Master's Thesis

Thoroughly Formalizing an Uncommon Construction of the Real Numbers

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Abstract

A thorough formalization of Norbert A'Campo's unusual construction of the real numbers — which defines real numbers using generalizations of endomorphisms on $\mathbb{Z}$ and does not rely on fractions — is presented. We start with the derivation of a method for obtaining models of suitable theories out of models of fragments of Zermelo-Fraenkel Set Theory and thereafter elaborate A'Campo's work by applying this method to the theories of natural, integer and real numbers. This process involves the definition and verification of welldefinedness of various relations, functions and constants, as well as proofs of the axioms of the respective theories within Set Theory. Eventually, a nonstandard model of $\mathbb{R}$ based on the previous construction is examined, yielding an intelligible characterization of real numbers in said model and, as a consequence, illustrating certain concepts of analysis in an unprecedented way.

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Contents

1 Introduction

While practicing mathematics, especially analysis, the concept of the real numbers — a set of objects forming a totally ordered, Dedekind-complete field usually denoted by $\mathbb{R}$ — plays a very important role. Unlike in $\mathbb{Q}$, desirable numbers such as π and $\sqrt{2}$ exist in $\mathbb{R}$, making the real numbers the foundation of modern calculus and analysis. Since their applications are countless, a rigorous definition — as well as the existence of models — of the real numbers are of interest to many mathematicians and in particular logicians.

An eligible basis for the formal construction of number systems is Zermelo-Fraenkel Set Theory (ZF), where numbers can be defined as sets, and the set of those numbers is again an object in this theory, i.e., a set. In the construction of the reals one usually starts with the natural numbers, then builds pairs of natural numbers to obtain integers and thereafter pairs of integers functioning as fractions. Here, Set Theory provides the tools for building pairs of numbers by allowing the definition of sets that are ordered pairs of given sets.

The crucial point in the axiomatization of $\mathbb{R}$ is its completeness, for which Richard Dedekind has proposed the idea of Dedekind cuts and that real numbers can be interpreted as such. There are other possibilities for obtaining completeness, such as viewing real numbers as equivalence classes of Cauchy sequences or nested intervals.

In this thesis we will concentrate on a less known construction which is due to Norbert A'Campo in "A natural construction for the real numbers" [A'C03]. The initial steps of defining the natural numbers and integers as pairs of natural numbers are as usual; what makes this construction special is that rational numbers are not needed to establish the reals. Instead, real numbers are defined as equivalence classes of special functions from $\mathbb{Z}$ to $\mathbb{Z}$ called *slopes*. One can picture such slopes as functions that are "almost" linear. Their name originates from the idea that a function represents the real number given by the limit of its overall slope.

Abraham Robinson points out the interestingness of nonstandard models of $\mathbb{R}$ in "Nonstandard Analysis" [Rob96]. The existence of infinitely big numbers, i.e. numbers greater than all "standard" numbers, implies, by the existence of multiplicative inverses of nonzero numbers, that there are infinitesimal real numbers, numbers smaller than all "standard" positive real numbers that are still greater than zero. Considering specific nonstandard objects, such as numbers, sequences or functions, as "extensions" of corresponding standard objects, many definitions and with them also proofs in analysis can be simplified compared to the well known proofs we are used to.

Outline

The aim of this thesis is to present the complete construction of a model of the real numbers given a model of Zermelo-Fraenkel Set Theory. We will first of all explicate how and under which circumstances models of arbitrary theories can be derived from models of (a certain fragment of) Zermelo-Fraenkel Set Theory. This provides insight into what exactly needs to be proven in ZF in order to being able to deduce the existence of a model of the desired theory (which in our case consists of the axioms of $\mathbb{R}$) from the existence of a model of ZF.

Thereafter we will, following A'Campo's construction as soon as the integers have been defined properly, step by step define the sets of numbers themselves, as well as addition, multiplication and order relations on ω, $\mathbb{Z}$ and $\mathbb{R}$, always proving their existence and, whenever it is needed, functionality. Moreover we will show for each defined number system that a slightly changed version of all according axioms — a version of which its provability in ZF will by then have turned out to be essential for the existence of a model of said number system — is indeed provable in ZF. Since in [A'C03] certain proofs are undetailed or completely left out, A'Campo's work will sometimes be altered or extended with own ideas.

In the last chapter, a nonstandard model of A'Campo's construction of the reals will be investigated by adding infinitely big and, as a result thereof, infinitely small numbers to the standard ones. We will see some interesting properties and characterizations of slopes representing such numbers and how they allow us to visualize certain concepts of analysis, such as convergence or continuity.

Notation

The notation underlying this thesis, as well as the definitions of concepts of logic such as proofs, models, completeness, etc., will be as in Lorenz Halbeisen's "Gödel's Theorems & Zermelo's Axioms" [Hal14]. All proofs will be mathematical, i.e. we will not give formal proofs of formulae out of a theory, but show that the formulae in question hold in every model of the respective theory. Although one should keep in mind that objects in models are not to be identified with their corresponding terms in the formal language, we will not use separate symbols for the respective cases; it should be clear from the context which one is meant. By [Hal14, pp. 52-53], the two kinds of proofs are equivalent and, remarkably, the proof of this equivalence which is given there does not involve the Axiom of Choice.

During the construction of the real numbers within Set Theory, we are going to show many statements to follow from the axioms of Set Theory. When solely writing a formula as the claim of a proposition or the like, we are actually claiming that the formula follows from Set Theory and some additionally introduced definitions.

2 Axioms

In this chapter we list all non-logical axioms that are needed for our construction of a model of the reals. There is a fundamental difference between the role played by the axioms of Set Theory and the role of all other axioms, i.e. those of Peano Arithmetic, the integers, and the real numbers. The axioms of Set Theory (which are formulated in Section 2.1) are the base of the construction, they make up the theory in which all other axioms will be shown. In Section 2.2, we present a method for deriving a model of an arbitrary theory from a model of Set Theory in order to know what "showing that the axioms hold" even means, before stating the axioms of the natural, integer and real numbers in Sections 2.3, 2.4 and 2.5 respectively.

But first, we will shortly recall the definitions of terms, formulae, models etc. out of [Hal14], since they will be of great importance for the rest of this chapter.

Definition 2.1. Let $\mathscr{L}$ be an arbitrary signature. An **$\mathscr{L}$-term** is a string of symbols resulting from applying the following rules *finitely* many times.

(T1) All variables are $\mathscr{L}$-terms.

(T2) All constant symbols out of $\mathscr{L}$ are $\mathscr{L}$-terms.

(T3) If $\tau_1, \dots, \tau_n$ are $\mathscr{L}$-terms and F is an n-ary function symbol out of $\mathscr{L}$, then $F\tau_1 \dots \tau_n$ is an $\mathscr{L}$-term.

The rules (T1)-(T3) are called the **term building rules**.
Similarly, an **$\mathscr{L}$-formula is** is a string of symbols resulting from applying the following rules *finitely* many times.

(F1) If τ_1 and τ_2 are $\mathscr{L}$-terms, then $\tau_1 = \tau_2$ is an $\mathscr{L}$-formula.

(F2) If $\tau_1, \dots, \tau_n$ are $\mathscr{L}$-terms and R is an n-ary relation symbol out of $\mathscr{L}$, then $R\tau_1 \dots \tau_n$ is an $\mathscr{L}$-formula.

(F3) If φ is an $\mathscr{L}$-formula, then $\neg\varphi$ is an $\mathscr{L}$-formula.

(F4) If φ_1 and φ_2 are $\mathscr{L}$-formulae, then $(\varphi_1 \wedge \varphi_2)$, $(\varphi_1 \vee \varphi_2)$ and $(\varphi_1 \rightarrow \varphi_2)$ are $\mathscr{L}$-formulae.

(F5) If φ is an $\mathscr{L}$-formula and ν is a variable, then $\forall\nu\varphi$ and $\exists\nu\varphi$ are $\mathscr{L}$-formulae.

(F1)-(F5) are called **formula building rules**, and formulae built using only (F1) or (F2) are **atomic formulae**.

Remark 2.2. For the definition of $\mathscr{L}$-structures, $\mathscr{L}$-interpretations and models, some concepts of naïve Set Theory such as a naïve membership relation and naïve cartesian products have to be assumed. Notice that these notions have to be distinguished from the symbols of ZFC which will be introduced in Section 2.1; while the latter are really nothing more than symbols, the former are ideas, translations of these symbols into conceptions of the "real world". However we will use the same symbols for both meanings, since otherwise we would have to introduce very complicated symbols and language.

Definition 2.3. An **$\mathscr{L}$-structure** M consists of a naïve set A (the **domain of** M) and a mapping which assigns an element $c^{\mathrm{M}} \in A$ to each constant symbol $c \in \mathscr{L}$, a subset $R^{\mathrm{M}} \subseteq A^n$ to each n-ary relation symbol $R \in \mathscr{L}$, and a (naïve) function F^{M} from A^n to A to each n-ary function symbol $F \in \mathscr{L}$. The elements of A are called **objects**.

An **$\mathscr{L}$-interpretation** I is a pair (M, j), where M is an $\mathscr{L}$-structure and j is an **assignment**, a mapping which assigns to each variable an element of the domain A of M. The assignment $j\frac{a}{\nu}$ for an assignment j, an object $a \in A$ and a variable ν coincides with j, except that to ν it assigns a. Furthermore, $\mathrm{I}\frac{a}{\nu} := (\mathrm{M}, j\frac{a}{\nu})$.

For an interpretation $\mathrm{I} = (\mathrm{M}, j)$ we then define $\mathrm{I}(\nu) := j(\nu)$ for a variable ν, $\mathrm{I}(c) := c^{\mathrm{M}}$ for a constant symbol $c \in \mathscr{L}$ and $\mathrm{I}(F\tau_1 \dots \tau_n) := F^{\mathrm{M}}(\mathrm{I}(\tau_1), \dots, \mathrm{I}(\tau_n))$ for an n-ary function symbol $F \in \mathscr{L}$ and $\mathscr{L}$-terms $\tau_1, \dots, \tau_n$. Like this, $\mathrm{I}(\tau)$ is a well-defined object of A for all $\mathscr{L}$-terms τ. For $\mathscr{L}$-formulae φ, we define $\mathrm{I} \models \varphi$, in words "$\varphi$ **holds in** I" or "φ **is true in** I", inductively as follows:

$$\begin{aligned}
\mathrm{I} \models \tau_1 = \tau_2 &\Longleftrightarrow \mathrm{I}(\tau_1) \text{ is the same object as } \mathrm{I}(\tau_2) \\
\mathrm{I} \models R\tau_1 \dots \tau_n &\Longleftrightarrow \langle \mathrm{I}(\tau_1), \dots, \mathrm{I}(\tau_n) \rangle \text{ belongs to } R^{\mathrm{M}} \\
\mathrm{I} \models \neg\varphi &\Longleftrightarrow \text{not } \mathrm{I} \models \varphi \\
\mathrm{I} \models \varphi_1 \wedge (\vee) \varphi_2 &\Longleftrightarrow \mathrm{I} \models \varphi_1 \text{ and (or) } \mathrm{I} \models \varphi_2 \\
\mathrm{I} \models \varphi_1 \rightarrow \varphi_2 &\Longleftrightarrow \text{if } \mathrm{I} \models \varphi_1 \text{ then } \mathrm{I} \models \varphi_2 \\
\mathrm{I} \models \exists\nu\varphi (\forall\nu\varphi) &\Longleftrightarrow \text{it exists (for all) } a \in A : \mathrm{I}\tfrac{a}{\nu} \models \varphi,
\end{aligned}$$

where $\tau_1, \dots, \tau_n$ are $\mathscr{L}$-terms, $R \in \mathscr{L}$ is an n-ary relation symbol, $\varphi, \varphi_1, \varphi_2$ are $\mathscr{L}$-formulae and ν is a variable. For an $\mathscr{L}$-theory T, a set of $\mathscr{L}$-formulae, an $\mathscr{L}$-structure M is now called a **model of** T, in symbols $\mathrm{M} \models \mathrm{T}$, if for every assignment j and every formula $\varphi \in \mathrm{T}$,

$$(\mathrm{M}, j) \models \varphi$$

holds.

2.1 Axioms of ZF(C)

The following are the axioms belonging to Zermelo-Fraenkel Set Theory (ZF), the base of our construction. In the order in which they appear below, they are called the Axiom of Empty Set (0), Extensionality (1), Pairing (2), Union (3), Infinity (4), Separation (5), Power Set (6), Replacement (7), and Foundation (8); the Axiom of Choice will be stated later.

ZF_0: $\exists x \forall z (z \notin x)$

ZF_1: $\forall x \forall y \forall z ((z \in x \leftrightarrow z \in y) \rightarrow x = y)$

ZF_2: $\forall x \forall y \exists u (u = \{x, y\})$

ZF_3: $\forall x \exists u (u = \bigcup x)$

ZF_4: $\exists x(\emptyset \in x \wedge \forall y \in x(y \cup \{y\} \in x))$

ZF_5: $\forall x \forall p \exists y\,(y = \{z \in x : \varphi(z,p)\})$, for any formula φ in which z, p occur freely with an n-tuple of parameters p.

ZF_6: $\forall x \exists y(y = \mathscr{P}(x))$

ZF_7: $\forall x \forall p\big(\forall z \in x \exists! w(\varphi(z,w,p)) \to \exists y \forall z \in x \exists w \in y(\varphi(z,w,p))\big)$, for any formula φ in which z, w, p occur freely with an n-tuple of parameters p.

ZF_8: $\forall x(x \neq \emptyset \to \exists z \in x(x \cap z = \emptyset))$

Most of these axioms are already stated in abbreviated form; $x \notin y$ is a shorter form of writing $\neg(x \in y)$. The terms $\{x, y\}$ and $\bigcup x$ are making use of the binary function $\{\cdot,\cdot\}$ and the unary function $\bigcup \cdot$ respectively, which first have to be defined in a way such that

$$\forall u(u = \{x, y\} \leftrightarrow \forall z(z \in u \leftrightarrow (z = x \vee z = y))) \text{ and}$$

$$\forall u(u = \bigcup x \leftrightarrow \forall z(z \in u \leftrightarrow \exists w \in x(z \in w))).$$

However, these abbreviations are not causing any trouble.

Remark 2.4. New relation, function and constant symbols can be defined in an $\mathscr{L}$-theory T, stipulating

(i) $Rv_1 \dots v_n :\Longleftrightarrow \varphi_R(v_1, \dots, v_n)$, for a new n-ary relation symbol R and any $\mathscr{L}$-formula φ_R for which $\mathrm{free}(\varphi_R) = \{v_1, \dots, v_n\}$.

(ii) $Fv_1 \dots v_n = y :\Longleftrightarrow \varphi_F(v_1, \dots, v_n, y)$, for a new n-ary function symbol F and any $\mathscr{L}$-formula φ_F for which $\mathrm{free}(\varphi_F) = \{v_1, \dots, v_n, y\}$ and for which $\mathrm{T} \vdash \forall v_1, \dots, v_n \exists! y(\varphi_F(v_1, \dots, v_n))$ holds.

(iii) $c = y :\Longleftrightarrow \varphi_c(y)$, for a new constant symbol c and any $\mathscr{L}$-formula φ_c for which $\mathrm{free}(\varphi_c) = \{y\}$ and for which $\mathrm{T} \vdash \exists! y(\varphi_c(y))$ holds.

It is described precisely in [Hal14, Chapter 7] what sentences need to be added to T (yielding T^*) and how formulae ψ^* containing the new symbols can be transformed into $\mathscr{L}$-formulae ψ such that $\mathrm{T}^* \vdash \psi \leftrightarrow \psi^*$ holds and $\mathrm{T}^* \vdash \psi^*$ implies $\mathrm{T} \vdash \psi$.

We call a formula φ **suitable to define a relation, function** or **constant symbol** if it satisfies the corresponding conditions stated above.

Remark 2.4 implies that the definition of new non-logical symbols by extending the language $\mathscr{L}_{\mathrm{ZF}}$ and the theory ZF does not actually change the consequences that can be drawn from the axioms of ZF; each model M of ZF can easily be turned into a model M^* (with the same domain as M) of ZF^*, the union of ZF with the formulae defining the new symbols [Hal14, Chapter 7]. However, such definitions are a huge benefit for the readability of the formulae, which is why we use abbreviations as in "Combinatorial Set Theory: with a gentle introduction to forcing" by Lorenz Halbeisen [Hal11].

One can easily see that according to Definition 2.1, the axioms $\mathrm{ZF}_0 - \mathrm{ZF}_8$ are $\mathscr{L}^*_{\mathrm{ZF}}$-formulae, where $\mathscr{L}^*_{\mathrm{ZF}}$ contains "$\in$" and all the newly added non-logical symbols. Notice that ZF_5 and ZF_7 are actually axiom schemes, i.e. they yield an axiom for every formula φ satisfying the stated conditions.

