tag{24}$$

where $\mathbf{q} \in Q_l$ and $v \in V_0 + u_0$.

Lemma 3.1

$$J(v) - I^*(\mathbf{q}) \le (1+\beta)D(\nabla v, \mathbf{y}) + \frac{1}{2}\left(1 + \frac{1}{\beta}\right) |\!|\!| \mathbf{y} - \mathbf{q} |\!|\!|_*^2\,, \tag{25}$$

where the functional D was defined in (5), v is an arbitrary function from $V_0 + u_0$, $\mathbf{q}$ is an arbitrary function from Q_l, $\mathbf{y}$ is an arbitrary function from Q and β is any positive number.

Proof. We have

$$\begin{aligned} J(v) - I^*(\mathbf{q}) &= \frac{1}{2} |\!|\!| \nabla v |\!|\!|^2 - l(v) + \frac{1}{2} |\!|\!| \mathbf{q} |\!|\!|_*^2 + l(u_0) - \int_\Omega \nabla u_0 \cdot \mathbf{q}\, dx \\ &= \int_\Omega \left(\frac{1}{2}\mathbf{A}\nabla v \cdot \nabla v + \frac{1}{2}\mathbf{A}^{-1}\mathbf{y}\cdot\mathbf{y} - \nabla v \cdot \mathbf{y}\right) dx + \int_\Omega \nabla(v - u_0)\cdot \mathbf{q}\, dx \\ &\quad + l(u_0 - v) + \frac{1}{2} |\!|\!| \mathbf{q} |\!|\!|_*^2 - \frac{1}{2} |\!|\!| \mathbf{y} |\!|\!|_*^2 + \int_\Omega \nabla v \cdot (\mathbf{y} - \mathbf{q})\, dx \\ &= D(\nabla v, \mathbf{y}) + \frac{1}{2} |\!|\!| \mathbf{y} - \mathbf{q} |\!|\!|_*^2 - \int_\Omega \mathbf{A}^{-1}\mathbf{y}\cdot\mathbf{y} + \int_\Omega \mathbf{A}^{-1}\mathbf{y}\cdot\mathbf{q} \end{aligned}$$

$$+ \int_\Omega \nabla v \cdot (\mathbf{y} - \mathbf{q})\, dx = D(\nabla v, \mathbf{y}) + \frac{1}{2} |\!|\!| \mathbf{y} - \mathbf{q} |\!|\!|_*^2 + \int_\Omega (\mathbf{A}^{-1}\mathbf{y} - \nabla v)\cdot(\mathbf{q} - \mathbf{y})\, dx\,.$$

Since

$$\int_\Omega \boldsymbol{\eta}_1 \cdot \boldsymbol{\eta}_2 \, dx \le \left(\int_\Omega \mathbf{A} \boldsymbol{\eta}_1 \cdot \boldsymbol{\eta}_1 \, dx \right)^{1/2} \left(\int_\Omega \mathbf{A}^{-1} \boldsymbol{\eta}_2 \cdot \boldsymbol{\eta}_2 \, dx \right)^{1/2} ,$$

we obtain

$$\begin{aligned} \int_\Omega (\mathbf{A}^{-1}\mathbf{y} - \nabla v) \cdot (\mathbf{q} - \mathbf{y}) \, dx &\le \left(\int_\Omega \mathbf{A}(\mathbf{A}^{-1}\mathbf{y} - \nabla v) \cdot (\mathbf{A}^{-1}\mathbf{y} - \nabla v) \right)^{1/2} \\ \cdot \left(\int_\Omega \mathbf{A}^{-1}(\mathbf{q} - \mathbf{y}) \cdot (\mathbf{q} - \mathbf{y}) \, dx \right)^{1/2} &= (2D(\nabla v, \mathbf{y}))^{1/2} \interleave \mathbf{q} - \mathbf{y} \interleave_* \\ &\le \beta D(\nabla v, \mathbf{y}) + \frac{1}{2\beta} \interleave \mathbf{q} - \mathbf{y} \interleave_*^2 \end{aligned}$$

with an arbitrary positive number β, which immediately implies the estimate (25).

