Approximate Solution of Problem on Viscous Flow with Evaporating Non-Compact Free Boundary

Anton A. Smolianski

Laboratory of Scientific Computing

Department of Mathematics

University of Jyväskylä

Abstract

The stationary problem on planar flow of viscous incompressible fluid with evaporating non–compact free boundary is considered. The solvability of this problem and smoothness of its solution are studied. The technique of artificial restriction of the domain is used for obtaining the approximate solution. It is shown that if the restricted domain is sufficiently wide, then the solution on it differs little from that on the original domain. A finite–element approximation of the problem on the restricted domain is constructed. For this the curvilinear C^1-elements are used along the free boundary to approximate velocity and temperature and C^0-elements — for pressure. Special algorithm for mesh refinement in the vicinity of the corner point of the boundary is presented to obtain optimal estimates of approximation. Also estimates for the rate of convergence of the finite–element solution to the exact one are given.

Keywords: Navier-Stokes and heat conductivity equations, free boundary, finite element method, mesh refinement, estimates of approximation, curvilinear elements.

AMS subject classification: 35R35, 65N30, 76D05.

We consider the problem on viscous flow with free evaporating surface, which has numerous applications in coating and drying processes encountered during the production of paper and polymers. The surface of the fluid is non-compact, and the gravitational field and heat sources are taken into account as external factors.

1.Introduction

The problem on steady flow of heavy viscous incompressible fluid leaking out of a narrow channel and spreading along an infinite bottom is studied. The bottom makes an angle α with the horizon. We assume that the motion of the fluid is plane-parallel and consider the problem in the plane $\mathbf{R}^2$ with fixed Cartesian coordinates. The Poiseuille flow is prescribed in the channel, and the bottom of the channel is moving with a given constant velocity. The flow takes place in a steady temperature field which is governed by a given distribution of heat sources and given temperature of the bottom, the upper wall of the channel and the environment in proximity to the free surface of the fluid.

There are two types of singularities: the corner point and non-compactness of the free boundary.

Our problem is, in essence, close to the one whose classical solvability was considered by K.Pileckas ([Pi1], [Pi2]), however, it is complicated by the presence of (steady) temperature field and evaporation of the free surface of the fluid.

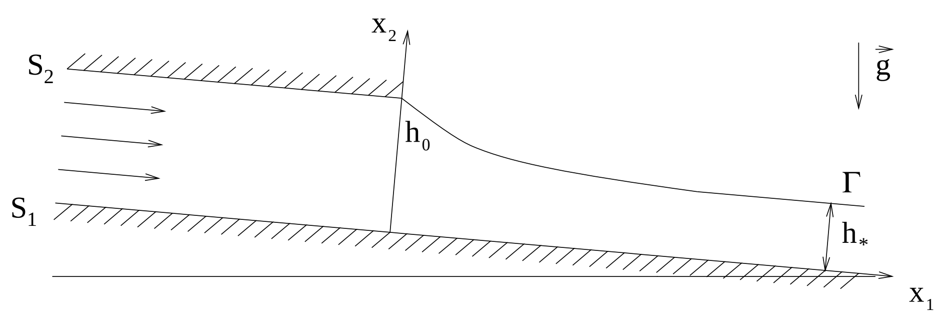

Figure 1.

2.Mathematical model and solvability of the problem

The setting of our problem is as follows. Let Ω be the domain bounded by the bottom of the channel S_1, its upper wall S_2 and the free surface of the fluid Γ (see Fig.1). Note that Ω has two "exits" to infinity: at $x_1 \to -\infty$ and at $x_1 \to +\infty$.