There is one remaining axiom belonging to Set Theory, the Axiom of Choice (AC). For being able to state it, the definition of what a function is, is necessary.

Definition 2.5. A **function** f from A to B, in symbols $f : A \to B$, where A and B are sets, is a subset of $A \times B$ such that for every $x \in A$ there is exactly one $y \in B$ such that $\langle x, y\rangle \in f$.
Instead of $\langle x, y\rangle \in f$ we will often write $f(x) = y$, since this is, according to Remark 2.4, a suitable way to define a function.
For $\langle x, y\rangle \in f$ we call y the **image of** x **under** f and x the **preimage of** y **under** f. For $A' \subseteq A$, we set

$$f[A'] := \{y \in B : \exists x \in A'(f(x) = y)\}$$

as the **image of** A' **under** f and the set of all functions $f : A \to B$ will be denoted by ${}^{A}B$.

Remark 2.6. Notice that we have in particular relied on the existence of a set $A \times B$, given two existing sets A and B. Due to ZF_2 it is possible in Set Theory to define the ordered pair $\langle x, y\rangle := \{\{x\}, \{x, y\}\}$ given two sets x and y, where $\langle x_0, y_0\rangle = \langle x_1, y_1\rangle$ if and only if $x_0 = x_1$ and $y_0 = y_1$ [Hal11, p. 47]. The set $A \times B$ can now be defined as

$$A \times B := \{t \in \mathscr{P}(\mathscr{P}(A \cup B)) : \exists x, y(t = \langle x, y\rangle \wedge x \in A \wedge y \in B)\},$$

i.e. as the set of all tuples $\langle x, y\rangle$ with $x \in A$ and $y \in B$. Here we have needed ZF_2 for ordered pairs, ZF_3 for the union of A and B, ZF_6 for the power set of $A \cup B$, and of course the Axiom of Separation, ZF_5.

What we now defined to be a function would usually be called the graph of a function, but there is no problem with identifying the two. The Axiom of Choice now reads as follows.

AC: $\forall \mathscr{F} \left(\emptyset \notin \mathscr{F} \to \exists f \in {}^{\mathscr{F}} \bigcup \mathscr{F} \left(\forall x \in \mathscr{F} \left(f(x) \in x\right)\right)\right)$

For the construction of the reals, only the axioms ZF_0-ZF_6 will be needed. In particular, it is independent of the Axiom of Choice, which is quite appeasing when having in mind the seemingly paradoxical consequences both accepting and denying it brings with it (see [Hal11, Chapter 6] and [Hal11, pp. 331-333]).

2.2 Deriving Models from Models of ZF

In ZF, given the required axioms, we have seen that the existence of functions from and to existing sets, the power set of an existing set, etc., follows. By viewing functions, relations and constants as well as the domains of the respective theories as sets, we want to derive models of other theories from a model of Set Theory. We therefore have to face the following problem.

In any model of Peano Arithmetic for example, the domain consists exclusively of natural numbers. Thus, a formula beginning with a universal quantifier is actually making a statement about all *natural numbers.* But when interpreting this formula in a model of Set Theory, it will suddenly be a statement about all *sets*, of which only some will be defined to be natural numbers. To make it stay an assertion about all natural numbers, one has to slightly adapt the axioms before "proving" them in ZF.

It is the goal of this section to show how the non-logical axioms of any theory have to be changed and why proving their adapted versions in ZF suffices to know that a model of said theory exists, provided a model of ZF.

Definition 2.7. Let T be an $\mathscr{L}$-theory and let $\mathscr{L}_{\mathrm{ZF}}^*$ be the language of Set Theory after having added helpful symbols through definitions as described in Remark 2.4, whereas ZF^* is the $\mathscr{L}_{\mathrm{ZF}}^*$-theory obtained by adding these definitions to ZF. Define

$$\begin{aligned} \mathscr{L}_{\mathrm{ZF_T}} &:= \mathscr{L}_{\mathrm{ZF}}^* \cup \boldsymbol{\mathscr{L}} \cup \{B\} \\ \mathscr{L}_{\mathrm{ZF_T}}^* &:= \mathscr{L}_{\mathrm{ZF}}^* \cup \mathscr{L} \cup \boldsymbol{\mathscr{L}} \cup \{B\}, \end{aligned}$$

where B is a constant symbol not contained in $\mathscr{L}_{\mathrm{ZF}}^*$ or $\mathscr{L}$, and $\boldsymbol{\mathscr{L}}$ contains bold copies of the symbols of $\mathscr{L}$, with the difference that in $\boldsymbol{\mathscr{L}}$ they are all constant symbols.

Definition 2.8. We say that a set Φ of $\mathscr{L}_{\mathrm{ZF_T}}$-formulae defining the newly added constant symbols out of $\boldsymbol{\mathscr{L}} \cup \{B\}$ (see Remark 2.4) is an **insertion of $\mathscr{L}$ into** ZF if for $\mathrm{ZF_T} := \mathrm{ZF}^* \cup \Phi$ we have

(i) $\mathrm{ZF_T} \vdash \boldsymbol{R} \subseteq B^n$, for all constant symbols $\boldsymbol{R} \in \boldsymbol{\mathscr{L}}$, where R is an n-ary relation symbol of $\mathscr{L}$.

(ii) $\mathrm{ZF_T} \vdash \boldsymbol{F}\colon B^n \to B$ is a function, for all constant symbols $\boldsymbol{F} \in \boldsymbol{\mathscr{L}}$, where F is an n-ary function symbol of $\mathscr{L}$.

(iii) $\mathrm{ZF_T} \vdash \boldsymbol{c} \in B$, for all constant symbols $\boldsymbol{c} \in \boldsymbol{\mathscr{L}}$, where c is a constant symbol of $\mathscr{L}$.

Remark 2.9. Notice that if we don't want to assume all of Set Theory, at least the axioms $\mathrm{ZF}_0 - \mathrm{ZF}_3$ and ZF_5 and ZF_6 are needed to obtain an insertion of a theory into ZF. Without ZF_0, it would be impossible to prove the existence of any set and hence, no formula could ever be suitable to define a constant symbol. But the constant symbol B inevitably needs to be defined, which is why we must include ZF_0 in the base axioms. Similarly, the uniqueness of sets, which is also essential for defining constant symbols, cannot be shown without ZF_1. As soon as the language of the theory in consideration includes relation symbols, ZF_2 is required to formulate what set theoretic relations are; and as we have seen earlier, the axioms $\mathrm{ZF}_2, \mathrm{ZF}_3, \mathrm{ZF}_5$ and ZF_6 are necessary to formulate whether or not a set is a set theoretic function. Since $\mathrm{ZF_T}$ must be able to prove that bold versions of relation or function symbols are set theoretic relations or functions respectively, these axioms should be assumed, as well.

Definition 2.10. Consider an $\mathscr{L}$-formula φ. From left to right we replace $\forall\nu$ and $\exists\nu$ by $\forall\nu \in B$ and $\exists\nu \in B$, respectively, for any variable ν, where

$$\forall\nu \in B(\psi) :\equiv \forall\nu(\nu \in B \to \psi) \text{ and}$$
$$\exists\nu \in B(\psi) :\equiv \exists\nu(\nu \in B \land \psi),$$

for any formula ψ. We call the $\mathscr{L}_{\mathrm{ZF_T}}$-formula $\widetilde{\varphi}$ we obtain in this way the **ZF-transformation of** φ.

Proposition 2.11. *For an insertion* Φ *of* $\mathscr{L}$ *into* ZF *consider the set* Ψ *of* $\mathscr{L}_{\mathrm{ZF_T}}$*-formulae consisting of*

(i) $\psi_R(v_1,\dots,v_n) :\equiv \langle v_1,\dots,v_n\rangle \in \boldsymbol{R}$*, if* R *is an* n*-ary relation symbol of* $\mathscr{L}$*.*

(ii) $\psi_F(v_1,\dots,v_n,y) :\equiv \langle v_1,\dots,v_n,y\rangle \in \boldsymbol{F} \lor (\langle v_1,\dots,v_n\rangle \notin B^n \land y = \emptyset)$*, if* F *is an* n*-ary function symbol of* $\mathscr{L}$*.*

(iii) $\psi_c(y) :\equiv \boldsymbol{c} = y$*, if* c *is a constant symbol of* $\mathscr{L}$*.*

A formula ψ *of* Ψ *is suitable for defining a new relation, function or constant symbol, if the symbol in* $\mathscr{L}$ *corresponding to the bold symbol of* $\boldsymbol{\mathscr{L}}$ *occurring in* ψ *is a relation, function or constant symbol, respectively.*

Proof.

Relation symbols: Since $\mathrm{free}(\psi_R) = \{v_1,\dots,v_n\}$, this is clear.

Function symbols: It gets a little more interesting for function symbols. Again, the free variables of ψ_F are as they should be, but additionally we have to show that

$$\mathrm{ZF_T} \vdash \forall v_1,\dots,v_n \exists! y(\psi_F(v_1,\dots,v_n,y)) \tag{1}$$

holds. We know that

$$\mathrm{ZF_T} \vdash \forall v_1,\dots,v_n(\langle v_1,\dots,v_n\rangle \in B^n \to \exists! y(\langle v_1,\dots,v_n,y\rangle \in \boldsymbol{F})) \tag{2}$$
$$\mathrm{ZF_T} \vdash \forall v_1,\dots,v_n(\langle v_1,\dots,v_n\rangle \notin B^n \to \exists! y(\langle v_1,\dots,v_n\rangle \notin B^n \land y = \emptyset))$$
$$\mathrm{ZF_T} \vdash \forall v_1,\dots,v_n(\langle v_1,\dots,v_n\rangle \in B^n \lor \langle v_1,\dots,v_n\rangle \notin B^n),$$

where (2) is a consequence of the fact that Φ is an insertion of $\mathscr{L}$ into ZF and therefore $\boldsymbol{F}$ is a function from B^n to B; the other two are clear. Together, they yield (1), which is what needed to be shown.

Constant symbols: This is easier again. The only free variable in $\boldsymbol{c} = y$ is y, and of course there is also exactly one y that can be equal to $\boldsymbol{c}$, namely $\boldsymbol{c}$.

□

Definition 2.12. For an insertion Φ of $\mathscr{L}$ into ZF consider the set Ψ^* of $\mathscr{L}^*_{\mathrm{ZF_T}}$-formulae defining the symbols out of $\mathscr{L}$ as follows.

(i) $Rv_1 \dots v_n :\Longleftrightarrow \langle v_1, \dots, v_n \rangle \in \boldsymbol{R}$

(ii) $Fv_1 \dots v_n = y :\Longleftrightarrow \langle v_1, \dots, v_n, y \rangle \in \boldsymbol{F} \vee (\langle v_1, \dots, v_n \rangle \notin B^n \wedge y = \emptyset)$

(iii) $c = y :\Longleftrightarrow \boldsymbol{c} = y$

These definitions are allowed due to what we just proved in Proposition 2.11. Let

$$\mathrm{ZF}^*_{\mathrm{T}} := \mathrm{ZF_T} \cup \Psi^* = \mathrm{ZF}^* \cup \Phi \cup \Psi^*.$$

We call an insertion Φ of $\mathscr{L}$ into ZF an **embedding of** T **into** ZF if

$$\mathrm{ZF}^*_{\mathrm{T}} \vdash \widetilde{\varphi}$$

for all ZF-transformations $\widetilde{\varphi}$ of non-logical axioms $\varphi \in \mathrm{T}$.

Theorem 2.13. *Let* T *be an arbitrary $\mathscr{L}$-theory and let Φ be an insertion of $\mathscr{L}$ into* ZF*. If* ZF *has a model and Φ is an embedding of* T *into* ZF*, then* T *has a model.*

Proof. If ZF has a model, $\mathrm{ZF}^*_{\mathrm{T}}$ automatically also has a model [Hal14, Chapter 7]. So let V be a model of $\mathrm{ZF}^*_{\mathrm{T}}$, and let $\mathbb{V}$ be its domain. We write $a \in^{\mathrm{V}} b$ for objects a, b in $\mathbb{V}$ if the tuple $\langle a, b \rangle$ belongs to $\in^{\mathrm{V}}$. Our goal is to construct a model M of T. For this purpose we denote by $\mathbb{B}$ the collection of all objects b in $\mathbb{V}$ for which $b \in^{\mathrm{V}} B^{\mathrm{V}}$ holds. $\mathbb{B}$ is the domain of our model. Now, a mapping needs to be defined in order to create an $\mathscr{L}$-structure.

- $R^{\mathrm{M}} := R^{\mathrm{V}}$, for n-ary relation symbols R of $\mathscr{L}$.
- $F^{\mathrm{M}}(b_1, \dots, b_n) := F^{\mathrm{V}}(b_1, \dots, b_n)$, for n-ary function symbols F of $\mathscr{L}$.
- $c^{\mathrm{M}} := c^{\mathrm{V}}$, for constant symbols c of $\mathscr{L}$.

Claim. This mapping indeed defines an $\mathscr{L}$-structure.

Proof of Claim. We want to show that R^{M} is a set of n-tuples of elements of $\mathbb{B}$, F^{M} is a function from n-tuples of $\mathbb{B}$ to $\mathbb{B}$ and c^{M} is an element of $\mathbb{B}$.

R^{M}: We have

$$\mathrm{V} \models \forall x_1, \dots, x_n \, (Rx_1 \dots x_n \leftrightarrow \langle x_1, \dots, x_n \rangle \in \boldsymbol{R}) \qquad (\Psi^*)$$

$$\mathrm{V} \models \forall x_1, \dots, x_n \, (\langle x_1, \dots, x_n \rangle \in \boldsymbol{R} \to \langle x_1, \dots, x_n \rangle \in B^n) \qquad (\mathrm{ZF_T})$$

$$\mathrm{V} \models \forall x_1, \dots, x_n \, (\langle x_1, \dots, x_n \rangle \in B^n \to (x_1 \in B \wedge \dots \wedge x_n \in B)) . \qquad (\mathrm{ZF}^*)$$

To the right of each formula we have named the subset of formulae of $\mathrm{ZF}^*_{\mathrm{T}}$ responsible for the corresponding statement. Together this implies

$$\mathrm{V} \models \forall x_1, \dots, x_n \, (Rx_1 \dots x_n \to (x_1 \in B \wedge \dots \wedge x_n \in B)) . \qquad (3)$$

Now let j be an arbitrary assignment which assigns objects of $\mathbb{V}$ to variables and consider $\mathrm{V}_j = (\mathrm{V}, j)$, the corresponding $\mathscr{L}_{\mathrm{ZF_T}}$-interpretation. (3) implies that

$$\mathrm{V}_j \models \forall x_1, \ldots, x_n \left(Rx_1 \ldots x_n \to (x_1 \in B \wedge \cdots \wedge x_n \in B) \right),$$

which in turn implies that for all objects $a_1, \ldots, a_n$ of $\mathbb{V}$, the following holds.

$$\begin{aligned} & \mathrm{V}_j^{\frac{a_1}{x_1}, \ldots, \frac{a_n}{x_n}} \models Rx_1 \ldots x_n \to (x_1 \in B \wedge \cdots \wedge x_n \in B) \\ \Longleftrightarrow\ & \text{if } \mathrm{V}_j^{\frac{a_1}{x_1}, \ldots, \frac{a_n}{x_n}} \models Rx_1 \ldots x_n \\ & \text{then } \mathrm{V}_j^{\frac{a_1}{x_1}, \ldots, \frac{a_n}{x_n}} \models (x_1 \in B \wedge \cdots \wedge x_n \in B) \\ \Longleftrightarrow\ & \text{if } \langle a_1, \ldots, a_n \rangle \text{ is in } R^{\mathrm{V}} \text{ then } a_1 \in^{\mathrm{V}} B^{\mathrm{V}} \text{ and } \ldots \text{ and } a_n \in^{\mathrm{V}} B^{\mathrm{V}}. \end{aligned}$$

Hence, we know that if a tuple of objects $\langle a_1, \ldots, a_n \rangle$ of $\mathbb{V}$ belongs to R^{V}, then the objects $a_1, \ldots, a_n$ all belong to $\mathbb{B}$. Therefore, $\langle a_1, \ldots, a_n \rangle$ is also a tuple of objects of $\mathbb{B}$. The fact that this holds for all objects $a_1, \ldots, a_n$ of $\mathbb{V}$ now makes $R^{\mathrm{M}} := R^{\mathrm{V}}$ a valid definition for R^{M}.