$\blacksquare$

Lemma 3.2 *Let* $\mathbf{y} \in \widehat{Q}^+$. *Then the following estimate holds true:*

$$\inf_{\mathbf{q} \in Q_l} \interleave \mathbf{q} - \mathbf{y} \interleave_*^2 \le C^2 \left(\|\operatorname{div} \mathbf{y} + f\|^2 + \|\mathbf{y} \cdot \mathbf{n} - F\|^2_{\partial_2 \Omega} \right) , \tag{26}$$

where C *is the constant in the inequality*

$$\|w\|^2 + \|w\|^2_{\partial_2 \Omega} \le C^2 \interleave \nabla w \interleave^2 \quad \forall w \in V_0 . \tag{27}$$

Proof. One can write

$$\inf_{\mathbf{q} \in Q_l} \frac{1}{2} \interleave \mathbf{q} - \mathbf{y} \interleave_*^2 = \inf_{\boldsymbol{\eta} \in \widetilde{Q}_l} \frac{1}{2} \interleave \boldsymbol{\eta} \interleave_*^2 , \tag{28}$$

where $\widetilde{Q}_l := \{ \boldsymbol{\eta} \in Q \mid \int_\Omega \nabla w \cdot \boldsymbol{\eta} \, dx = \int_\Omega \widetilde{f} w \, dx + \int_{\partial_2 \Omega} \widetilde{F} w \, ds \; \forall w \in V_0 \}$ with the notation $\widetilde{f} = f + \operatorname{div} \mathbf{y}$ and $\widetilde{F} = F - \mathbf{y} \cdot \mathbf{n}$.

Note that under the assumptions made we have $\widetilde{f} \in L_2(\Omega)$ and $\widetilde{F} \in L_2(\partial_2 \Omega)$. The problem on the right-hand side of (28) can be transformed by virtue of the duality relations (see (8)), namely

$$\inf_{\boldsymbol{\eta} \in \widetilde{Q}_l} \frac{1}{2} \interleave \boldsymbol{\eta} \interleave_*^2 = - \sup_{\boldsymbol{\eta} \in \widetilde{Q}_l} \{ -\frac{1}{2} \interleave \boldsymbol{\eta} \interleave_*^2 \} = - \inf_{w \in V_0} \{ \frac{1}{2} \interleave \nabla w \interleave^2 - \widetilde{l}(w) \} , \tag{29}$$

where $\widetilde{l}(w) = \int_\Omega \widetilde{f} w \, dx + \int_{\partial_2\Omega} \widetilde{F} w \, ds$.

Further, we use the inequality (27), where the constant C depends only on the geometry of Ω and $\partial_2\Omega$ and on the bounds of the matrix $\mathbf{A}$ (see (4)). In fact, the constant can be presented as a combination of the constants from Friedrichs' and the trace inequalities (see [19] for the discussion on these constants and on the ways of their evaluation).
Since

$$\widetilde{l}(w) \leq \|\widetilde{f}\|\|w\| + \|\widetilde{F}\|_{\partial_2\Omega}\|w\|_{\partial_2\Omega} \leq (\|w\|^2 + \|w\|^2_{\partial_2\Omega})^{1/2}(\|\widetilde{f}\|^2 + \|\widetilde{F}\|^2_{\partial_2\Omega})^{1/2}$$
$$\leq C \mid\!\mid\!\mid \nabla w \mid\!\mid\!\mid (\|\widetilde{f}\|^2 + \|\widetilde{F}\|^2_{\partial_2\Omega})^{1/2} ,$$

we find that

$$\inf_{w\in V_0}\{\frac{1}{2} \mid\!\mid\!\mid \nabla w \mid\!\mid\!\mid^2 - \widetilde{l}(w)\} \geq \inf_{w\in V_0}\{\frac{1}{2} \mid\!\mid\!\mid \nabla w \mid\!\mid\!\mid^2 - C \mid\!\mid\!\mid \nabla w \mid\!\mid\!\mid (\|\widetilde{f}\|^2 + \|\widetilde{F}\|^2_{\partial_2\Omega})^{1/2}\}$$
$$\geq -\frac{1}{2}C^2(\|\widetilde{f}\|^2 + \|\widetilde{F}\|^2_{\partial_2\Omega}) .$$