It is required to find the velocity vector $\vec{v}(x)$, the pressure $p(x)$ and the temperature $\theta(x)$ satisfying in Ω the system of Navier-Stokes equations

$$-\nu\Delta\vec{v}+(\vec{v}\cdot\nabla)\vec{v}+\frac{1}{\rho}\nabla p=-\nabla G, \qquad \nabla\cdot\vec{v}=0, \tag{1}$$

the heat conductivity equation

$$-\lambda\Delta\theta+\vec{v}\cdot\nabla\theta=f \tag{2}$$

and the boundary conditions

$$\vec{v}|_{S_1}=(R,0); \qquad \vec{v}|_{S_2}=(0,0);$$

$$\vec{v}\cdot\vec{n}|_\Gamma=\lambda_1\frac{\partial\theta}{\partial n}\bigg|_\Gamma; \qquad \vec{\tau}\cdot\mathbf{S}(\vec{v})\cdot\vec{n}|_\Gamma=0 \tag{3}$$

$$\theta|_{S_1}=\theta_1+\theta_\infty; \qquad \theta|_{S_2\cup\Gamma}=\theta_2+\theta_\infty \tag{4}$$

Here $G=g(-x_1\sin\alpha+x_2\cos\alpha)$ is the potential of gravitational field, $\mathbf{S}(\vec{v})$ is the strain tensor; R is a given constant velocity of the bottom of the channel; f, θ_1 and θ_2 are given functions; θ_∞ is a given constant temperature at infinity.

Note that the condition for the normal component of velocity on the free boundary (the so-called kinematic condition) simulates evaporation of the fluid.

It is also required to find the function $x_2=\varphi(x_1),\ x_1>0$ specifying the free boundary Γ and satisfying the equation (the so-called dynamic condition)

$$\sigma(\theta)\frac{d}{dx_1}\frac{\varphi'(x_1)}{\sqrt{1+(\varphi'(x_1))^2}}=(-p(x)+\rho\nu\vec{n}\cdot\mathbf{S}(v(\vec{x}))\cdot\vec{n})|_{x_2=\varphi(x_1)} \tag{5}$$

where $\sigma(\theta)$ is the coefficient of surface tension, and the left-hand side contains the curvature of Γ.

At that the boundary conditions for the function $\varphi(x_1)$ read

$$\varphi(0)=h_0, \quad \text{where } h_0 \text{ is the height of the channel}$$

$$\lim_{x_1\to+\infty}\varphi(x_1)=h_*, \quad \text{where } h_* \text{ is the elevation of fluid at infinity} \tag{6}$$

h_* is a priori unknown and is derived from a physically justified requirement that the pressure be bounded at $x_1\to+\infty$.

To prove the unique solvability of problem (1)–(6) in weighted Sobolev spaces, we use the method of splitting the complete problem into two auxiliary problems: (1)–(4) with a fixed boundary and (5)–(6) for finding the uknown boundary Γ (see, for example, [Pu1], [Pu2], [Riv1], [Riv2], [Riv3], [Sol1], [Sol2], [Soc], [NeRi], [Er]). The analysis of solvability is carried out by a method proposed in [Pu1], [RiFr], [SoSh] and is based on the coercive estimates for solutions of Stokes' problem and linearized problems for determining the temperature and the uknown boundary. This approach enables us to obtain the theorem below. Here we use the weighted Sobolev spaces $H^{(i)}_\mu(\Omega;b)$ $(i=1,2,\ldots)$ with power weight in vicinity of the corner point and exponential weight at infinity:

$$\|u\|_{H^{(i)}_\mu(\Omega;b)} = \|u\|_{H^{(i)}_\mu(\Omega_0)} + \|u\cdot\exp(-bx_1)\|_{H^i(\Omega_-)} + \|u\cdot\exp(bx_1)\|_{H^i(\Omega_+)}, \quad (7)$$

where $\Omega_0=\{x\in\Omega|\ |x_1|<2\}, \Omega_+=\{x\in\Omega|\ x_1>1\}, \Omega_-=\{x\in\Omega|\ x_1<-1\}$ and

$$\|u\|_{H^{(i)}_\mu(\Omega_0)} = \|u\|_{H^{i-1}(\Omega_0)} + \Big(\sum_{|\gamma|=i}\int_{\Omega_0} d^\mu(x)|D^\gamma u(x)|^2\,dx\Big)^{1/2},$$

$d(x)$ is a distance of point $x=(x_1;x_2)$ from the corner point $A=(0;h_0)$. Also we use Slobodetski-Sobolev spaces $W^{l/2}_2(S_1;b)$, $W^{l/2}_{2,\mu}(S_2\cup\Gamma;b)$, $W^{l/2}_{2,\mu}(\mathbf{R}^1_+;b)$ $(l=1,3,5)$ the norms in which are defined analoguosly.