F^{M}: For function symbols, the proof is similar.

$$\mathrm{V} \models \forall x_1, \ldots, x_n, y \Big(y = F(x_1, \ldots, x_n) \leftrightarrow \big(\langle x_1, \ldots, x_n, y \rangle \in \boldsymbol{F} \vee (\langle x_1, \ldots, x_n \rangle \notin B^n \wedge y = \emptyset) \big) \Big) \qquad (\Psi^*)$$

$$\mathrm{V} \models \forall x_1, \ldots, x_n, y \left((x_1 \in B \wedge \cdots \wedge x_n \in B) \to \langle x_1, \ldots, x_n \rangle \in B^n \right) \qquad (\mathrm{ZF}^*)$$

$$\mathrm{V} \models \forall x_1, \ldots, x_n, y \left(\langle x_1, \ldots, x_n, y \rangle \in \boldsymbol{F} \to y \in B \right) \qquad (\mathrm{ZF_T})$$

From this it follows that

$$\mathrm{V} \models \forall x_1, \ldots, x_n, y((x_1 \in B \wedge \cdots \wedge x_n \in B) \wedge y = F(x_1, \ldots, x_n) \to y \in B). \qquad (4)$$

Consider the $\mathscr{L}_{\mathrm{ZF_T}}$-interpretation V_j as above for an arbitrary assignment j. From (4) we now know that for all objects $a_1, \ldots, a_n, b$ of $\mathbb{V}$

$$\begin{aligned} & \mathrm{V}_j^{\frac{a_1}{x_1}, \ldots, \frac{a_n}{x_n}, \frac{b}{y}} \models (x_1 \in B \wedge \cdots \wedge x_n \in B) \wedge y = F(x_1, \ldots, x_n) \\ & \qquad \to y \in B \\ \Longleftrightarrow\ & \text{if } \mathrm{V}_j^{\frac{a_1}{x_1}, \ldots, \frac{a_n}{x_n}, \frac{b}{y}} \models (x_1 \in B \wedge \cdots \wedge x_n \in B) \wedge y = F(x_1, \ldots, x_n) \\ & \text{then } \mathrm{V}_j^{\frac{a_1}{x_1}, \ldots, \frac{a_n}{x_n}, \frac{b}{y}} \models y \in B \\ \Longleftrightarrow\ & \text{if } a_1 \in^{\mathrm{V}} B^{\mathrm{V}} \text{ and } \ldots \text{ and } a_n \in^{\mathrm{V}} B^{\mathrm{V}} \\ & \qquad \text{and } b \text{ is the same object as } F^{\mathrm{V}}(a_1, \ldots, a_n) \\ & \text{then } b \in^{\mathrm{V}} B^{\mathrm{V}}. \end{aligned}$$

Thus, for arbitrary objects $a_1, \ldots, a_n$ of $\mathbb{B}$, the object $F^{\mathrm{V}}(a_1, \ldots, a_n)$ is also an object of $\mathbb{B}$. This shows that $F^{\mathrm{M}}(b_1, \ldots, b_n) := F^{\mathrm{V}}(b_1, \ldots, b_n)$ defines a function from $\mathbb{B}^n$ to $\mathbb{B}$.

c: What remains for finishing the proof of our claim is to show that c^{V} is an object of $\mathbb{B}$. V being a model of $\mathrm{ZF}^*_{\mathrm{T}}$ implies the following.

$$\mathrm{V} \models \forall y (y = c \leftrightarrow y = \boldsymbol{c}) \qquad (\Psi^*)$$
$$\mathrm{V} \models \forall y (\boldsymbol{c} \in B) \qquad (\mathrm{ZF_T})$$

Thus,

$$\mathrm{V} \models \forall y (y = c \to y \in B). \tag{5}$$

Now for any assignment j and any object b of V, the interpretation V_j satisfies

$$\begin{aligned} & \mathrm{V}_j^{\frac{b}{y}} \models y = c \to y \in B \\ \Longleftrightarrow\ & \text{if } \mathrm{V}_j^{\frac{b}{y}} \models y = c \text{ then } \mathrm{V}_j^{\frac{b}{y}} \models y \in B \\ \Longleftrightarrow\ & \text{if } b \text{ is the same object as } c^{\mathrm{V}} \text{ then } b \in^{\mathrm{V}} B^{\mathrm{V}}. \end{aligned}$$

We can deduce that c^{V} is indeed an object of $\mathbb{B}$, completing the proof of the claim.

□

M, the domain $\mathbb{B}$ together with the mapping stated above is therefore an $\mathscr{L}$-structure. We want to show that this structure is a model of T.

For this purpose, let φ_0 be an arbitrary formula contained in T. We need to prove that for any assignment j, the $\mathscr{L}$-interpretation $\mathrm{M}_j = (\mathrm{M}, j)$ satisfies

$$\mathrm{M}_j \models \varphi_0.$$

So if j is an assignment which assigns objects of $\mathbb{B}$ to variables, we denote by V_j the $\mathscr{L}^*_{\mathrm{ZF_T}}$-interpretation (V, j) corresponding to the same assignment j which assigns objects b of $\mathbb{V}$ satisfying $b \in^{\mathrm{V}} B^{\mathrm{V}}$ to variables. Let us call these assignments which assign objects of $\mathbb{B}$ to variables "$\mathbb{B}$-assignments". Now we can deduce the following.

$$\mathrm{M}_j(\tau) \text{ is the same object as } \mathrm{V}_j(\tau) \tag{6}$$

for every $\mathscr{L}$-term τ and every $\mathbb{B}$-assignment j. This can be verified by running through the term building rules quickly.

(T1) $\mathrm{M}_j(\nu) = j(\nu) = \mathrm{V}_j(\nu)$ for all variables ν.

(T2) $\mathrm{M}_j(c) = c^{\mathrm{M}} = c^{\mathrm{V}} = \mathrm{V}_j(c)$ for all constant symbols c of $\mathscr{L}$.

(T3) $\mathrm{M}_j(F\tau_1 \ldots \tau_n) = F^{\mathrm{M}}(\mathrm{M}_j(\tau_1), \ldots, \mathrm{M}_j(\tau_n)) = F^{\mathrm{V}}(\mathrm{V}_j(\tau_1), \ldots, \mathrm{V}_j(\tau_n)) = \mathrm{V}_j(F\tau_1 \ldots \tau_n)$ for all functions symbols F of $\mathscr{L}$ and $\mathscr{L}$-terms $\tau_1, \ldots, \tau_n$.

We know that $\mathrm{ZF}^*_\mathrm{T} \vdash \widetilde{\varphi_0}$ since Φ is, by assumption, an embedding of T into ZF.

Claim. $\mathrm{V}_j \models \widetilde{\varphi_0}$ if and only if $\mathrm{M}_j \models \varphi_0$, for all $\mathbb{B}$-assignments j.

If the claim is true, it follows that $\mathrm{M}_j \models \varphi_0$ since $\mathrm{V} \models \mathrm{ZF}^*_\mathrm{T}$ implies $\mathrm{V}_j \models \widetilde{\varphi_0}$ for all possible $\mathbb{B}$-assignments j.

Proof of Claim. We prove this for arbitrary $\mathscr{L}$-formulae ψ by running through the formula building rules. The statement then follows for $\varphi_0 \in \mathrm{T}$ immediately.

Atomic formulae, i.e. formulae built by using (F1) or (F2), don't contain quantifiers, hence if ψ is an atomic formula, ψ and $\widetilde{\psi}$ are equal. We will therefore show $\mathrm{V}_j \models \psi \iff \mathrm{M}_j \models \psi$ for atomic $\mathscr{L}$-formulae.

(F1) For $\mathscr{L}$-terms τ_1 and τ_2 and a $\mathbb{B}$-assignment j we have

$$\begin{aligned} & \mathrm{V}_j \models \tau_1 = \tau_2 \\ \iff & \mathrm{V}_j(\tau_1) \text{ is the same object as } \mathrm{V}_j(\tau_2) \\ \overset{(6)}{\iff} & \mathrm{M}_j(\tau_1) \text{ is the same object as } \mathrm{M}_j(\tau_2) \\ \iff & \mathrm{M}_j \models \tau_1 = \tau_2. \end{aligned}$$

(F2) For an n-ary relation symbol R out of $\mathscr{L}$, $\mathscr{L}$-terms $\tau_1, \ldots, \tau_n$ and a $\mathbb{B}$-assignment j we have

$$\begin{aligned} & \mathrm{V}_j \models R\tau_1 \ldots \tau_n \\ \iff & \langle \mathrm{V}_j(\tau_1), \ldots, \mathrm{V}_j(\tau_n) \rangle \in R^\mathrm{V} \\ \iff & \langle \mathrm{V}_j(\tau_1), \ldots, \mathrm{V}_j(\tau_n) \rangle \in R^\mathrm{M} \\ \overset{(6)}{\iff} & \langle \mathrm{M}_j(\tau_1), \ldots, \mathrm{M}_j(\tau_n) \rangle \in R^\mathrm{M} \\ \iff & \mathrm{M}_j \models R\tau_1 \ldots \tau_n. \end{aligned}$$

(F3)&(F4) By the definitions of $\mathrm{I} \models \neg\psi$, $\mathrm{I} \models \psi_1 \wedge \psi_2$, $\mathrm{I} \models \psi_1 \vee \psi_2$ and $\mathrm{I} \models \psi_1 \to \psi_2$ for interpretations I and $\mathscr{L}$-formulae ψ, ψ_1, ψ_2, it is clear that if

$$\mathrm{V}_j \models \widetilde{\psi} \iff \mathrm{M}_j \models \psi$$

holds and the analogous is true for ψ_1 and ψ_2, then also

$$\begin{aligned} \mathrm{V}_j \models \neg\widetilde{\psi} &\iff \mathrm{M}_j \models \neg\psi \\ \mathrm{V}_j \models \widetilde{\psi}_1 \wedge \widetilde{\psi}_2 &\iff \mathrm{M}_j \models \psi_1 \wedge \psi_2 \\ \mathrm{V}_j \models \widetilde{\psi}_1 \vee \widetilde{\psi}_2 &\iff \mathrm{M}_j \models \psi_1 \vee \psi_2 \\ \mathrm{V}_j \models \widetilde{\psi}_1 \to \widetilde{\psi}_2 &\iff \mathrm{M}_j \models \psi_1 \to \psi_2 \end{aligned}$$

must be true.

(F5) Now we arrive at the crucial point, (F5) is the formula building rule concerning quantifiers. What we want to show is that if an $\mathscr{L}$-formula ψ satisfies

$$\mathrm{V}_j \models \widetilde{\psi} \Longleftrightarrow \mathrm{M}_j \models \psi$$

for any $\mathbb{B}$-assignment j, then

$$\mathrm{V}_j \models \underbrace{\forall\nu(\nu \in B \to \widetilde{\psi})}_{\equiv\widetilde{\forall\nu(\psi)}} \Longleftrightarrow \mathrm{M}_j \models \forall\nu(\psi) \text{ and}$$
$$\mathrm{V}_j \models \underbrace{\exists\nu(\nu \in B \wedge \widetilde{\psi})}_{\equiv\widetilde{\exists\nu(\psi)}} \Longleftrightarrow \mathrm{M}_j \models \exists\nu(\psi)$$

follows (for every $\mathbb{B}$-assignment j and every variable ν). So for the universal quantifier consider

$$\begin{aligned}
&\mathrm{V}_j \models \forall\nu(\nu \in B \to \widetilde{\psi})\\
\Longleftrightarrow\ & \text{for all objects } a \text{ of } \mathbb{V},\ \mathrm{V}_j^{\frac{a}{\nu}} \models \nu \in B \to \widetilde{\psi}\\
\Longleftrightarrow\ & \text{for all objects } a \text{ of } \mathbb{V}, \text{ if } \mathrm{V}_j^{\frac{a}{\nu}} \models \nu \in B \text{ then } \mathrm{V}_j^{\frac{a}{\nu}} \models \widetilde{\psi}\\
\Longleftrightarrow\ & \text{for all objects } a \text{ of } \mathbb{V}, \text{ if } a \text{ belongs to } \mathbb{B}, \text{ then } \mathrm{V}_j^{\frac{a}{\nu}} \models \widetilde{\psi}\\
\Longleftrightarrow\ & \text{for all objects } a \text{ of } \mathbb{B},\ \mathrm{V}_j^{\frac{a}{\nu}} \models \widetilde{\psi}\\
\overset{\text{ass.}}{\Longleftrightarrow}\ & \text{for all objects } a \text{ of } \mathbb{B},\ \mathrm{M}_j^{\frac{a}{\nu}} \models \psi\\
\Longleftrightarrow\ & \mathrm{M}_j \models \forall\nu(\psi).
\end{aligned}$$

We proceed similarly for the existential quantifier.

$$\begin{aligned}
&\mathrm{V}_j \models \exists\nu(\nu \in B \wedge \widetilde{\psi})\\
\Longleftrightarrow\ & \text{there is an object } a \text{ of } \mathbb{V} \text{ such that } \mathrm{V}_j^{\frac{a}{\nu}} \models \nu \in B \wedge \widetilde{\psi}\\
\Longleftrightarrow\ & \text{there is an object } a \text{ of } \mathbb{V} \text{ such that } \mathrm{V}_j^{\frac{a}{\nu}} \models \nu \in B \text{ and } \mathrm{V}_j^{\frac{a}{\nu}} \models \widetilde{\psi}\\
\Longleftrightarrow\ & \text{there is an object } a \text{ of } \mathbb{V} \text{ such that } a \text{ belongs to } \mathbb{B} \text{ and } \mathrm{V}_j^{\frac{a}{\nu}} \models \widetilde{\psi}\\
\Longleftrightarrow\ & \text{there is an object } a \text{ of } \mathbb{B} \text{ such that } \mathrm{V}_j^{\frac{a}{\nu}} \models \widetilde{\psi}\\
\overset{\text{ass.}}{\Longleftrightarrow}\ & \text{there is an object } a \text{ of } \mathbb{B} \text{ such that } \mathrm{M}_j^{\frac{a}{\nu}} \models \psi\\
\Longleftrightarrow\ & \mathrm{M}_j \models \exists\nu(\psi).
\end{aligned}$$

This proves our claim and simultaneously, the theorem. □

□

This theorem is the foundation of the rest of this thesis. It assures us of the fact that the following procedure is enough to prove the existence of a model of the $\mathscr{L}$-theory T. The theory from which we start is ZF^*, the extension of ZF containing the definitions of useful symbols, together with its language $\mathscr{L}^*_{\mathrm{ZF}}$. Notice that instead of with ZF^*, one could also start with a subset of ZF^* containing at least the axioms $\mathrm{ZF}_0 - \mathrm{ZF}_3$ and ZF_5 and ZF_6. For convenience however, we mostly won't distinguish these cases and work with ZF^* as our base theory.

Algorithm 2.14.

1. Add a new constant symbol B to $\mathscr{L}^*_{\mathrm{ZF}}$ and define it according to Remark 2.4.
2. Add new constant symbols which are bold copies of the symbols of $\mathscr{L}$ to the resulting language and also define them according to Remark 2.4. The now resulting language is called $\mathscr{L}_{\mathrm{ZF_T}}$, the corresponding theory is $\mathrm{ZF_T}$.
3. Show that
   - $\mathrm{ZF_T} \vdash \boldsymbol{R} \subseteq B^n$, for n-ary relation symbols R of $\mathscr{L}$.
   - $\mathrm{ZF_T} \vdash \boldsymbol{F} : B^n \to B$ is a function, for n-ary function symbols F of $\mathscr{L}$.
   - $\mathrm{ZF_T} \vdash \boldsymbol{c} \in B$, for constant symbols c of $\mathscr{L}$.
4. Add the symbols of $\mathscr{L}$ to $\mathscr{L}_{\mathrm{ZF_T}}$ and call the resulting language $\mathscr{L}^*_{\mathrm{ZF_T}}$.
5. Define the new symbols (according to Remark 2.4) as follows.
   - $Rv_1 \dots v_n :\Longleftrightarrow \langle v_1, \dots, v_n \rangle \in \boldsymbol{R}$, for relation symbols R of $\mathscr{L}$.
   - $Fv_1 \dots v_n = y :\Longleftrightarrow \langle v_1, \dots, v_n, y \rangle \in \boldsymbol{F} \vee (\langle v_1, \dots, v_n \rangle \notin B^n \wedge y = \emptyset)$, for function symbols F of $\mathscr{L}$.
   - $c = y :\Longleftrightarrow \boldsymbol{c} = y$, for constant symbols c of $\mathscr{L}$.
6. Show that $\mathrm{ZF}^*_{\mathrm{T}} \vdash \widetilde{\varphi}$ for all ZF-transformations $\widetilde{\varphi}$ of non-logical axioms of T.

Now that we know what needs to be shown to obtain models of arbitrary theories from a model of Set Theory, we concentrate on those theories we are interested in for constructing a model of the real numbers. The rest of this chapter consists of three lists of non-logical axioms, that will each be examined in the following two chapters by applying the above algorithm.

2.3 Peano Axioms

To begin with our construction, we will show that a model of the natural numbers can be derived from any model of ZF, meaning that we want to show with the axioms of ZF the existence of a set on which an element called 0, addition, multiplication and a successor function can be defined such that all Peano Axioms hold. There are seven of them, two to determine the usage of the successor function, two for addition and two for multiplication. The last one, which is rather an axiom scheme than a single axiom, is called the Axiom Scheme of Induction and basically states that if $\varphi(x)$ holds for $x = 0$, and that if it holds for any natural number x then it also holds for the successor of x, then it must hold for all natural numbers.