Therefore,

$$\inf_{\boldsymbol{\eta}\in\widetilde{Q}_l} \frac{1}{2} \mid\!\mid\!\mid \boldsymbol{\eta} \mid\!\mid\!\mid^2_* \leq \frac{C^2}{2}(\|\widetilde{f}\|^2 + \|\widetilde{F}\|^2_{\partial_2\Omega}) ,$$

which yields estimate (26).

■

Combining formula (24) and the results of Lemmas 3.1 and 3.2 we arrive at the following estimate:

$$\mid\!\mid\!\mid \mathbf{p} - \mathbf{q} \mid\!\mid\!\mid^2_* + \mid\!\mid\!\mid \nabla(u - v) \mid\!\mid\!\mid^2 \leq 2(1 + \beta)D(\nabla v, \mathbf{y}) + \left(1 + \frac{1}{\beta}\right) C^2 \left(\|\mathrm{div}\, \mathbf{y} + f\|^2 + \|\mathbf{y}\cdot\mathbf{n} - F\|^2_{\partial_2\Omega}\right) , \quad (30)$$

where $\mathbf{y} \in \widehat{Q}^+$, $\mathbf{q} \in Q_l$ and $v \in V_0 + u_0$ are arbitrary functions and β is any positive number. Thus, we have the general estimate for the error in the primal variable:

$$\mid\!\mid\!\mid \nabla(u - v) \mid\!\mid\!\mid^2 \leq 2(1 + \beta)D(\nabla v, \mathbf{y}) + \left(1 + \frac{1}{\beta}\right) C^2 \left(\|\mathrm{div}\, \mathbf{y} + f\|^2 + \|\mathbf{y}\cdot\mathbf{n} - F\|^2_{\partial_2\Omega}\right) . \quad (31)$$

It is worth noticing that for any "approximate solution" $v \in V_0 + u_0$ the estimate is exact in the sense that there is a function $\mathbf{y} \in \widehat{Q}^+$ (namely,

$\mathbf{y} = \mathbf{A}\nabla u$) such that the inequality becomes the equality (the last two terms will then be zero and β can be taken arbitrarily small).

Since $u_h \in V_0 + u_0$, we can substitute it into (31) instead of v and obtain an upper bound for $\| \nabla(u - u_h) \|^2$; however, the "free function" $\mathbf{y} \in \widehat{Q}^+$ remains still to be chosen. There are several possibilities to find a good approximation of the exact flux $\mathbf{A}\nabla u$ in $\widehat{Q}^+$, i.e. a candidate for the role of $\mathbf{y}$ in (31). The cheapest way is to use the function $\mathbf{p}_h \in Q_h$ available from the solution of the primal mixed problem and to construct the projection operator $\mathcal{D}_h : Q_h \to \widehat{Q}^+$, so that $\mathcal{D}_h p_h$ could be inserted into (31) in place of $\mathbf{y}$. This projection as well as other methods for finding $\mathbf{y} \in \widehat{Q}^+$ will be discussed below.

Now, we derive the estimate for the error in the dual variable. If $\mathbf{p}_h$ were in Q_l, we could directly apply (30) to obtain an upper bound for $\| \mathbf{p} - \mathbf{p}_h \|_*^2$. However, this is not the case, as the function $\mathbf{p}_h$ satisfies the relation

$$\int_\Omega \mathbf{p}_h \cdot \nabla w_h \, dx = l(w_h) \quad \forall w_h \in V_{0h} \subset V_0$$

which defines a set of vector-valued functions wider than Q_l. This difficulty is circumvented as follows.