Theorem 1

Let the numbers $h_0,h_*>0$ satisfy the inequality $|h_0-h_*|<\sqrt{\frac{2}{\beta_0}}$, where $\beta_0=\frac{\rho g}{\sigma(\theta_\infty)}$.
Let $f\in H^{(1)}_\mu(\Omega;b)$, $\theta_1\in W^{5/2}_2(S_1;b)$, $\theta_2\in W^{5/2}_{2,\mu}(S_2\cup\Gamma;b)$.

Then there exist such numbers $b_*>0$, $\delta_*\in(0;1)$, R_*,Q_*, α_*, f_*, θ_{1*}, θ_{2*} – all positive, that at

$$|R|<R_*,\quad |Q|<Q_*,\quad \alpha\in(0;\alpha_*),\quad \|f\|_{H^{(1)}_\mu(\Omega;b)}<f_*,$$
$$\|\theta_1\|_{W^{5/2}_2(S_1;b)}<\theta_{1*},\quad \|\theta_2\|_{W^{5/2}_{2,\mu}(S_2\cup\Gamma;b)}<\theta_{2*},$$

where $b\in(0;b_*)$, $\quad\mu>2(1-\delta)$, $\quad\delta\in(0;\delta_*)$
the problem (1)–(6) has a unique solution

$$\{\vec v(x),p(x),\theta(x),\varphi(x)\}\in H^{(2)}_\mu(\Omega;b)\times H^{(1)}_\mu(\Omega;b)\times H^{(2)}_\mu(\Omega;b)\times W^{5/2}_{2,\mu}(\mathbf{R}^1_+;b).$$

Note that the exact value of parameter δ determining the behaviour of solution in vicinity of the corner point is a priori uknown and depends on the magnitude of the angle.

3.Restriction of the physical domain and connection between solutions on the restricted and original domains

For solving our problem approximately, we restrict the original domain, so as to eliminate the exits to infinity. It can be shown that if we choose the bounded domain to be sufficiently wide, the solution will be close to the solution of the problem in the original unbounded domain.

So, let us consider in the domain $\Omega_L = \{x \in \Omega |\ |x_1| < L\}$ the problem: find the velocity vector $\vec{v}_L(x)$, the pressure $p_L(x)$, the temperature $\theta_L(x)$ and the function $\varphi_L(x)$ specifying the free boundary which satisfy the system of Navier-Stockes equations

$$-\nu\Delta\vec{v}_L + (\vec{v}_L \cdot \nabla)\vec{v}_L + \frac{1}{\rho}\nabla p_L = -\nabla G; \qquad \nabla \cdot \vec{v}_L = 0 \tag{8}$$

the heat conductivity equation

$$-\lambda\Delta\theta_L + \vec{v}_L \cdot \nabla\theta_L = f \tag{9}$$

and the boundary conditions

$$\begin{aligned} &\vec{v}_L|_{\Sigma_1} = (R,0); \quad \vec{v}_L|_{\Sigma_2} = (0,0); \quad \vec{v}_L \cdot \vec{n}|_{\Gamma_L} = \lambda_1 \left.\frac{\partial\theta_L}{\partial n}\right|_{\Gamma_L}; \\ &\vec{v}_L|_{\Sigma_3} = \vec{v}^-; \quad \vec{v}_L|_{\Sigma_4} = \vec{v}^+; \quad \vec{\tau} \cdot \mathbf{S}(\vec{v}_L) \cdot \vec{n}|_{\Gamma_L} = 0 \qquad (10) \\ &\theta_L|_{\Sigma_1} = \tilde{\theta}_1 + \theta_\infty; \quad \theta_L|_{\Sigma_2 \cup \Gamma_L} = \tilde{\theta}_2 + \theta_\infty; \\ &\theta_L|_{\Sigma_3} = \theta_\infty; \quad \theta_L|_{\Sigma_4} = \theta_\infty \qquad (11) \end{aligned}$$