PA_1: $\neg\exists x(\mathrm{s}(x) = 0)$

PA_2: $\forall x \forall y(\mathrm{s}(x) = \mathrm{s}(y) \to x = y)$

PA_3: $\forall x(x + 0 = x)$

PA_4: $\forall x \forall y(x + \mathrm{s}(y) = \mathrm{s}(x + y))$

PA_5: $\forall x(x \cdot 0 = 0)$

PA_6: $\forall x \forall y(x \cdot \mathrm{s}(y) = (x \cdot y) + x)$

PA_7: $\big(\varphi(0) \wedge \forall x\big(\varphi(x) \to \varphi(\mathrm{s}(x))\big)\big) \to \forall x(\varphi(x))$ for any formula φ in which x occurs freely.

2.4 Axioms of $\mathbb{Z}$

Compared to the axioms of ZF(C) and PA, the axiomatization of the integers is more descriptive. It basically states that $\mathbb{Z}$ is an ordered ring with neutral elements 0 and 1.

- $(\mathbb{Z}, +)$ is an abelian group:

  IA_1: $\forall x, y, z((x + y) + z = x + (y + z))$
  IA_2: $\forall x(x + 0 = 0 + x = x)$
  IA_3: $\forall x, y(x + y = y + x)$
  IA_4: $\forall x \exists y(x + y = 0)$

- $(\mathbb{Z}, \cdot)$ is an abelian monoid:

  IM_1: $\forall x, y, z((x \cdot y) \cdot z = x \cdot (y \cdot z))$
  IM_2: $\forall x(x \cdot 1 = 1 \cdot x = x)$
  IM_3: $\forall x, y(x \cdot y = y \cdot x)$

- The distributive property holds:

  ID_1: $\forall x, y, z(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$

- $<$ is a total order:

  IO_1: $\forall x \neg(x < x)$
  IO_2: $\forall x, y, z(x < y \wedge y < z \to x < z)$
  IO_3: $\forall x, y(x < y \vee y < x \vee x = y)$

- Addition and multiplication are monotone:

  IO_4: $\forall x, y, z(x < y \to x + z < y + z)$
  IO_5: $\forall x, y(x < y \wedge 0 < z \to x \cdot z < y \cdot z)$

The integer numbers will be defined as ordered pairs of natural numbers, hence the fact that all these axioms hold in our setting will have to be shown using corresponding properties of the natural numbers. Although it may not be obvious at first sight, commutativity, associativity and distributivity are all consequences of the Peano Axioms.

2.5 Axioms of $\mathbb{R}$

The real numbers are characterized through being a Dedekind complete, totally ordered field, i.e. the underlying axioms are the following:

- $(\mathbb{R}, +)$ is an abelian group:

  RA_1: $\forall x, y, z((x + y) + z = x + (y + z))$

  RA_2: $\forall x(x + 0 = 0 + x = x)$

  RA_3: $\forall x, y(x + y = y + x)$

  RA_4: $\forall x \exists y(x + y = 0)$

- $(\mathbb{R}\backslash\{0\}, \cdot)$ is an abelian group:

  RM_1: $\forall x, y, z((x \cdot y) \cdot z = x \cdot (y \cdot z))$

  RM_2: $\forall x(x \cdot 1 = 1 \cdot x = x)$

  RM_3: $\forall x, y(x \cdot y = y \cdot x)$

  RM_4: $\forall x \neq 0 \exists y(x \cdot y = 1)$

- The distributive property holds:

  RD_1: $\forall x, y, z(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$

- $<$ is a total order:

  RO_1: $\forall x \neg(x < x)$

  RO_2: $\forall x, y, z(x < y \wedge y < z \rightarrow x < z)$

  RO_3: $\forall x, y(x < y \vee y < x \vee x = y)$

- Addition and multiplication are monotone:

  RO_4: $\forall x, y, z(x < y \rightarrow x + z < y + z)$

  RO_5: $\forall x, y, z(x < y \wedge 0 < z \rightarrow x \cdot z < y \cdot z)$

- $\mathbb{R}$ is Dedekind complete:

  RC_1: Every nonempty subset of $\mathbb{R}$ having an upper bound also has a least upper bound.

Although we will have to prove the ZF-transformations of these axioms in order to derive models of the natural, integer and real numbers from a model of Set Theory as described in Theorem 2.13, we do not state the transformations here; the axioms are transformed very easily.

Notice that RC_1 is an axiom that already uses the concept of subsets, i.e. it must be formulated in the language of Set Theory. So in some sense, it already is ZF-transformed, even though it is not the transformation of another formula. It is indeed the case that the axiomatization of the real numbers requires either a theory of higher order or it must be embedded into ZF, which is what we are going to do.

3 Construction of Number Systems

In this chapter we will, starting with the definition of the set of natural numbers, step by step add new symbols to our language for every function, relation or constant appearing in the axioms stated in Chapter 2 by adding formulae to the theory as described in Theorem 2.13. Following Algorithm 2.14, it is necessary to show for each newly introduced bold symbol that it describes a relation, function or constant in the set theoretic sense, upon which the corresponding non-bold relation, function and constant symbols can be defined. Proceeding chronologically, the truth of the desired ZF-transformed axioms concerning that number system will then be shown, before afterwards defining the next number system with its corresponding functions, relations and constants. During this process, we will try to avoid the usage of as many axioms of Set Theory as possible.

Until being able to define the set of real numbers within ZF, there is a lot of work that needs to be done, i.e. not only axioms have to be shown, but all relations, functions and constants need to be defined and their welldefinedness has to be proven. Furthermore, proving certain axioms of the real numbers needs quite some preparation and is generally a lot more extensive than proving the axioms of the natural or integer numbers. For this reason, the verification of the axioms of the real numbers will be treated in the separate Chapter 4.

3.1 Natural Numbers (ω)

In [Hal11, pp. 50-51], the set of natural numbers, ω, is defined to be the smallest nonempty limit ordinal. It is shown there, as well, that the following are some of its properties:

(i) $\emptyset \in \omega$

(ii) ω is inductive, that is, if $x \in \omega$, then $x \cup \{x\} \in \omega$.

(iii) ω does not have a proper subset which contains $\emptyset$ and is inductive.

We adopt this definition of ω, where ω is serving as the constant symbol B from Algorithm 2.14; however we will define addition and multiplication differently to avoid using the Axiom of Replacement, ZF_7. So far we have needed the axioms ZF_0-ZF_5 [Hal11, pp. 44-51].

To obtain a model of PA, we start by defining the constant symbol $\mathbf{0}^{\mathbb{N}}$ in such a way that $\mathbf{0}^{\mathbb{N}} \in \omega$ follows. The superscript "$^{\mathbb{N}}$" has been added to the symbols since the languages of both the integers and the reals also contain the symbol "0". Similarly we will later add "$^{\mathbb{Z}}$" and "$^{\mathbb{R}}$" to the symbols when we are working with the integers and reals, respectively.

Definition 3.1.

$$\mathbf{0}^{\mathbb{N}} := \emptyset.$$

$\mathbf{0}^{\mathbb{N}} \in \omega$ is nothing else than the statement of property (i) of ω and thus, we are allowed to define $0^{\mathbb{N}}$ as follows.

Definition 3.2.

$$0^{\mathbb{N}} := \mathbf{0}^{\mathbb{N}} = \emptyset.$$

We continue with the definition of the successor function. The aim is to define a set for which it is provable from ZF that it is a function from ω to ω, and, as already becomes clear in the definition of ω, we interpret $x \cup \{x\}$ as the successor of x.

Definition 3.3.

$$\mathbf{s}^{\mathbb{N}} := \{\langle x, y\rangle \in \omega \times \omega : \ y = x \cup \{x\}\}.$$

Remark 3.4. Notice that we have in particular made use of ZF_6 now by relying on the existence of $\omega \times \omega$.

Proposition 3.5.

$$\forall x \in \omega \exists! y \in \omega(\langle x, y\rangle \in \mathbf{s}^{\mathbb{N}}),$$

i.e. $\mathbf{s}^{\mathbb{N}} : \omega \to \omega$ *is a function.*

Proof.

∃: As we already know from property (ii) of ω, for any $x \in \omega$, the set $x \cup \{x\}$ is again contained in ω. Therefore there exists a tuple $\langle x, x \cup \{x\}\rangle \in \mathbf{s}^{\mathbb{N}}$.

!: Let $\langle x, y\rangle$ and $\langle x, y'\rangle$ be two elements of $\mathbf{s}^{\mathbb{N}}$. Then, by the definition of $\mathbf{s}^{\mathbb{N}}$ we know that $y = x \cup \{x\}$ and $y' = x \cup \{x\}$. By extensionality, y and y' must be identical.

□

This justifies the following definition.

Definition 3.6.

$$\mathrm{s}^{\mathbb{N}}(x) = y :\Longleftrightarrow \langle x, y\rangle \in \mathbf{s}^{\mathbb{N}}.$$

Proposition 3.7. $\widetilde{\mathrm{PA}_1}$, $\widetilde{\mathrm{PA}_2}$ *and* $\widetilde{\mathrm{PA}_7}$ *hold, that is,*

(i) $\neg\exists x \in \omega(\mathrm{s}^{\mathbb{N}}(x) = 0^{\mathbb{N}})$

(ii) $\forall x, y \in \omega(\mathrm{s}^{\mathbb{N}}(x) = \mathrm{s}^{\mathbb{N}}(y) \to x = y)$

(iii) $\big(\varphi(0^{\mathbb{N}}) \wedge \forall x \in \omega\big(\varphi(x) \to \varphi(\mathrm{s}^{\mathbb{N}}(x))\big)\big) \to \forall x \in \omega(\varphi(x))$ *for any formula* φ *in which* x *occurs freely.*

Proof.

(i) Suppose there exists $x \in \omega$ such that $\mathrm{s}^{\mathbb{N}}(x) = 0^{\mathbb{N}}$. Then, $\emptyset = 0^{\mathbb{N}} = \mathrm{s}^{\mathbb{N}}(x) = x \cup \{x\}$. But since $x \in x \cup \{x\}$ and $x \notin \emptyset$, this is a contradiction to the Axiom of Extensionality and therefore no such x can exist.

(ii) Let $x \neq y$ be such that $\mathrm{s}^{\mathbb{N}}(x) = \mathrm{s}^{\mathbb{N}}(y)$. This means that $x \cup \{x\} = y \cup \{y\}$ and hence we know that $x \in x \cup \{x\} = y \cup \{y\}$. If $x \neq y$, then x must be an element of y. Analogously it follows that $y \in x$. Since ω is an ordinal, x and y are ordinals as well by [Hal11, Theorem 3.12 (b)], and thus by [Hal11, Theorem 3.12 (a)], $x \in y$ and $y \in x$ exclude each other, yielding a contradiction. Therefore, $\mathrm{s}^{\mathbb{N}}(x) = \mathrm{s}^{\mathbb{N}}(y)$ implies that $x = y$.

(iii) Suppose φ is a formula such that x occurs freely in φ and

$$\varphi(0^{\mathbb{N}}) \wedge \forall x \in \omega(\varphi(x) \rightarrow \varphi(\mathrm{s}^{\mathbb{N}}(x)) \tag{7}$$

holds. Then, assuming

$$\exists x \in \omega(\neg\varphi(x)) \tag{8}$$

to be true, we can define $\omega' := \{x \in \omega : \varphi(x)\}$. Now we have $0^{\mathbb{N}} \in \omega'$ and $\forall x(x \in \omega' \rightarrow \mathrm{s}^{\mathbb{N}}(x) \in \omega')$ by (7) and $\omega' \subsetneq \omega$ by (8).
We have found an inductive set ω' containing $0^{\mathbb{N}} = \emptyset$ which is strictly smaller than ω, yielding a contradiction to property (iii) of ω.

□

From now on, the symbol $\mathrm{s}^{\mathbb{N}}$ will be replaced by s and we write 0 instead of $0^{\mathbb{N}}$. Still, our construction is independent of ZF_7, the Axiom of Replacement, ZF_8, the Axiom of Foundation, and AC, the Axiom of Choice.

Corollary 3.8.

$$\forall x \in \omega(x = 0 \vee \exists x' \in \omega(x = \mathrm{s}(x'))).$$

Proof. This is a simple consequence of PA_7. If we set $x = 0$, obviously $0 = 0 \vee \exists x' \in \omega(0 = \mathrm{s}(x'))$ is true. And if we know that $x = 0 \vee \exists x' \in \omega(x = \mathrm{s}(x'))$, then of course $\mathrm{s}(x)$ is the successor of x, which implies that $\mathrm{s}(x) = 0 \vee \exists x' \in \omega(\mathrm{s}(x) = \mathrm{s}(x'))$ is true as well. □

We can also define addition on ω, which we will do in a way making PA_3 and PA_4 automatically true and, as already mentioned, not using ZF_7 in contrast to [Hal11, p. 56].

Definition 3.9.

$$+^{\mathbb{N}} := \bigcup\{f \in \mathscr{P}((\omega \times \omega) \times \omega) : \varphi_{+^{\mathbb{N}}}(f)\}, \text{ where}$$

$$\varphi_{+^{\mathbb{N}}}(f) :\equiv \forall x \in \omega[\langle\langle x, 0\rangle, x\rangle \in f] \tag{9}$$

$$\wedge \forall x \forall y \forall z[\langle\langle x, y\rangle, z\rangle \in f \wedge \exists y'(y = \mathrm{s}(y')) \rightarrow \exists y' \exists z'(y = \mathrm{s}(y') \wedge z = \mathrm{s}(z') \wedge \langle\langle x, y'\rangle, z'\rangle \in f)] \tag{10}$$

$$\wedge \forall x \forall z[\langle\langle x, 0\rangle, z\rangle \in f \rightarrow z = x]. \tag{11}$$

$\varphi_{+^{\mathbb{N}}}(f)$ can be divided into three parts. (9) ensures that PA_3 holds, (10) is responsible for PA_4, and as we will see just below, (11) is needed to show that $+^{\mathbb{N}}$ is a function.

Proposition 3.10.

$$\forall x, y \in \omega \exists! z \in \omega\left(\langle\langle x, y\rangle, z\rangle \in +^{\mathbb{N}}\right).$$

Proof.

$\exists$: We are proving this using the Axiom of Induction for the formula

$$\varphi_{\exists}(y) :\equiv \exists f \in \mathscr{P}((\omega \times \omega) \times \omega)\forall x \in \omega \exists z \in \omega(\varphi_{+^{\mathbb{N}}}(f) \wedge \langle\langle x, y\rangle, z\rangle \in f).$$

First, we consider the case $y = 0$. Let $f_0 = \{\langle\langle x_0, 0\rangle, x_0\rangle : x_0 \in \omega\}$. Then, $\langle\langle x, 0\rangle, x\rangle \in f_0$ for all $x \in \omega$ and, since PA_1 holds, part (10) of $\varphi_{+^{\mathbb{N}}}(f_0)$ is also true. The set f_0 only has elements of the form $\langle\langle x, 0\rangle, x\rangle$, which assures the uniqueness (11) required for $\varphi_{+^{\mathbb{N}}}(f_0)$ to be entirely true. Therefore, $\varphi_{+^{\mathbb{N}}}(f_0)$ holds and so does $\varphi_{\exists}(0)$.

Now we assume $\varphi_{\exists}(y)$ to be true. We therefore have f such that $\varphi_{+^{\mathbb{N}}}(f)$ holds and $\forall x \in \omega \exists z(\langle\langle x, y\rangle, z\rangle \in f)$. Define

$$f' := f \cup \{\langle\langle x, \mathrm{s}(y)\rangle, \mathrm{s}(z)\rangle : \langle\langle x, y\rangle, z\rangle \in f\}.$$

Again, $\varphi_{+^{\mathbb{N}}}(f')$ holds: We have $f \subseteq f'$ and $\forall x \in \omega\,(\langle\langle x, 0\rangle, x\rangle \in f)$, so we also have $\forall x \in \omega(\langle\langle x, 0\rangle, x\rangle \in f')$. The elements of $f' \backslash f$ are of the form $\langle\langle x, \mathrm{s}(y)\rangle, \mathrm{s}(z)\rangle$, where $\langle\langle x, y\rangle, z\rangle$ already is in f and therefore also in f', making part (10) of $\varphi_{+^{\mathbb{N}}}(f')$ true for all the "new" elements of f'. Together with $\varphi_{+^{\mathbb{N}}}(f)$ we also know it is true for the "old" elements, and therefore it holds for the entire set f'. The fact that we only added elements of the form $\langle\langle x, \mathrm{s}(y)\rangle, \mathrm{s}(z)\rangle$ to f makes sure that there has not been added anything of the form $\langle\langle x, 0\rangle, z\rangle$, since 0 is not the successor of any natural number. We therefore know that the last part, (11), of $\varphi_{+^{\mathbb{N}}}(f')$ holds as well.