Let $\mathbf{y} \in \widehat{Q}^+$ and $\mathbf{q} \in Q_l$, then

$$\begin{aligned} \| \mathbf{p} - \mathbf{p}_h \|_* &\leq \| \mathbf{p} - \mathbf{q} \|_* + \| \mathbf{q} - \mathbf{y} \|_* + \| \mathbf{y} - \mathbf{p}_h \|_* \\ &\leq \sqrt{2}(J(v) - I^*(\mathbf{q}))^{1/2} + \| \mathbf{q} - \mathbf{y} \|_* + \| \mathbf{y} - \mathbf{p}_h \|_* \ . \end{aligned} \tag{32}$$

By Lemma 3.1 (after minimizing the right-hand side of (25) with respect to the positive scalar parameter β)

$$\sqrt{2}(J(v) - I^*(\mathbf{q}))^{1/2} \leq \sqrt{2} D^{1/2}(\nabla v, \mathbf{y}) + \| \mathbf{q} - \mathbf{y} \|_* \ . \tag{33}$$

Thus,

$$\| \mathbf{p} - \mathbf{p}_h \|_* \leq \sqrt{2} D^{1/2}(\nabla v, \mathbf{y}) + 2 \| \mathbf{q} - \mathbf{y} \|_* + \| \mathbf{y} - \mathbf{p}_h \|_*$$

and, taking the infimum of the right-hand side over all $\mathbf{q} \in Q_l$ and using Lemma 3.2, one obtains

$$\begin{aligned} \| \mathbf{p} - \mathbf{p}_h \|_* \leq{}& \sqrt{2} D^{1/2}(\nabla v, \mathbf{y}) + \| \mathbf{y} - \mathbf{p}_h \|_* \\ &+ 2C \left(\|\operatorname{div} \mathbf{y} + f\|^2 + \|\mathbf{y} \cdot \mathbf{n} - F\|^2_{\partial_2 \Omega} \right)^{1/2} \ . \end{aligned} \tag{34}$$

Here v is an arbitrary function from $V_0 + u_0$ and $\mathbf{y}$ is an arbitrary function from $\widehat{Q}^+$. We note that, if $\mathbf{y} = \mathbf{A}\nabla u$ and $v = u$, then the right-hand side of (34) coincides with the left-hand side, i.e. estimate (34) is exact in the sense that one can always find the "free variables" in such a way that the inequality becomes the equality.

A directly computable upper bound of $\| \mathbf{p} - \mathbf{p}_h \|_*$ is given by (34), if we set $v = u_h$ and $\mathbf{y} = \mathcal{D}_h p_h$, where $\mathcal{D}_h : Q_h \to \widehat{Q}^+$ is the projection operator mentioned above. It is clear that, for obtaining computable bounds of the errors in both the primal and the dual variables, we need to build the projection of $\mathbf{p}_h$ onto the space $\widehat{Q}^+$. This can be easily done, for instance, by a simple averaging, as we are about to see.

Projection from Q_h onto $\widehat{Q}^+$

If $\mathbf{p}_h$ is a piecewise-constant vector field on a simplicial mesh $\mathcal{T}_h$, we can readily project it onto the subspace of $\widehat{Q}^+$ that is formed by the Raviart-Thomas elements of the lowest order $\mathbf{RT}_0$ (see [15], [6]). Below, we consider the two-dimensional case for the sake of simplicity, but the extension to the three-dimensional case is straightforward. We also assume that the domain Ω has a polygonal boundary, and the latter is exactly matched by the triangulation $\mathcal{T}_h$.