$$\frac{d}{dx_1}\frac{\varphi_L'(x_1)}{\sqrt{1 + (\varphi_L'(x_1))^2}} = \frac{1}{\sigma(\theta_L)}(-p_L(x) + \rho\nu\vec{n} \cdot \mathbf{S}(\vec{v_L}) \cdot \vec{n})|_{\Gamma_L} \tag{12}$$

$$\varphi_L(0) = h_0; \ \varphi_L'(L) = 0 \tag{13}$$

Here $\Sigma_1 = S_1 \cap \partial\Omega_L$ is a part of the bottom of the cannel, $\Sigma_2 = S_2 \cap \partial\Omega_L$ is a part of its upper wall, Σ_3 and Σ_4 are "left" and "right" newly formed boundaries of our domain. Also $\Gamma_L = \Gamma \cap \partial\Omega_L$ is a part of free boundary. The functions $\vec{v}^-$ and $\vec{v}^+$ specify the asymptotics of velocity at $x_1 \to -\infty$ and $x_1 \to +\infty$, respectively. These functions as well as the asymptotics for pressure are found from the problem (1)–(4) on original domain with fixed boundary. The functions $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are smooth truncations of θ_1 and θ_2, respectively, so that $\tilde{\theta}_1 = 0$ and $\tilde{\theta}_2 = 0$ when $|x_1| \geq L$.

The unique solvability of this problem under the assumption of smallness of initial data is proved as for the unbounded domain.

Let us now study the connection of solution of the problem in restricted domain with that in the original one.

The following theorem holds:

Theorem 2

Let the conditions of Theorem 1 on unique solvability of the problem on unbounded domain hold.

Then there exist such positive numbers

$$f^0,\ \theta_1^0,\ \theta_2^0,\ R^0,\ Q^0 \text{ and } \alpha^0$$

that at

$$|R| < R^0,\ |Q| < Q^0,\ \alpha \in (0;\alpha^0),\ \|f\|_{H_\mu^{(1)}(\Omega;b)} < f^0,$$
$$\|\theta_1\|_{W_2^{5/2}(S_1;b)} < \theta_1^0,\ \|\theta_2\|_{W_{2,\mu}^{5/2}(S_2\cup\Gamma;b)} < \theta_2^0,$$

for any $\epsilon > 0$ there can be found such sufficiently large $L = L(\epsilon)$ that we'll have

$$\|\vec{v}_L - \vec{v}\|_{H_\mu^{(2)}(\Omega_L)} + \|p_L - p\|_{H_\mu^{(1)}(\Omega_L)} + \|\theta_L - \theta\|_{H_\mu^{(2)}(\Omega_L)} < \epsilon$$

and

$$\|\varphi_L - \varphi\|_{W_{2,\mu}^{5/2}((0;L))} < \epsilon$$

where $\{\vec{v}_L, p_L, \theta_L, \varphi_L\}$ and $\{\vec{v}, p, \theta, \varphi\}$ are the solutions of problems in the bounded and original unbounded domain, respectively.

4.Finite–element solution of problem on the restricted domain

The last theorem enables us to limit ourselves to constructing an approximate solution in the restricted domain. First, we construct an approximation of the space $H_\mu^{(2)}(\Omega_L) \times H_\mu^{(1)}(\Omega_L) \times H_\mu^{(2)}(\Omega_L) \times W_{2,\mu}^{5/2}((0;L))$ containing the exact solution $\{\vec{v}_L, p_L, \theta_L, \varphi_L\}$ with appropriate finite element spaces. At that the space $H_\mu^{(2)}(\Omega_L)$ (i.e. the velocity and temperature) is approximated by the space $(H_\mu^{(2)})_h$ of C^1-elements which are Bell's triangular elements inside of the domain and curvilinear Ženišek's triangular elements along the free boundary; the space $H_\mu^{(1)}(\Omega_L)$,i.e. the pressure, — by the space $(H_\mu^{(1)})_h$ of C^0-elements which are piecewise-linear inside of the domain (see [Žen1], [Žen2]). The free boundary is approximated by Hermite polynomials of 5-th degree.