Hence, $\forall x \in \omega \exists z \in \omega(\langle\langle x, \mathrm{s}(y)\rangle, z\rangle \in f'$ is true. This means that $\varphi_{\exists}(\mathrm{s}(y))$ is true as well, implying that $\varphi_{\exists}(y)$ holds for every $y \in \omega$ by PA_7.

We now know that for every pair $\langle x, y\rangle \in \omega \times \omega$ there are f and z such that $\varphi_{+^{\mathbb{N}}}(f)$ holds and $\langle\langle x, y\rangle, z\rangle \in f$, which implies that for every pair $\langle x, y\rangle \in \omega \times \omega$ there is a z for which $\langle\langle x, y\rangle, z\rangle \in +^{\mathbb{N}}$.

!: Suppose there are $x, y, z_0, z_1 \in \omega$ such that both $\langle\langle x, y\rangle, z_0\rangle \in +^{\mathbb{N}}$ and $\langle\langle x, y\rangle, z_1\rangle \in +^{\mathbb{N}}$. We can prove that $z_0 = z_1$ using the Axiom of Induction, PA_7, applied to the formula

$$\varphi_{!}(y) :\equiv \forall x, z_0, z_1 \in \omega((\langle\langle x, y\rangle, z_0\rangle \in +^{\mathbb{N}} \wedge \langle\langle x, y\rangle, z_1\rangle \in +^{\mathbb{N}}) \rightarrow z_0 = z_1).$$

$\langle\langle x, y\rangle, z_0\rangle \in +^{\mathbb{N}}$ and $\langle\langle x, y\rangle, z_1\rangle \in +^{\mathbb{N}}$ means there exist f_0 and f_1 with $\varphi_{+^{\mathbb{N}}}(f_0)$ and $\varphi_{+^{\mathbb{N}}}(f_1)$ such that $\langle\langle x, y\rangle, z_0\rangle \in f_0$ and $\langle\langle x, y\rangle, z_1\rangle \in f_1$.

Let $y = 0$. Part (9) of $\varphi_{+^{\mathbb{N}}}(f_0)$ implies $\langle\langle x, 0\rangle, x\rangle \in f_0$ and together with $\langle\langle x, 0\rangle, z_0\rangle \in f_0$, we know by (11) that $x = z_0$. The same argument works for z_1, yielding $z_0 = 0 = z_1$. We have shown that $\varphi_{!}(0)$ holds.

Now, assuming $\varphi_{!}(y)$ holds, we show that $\varphi_{!}(\mathrm{s}(y))$ holds as well. For this purpose, suppose that $\langle\langle x, \mathrm{s}(y)\rangle, z_0\rangle \in f_0$ and $\langle\langle x, \mathrm{s}(y)\rangle, z_1\rangle \in f_1$. Part (10) of $\varphi_{+^{\mathbb{N}}}(f_0)$ implies the existence of a z_0' such that $z_0 = \mathrm{s}(z_0')$ and $\langle\langle x, y\rangle, z_0'\rangle \in f_0$. Therefore we have $\langle\langle x, y\rangle, z_0'\rangle \in +^{\mathbb{N}}$. In the same way we also get a z_1' such that $z_1 = \mathrm{s}(z_1')$ and

$\langle\langle x, y\rangle, z_1'\rangle \in +^{\mathbb{N}}$. By our assumption, z_0' and z_1' must be identical, meaning that $\mathrm{s}(z_0') = \mathrm{s}(z_1')$, i.e. $z_0 = z_1$.

This completes our proof.

□

Definition 3.11.

$$x +^{\mathbb{N}} y = z :\Longleftrightarrow \langle\langle x, y\rangle, z\rangle \in +^{\mathbb{N}}.$$

Proposition 3.12. $\widetilde{\mathrm{PA}_3}$ *and* $\widetilde{\mathrm{PA}_4}$ *hold:*

(i) $\forall x \in \omega(x +^{\mathbb{N}} 0 = x)$

(ii) $\forall x, y \in \omega(x +^{\mathbb{N}} \mathrm{s}(y) = \mathrm{s}(x +^{\mathbb{N}} y))$.

Proof.

(i) f_0, as defined in the proof of Proposition 3.10, contains $\langle\langle x, 0\rangle, x\rangle$ for all $x \in \omega$. This already proves that $\forall x \in \omega\ (x +^{\mathbb{N}} 0 = x)$.

(ii) $x +^{\mathbb{N}} \mathrm{s}(y) = z$ implies the existence of $f \ni \langle\langle x, \mathrm{s}(y)\rangle, z\rangle$ such that $\varphi_{+^{\mathbb{N}}}(f)$ holds. This means there exists z' such that $z = \mathrm{s}(z')$ and $\langle\langle x, y\rangle, z'\rangle \in f$, and as a consequence, $x +^{\mathbb{N}} y = z'$. So now we have $\mathrm{s}(x +^{\mathbb{N}} y) = \mathrm{s}(z') = z = x +^{\mathbb{N}} \mathrm{s}(y)$ for all x and y in ω.

□

The following corollary shows further properties of the addition on ω; they will later be needed during the construction of the integers.

Corollary 3.13.

(i) $\forall x \in \omega(0 +^{\mathbb{N}} x = x)$

(ii) $\forall x, y, z \in \omega((x +^{\mathbb{N}} y) +^{\mathbb{N}} z = x +^{\mathbb{N}} (y +^{\mathbb{N}} z))$

(iii) $\forall x \in \omega(\mathrm{s}(0) +^{\mathbb{N}} x = \mathrm{s}(x) = x +^{\mathbb{N}} \mathrm{s}(0))$

(iv) $\forall x, y \in \omega(x +^{\mathbb{N}} y = y +^{\mathbb{N}} x)$

(v) $\forall x, y \in \omega(\mathrm{s}(x) +^{\mathbb{N}} y = \mathrm{s}(x +^{\mathbb{N}} y))$

(vi) $\forall x, y, z \in \omega(x +^{\mathbb{N}} y = x +^{\mathbb{N}} z \to y = z)$

Proof. All of these statements, except (v), will be proven by induction.

(i) Let $x = 0$. Then,

$$0 +^{\mathbb{N}} 0 \stackrel{\mathrm{PA}_3}{=} 0.$$

Assuming $0 +^{\mathbb{N}} x = x$, we have

$$0 +^{\mathbb{N}} \mathrm{s}(x) \stackrel{\mathrm{PA}_4}{=} \mathrm{s}(0 +^{\mathbb{N}} x) \stackrel{\text{ass.}}{=} \mathrm{s}(x).$$

(ii) We will prove this by induction on z. Let $z = 0$. Then,

$$(x +^{\mathbb{N}} y) +^{\mathbb{N}} 0 \overset{\text{PA}_3}{=} x +^{\mathbb{N}} y \overset{\text{PA}_3}{=} x +^{\mathbb{N}} (y +^{\mathbb{N}} 0).$$

Assuming $(x +^{\mathbb{N}} y) +^{\mathbb{N}} z = x +^{\mathbb{N}} (y +^{\mathbb{N}} z)$, we have

$$\begin{aligned}(x +^{\mathbb{N}} y) +^{\mathbb{N}} \mathrm{s}(z) &\overset{\text{PA}_4}{=} \mathrm{s}((x +^{\mathbb{N}} y) +^{\mathbb{N}} z) \overset{\text{ass.}}{=} \mathrm{s}(x +^{\mathbb{N}} (y +^{\mathbb{N}} z)) \\ &\overset{\text{PA}_4}{=} x +^{\mathbb{N}} \mathrm{s}(y +^{\mathbb{N}} z) \overset{\text{PA}_4}{=} x +^{\mathbb{N}} (y +^{\mathbb{N}} \mathrm{s}(z)).\end{aligned}$$

(iii) This is proven by induction on x. Let $x = 0$. Then,

$$\mathrm{s}(0) +^{\mathbb{N}} 0 \overset{\text{PA}_3}{=} \mathrm{s}(0) \overset{(i)}{=} 0 +^{\mathbb{N}} \mathrm{s}(0).$$

Assuming $\mathrm{s}(0) +^{\mathbb{N}} x = \mathrm{s}(x) = x +^{\mathbb{N}} \mathrm{s}(0)$, we have

$$\begin{aligned}\mathrm{s}(0) +^{\mathbb{N}} \mathrm{s}(x) &\overset{\text{PA}_4}{=} \mathrm{s}(\mathrm{s}(0) +^{\mathbb{N}} x) \overset{\text{ass.}}{=} \mathrm{s}(\mathrm{s}(x)) \\ &\overset{\text{PA}_3}{=} \mathrm{s}(\mathrm{s}(x) +^{\mathbb{N}} 0) \overset{\text{PA}_4}{=} \mathrm{s}(x) +^{\mathbb{N}} \mathrm{s}(0).\end{aligned}$$

(iv) Let $x = 0$. Then,

$$0 +^{\mathbb{N}} y \overset{(i)}{=} y \overset{\text{PA}_3}{=} y +^{\mathbb{N}} 0.$$

Assuming $x +^{\mathbb{N}} y = y +^{\mathbb{N}} x$, we have

$$\begin{aligned}\mathrm{s}(x) +^{\mathbb{N}} y &\overset{(iii)}{=} (x +^{\mathbb{N}} \mathrm{s}(0)) +^{\mathbb{N}} y \overset{(ii)}{=} x +^{\mathbb{N}} (\mathrm{s}(0) +^{\mathbb{N}} y) \\ &\overset{(iii)}{=} x +^{\mathbb{N}} (y +^{\mathbb{N}} \mathrm{s}(0)) \overset{(ii)}{=} (x +^{\mathbb{N}} y) +^{\mathbb{N}} \mathrm{s}(0) \\ &\overset{\text{ass.}}{=} (y +^{\mathbb{N}} x) +^{\mathbb{N}} \mathrm{s}(0) \overset{(ii)}{=} y +^{\mathbb{N}} (x +^{\mathbb{N}} \mathrm{s}(0)) \\ &\overset{(iii)}{=} y +^{\mathbb{N}} \mathrm{s}(x).\end{aligned}$$

(v) Here, we don't need induction.

$$\begin{aligned}\mathrm{s}(x) +^{\mathbb{N}} y &\overset{(iv)}{=} y +^{\mathbb{N}} \mathrm{s}(x) \overset{\text{PA}_4}{=} \mathrm{s}(y +^{\mathbb{N}} x) \\ &\overset{(iv)}{=} \mathrm{s}(x +^{\mathbb{N}} y)\end{aligned}$$

proves it already.

(vi) Again, we use induction on x. Let $x = 0$. Then,

$$0 +^{\mathbb{N}} y = 0 +^{\mathbb{N}} z \overset{(i)}{\Longrightarrow} y = z.$$

Assuming $x +^{\mathbb{N}} y = x +^{\mathbb{N}} z \Longrightarrow y = z$, we have

$$\begin{aligned} \mathrm{s}(x) +^{\mathbb{N}} y = \mathrm{s}(x) +^{\mathbb{N}} z &\overset{(v)}{\Longrightarrow} \mathrm{s}(x +^{\mathbb{N}} y) = \mathrm{s}(x +^{\mathbb{N}} z) \\ &\overset{\mathrm{PA}_2}{\Longrightarrow} x +^{\mathbb{N}} y = x +^{\mathbb{N}} z \overset{\text{ass.}}{\Longrightarrow} y = z. \end{aligned}$$

□

(ii) legitimates omitting the brackets in terms of the form $(x + y) + z$ or, what we have now shown to be the same, $x + (y + z)$. From now on we will also simply write $+$ instead of $+^{\mathbb{N}}$.

We continue with the definition of multiplication on ω, which is very similar to the definition of addition.

Definition 3.14.

$$\begin{aligned} \cdot^{\mathbb{N}} :=& \bigcup\{f \in \mathscr{P}((\omega \times \omega) \times \omega) : \varphi_{\cdot^{\mathbb{N}}}(f)\}, \text{ where} \\ \varphi_{\cdot^{\mathbb{N}}}(f) :\equiv& \forall x \in \omega[\langle\langle x, 0\rangle, 0\rangle \in f] & (12) \\ &\wedge \forall x \forall y \forall z[\langle\langle x, y\rangle, z\rangle \in f \wedge \exists y'(y = \mathrm{s}(y')) \\ &\quad \to \exists y' \exists z'(y = \mathrm{s}(y') \wedge z = z' + x \wedge \langle\langle x, y'\rangle, z'\rangle \in f)] & (13) \\ &\wedge \forall x \forall z[\langle\langle x, 0\rangle, z\rangle \in f \to z = 0] & (14) \end{aligned}$$

Proposition 3.15.

$$\forall x, y \in \omega \exists! z \in \omega(\langle\langle x, y\rangle, z\rangle \in \cdot^{\mathbb{N}}).$$

Proof.

$\exists$: To show the existence of such a z we will prove by induction that

$$\varphi_{\exists}(y) :\equiv \exists f \in \mathscr{P}((\omega \times \omega) \times \omega) \forall x \in \omega \exists z \in \omega(\varphi_{\cdot^{\mathbb{N}}}(f) \wedge \langle\langle x, y\rangle, z\rangle \in f)$$

holds for all y.

Let first $y = 0$. We define $f_0 := \{\langle\langle x, 0\rangle, 0\rangle : x \in \omega\}$. By definition, $\langle\langle x, 0\rangle, 0\rangle \in f_0$ for all $x \in \omega$, and like in the proof of Proposition 3.10, the second part of $\varphi_{\cdot^{\mathbb{N}}}$, (13), is automatically true since 0 is not the successor of any element of ω. The uniqueness part (14) of $\varphi_{\cdot^{\mathbb{N}}}$ again is a direct consequence of the definition of f_0. So, $\varphi_{\exists}(0)$ is true.

Now assume $\varphi_{\exists}(y)$ holds. We then have f such that $\varphi_{\cdot^{\mathbb{N}}}(f)$ holds, as well as $\forall x \in \omega \exists z \in \omega(\langle\langle x, y\rangle, z\rangle \in f)$. Define

$$f' := f \cup \{\langle\langle x, \mathrm{s}(y)\rangle, z + x\rangle : \langle\langle x, y\rangle, z\rangle \in f\}.$$

Since $\langle\langle x, 0\rangle, 0\rangle \in f$ for all $x \in \omega$, we know that $\langle\langle x, 0\rangle, 0\rangle$ is also in f' for all $x \in \omega$. The elements of $f' \setminus f$ are of the form $\langle\langle x, \mathrm{s}(y)\rangle, z + x\rangle$, where $\langle\langle x, y\rangle, z\rangle$ is in f. Thus the "new" elements of f' fulfill part (13) of $\varphi_{\cdot^{\mathbb{N}}}(f')$ and the "old" ones already fulfill it because $\varphi_{\cdot^{\mathbb{N}}}(f)$ holds. Since we didn't add any elements of the form $\langle\langle x, 0\rangle, z\rangle$ to f when constructing f' because by PA_1, 0 is no successor, we know that part (14) of $\varphi_{\cdot^{\mathbb{N}}}(f')$ still holds, too.

We have found f' such that $\forall x \in \omega \exists z \in \omega(\langle\langle x, \mathrm{s}(y)\rangle, z\rangle \in f')$ and $\varphi_{\cdot^{\mathbb{N}}}(f')$, meaning that $\varphi_{\exists}(\mathrm{s}(y))$ holds. So now, $\varphi_{\exists}(y)$ is true for all $y \in \omega$ and we therefore have that for every pair $\langle x, y\rangle \in \omega \times \omega$ there is f such that $(\varphi_{\cdot^{\mathbb{N}}}(f) \wedge \langle\langle x, y\rangle, z\rangle \in f)$, implying the existence of a z for each pair $\langle x, y\rangle \in \omega \times \omega$ such that $\langle\langle x, y\rangle, z\rangle \in \cdot^{\mathbb{N}}$.

!: Suppose that there are $x, y, z_0, z_1 \in \omega$ such that $\langle\langle x, y\rangle, z_0\rangle \in \cdot^{\mathbb{N}}$ and $\langle\langle x, y\rangle, z_1\rangle \in \cdot^{\mathbb{N}}$. We want to show that $z_0 = z_1$, which we will do using induction applied to the formula

$$\varphi_!(y) :\equiv \forall x, z_0, z_1 \in \omega(\langle\langle x, y\rangle, z_0\rangle \in \cdot^{\mathbb{N}} \wedge \langle\langle x, y\rangle, z_1\rangle \in \cdot^{\mathbb{N}} \rightarrow z_0 = z_1).$$

That $\langle\langle x, y\rangle, z_0\rangle \in \cdot^{\mathbb{N}}$ and $\langle\langle x, y\rangle, z_1\rangle \in \cdot^{\mathbb{N}}$ hold means there are f_0 and f_1 which fulfill $\varphi_{\cdot^{\mathbb{N}}}(f_0)$, $\varphi_{\cdot^{\mathbb{N}}}(f_1)$, $\langle\langle x, y\rangle, z_0\rangle \in f_0$ and $\langle\langle x, y\rangle, z_1\rangle \in f_1$.