Let T_i and T_j be two neighbouring simplexes with the common edge E_{ij}. If $\mathbf{q}_h$ is a piecewise constant vector-valued function that has the values $\mathbf{q}_i$ and $\mathbf{q}_j$ on T_i and T_j respectively, we set on E_{ij} the value of the normal component of the "averaged flux" as follows:

$$\widetilde{\mathbf{q}}_{ij} \cdot \mathbf{n}_{ij} = \frac{1}{2}(\mathbf{q}_i + \mathbf{q}_j) \cdot \mathbf{n}_{ij} \,,$$

where $\mathbf{n}_{ij}$ is the unit normal to the edge E_{ij} oriented from T_i to T_j if $i > j$. Another option is

$$\widetilde{\mathbf{q}}_{ij} \cdot \mathbf{n}_{ij} = \frac{|T_i|\mathbf{q}_i + |T_j|\mathbf{q}_j}{|T_i| + |T_j|} \cdot \mathbf{n}_{ij} \,,$$

where $|T_i|$ and $|T_j|$ are the areas of T_i and T_j. We repeat this procedure for all internal edges of $\mathcal{T}_h$.

If $E_{i0} \in \partial_1\Omega$, then we set $\widetilde{\mathbf{q}}_{i0} \cdot \mathbf{n}_{i0} = \mathbf{q}_{i0} \cdot \mathbf{n}_{i0}$. If $E_{i0} \in \partial_2\Omega$, then

$$\widetilde{\mathbf{q}}_{i0} \cdot \mathbf{n}_{i0} = \frac{1}{|E_{i0}|} \int_{E_{i0}} F \, ds \,.$$

Here $|E_{i0}|$ is the length of the edge E_{i0}.

Thus, all normal components $\widetilde{\mathbf{q}}_{ij} \cdot \mathbf{n}_{ij}$ on internal and external edges are defined. By prolongation inside all T_i, $i = \overline{1, N}$, with the help of $\mathbf{RT}_0$-approximation we obtain the function

$$\widetilde{\mathbf{q}}_h = \mathcal{D}_h \mathbf{q}_h \in \widehat{Q}^+ \,.$$

■

Now, we are ready to show the a posteriori estimates for the errors in the primal and the dual variables for the primal mixed finite element method.

Theorem 3.1 *Let $(u, \mathbf{p}) \in (V_0 + u_0) \times Q$ be the exact solution of the primal mixed problem (6)–(7) and $(u_h, \mathbf{p}_h) \in (V_{0h} + u_0) \times Q_h$ the solution of the discrete primal mixed problem (10)–(11).*
Then, the following a posteriori estimates hold true:

$$\begin{aligned} \| \nabla(u-u_h) \| &\leq \| \mathbf{A}\nabla u_h - \mathcal{D}_h \mathbf{p}_h \|_* \\ &+ C(\|\mathrm{div}\,(\mathcal{D}_h \mathbf{p}_h) + f\|^2 + \|(\mathcal{D}_h \mathbf{p}_h)\cdot \mathbf{n} - F\|^2_{\partial_2\Omega})^{1/2}, \quad (35) \\ \| \mathbf{p} - \mathbf{p}_h \|_* &\leq \| \mathbf{A}\nabla u_h - \mathcal{D}_h \mathbf{p}_h \|_* + \| \mathcal{D}_h \mathbf{p}_h - \mathbf{p}_h \|_* \\ &+ 2C \left(\|\mathrm{div}\,(\mathcal{D}_h \mathbf{p}_h) + f\|^2 + \|(\mathcal{D}_h \mathbf{p}_h)\cdot \mathbf{n} - F\|^2_{\partial_2\Omega}\right)^{1/2} \quad (36) \end{aligned}$$

where $\mathcal{D}_h : Q_h \to \widehat{Q}^+$ is the projection (averaging) operator introduced above and C is the constant from Lemma 3.2.

Proof. The first estimate immediately follows from (31), if we set there $v = u_h$, $\mathbf{y} = \mathcal{D}_h \mathbf{p}_h$, use definition (5) of the functional D and, then, minimize the right-hand side of (31) with respect to the scalar parameter β.

Estimate (36) is the direct consequence of (34).

■

Remarks.
1) The first term in estimates (35), (36), being computed elementwise, can serve as a local error indicator (see also [18], [19]).
2) A better estimate for $\| \nabla(u-u_h) \|$ can be obtained by the minimization of the right-hand side of (31) with respect to $\beta > 0$ and $\mathbf{y} \in \widehat{Q}^+$. With respect to $\mathbf{y}$ one has to minimize over some finite-dimensional subspace $\widehat{Q}^+_h \subset \widehat{Q}^+$, for instance, the Raviart-Thomas space $\mathbf{RT}_0$. The detailed discussion on the minimization of the functional of form (31) can be found in [18].