For obtaining the optimal estimates of approximation, the grid must be constructed in a special way with mesh refinement in the vicinity of the corner point of the boundary.

The following theorem was proved in correspondence with the A.Schatz's analysis ([STW]):

Theorem 3

Let

$h \in (0;1)$ be the grid discretization parameter;
T — any triangle of the triangulation;
h_T — the length of the longest side of the triangle T;
d_T — the distance of the triangle T from the corner point $(0;h_0)$;
$\delta \in (0;1)$ — the parameter determining the behaviour of solution in vicinity of the corner point (see Th. 1).

Let $v \in H^5(\Omega_L)$.

1. (Optimal grid in $L_2(\Omega_L)$)

$$\text{If} \qquad h_T \le \begin{cases} C h^{\frac{2}{1+\delta/2}} & \text{when} \quad d_T = 0 \\ C h d_T^{\frac{1-\delta/2}{2}} & \text{when} \quad d_T > 0 \end{cases}$$

then

$$\|v - v_I\|_{L_2(\Omega_L)} \leq Ch^2 \|v\|_{H^{(2)}_\mu(\Omega_L)}$$

2. (Optimal grid in $H^1(\Omega_L)$)

$$\text{If} \qquad h_T \leq \begin{cases} Ch^{\frac{2}{\delta}} & \text{when} \quad d_T = 0 \\ Chd_T^{1-\frac{\delta}{2}} & \text{when} \quad d_T > 0 \end{cases}$$

then

$$\|v - v_I\|_{H^1(\Omega_L)} \leq Ch \|v\|_{H^{(2)}_\mu(\Omega_L)}$$

Here v_I is a $(H^{(2)}_\mu)_h$-interpolant of function v and constants in the estimates are independent of both the longitudinal dimension L of the domain Ω_L and of v.

Note that the grid optimal in H^1 will also be optimal for L_2 but not vice versa.

Analoguosly one can obtain the interpolation estimates of first order with respect to h for $(H^{(1)}_\mu)_h$-interpolant and $(W^{5/2}_{2,\mu})_h$-interpolant in L_2-norm and $W^{3/2}_2$-norm, respectively, considering the grid 2. of Theorem 3.

Using the result of Theorem 3 (see clause 2.) we can construct the "optimal" grid on Ω_L.

At first, we describe the splitting of the interval $(0; L)$ with intervals Δ_j $(j = 1, 2, \ldots)$ to define an approximation of free boundary :

$$\begin{aligned} &\Delta_0 = (0; h^{\frac{2}{\delta}}), \quad \Delta_j = (x_1^{(j)}; x_1^{(j+1)}), \; j = 1, 2, \ldots, \\ &x_1^{(0)} = 0, \; x_1^{(1)} = h^{\frac{2}{\delta}}, \; x_1^{(j+1)} = x_1^{(j)} + h(x_1^{(j)})^{1-\frac{\delta}{2}} \end{aligned} \tag{14}$$

We go on constructing Δ_j in this way until $x_1^{(j+1)}$ becomes greater than h_0, after which we cover the remaining part $(0; L)$ with intervals of length h. Then every function from $(W^{5/2}_{2,\mu})_h$ will be a Hermite polynomial of 5-th degree on each Δ_j.

Given the function $\varphi_h \in (W^{5/2}_{2,\mu})_h$ determining the free boundary of Ω_L, we construct the desired grid by the following scheme:
1. $d_0 = h^{\frac{2}{\delta}}$. We draw a circle of radius d_0 with center at the point $A = (0; h_0)$ and triangulate the domain $G_0 = \{x \in \Omega_L | \; \mathrm{dist}(x, A) < d_0\}$ with triangles of size $h^{\frac{2}{\delta}}$.