Let $y = 0$. Then, $\langle\langle x, 0\rangle, 0\rangle \in f_0$, since $\varphi_{\cdot^{\mathbb{N}}}(f_0)$ holds and $\langle\langle x, 0\rangle, z_0\rangle \in f_0$. The uniqueness part of $\varphi_{\cdot^{\mathbb{N}}}(f_0)$ implies that $z_0 = 0$ and using the same argument for z_1, we get that $z_0 = 0 = z_1$, proving that $\varphi_!(0)$ holds.

Now suppose that $\varphi_!(y)$ holds and let once again $\langle\langle x, \mathrm{s}(y)\rangle, z_0\rangle \in f_0$ and $\langle\langle x, \mathrm{s}(y)\rangle, z_1\rangle \in f_1$. Then, $\varphi_{\cdot^{\mathbb{N}}}(f_0)$ implies that there is z_0' such that $z_0 = z_0' + x$ and $\langle\langle x, y\rangle, z_0'\rangle \in f_0$. On the other hand we get z_1' in the same way which satisfies $z_1 = z_1' + x$ and $\langle\langle x, y\rangle, z_1'\rangle \in f_1$. The truth of $\varphi_!(y)$ now tells us that $z_0' = z_1'$, which implies that $z_0 = z_0' + x = z_1' + x = z_1$.

□

Definition 3.16.

$$x \cdot^{\mathbb{N}} y = z :\Longleftrightarrow \langle\langle x, y\rangle, z\rangle \in \cdot^{\mathbb{N}}.$$

Proposition 3.17.

(i) $\forall x \in \omega(x \cdot^{\mathbb{N}} 0 = 0)$

(ii) $\forall x, y \in \omega(x \cdot^{\mathbb{N}} \mathrm{s}(y) = (x \cdot^{\mathbb{N}} y) + x)$,

meaning that $\widetilde{\mathrm{PA}_5}$ *and* $\widetilde{\mathrm{PA}_6}$ *hold.*

Proof.

(i) f_0, as defined in the proof of Proposition 3.15, contains $\langle\langle x, 0\rangle, 0\rangle$ for all $x \in \omega$. This is already enough to know that $x \cdot^{\mathbb{N}} 0 = 0$ for all $x \in \omega$.

(ii) $x \cdot^{\mathbb{N}} \mathrm{s}(y) = z$ implies that there is $f \ni \langle\langle x, \mathrm{s}(y)\rangle, z\rangle$ such that $\varphi_{\cdot^{\mathbb{N}}}(f)$ is true. Since $\mathrm{s}(y)$ is a successor, there must be z' such that $z = z' + x$ and $\langle\langle x, y\rangle, z'\rangle \in f$. Thus, $x \cdot^{\mathbb{N}} y = z'$ and therefore $x \cdot^{\mathbb{N}} \mathrm{s}(y) = z = z' + x = (x \cdot^{\mathbb{N}} y) + x$ for all $x, y \in \omega$. □

Again, we are interested in some more properties of the multiplication on ω, which will turn out to be useful in the construction of the integers.

Corollary 3.18.

(i) $\forall x \in \omega(0 \cdot^{\mathbb{N}} x = 0)$

(ii) $\forall x \in \omega(x \cdot^{\mathbb{N}} \mathrm{s}(0) = x = \mathrm{s}(0) \cdot^{\mathbb{N}} x)$

(iii) $\forall x, y, z \in \omega((x + y) \cdot^{\mathbb{N}} z) = (x \cdot^{\mathbb{N}} z) + (y \cdot^{\mathbb{N}} z))$

(iv) $\forall x, y \in \omega(x \cdot^{\mathbb{N}} y = y \cdot^{\mathbb{N}} x)$

(v) $\forall x, y, z \in \omega(x \cdot^{\mathbb{N}} (y + z) = (x \cdot^{\mathbb{N}} y) + (x \cdot^{\mathbb{N}} z))$

(vi) $\forall x, y, z \in \omega((x \cdot^{\mathbb{N}} y) \cdot^{\mathbb{N}} z = x \cdot^{\mathbb{N}} (y \cdot^{\mathbb{N}} z))$

Proof. Like in the proof of Corollary 3.13, all of these statements can be proven directly or by induction, where the order in which they should be proven is — due to dependency between each other — the one in which they are stated above. □

Again for readability's sake, we replace $\cdot^{\mathbb{N}}$ by $\cdot$.

We have now defined the natural numbers ω, with an element $0^{\mathbb{N}}$, a successor function, addition and multiplication, and have shown that in this setting, all Peano Axioms hold. In other words, we have shown that the formulae defining these symbols form an embedding of PA into ZF and therefore, the existence of a model of PA follows from the existence of a model of $\mathrm{ZF}_0 - \mathrm{ZF}_6$ by Theorem 2.13. Also we have seen some important properties like commutativity of addition and multiplication and distributivity. To define an order relation on the integers needed for the axioms IO_1-IO_4, it will be useful to have a relation defining an order on ω, which is the reason for the next definition.

Definition 3.19.

$$\begin{aligned} \leq^{\mathbb{N}} &:= \{\langle x, y\rangle \in \omega \times \omega : \exists z \in \omega(x + z = y)\} \\ x \leq^{\mathbb{N}} y &:\Longleftrightarrow \langle x, y\rangle \in \leq^{\mathbb{N}} \\ <^{\mathbb{N}} &:= \{\langle x, y\rangle \in \omega \times \omega : \langle x, y\rangle \in \leq^{\mathbb{N}} \wedge x \neq y\} \\ x <^{\mathbb{N}} y &:\Longleftrightarrow \langle x, y\rangle \in <^{\mathbb{N}} . \end{aligned}$$

Remark 3.20. Obviously, $\leq^{\mathbb{N}}$ and $<^{\mathbb{N}}$ both are subsets of $\omega \times \omega$, justifying the definitions of $\leq^{\mathbb{N}}$ and $<^{\mathbb{N}}$.

Proposition 3.21.

(i) $\forall x \in \omega(x \leq^{\mathbb{N}} x)$

(ii) $\forall x, y \in \omega(x \leq^{\mathbb{N}} y \wedge y \leq^{\mathbb{N}} x \rightarrow x = y)$

(iii) $\forall x, y, z \in \omega(x \leq^{\mathbb{N}} y \wedge y \leq^{\mathbb{N}} z \rightarrow x \leq^{\mathbb{N}} z)$

(iv) $\forall x, y \in \omega(x \leq^{\mathbb{N}} y \vee y \leq^{\mathbb{N}} x)$

Proof.

(i) Since $x + 0 = x$, we have $x \leq^{\mathbb{N}} x$.

(ii) If $x \leq^{\mathbb{N}} y$ and $y \leq^{\mathbb{N}} x$, then there are z_0, z_1 such that $x + z_0 = y$ and $y + z_1 = x$. Therefore we have that $x + z_0 + z_1 = x(+0)$, which yields that $z_0 + z_1 = 0$ by Corollary 3.13 (vi). Now both z_0 and z_1 have to be 0, since otherwise, one would be a successor, making their sum a successor, which is a contradiction to PA_1. So now, $x + 0 = y$ and therefore $x = y$.

(iii) If there are z_0 and z_1 such that $x + z_0 = y$ and $y + z_1 = z$, then $x + z_0 + z_1 = z$. This already implies $x \leq^{\mathbb{N}} z$.

(iv) This can be proven by induction on x. First, consider the case $x = 0$. Then, $x + y = y$ and so $x \leq^{\mathbb{N}} y$. We now assume $x \leq^{\mathbb{N}} y \vee y \leq^{\mathbb{N}} x$ and want to prove the statement for $\mathrm{s}(x)$ as well. We discuss the two possibilities separately.

$x \leq^{\mathbb{N}} y$: We have $z \in \omega$ such that $x + z = y$ and again distinguish two cases:

$z = 0$: $x + 0 = y$ implies $x = y$, which then implies that $y + \mathrm{s}(0) = \mathrm{s}(x)$. It follows that $y \leq^{\mathbb{N}} \mathrm{s}(x)$.

$z \neq 0$: In this case, z is the successor of some $z' \in \omega$. Hence $x + \mathrm{s}(z') = y$, which yields $\mathrm{s}(x) + z' = y$, meaning that $\mathrm{s}(x) \leq^{\mathbb{N}} y$.

$y \leq^{\mathbb{N}} x$: This is the easier case. There exists a natural number z such that $y + z = x$. Thus, $y + \mathrm{s}(z) = \mathrm{s}(x)$ and therefore $y \leq^{\mathbb{N}} \mathrm{s}(x)$.

□

Proposition 3.22.

(i) $\forall x \in \omega(\neg(x <^{\mathbb{N}} x))$

(ii) $\forall x, y, z \in \omega(x <^{\mathbb{N}} y \wedge y <^{\mathbb{N}} z \rightarrow x <^{\mathbb{N}} z)$

(iii) $\forall x, y \in \omega(x <^{\mathbb{N}} y \vee y <^{\mathbb{N}} x \vee x = y)$

Proof.

(i) Since $x = x$, $\langle x, x \rangle \notin <^{\mathbb{N}}$ by definition.

(ii) $x <^{\mathbb{N}} y \wedge y <^{\mathbb{N}} z$ implies that there are z_0 and z_1 such that $x + z_0 = y$ and $y + z_1 = z$, hence $x + z_0 + z_1 = z$. Furthermore, $x \neq y$ and $y \neq z$. Now it follows that $z_0 \neq 0 \neq z_1$ and consequently $z = x + z_0 + z_1 \neq x$. Thus, $x \leq^{\mathbb{N}} z$ and $x \neq z$, yielding $x <^{\mathbb{N}} z$.

(iii) In Proposition 3.21 (iv) we have already proven that $x \leq^{\mathbb{N}} y$ or $y \leq^{\mathbb{N}} x$ must hold. Let's now assume neither $x <^{\mathbb{N}} y$ nor $y <^{\mathbb{N}} x$ are true. If $x \leq^{\mathbb{N}} y$ but $x \not<^{\mathbb{N}} y$, x and y must be identical. The same follows in the case where $y \leq^{\mathbb{N}} x$ but $y \not<^{\mathbb{N}} x$, which proves the proposition.

□

From now on we will write $\leq$ and $<$ instead of $\leq^{\mathbb{N}}$ and $<^{\mathbb{N}}$.

The following is an interesting property of the newly defined order relation. It nicely points out the connection between our definition of ω and the membership relation of Set Theory.

Lemma 3.23.

$$\forall x, y \in \omega(x < y \leftrightarrow x \in y).$$

Proof. For the first implication, "$\rightarrow$", we use induction on z for the formula

$$\forall x, y \in \omega(x + z = y \rightarrow x \in y),$$

but we start with the case $z = \mathrm{s}(0)$ instead of $z = 0$, since $x < y \iff \exists z \neq 0(x + z = y)$. So let $z = \mathrm{s}(0)$. Then, $x + z = x + \mathrm{s}(0) = \mathrm{s}(x)$. By the definition of s, we have indeed that $x \in \mathrm{s}(x)$.
Now assume that $\forall x, y \in \omega(x + z = y \rightarrow x \in y)$ is given. In particular if for arbitrary $x, y' \in \omega$ we have $x + z = y'$, then $x \in y'$. So if now $x + \mathrm{s}(z) = y$, this implies that

$$y = \mathrm{s}(x + z) = \mathrm{s}(y') = y' \cup \{y'\} \supseteq y' \overset{\text{ass.}}{\ni} x.$$

For the other direction, "$\leftarrow$", we use induction on y for the formula

$$\forall x \in \omega(x \in y \rightarrow x < y).$$

Let $y = 0 = \emptyset$. Then of course, $x \notin y$ for every x and so $x \in 0 \rightarrow x < 0$ holds for every $x \in \omega$. So assume $x \in y \rightarrow x < y$ holds for every $x \in \omega$. Now, $x \in \mathrm{s}(y)$ means $x \in y \cup \{y\}$ which implies

$$x = y \vee x \in y.$$

If $x = y$, then $x + \mathrm{s}(0) = \mathrm{s}(y)$, hence $x < \mathrm{s}(y)$. If $x \in y$, by our assumption we have $x < y$. Surely $y + \mathrm{s}(0) = \mathrm{s}(y)$, implying $y < \mathrm{s}(y)$ and by the transitivity of $<$, we know that $y < \mathrm{s}(y)$. □

Corollary 3.24.

(i) $\forall x_0, x_1, y_0, y_1 \in \omega(x_0 < y_0 \wedge x_1 \leq y_1 \rightarrow x_0 + x_1 < y_0 + y_1)$

(ii) $\forall x, y, z \in \omega(x + z < y + z \rightarrow x < y)$

(iii) $\forall x, y, z \in \omega(x < z \wedge 0^{\mathbb{N}} < y \rightarrow x \cdot y < z \cdot y)$

Proof.

(i) If $x_0 < y_0$ and $x_1 \leq y_1$, this implies that there are z_0 and z_1 such that $x_0 + z_0 = y_0$ and $x_1 + z_1 = y_1$, where $z_0 \neq 0$. Therefore, $x_0 + x_1 + z_0 + z_1 = y_0 + y_1$, meaning that $x_0 + x_1 \leq y_0 + y_1$. Additionally we know that $z_0 + z_1 \neq 0$, hence $x_0 + x_1 < y_0 + y_1$.

(ii) If $x + z < y + z$ it follows that there is $w \neq 0$ such that $x + z + w = y + z$. Therefore, by Corollary 3.13 (vi), $x + w = y$, immediately implying $x < y$.

(iii) That $x < z$ holds, implies the existence of $w \neq 0$ such that $x + w = z$ and since $w \neq 0$, we know there is w' such that $w = \mathrm{s}(w')$. Hence, $x + \mathrm{s}(w') = z$ and furthermore, since $y \neq 0$, there exists y' for which $\mathrm{s}(y') = y$. Together, this yields

$$\begin{aligned} z \cdot y &= (x + \mathrm{s}(w')) \cdot \mathrm{s}(y') \\ &= (x \cdot \mathrm{s}(y')) + (\mathrm{s}(w') \cdot \mathrm{s}(y')) \\ &= (x \cdot y) + (\mathrm{s}(w') \cdot y') + \mathrm{s}(w'). \end{aligned}$$

Clearly, $(\mathrm{s}(w') \cdot y') + \mathrm{s}(w')$ is different from 0 and thus, $x \cdot y < z \cdot y$ follows.

□

3.2 Integers ($\mathbb{Z}$)

Now we are done defining the natural numbers. Our next step is the construction of the integers, which will later lead us to the construction of the reals. We will view integers as representatives of equivalence classes of ordered pairs of natural numbers, where the pair $\langle x, y\rangle$ stands for the integer "$x - y$". For this, it is necessary to have an equivalence relation on $\omega \times \omega$ such that the pairs $\langle x_0, x_1\rangle$ and $\langle y_0, y_1\rangle$ are equivalent iff "$x_0 - x_1 = y_0 - y_1$", i.e., in terms of natural numbers and their as yet defined functions, $x_0 + y_1 = y_0 + x_1$.

Definition 3.25.

$$\begin{aligned} \sim^{\mathbb{Z}} :&= \{\langle \langle x_0, x_1\rangle, \langle y_0, y_1\rangle\rangle \in (\omega \times \omega) \times (\omega \times \omega) : \; x_0 + y_1 = y_0 + x_1\} \\ x \sim^{\mathbb{Z}} y &:\Longleftrightarrow \langle x, y\rangle \in \sim^{\mathbb{Z}} \end{aligned}$$

Proposition 3.26. $\sim^{\mathbb{Z}}$ *is an equivalence relation.*

Proof. Let $x_0, x_1, y_0, y_1, z_0, z_1$ be arbitrary elements of ω. We have to show reflexivity, symmetry and transitivity of $\sim^{\mathbb{Z}}$.

reflexivity: We know that $\langle \langle x_0, x_1\rangle, \langle x_0, x_1\rangle\rangle \in \sim^{\mathbb{Z}}$, since obviously $x_0 + x_1 = x_0 + x_1$.

symmetry: If $\langle \langle x_0, x_1\rangle, \langle y_0, y_1\rangle\rangle \in \sim^{\mathbb{Z}}$, this means that $x_0 + y_1 = y_0 + x_1$. Then of course, $y_0 + x_1 = x_0 + y_1$, which ensures that $\langle \langle y_0, y_1\rangle, \langle x_0, x_1\rangle\rangle \in \sim^{\mathbb{Z}}$ as well.

transitivity: Suppose $\langle x_0, x_1\rangle \sim^{\mathbb{Z}} \langle y_0, y_1\rangle$ and $\langle y_0, y_1\rangle \sim^{\mathbb{Z}} \langle z_0, z_1\rangle$. Then we have $x_0 + y_1 = y_0 + x_1$ and $y_0 + z_1 = z_0 + y_1$, yielding

$$(x_0 + y_1) + (y_0 + z_1) = (y_0 + x_1) + (z_0 + y_1).$$

After applying Corollary 3.13 (ii) and (iv) several times, we get

$$(x_0 + z_1) + (y_0 + y_1) = (z_0 + x_1) + (y_0 + y_1).$$

Finally, with Corollary 3.13 (vi) we obtain the desired result

$$x_0 + z_1 = z_0 + x_1,$$

implying that $\langle x_0, x_1 \rangle \sim^{\mathbb{Z}} \langle z_0, z_1 \rangle$, as well.