4 A posteriori estimates for the dual mixed formulation

An a posteriori estimate for the flux $\widehat{\mathbf{p}}_h$ readily follows from (34), if we set $\mathbf{y} = \widehat{\mathbf{p}}_h \in \widehat{Q}^+$:

$$\| \mathbf{p} - \widehat{\mathbf{p}}_h \|_* \leq \sqrt{2} D^{1/2}(\nabla v, \widehat{\mathbf{p}}_h) + 2C \left(\|\mathrm{div}\,\widehat{\mathbf{p}}_h + f\|^2 + \|\widehat{\mathbf{p}}_h \cdot \mathbf{n} - F\|^2_{\partial_2\Omega}\right)^{1/2}, \quad (37)$$

where v is an arbitrary function from $V_0 + u_0$.

Suppose that Ω is a polygonal domain whose boundary is exactly resolved by the triangulation $\mathcal{T}_h$. If $\widehat{\mathbf{p}}_h$ is constructed by means of $\mathbf{RT}_0$-elements, then

$$\int_\Omega (\operatorname{div} \widehat{\mathbf{p}}_h + f) w_h \, dx = 0 \quad \forall w_h \in \widehat{V}_h \subset \widehat{V}\,, \tag{38}$$

where the subspace $\widehat{V}_h$ contains piecewise constant functions. Therefore, on each element T_i

$$\operatorname{div} \widehat{\mathbf{p}}_h = -\frac{1}{|T_i|} \int_{T_i} f \, dx\,. \tag{39}$$

Let us define by $[f]$ the function that belongs to $\widehat{V}_h$ and whose values on T_i coincide with the mean values of f on T_i. Then, we have $\operatorname{div} \widehat{\mathbf{p}}_h = -[f]$ on every T_i.

Now, we observe that estimate (37) is valid for any approximate flux $\widehat{\mathbf{p}}_h$ from $\widehat{Q}^+$. If $\widehat{\mathbf{p}}_h$ were in the narrower set $\widehat{Q}_F$ (as it is supposed to be in the discrete dual mixed method (22)–(23)), the last norm in (37) would be identically zero. It cannot, however, be expected, when $\widehat{\mathbf{p}}_h$ is constructed in the space $\mathbf{RT}_0$, unless the function F is a constant on $\partial_2\Omega$.

The problem of taking into account the essential boundary condition for the flux variable ($\widehat{\mathbf{p}} \cdot \mathbf{n} = F$ on $\partial_2\Omega$) in the dual mixed method is not easy and, usually, leads to a non-conforming approximation $\widehat{\mathbf{p}}_h$ (see, e.g., [3] for the use of Lagrange multiplier technique in this case). Since our estimate (37) still works for such approximations of the flux, we propose a simple modification of the discrete dual method (22)–(23), particularly suited for the lowest-order Raviart-Thomas approximation.

Namely, instead of requiring $\widehat{\mathbf{p}}_h \in \widehat{Q}_F$, we impose a weaker condition

$$\widehat{\mathbf{p}}_h \cdot \mathbf{n}\big|_{E_{i0}} = \frac{1}{|E_{i0}|} \int_{E_{i0}} F \, ds \tag{40}$$

on every edge $E_{i0} \in \partial_2\Omega$. The space of test functions $\widehat{Q}_{0h} \subset \widehat{Q}_0$ will obviously consist of the $\mathbf{RT}_0$-approximations $\widehat{\mathbf{q}}_h$ such that $\widehat{\mathbf{q}}_h \cdot \mathbf{n} = 0$ on each edge $E_{i0} \in \partial_2\Omega$.