2. For $j = 1, 2, 3 \ldots$ we assume $d_j = d_{j-1} + h(d_{j-1})^{1-\frac{\delta}{2}}$ and triangulate the domain $G_j = \{x \in \Omega_L |\ d_{j-1} < \text{dist}(x, A) < d_j\}$ with triangles of size $h(d_{j-1})^{1-\frac{\delta}{2}}$. At that we substitute chords for all the circular arcs. We continue this process until d_j becomes greater than h_0, after which we cover the remaining part of Ω_L with rectilinear triangles of size h.
3. In the proximity to φ_h we alter the triangulation so that each point $(x_1; \varphi_h(x_1^{(j)}))$, where $x_1^{(j)}$ is an endpoint of Δ_j, becomes a node of the triangulation.

It is easy to see that all the curvilinear sides of the triangles located along the boundary $x_2 = \varphi_h(x_1)$ will be exact polynomials of degree no greater than 5-th that are known. This allows us to use an exact mapping of each curvilinear triangle onto the reference one.

It is also not difficult to ascertain that the total number of triangles will be of order Ch^{-2}, i.e. proportional to the number of triangles of a uniform triangulation.

Figures 2–5 corresponding to different values of discretization parameter h and refinement parameter δ illustrate the mesh refinement near the corner point.

Finally, for any function $v \in H_\mu^{(k)}(\Omega_L)$, $k = 1, 2$ we can obtain the estimate of approximation ([OgRu], [Riv2], [Riv3])

$$\|v - v_h\|_{H^{k-1}(\Omega_L)} \leq Ch\|v\|_{H_\mu^{(k)}(\Omega_L)} \tag{15}$$

and also the estimate

$$\|\varphi - \varphi_h\|_{W_2^{3/2}((0;L))} \leq Ch\|\varphi\|_{W_{2,\mu}^{5/2}((0;L))} \tag{16}$$

for any $\varphi \in W_{2,\mu}^{5/2}((0; L))$, where v_h and φ_h are projections on the corresponding finite element spaces.

Approximate solution of our problem splits into two stages: approximate solution of auxiliary problem given a fixed boundary and approximate solution of the boundary-value problem for the free boundary itself.

The investigation of existence of the approximate solution and the rate of its convergence to the exact one was conducted by a theory worked out by V.Rivkind ([Riv2], [Riv3], [Riv4]).
So, the following theorem holds

Theorem 4

Let

$$\{\vec{v}, p, \theta, \varphi\} \in H_\mu^{(2)}(\Omega_L) \times H_\mu^{(1)}(\Omega_L) \times H_\mu^{(2)}(\Omega_L) \times W_{2,\mu}^{5/2}((0;L))$$

be the exact solution of the problem in the restricted domain Ω_L.

Then for h sufficiently small, there exists the approximate solution

$$\{\vec{v}_h, p_h, \theta_h, \varphi_h\} \in (H_\mu^{(2)})_h \times (H_\mu^{(1)})_h \times (H_\mu^{(2)})_h \times (W_{2,\mu}^{5/2})_h$$

such that

$$\|\vec{v} - \vec{v}_h\|_{H_\mu^{(2)}(\Omega_L)} + \|p - p_h\|_{H_\mu^{(1)}(\Omega_L)} + \|\theta - \theta_h\|_{H_\mu^{(2)}(\Omega_L)} + \|\varphi - \varphi_h\|_{W_{2,\mu}^{5/2}((0;L))} \le C_h\,, \quad C_h \to 0 \quad (h \to 0)$$

and

$$\|\vec{v} - \vec{v}_h\|_{H_\mu^{1}(\Omega_L)} + \|p - p_h\|_{L_2(\Omega_L)} + \|\theta - \theta_h\|_{H_\mu^{1}(\Omega_L)} + \|\varphi - \varphi_h\|_{W_2^{3/2}((0;L))} \le Ch\,,$$

where constants are independent of the longitudinal dimension L of the domain Ω_L.

The proof is based on the fact that the operator of the exact problem is contractive in some ball of the Banach space and approximate operator is also contractive in the finite–dimensional space. This fact allows us to make use of the method of successive approximations to solve our problem.