□

Before defining $\mathbb{Z}$, we show another helpful proposition which gives us a unique representative of each equivalence class of $\sim^{\mathbb{Z}}$.

Proposition 3.27.

$$\forall x, y \in \omega \exists! n \in \omega \left(\langle x, y \rangle \sim^{\mathbb{Z}} \langle n, 0 \rangle \vee \langle x, y \rangle \sim^{\mathbb{Z}} \langle 0, n \rangle \right),$$

or, informally speaking, every integer can be written as "n" or "$-n$" with a unique natural number n.

Proof.

$\exists$: There are two cases to consider:

$x \leq y$: By definition of $\leq$, there exists a $z \in \omega$ such that $x + z = y (= 0 + y)$. Therefore, $\langle x, y \rangle \sim^{\mathbb{Z}} \langle 0, z \rangle$.

$y \leq x$: Similarly to before, there exists a $z \in \omega$ such that $y + z = x$, or, equivalently, $x + 0 = z + y$. Hence, $\langle x, y \rangle \sim^{\mathbb{Z}} \langle z, 0 \rangle$.

By Proposition 3.21 (iv), these are the only possible cases and hence we have proven the statement for all choices of x, y.

!: Here, we have to consider four cases:

$\langle n, 0 \rangle \sim^{\mathbb{Z}} \langle m, 0 \rangle$: $n + 0 = m + 0$ implies $n = m$.

$\langle 0, n \rangle \sim^{\mathbb{Z}} \langle 0, m \rangle$: $0 + m = 0 + n$ implies $m = n$.

$\langle n, 0 \rangle \sim^{\mathbb{Z}} \langle 0, m \rangle$: $n + m = 0 + 0$ implies $n = m = 0$, since otherwise one of them would be a successor, and then their sum would also be a successor, which contradicts PA_1.

$\langle 0, n \rangle \sim^{\mathbb{Z}} \langle m, 0 \rangle$: $0 + 0 = m + n$ implies $m = n = 0$ in the same way as before.

□

At this point we are ready to define $\mathbb{Z}$, playing the role of the symbol B from Theorem 2.13 for the integers.

Definition 3.28.

$$\mathbb{Z} := \{\langle x, y\rangle \in \omega \times \omega : \; x = 0 \vee y = 0\}$$
$$[x]^{\mathbb{Z}} = y :\Longleftrightarrow y \in \mathbb{Z} \wedge x \sim^{\mathbb{Z}} y.$$

Like for ω, we need addition and multiplication for the integers, as well as an order relation. They are all defined through their corresponding functions and relations on ω. Additionally, we will define the additive inverse of the elements of $\mathbb{Z}$. But we start with the definition of the constant symbols.

Definition 3.29.

$$\mathbf{0}^{\mathbb{Z}} := \langle 0, 0\rangle$$
$$\mathbf{1}^{\mathbb{Z}} := \langle \mathrm{s}(0), 0\rangle$$

Clearly, $\mathbf{0}^{\mathbb{Z}}$ and $\mathbf{1}^{\mathbb{Z}}$ are elements of $\mathbb{Z}$, allowing us to define the following.

Definition 3.30.

$$0^{\mathbb{Z}} := \mathbf{0}^{\mathbb{Z}} = \langle 0, 0\rangle$$
$$1^{\mathbb{Z}} := \mathbf{1}^{\mathbb{Z}} = \langle \mathrm{s}(0), 0\rangle$$

Our goal is to show that $(\mathbb{Z}, +^{\mathbb{Z}}, \cdot^{\mathbb{Z}})$ is a ring where $0^{\mathbb{Z}}$ and $1^{\mathbb{Z}}$ are the neutral elements with respect to addition and multiplication, respectively.

Definition 3.31.

$$+^{\mathbb{Z}} := \{ \langle\langle\langle x_0, x_1\rangle, \langle y_0, y_1\rangle\rangle, \langle z_0, z_1\rangle\rangle \in (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z} : \langle z_0, z_1\rangle = [\langle x_0 + y_0, x_1 + y_1\rangle]^{\mathbb{Z}}\}.$$

Proposition 3.32.

$$\forall x, y \in \mathbb{Z} \exists! z \in \mathbb{Z}(\langle\langle x, y\rangle, z\rangle \in +^{\mathbb{Z}}).$$

Proof. This basically follows from Proposition 3.27. Clearly, $\langle x_0 + y_0, x_1 + y_1\rangle$ is an element of $\omega \times \omega$, which, by Proposition 3.27, has a unique representation in what we have defined to be $\mathbb{Z}$. □

Definition 3.33.

$$x +^{\mathbb{Z}} y = z :\Longleftrightarrow \langle\langle x, y\rangle, z\rangle \in +^{\mathbb{Z}}.$$

With this definition, we can add integers, i.e. tuples of natural numbers of which at least one is a 0, to each other. Since being able to add tuples of natural numbers without first having to determine their representatives in $\mathbb{Z}$ is desirable, the following is a favourable proposition.

Proposition 3.34. *For natural numbers $x_0, x_1, y_0, y_1 \in \omega$ we have*

$$[\langle x_0, x_1\rangle]^{\mathbb{Z}} +^{\mathbb{Z}} [\langle y_0, y_1\rangle]^{\mathbb{Z}} = [\langle x_0 + y_0, x_1 + y_1\rangle]^{\mathbb{Z}}.$$

Proof. Let $a_0, a_1, b_0, b_1 \in \omega$ be natural numbers such that

$$\langle a_0, a_1 \rangle = [\langle x_0, x_1 \rangle]^{\mathbb{Z}}$$
$$\langle b_0, b_1 \rangle = [\langle y_0, y_1 \rangle]^{\mathbb{Z}}.$$

This implies

$$a_0 + x_1 = x_0 + a_1 \tag{15}$$
$$b_0 + y_1 = y_0 + b_1, \tag{16}$$

which yields

$$\begin{aligned}
[\langle x_0, x_1 \rangle]^{\mathbb{Z}} +^{\mathbb{Z}} [\langle y_0, y_1 \rangle]^{\mathbb{Z}} &= \langle a_0, a_1 \rangle +^{\mathbb{Z}} \langle b_0, b_1 \rangle \\
&= [\langle a_0 + b_0, a_1 + b_1 \rangle]^{\mathbb{Z}} \\
&= [\langle a_0 + b_0 + x_1 + y_1, a_1 + b_1 + x_1 + y_1 \rangle]^{\mathbb{Z}} \\
&\overset{(15),(16)}{=} [\langle x_0 + a_1 + y_0 + b_1, a_1 + b_1 + x_1 + y_1 \rangle]^{\mathbb{Z}} \\
&= [\langle x_0 + y_0, x_1 + y_1 \rangle]^{\mathbb{Z}}.
\end{aligned}$$

□

Proposition 3.35. $\widetilde{\mathrm{IA}_1}, \widetilde{\mathrm{IA}_2}$ *and* $\widetilde{\mathrm{IA}_3}$ *hold, i.e.*

(i) $\forall x, y, z \in \mathbb{Z}((x +^{\mathbb{Z}} y) +^{\mathbb{Z}} z = x +^{\mathbb{Z}} (y +^{\mathbb{Z}} z))$

(ii) $\forall x \in \mathbb{Z}(x +^{\mathbb{Z}} 0^{\mathbb{Z}} = 0^{\mathbb{Z}} +^{\mathbb{Z}} x = x)$

(iii) $\forall x, y \in \mathbb{Z}(x +^{\mathbb{Z}} y = y +^{\mathbb{Z}} x)$.

Proof. Consider $x = \langle x_0, x_1 \rangle, y = \langle y_0, y_1 \rangle, z = \langle z_0, z_1 \rangle$.

(i) We can deduce this property of $+^{\mathbb{Z}}$ from properties of $+^{\mathbb{N}}(+)$ we have already proven.

$$\begin{aligned}
(x +^{\mathbb{Z}} y) +^{\mathbb{Z}} z &= (\langle x_0, x_1 \rangle +^{\mathbb{Z}} \langle y_0, y_1 \rangle) +^{\mathbb{Z}} \langle z_0, z_1 \rangle \\
&= [\langle x_0 + y_0, x_1 + y_1 \rangle]^{\mathbb{Z}} +^{\mathbb{Z}} \langle z_0, z_1 \rangle \\
&\overset{3.34}{=} [\langle (x_0 + y_0) + z_0, (x_1 + y_1) + z_1 \rangle]^{\mathbb{Z}} \\
&= [\langle x_0 + (y_0 + z_0), x_1 + (y_1 + z_1) \rangle]^{\mathbb{Z}} \\
&= \ldots = x +^{\mathbb{Z}} (y +^{\mathbb{Z}} z).
\end{aligned}$$

(ii) We have

$$x +^{\mathbb{Z}} 0^{\mathbb{Z}} = \langle x_0, x_1 \rangle +^{\mathbb{Z}} \langle 0, 0 \rangle = [\langle x_0 + 0, x_1 + 0 \rangle]^{\mathbb{Z}} = \langle x_0, x_1 \rangle = x.$$

On the other hand,

$$0^{\mathbb{Z}} +^{\mathbb{Z}} x = \langle 0, 0 \rangle +^{\mathbb{Z}} \langle x_0, x_1 \rangle = [\langle 0 + x_0, 0 + x_1 \rangle]^{\mathbb{Z}} = \langle x_0, x_1 \rangle = x.$$

(iii) And by proceeding in the same way we get

$$\begin{aligned} x +^{\mathbb{Z}} y &= \langle x_0, x_1 \rangle +^{\mathbb{Z}} \langle y_0, y_1 \rangle \\ &= [\langle x_0 + y_0, x_1 + y_1 \rangle]^{\mathbb{Z}} \\ &= [\langle y_0 + x_0, y_1 + x_1 \rangle]^{\mathbb{Z}} \\ &= \ldots = y +^{\mathbb{Z}} x. \end{aligned}$$

□

The next operation we define on $\mathbb{Z}$ is the inverse with respect to addition. It is needed to show that $(\mathbb{Z}, +^{\mathbb{Z}})$ is an abelian group.

Definition 3.36.

$$-^{\mathbb{Z}} := \{\langle \langle x_0, x_1 \rangle, \langle y_0, y_1 \rangle \rangle \in \mathbb{Z} \times \mathbb{Z} : \ x_0 = y_1 \wedge x_1 = y_0\}.$$

Proposition 3.37.

$$\forall x \in \mathbb{Z} \exists! y \in \mathbb{Z}(\langle x, y \rangle \in -^{\mathbb{Z}}).$$

Proof. If $x = \langle x_0, x_1 \rangle \in \mathbb{Z}$, then $y := \langle x_1, x_0 \rangle$ is in $\mathbb{Z}$ as well, since the only requirement for an ordered pair of natural numbers to be in $\mathbb{Z}$ is to have at least one entry equal to 0, which is certainly still the case if we just flip the entries of an ordered pair with a 0-entry. Also, y, as we defined it, satisfies $x_0 = y_1$ and $x_1 = y_0$. Uniqueness follows directly from the definition of $-^{\mathbb{Z}}$. □

Definition 3.38.

$$\begin{aligned} -^{\mathbb{Z}} x = y &:\Longleftrightarrow \langle x, y \rangle \in -^{\mathbb{Z}} \\ x -^{\mathbb{Z}} y = z &:\Longleftrightarrow x +^{\mathbb{Z}} -^{\mathbb{Z}} y = z. \end{aligned}$$

Just like with the addition on ω, Proposition 3.35 (i) justifies omitting the brackets in terms of the form $(x +^{\mathbb{Z}} y) +^{\mathbb{Z}} z$ and $x +^{\mathbb{Z}} (y +^{\mathbb{Z}} z)$. Accordingly, this also works if one or both additions are replaced by subtractions ($-^{\mathbb{Z}}$), since the latter are just abbreviations for particular additions.

Proposition 3.39.

$$\forall x \in \mathbb{Z}(-^{\mathbb{Z}} x +^{\mathbb{Z}} x = x -^{\mathbb{Z}} x = 0^{\mathbb{Z}}).$$

Proof. Let $x = \langle x_0, x_1 \rangle$.

$$\begin{aligned} -^{\mathbb{Z}} x +^{\mathbb{Z}} x &= \langle x_1, x_0 \rangle +^{\mathbb{Z}} \langle x_0, x_1 \rangle \\ &= [\langle x_1 + x_0, x_0 + x_1 \rangle]^{\mathbb{Z}} \\ &= [\langle x_0 + x_1, x_0 + x_1 \rangle]^{\mathbb{Z}} = 0^{\mathbb{Z}}, \end{aligned}$$

since $\langle x_0 + x_1, x_0 + x_1 \rangle$ and $0^{\mathbb{Z}}$ are $\sim^{\mathbb{Z}}$-equivalent by

$$(x_0 + x_1) + 0 = 0 + (x_0 + x_1).$$

The other identity follows from the commutativity of $+^{\mathbb{Z}}$ shown in Proposition 3.35 (iii). □

Corollary 3.40. *As a direct consequence of this,*

$$\widetilde{\mathrm{IA}_4} \equiv \forall x \in \mathbb{Z} \exists y \in \mathbb{Z}(x +^{\mathbb{Z}} y = 0)$$

follows.

Corollary 3.41.

(i) $\forall x \in \mathbb{Z}(-^{\mathbb{Z}} -^{\mathbb{Z}} x = x)$

(ii) $\forall x, y \in \mathbb{Z}(-^{\mathbb{Z}}(x +^{\mathbb{Z}} y) = -^{\mathbb{Z}} x -^{\mathbb{Z}} y)$

Proof. Consider $x = \langle x_0, x_1 \rangle$ and $y = \langle y_0, y_1 \rangle$.

(i) The proof is very simple:

$$-^{\mathbb{Z}} -^{\mathbb{Z}} x = -^{\mathbb{Z}} -^{\mathbb{Z}} \langle x_0, x_1 \rangle = -^{\mathbb{Z}} \langle x_1, x_0 \rangle = \langle x_0, x_1 \rangle = x.$$

(ii) This also follows from the definition of addition and the additive inverse.

$$\begin{aligned} -^{\mathbb{Z}}(x +^{\mathbb{Z}} y) &= -^{\mathbb{Z}}[\langle x_0 + y_0, x_1 + y_1 \rangle]^{\mathbb{Z}} \\ &= [\langle x_1 + y_1, x_0 + y_0 \rangle]^{\mathbb{Z}} \\ &= [\langle x_1, x_0 \rangle]^{\mathbb{Z}} +^{\mathbb{Z}} [\langle y_1, y_0 \rangle]^{\mathbb{Z}} \\ &= -^{\mathbb{Z}} x +^{\mathbb{Z}} -^{\mathbb{Z}} y \\ &= -^{\mathbb{Z}} x -^{\mathbb{Z}} y. \end{aligned}$$

□

Now that we have finished proving that $(\mathbb{Z}, +^{\mathbb{Z}})$ is an abelian group, we can introduce multiplication on $\mathbb{Z}$.

Definition 3.42.

$$\begin{aligned} \cdot^{\mathbb{Z}} := \{ \langle \langle \langle x_0, x_1 \rangle, \langle y_0, y_1 \rangle \rangle, \langle z_0, z_1 \rangle \rangle \in (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z} : \\ \langle z_0, z_1 \rangle = [\langle (x_0 \cdot y_0) + (x_1 \cdot y_1), (x_0 \cdot y_1) + (x_1 \cdot y_0) \rangle]^{\mathbb{Z}} \}. \end{aligned}$$

Proposition 3.43.

$$\forall x, y \in \mathbb{Z} \exists! z \in \mathbb{Z}(\langle \langle x, y \rangle, z \rangle \in \cdot^{\mathbb{Z}}).$$

Proof. Like in Proposition 3.32, this is an immediate consequence of Proposition 3.27, since $\langle (x_0 \cdot y_0) + (x_1 \cdot y_1), (x_0 \cdot y_1) + (x_1 \cdot y_0) \rangle$ is indeed a tuple of natural numbers, which has one and only one representation in $\mathbb{Z}$. □

Definition 3.44.

$$x \cdot^{\mathbb{Z}} y = z :\Longleftrightarrow \langle \langle x, y \rangle , z \rangle \in \boldsymbol{\cdot}^{\mathbb{Z}}.$$

Proposition 3.45. $\widetilde{\mathrm{IM}_1} - \widetilde{\mathrm{IM}_3}$ *and* $\widetilde{\mathrm{ID}_1}$ *hold, that is*

(i) $\forall x, y, z \in \mathbb{Z}((x \cdot^{\mathbb{Z}} y) \cdot^{\mathbb{Z}} z = x \cdot^{\mathbb{Z}} (y \cdot^{\mathbb{Z}} z))$

(ii) $\forall x \in \mathbb{Z}(x \cdot^{\mathbb{Z}} 1^{\mathbb{Z}} = 1^{\mathbb{Z}} \cdot^{\mathbb{Z}} x = x)$

(iii) $\forall x, y \in \mathbb{Z}(x \cdot^{\mathbb{Z}} y = y \cdot^{\mathbb{Z}} x)$

(iv) $\forall x, y, z \in \mathbb{Z}(x \cdot^{\mathbb{Z}} (y +^{\mathbb{Z}} z) = (x \cdot^{\mathbb{Z}} y) +^{\mathbb{Z}} (x \cdot^{\mathbb{Z}} z))$.