If now we denote by $[F]$ the piecewise constant function defined on the set of edges forming $\partial_2\Omega$ and whose value on every edge $E_{i0} \in \partial_2\Omega$ is equal to the mean value of F on that edge, we can write that $\widehat{\mathbf{p}}_h \cdot \mathbf{n} = [F]$ for all $E_{i0} \in \partial_2\Omega$.

As a result, we obtain from (37)

$$|\!|\!|\, \mathbf{p} - \widehat{\mathbf{p}}_h \,|\!|\!|_* \leq \sqrt{2} D^{1/2}(\nabla v, \widehat{\mathbf{p}}_h) + 2C \left(\|f - [f]\|^2 + \|F - [F]\|^2_{\partial_2\Omega} \right)^{1/2}\,. \tag{41}$$

The question that now arises is how to choose in (41) the function $v \in V_0 + u_0$. The simplest way is to use the function $\widehat{u}_h \in \widehat{V}_h$ available from the solution of the discrete dual mixed problem and to construct a suitable projection operator $\mathcal{P}_h : \widehat{V}_h \to V_0 + u_0$. Again, the projection can be easily accomplished with a simple averaging.

Projection from $\widehat{V}_h$ onto $V_0 + u_0$

In order to find $v \in V_0 + u_0$, it is sufficient to find $w \in V_0$ in the representation $v = w + u_0$ (the function u_0 is given). Using the computed piecewise-constant function $\widehat{u}_h$, we define $w_h \in V_0$ as follows.

We set

$$w_h(x_k) = \frac{\sum\limits_{s=1}^{N_k} |T_s^{(k)}| \cdot \widehat{u}_h\big|_{T_s^{(k)}}}{\sum\limits_{s=1}^{N_k} |T_s^{(k)}|} - u_0(x_k) \tag{42}$$

for any internal node x_k and when $x_k \in \partial_2\Omega$. Here $T_s^{(k)}$, $s = \overline{1, N_k}$, are the elements containing the vertex x_k, and we have assumed that the function u_0 has a sufficient regularity, so that its point values are defined.

If the node $x_k \in \partial_1\Omega$, we simply set $w_h(x_k) = 0$.

Thus, using the nodal values of w_h and the piecewise-linear continuous finite element approximation on the mesh $\mathcal{T}_h$ we define the function

$$w_h + u_0 = \mathcal{P}_h \widehat{u}_h \in V_0 + u_0 \,.$$

■

Hence, from (41) one obtains

$$\| \, \mathbf{p} - \widehat{\mathbf{p}}_h \, \|_* \leq \sqrt{2} D^{1/2}(\nabla(\mathcal{P}_h \widehat{u}_h), \widehat{\mathbf{p}}_h) + 2C \left(\|f - [f]\|^2 + \|F - [F]\|^2_{\partial_2\Omega} \right)^{1/2} , \tag{43}$$

which, together with the obvious relation

$$\|\mathrm{div}\,(\widehat{\mathbf{p}} - \widehat{\mathbf{p}}_h)\| = \| - f - \mathrm{div}\,\widehat{\mathbf{p}}_h\| = \|f - [f]\| \tag{44}$$

leads to the upper bound for $\|\widehat{\mathbf{p}} - \widehat{\mathbf{p}}_h\|_{\mathrm{div}}$:

Theorem 4.1 *Let $(\widehat{u}, \widehat{\mathbf{p}}) \in \widehat{V} \times \widehat{Q}_F$ be the exact solution of the dual mixed problem (14)–(15) and $(\widehat{u}_h, \widehat{\mathbf{p}}_h) \in \widehat{V}_h \times \widehat{Q}_{Fh}$ the solution of the discrete dual mixed problem (22)–(23) with $\widehat{Q}_{Fh}$ being the Raviart-Thomas space* $\mathbf{RT}_0$

equipped with the conditions (40).
Then, the following estimate holds true:

$$\|\widehat{\mathbf{p}} - \widehat{\mathbf{p}}_h\|_{\mathrm{div}} \leq \|\!| \mathbf{A}\nabla(\mathcal{P}_h\widehat{u}_h) - \widehat{\mathbf{p}}_h \|\!|_* + (2C+1)\|f - [f]\| + 2C\|F - [F]\|_{\partial_2\Omega}\,, \tag{45}$$

where $\mathcal{P}_h : \widehat{V}_h \to V_0 + u_0$ *is the projection (averaging) operator introduced above,* C *is the constant from Lemma 3.2, and* $[f]$ *and* $[F]$ *are the averaged functions (see the right-hand sides of (39) and (40)).*