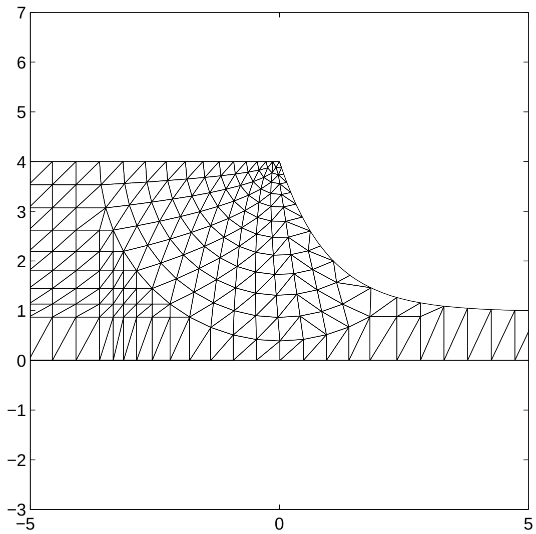

Figure 2. $h = 0.3, \quad \delta = 0.8$.

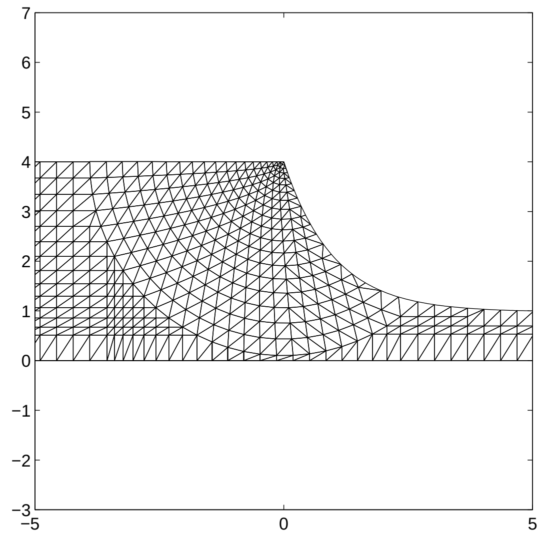

Figure 3. $h = 0.1, \quad \delta = 0.8$.

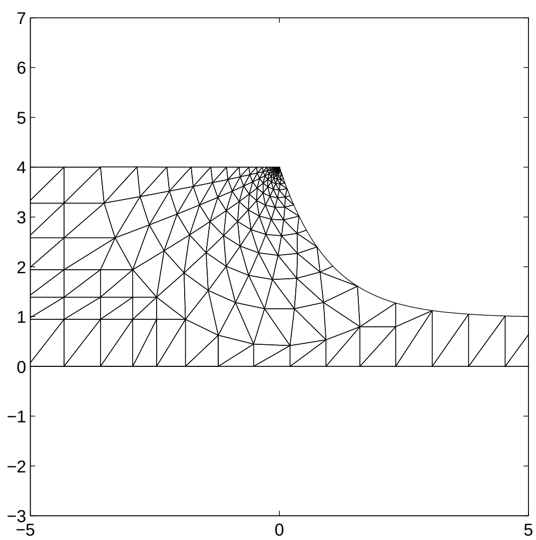

Figure 4. $h = 0.3, \quad \delta = 0.3$.

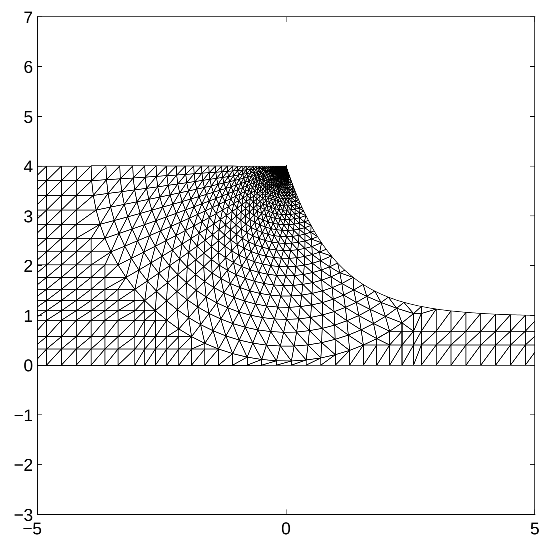

Figure 5. $h = 0.1, \quad \delta = 0.3$.

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