Remarks.
1) The first and the second terms in (45), being computed elementwise, can be used as local indicators for the L_2-error and the error in the equation $\mathrm{div}\,\widehat{\mathbf{p}} + f = 0$, respectively.
2) A possibly sharper estimate may be obtained, if one has the conforming finite element Galerkin solution to original problem (1)–(3) and inserts it in (45) instead of $\mathcal{P}_h\widehat{u}_h$.
3) Yet better estimate may be obtained by the minimization of the right-hand side of (41) with respect to v. Here, we can restrict ourselves to the minimization of the functional $D(\nabla v, \widehat{\mathbf{p}}_h)$ over the set $V_{0h} + u_0$, i.e. to the finite-dimensional problem

$$\min_{w_h \in V_{0h}} \left(\int_\Omega (\frac{1}{2}\mathbf{A}\nabla w_h \cdot \nabla w_h + (\mathbf{A}\nabla u_0 - \widehat{\mathbf{p}}_h) \cdot \nabla w_h)\, dx \right) . \tag{46}$$

Under the assumption on a proper regularity of u_0, this problem is transformed as follows

$$\min_{w_h \in V_{0h}} \left(\int_\Omega (\frac{1}{2}\mathbf{A}\nabla w_h \cdot \nabla w_h - f_0 w_h)\, dx - \int_{\partial_2\Omega} F_0 w_h\, ds \right) , \tag{47}$$

where $f_0 = \mathrm{div}\,(\mathbf{A}\nabla u_0) + [f]$ and $F_0 = [F] - \mathbf{A}\nabla u_0 \cdot \mathbf{n}$. In the particular case $u_0 = 0$, the best choice of w_h is to take it as the Galerkin approximation of original problem (1)–(3) with averaged right-hand sides $[f]$ and $[F]$.

Although the primal variable $\widehat{u}$ is sought in the space $L_2(\Omega)$ within the framework of the dual mixed method, the exact solution u of problem (1)–(3) belongs to the space $V_0 + u_0 \subset H^1(\Omega)$. That is why it does not seem to be non-natural to measure the error of the primal variable approximation in the energy norm $\|\!| \nabla \cdot \|\!|$ also in the dual mixed approach. In fact, with estimate (31) we can assess the squared norm of the error of the averaged solution $\mathcal{P}_h\widehat{u}_h$ using the computed flux approximation $\widehat{\mathbf{p}}_h$:

$$\| \nabla(u-\mathcal{P}_h\widehat{u}_h) \|^2 \leq 2(1+\beta)D(\nabla(\mathcal{P}_h\widehat{u}_h),\widehat{\mathbf{p}}_h) + \left(1+\frac{1}{\beta}\right) C^2(\|f-[f]\|^2+\|F-[F]\|^2_{\partial_2\Omega}) , \quad (48)$$

where $\beta > 0$ is an arbitrary number that can be used to minimize the right-hand side of (48) and to obtain the estimate for the norm of the error.

A sharper estimate may be obtained, if one spends some time on the minimization of the right-hand side of (31) with respect to the dual variable $\mathbf{y}$ over some finite-dimensional subspace of $\widehat{Q}^+$.

Remark.
If one has the solutions of both the primal and the dual mixed problems, the flux approximation $\widehat{\mathbf{p}}_h$ can be substituted into (31) to immediately yield the error estimate for the primal variable (which is the most important in the primal mixed method), while the approximation u_h can be used in (41) to bring the error estimate for the dual variable (which is the most important in the dual mixed method).

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