Proof. Consider $x = \langle x_0, x_1 \rangle$, $y = \langle y_0, y_1 \rangle$ and $z = \langle z_0, z_1 \rangle$.

(i) Like in the proof of Proposition 3.35 this follows from properties of $+^{\mathbb{N}}(+)$ and $\cdot^{\mathbb{N}}(\cdot)$ shown in Corollary 3.13 and Corollary 3.18, such as associativity of both $+$ and $\cdot$, as well as distributivity of $\cdot$ over $+$.

(ii) We have

$$\begin{aligned} x \cdot^{\mathbb{Z}} 1^{\mathbb{Z}} &= \langle x_0, x_1 \rangle \cdot^{\mathbb{Z}} \langle \mathrm{s}(0), 0 \rangle \\ &= [\langle (x_0 \cdot \mathrm{s}(0)) + (x_1 \cdot 0), (x_0 \cdot 0) + (x_1 \cdot \mathrm{s}(0)) \rangle]^{\mathbb{Z}} \\ &= \langle x_0, x_1 \rangle = x. \end{aligned}$$

Analogously, $1^{\mathbb{Z}} \cdot^{\mathbb{Z}} x = x$ is shown.

(iii) From the commutativity of $+^{\mathbb{N}}(+)$ and $\cdot^{\mathbb{N}}(\cdot)$ we can deduce

$$\begin{aligned} x \cdot^{\mathbb{Z}} y &= \langle x_0, x_1 \rangle \cdot^{\mathbb{Z}} \langle y_0, y_1 \rangle \\ &= [\langle (x_0 \cdot y_0) + (x_1 \cdot y_1), (x_0 \cdot y_1) + (x_1 \cdot y_0) \rangle]^{\mathbb{Z}} \\ &= [\langle (y_0 \cdot x_0) + (y_1 \cdot x_1), (y_0 \cdot x_1) + (y_1 \cdot x_0) \rangle]^{\mathbb{Z}} \\ &= \langle y_0, y_1 \rangle \cdot^{\mathbb{Z}} \langle x_0, x_1 \rangle \\ &= y \cdot^{\mathbb{Z}} x. \end{aligned}$$

(iv) This is another property of the integers that follows from the analogous property of the natural numbers, Corollary 3.18 (v).

$$\begin{aligned} x \cdot^{\mathbb{Z}} (y +^{\mathbb{Z}} z) &= \langle x_0, x_1 \rangle \cdot^{\mathbb{Z}} [\langle y_0 + z_0, y_1 + z_1 \rangle]^{\mathbb{Z}} \\ &= [\langle x_0 \cdot (y_0 + z_0) + x_1 \cdot (y_1 + z_1), \\ &\qquad x_0 \cdot (y_1 + z_1) + x_1 \cdot (y_0 + z_0) \rangle]^{\mathbb{Z}} \\ &= [\langle (x_0 \cdot y_0) + (x_0 \cdot z_0) + (x_1 \cdot y_1) + (x_1 \cdot z_1), \\ &\qquad (x_0 \cdot y_1) + (x_0 \cdot z_1) + (x_1 \cdot y_0) + (x_1 \cdot z_0) \rangle]^{\mathbb{Z}} \\ &= [\langle (x_0 \cdot y_0) + (x_1 \cdot y_1), (x_0 \cdot y_1) + (x_1 \cdot y_0) \rangle]^{\mathbb{Z}} \\ &\qquad +^{\mathbb{Z}} [\langle (x_0 \cdot z_0) + (x_1 \cdot z_1), (x_0 \cdot z_1) + (x_1 \cdot z_0) \rangle]^{\mathbb{Z}} \\ &= (x \cdot^{\mathbb{Z}} y) +^{\mathbb{Z}} (x \cdot^{\mathbb{Z}} z). \end{aligned}$$

□

From this point, $+^{\mathbb{Z}}$ will be replaced by $+$ and $-^{\mathbb{Z}}$ by $-$. There should not occur any confusion between $+^{\mathbb{N}}$ and $+^{\mathbb{Z}}$, since it will always be clear whether the two numbers being added are elements of ω or of $\mathbb{Z}$. Furthermore, we will from now on write $\cdot, 0, 1$ instead of $\cdot^{\mathbb{Z}}, 0^{\mathbb{Z}}, 1^{\mathbb{Z}}$ and occasionally omit the symbol $\cdot$ and brackets around products.

Definition 3.46.

$$\begin{aligned} <^{\mathbb{Z}} &:= \{\langle\langle x_0, x_1\rangle, \langle y_0, y_1\rangle\rangle \in \mathbb{Z}\times\mathbb{Z} :\ x_0 + y_1 < y_0 + x_1\} \\ x <^{\mathbb{Z}} y &:\Longleftrightarrow \langle x, y\rangle \in <^{\mathbb{Z}} \\ x \leq^{\mathbb{Z}} y &:\Longleftrightarrow x <^{\mathbb{Z}} y \vee x = y. \end{aligned}$$

Proposition 3.47.

(i) $\forall x \in \mathbb{Z}\neg(x <^{\mathbb{Z}} x)$

(ii) $\forall x, y, z \in \mathbb{Z}(x <^{\mathbb{Z}} y \wedge y <^{\mathbb{Z}} z \to x <^{\mathbb{Z}} z)$

(iii) $\forall x, y \in \mathbb{Z}(x <^{\mathbb{Z}} y \vee y <^{\mathbb{Z}} x \vee x = y)$

(iv) $\forall x \in \mathbb{Z}\exists y, z \in \mathbb{Z}(y <^{\mathbb{Z}} x \wedge x <^{\mathbb{Z}} z)$

The formulae (i)-(iii) *correspond to* $\widetilde{\mathrm{IO}_1} - \widetilde{\mathrm{IO}_3}$*, whereas* (iv) *is an interesting additional property of* $<^{\mathbb{Z}}$.

Proof. Let $x = \langle x_0, x_1\rangle, y = \langle y_0, y_1\rangle$ and $z = \langle z_0, z_1\rangle$.

(i) From Proposition 3.22 (i) we know that

$$\begin{aligned} & x_0 + x_1 \nless x_0 + x_1 \\ \Longleftrightarrow\ & x \nless^{\mathbb{Z}} x. \end{aligned}$$

(ii)

$$\begin{aligned} x <^{\mathbb{Z}} y \wedge y <^{\mathbb{Z}} z &\Longleftrightarrow (x_0 + y_1 < y_1 + x_0) \wedge (y_0 + z_1 < z_0 + y_1) \\ &\overset{3.24(i)}{\Longrightarrow} (x_0 + y_1) + (y_0 + z_1) < (y_1 + x_0) + (z_0 + y_1) \\ &\Longleftrightarrow (x_0 + z_1) + (y_0 + y_1) < (z_0 + x_1) + (y_0 + y_1) \\ &\overset{3.24(ii)}{\Longrightarrow} x_0 + z_1 < z_0 + x_1 \Longleftrightarrow x <^{\mathbb{Z}} z. \end{aligned}$$

(iii) This is a consequence of the corresponding property of $<^{\mathbb{N}}$ ($<$). For $x, y \in \mathbb{Z}$ we have that $x_0 + y_1 \in \omega$ and $y_0 + x_1 \in \omega$. Thus, by Proposition 3.22 (iii)

$$\begin{aligned} &\underbrace{(x_0 + y_1 < y_0 + x_1)}_{\Longleftrightarrow x <^{\mathbb{Z}} y} \vee \underbrace{(y_0 + x_1 < x_0 + y_1)}_{\Longleftrightarrow y <^{\mathbb{Z}} x} \vee \underbrace{(x_0 + y_1 = y_0 + x_1)}_{\Longleftrightarrow x \sim^{\mathbb{Z}} y (\Longrightarrow x = y)} \\ \Longrightarrow &(x <^{\mathbb{Z}} y) \vee (y <^{\mathbb{Z}} x) \vee (x = y). \end{aligned}$$

(iv) For $x = \langle x_0, x_1 \rangle$ we can always consider

$$y := [\langle x_0, \mathrm{s}(x_1) \rangle]^{\mathbb{Z}}, z := [\langle \mathrm{s}(x_0), x_1 \rangle]^{\mathbb{Z}}.$$

Then,

$$\begin{aligned} y_0 + x_1 &< \mathrm{s}(y_0 + x_1) \\ &= y_0 + \mathrm{s}(x_1) = x_0 + y_1 \end{aligned}$$

$$\Longrightarrow y <^{\mathbb{Z}} x.$$

And in the same way, $x <^{\mathbb{Z}} z$ follows.

□

When comparing integers, we will use $\leq$ and $<$ instead of $\leq^{\mathbb{Z}}$ and $<^{\mathbb{Z}}$ from now on, which again won't make any trouble for the same reason as before. Furthermore, it will occasionally be more convenient to write $y > x, y \geq x$ for $x < y$ and $x \leq y$, respectively.

There are two axioms remaining to be shown until we can derive a model of the integers from a model of ZF according to Theorem 2.13: monotonicity of addition and multiplication.

Proposition 3.48.

(i) $\forall x, y, z \in \mathbb{Z}(x < y \to x + z < y + z)$

(ii) $\forall x, y, z \in \mathbb{Z}(x < y \wedge 0 < z \to x \cdot z < y \cdot z)$,

meaning that $\widetilde{\mathrm{IO}_4}$ *and* $\widetilde{\mathrm{IO}_5}$ *hold.*

Proof. Consider $x = \langle x_0, x_1 \rangle$, $y = \langle y_0, y_1 \rangle$ and $z = \langle z_0, z_1 \rangle$.

(i) $x < y$ implies $x_0 + y_1 < y_0 + x_1$. Therefore we have by Corollary 3.24 (i)

$$\begin{aligned} x_0 + y_1 + z_0 + z_1 &< y_0 + x_1 + z_0 + z_1 \\ \Longrightarrow [\langle x_0 + z_0, x_1 + z_1 \rangle]^{\mathbb{Z}} &< [\langle y_0 + z_0, y_1 + z_1 \rangle]^{\mathbb{Z}} \\ \Longrightarrow x + z &< y + z. \end{aligned}$$

(ii) Again we have $x_0 + y_1 < y_0 + x_1$. And, since $0 < z = \langle z_0, z_1 \rangle$, we know that $z_1 < z_0$ and therefore $z_1 = 0$. So consider

$$\begin{aligned} x_0 + y_1 &< y_0 + x_1 \\ \overset{3.24(iii)}{\Longrightarrow} (x_0 + y_1) \cdot z_0 &< (y_0 + x_1) \cdot z_0 \\ \overset{z_1=0}{\Longleftrightarrow} (x_0 + y_1) \cdot z_0 + (x_1 + y_0) \cdot z_1 &< (y_0 + x_1) \cdot z_0 + (y_1 + x_0) \cdot z_1 \\ \Longleftrightarrow (x_0 z_0 + x_1 z_1) + (y_0 z_1 + y_1 z_0) &< (y_0 z_0 + y_1 z_1) + (x_0 z_1 + x_1 z_0) \\ \Longleftrightarrow x \cdot z &< y \cdot z. \end{aligned}$$

□

3.3 Real Numbers ($\mathbb{R}$)

We have now shown all characteristic properties of $\mathbb{Z}$; a model of the integers therefore exists by Theorem 2.13 given a model of ZF. Now, the next step towards our primary goal of constructing the system of real numbers can be approached. Unlike in the usual construction, we won't need fractions and later Cauchy sequences or Dedekind cuts but will obtain the reals as equivalence classes of special functions from $\mathbb{Z}$ to $\mathbb{Z}$. In Definition 2.5 we have already seen what functions are; now we need to develop the concept of finiteness via bijective functions and only then specific functions we will later call "slopes" can be established.

3.3.1 Finite Sets

Definition 3.49. A function $f : A \to B$ is called **bijective** if for every $y \in B$ there is exactly one $x \in A$ such that $\langle x, y\rangle \in f$. To indicate that f is a bijective function from A to B we will write $f : A \hookrightarrow\!\!\!\!\rightarrow B$.

Definition 3.50. For a bijective function $f : A \hookrightarrow\!\!\!\!\rightarrow B$, the **inverse of** f is defined as

$$f^{-1} := \{\langle y, x\rangle \in B \times A : \langle x, y\rangle \in f\}.$$

Corollary 3.51. *The inverse of a bijective function from A to B is a bijective function from B to A.*

Proof. Consider a bijective function $f : A \hookrightarrow\!\!\!\!\rightarrow B$. Since it is a function we know that for every $x \in A$ there is exactly one $y \in B$ such that $\langle x, y\rangle \in f$, and since it is bijective, there is exactly one $x \in A$ for every $y \in B$ such that the same is true. As this property is "symmetric" with respect to A and B, switching the roles of A and B leaves the property unchanged. □

Definition 3.52. The **composition** $g \circ f$ of two functions $f : A \to B$ and $g : B \to C$ is defined by

$$g \circ f := \{\langle x, z\rangle : \exists y(\langle x, y\rangle \in f \wedge \langle y, z\rangle \in g)\}.$$

Proposition 3.53. *The composition $g \circ f$ of two functions $f : A \to B$ and $g : B \to C$ is a function from A to C.*

Proof. Consider $f : A \to B$ and $g : B \to C$. We want to show that for each $x \in A$ there is exactly one $z \in C$ such that $\langle x, z\rangle \in g \circ f$. For $x \in A$ there is exactly one $y \in B$ such that $\langle x, y\rangle \in f$. And for this $y \in B$, there is exactly one $z \in C$ such that $\langle y, z\rangle \in g$. So for each $x \in A$ there remains exactly one z such that there is a $y \in B$ such that both $\langle x, y\rangle \in f$ and $\langle y, z\rangle \in g$, which is the requirement for $\langle x, z\rangle$ being in $g \circ f$. Therefore, $g \circ f$ is a function. □

Proposition 3.54. *The composition of two bijective functions is again a bijective function.*

Proof. Consider $f : A \hookrightarrow\!\!\!\!\rightarrow B$ and $g : B \hookrightarrow\!\!\!\!\rightarrow C$. We aim to prove that for each $z \in C$ there is exactly one $x \in A$ such that $\langle x, z\rangle \in g \circ f$. Since g is by assumption bijective, there is exactly one $y \in B$ for each $z \in C$ such that $\langle y, z\rangle \in g$. For this y, there is by the bijectivity of f again exactly one $x \in A$ for which $\langle x, y\rangle \in f$. So, like before, we can deduce that there is exactly one $x \in A$ for each $z \in C$ such that there is a $y \in B$ such that both $\langle x, y\rangle \in f$ and

$\langle y, z\rangle \in g$. In other words, there is exactly one $x \in A$ for each $z \in C$ such that $\langle x, z\rangle \in g \circ f$. Hence, $g \circ f$ is bijective. □

Example 3.55. The following are bijective functions from $\mathbb{Z}$ to $\mathbb{Z}$:

(i) $-^{\mathbb{Z}}$

(ii) $+_a := \{\langle x, x+a\rangle : x \in \mathbb{Z}\}$, for any $a \in \mathbb{Z}$.

Proof.

(i) The fact that $-^{\mathbb{Z}}$ is a function has been proven in Proposition 3.37, whereas its bijectivity becomes evident in the exact same way.

(ii) Again, $+_a$ being a function is an immediate consequence of Proposition 3.32. Now to the bijectivity of $+_a$. Let $y = \langle y_0, y_1\rangle$ and $a = \langle a_0, a_1\rangle$ for natural numbers y_0, y_1, a_0, a_1. Then, if we choose $x = \langle x_0, x_1\rangle := [\langle y_0 + a_1, y_1 + a_0\rangle]^{\mathbb{Z}}$, we have that

$$\begin{aligned}\langle x, x+a\rangle &= \left\langle x, [\langle y_0 + a_1 + a_0, y_1 + a_0 + a_1\rangle]^{\mathbb{Z}}\right\rangle \\ &= \left\langle x, [\langle y_0, y_1\rangle]^{\mathbb{Z}}\right\rangle \\ &= \langle x, y\rangle ,\end{aligned}$$

hence, $\langle x, y\rangle \in +_a$ holds. So we already have the existence of a preimage for every $y \in \mathbb{Z}$. The uniqueness of this preimage is a consequence of Corollary 3.13 (vi). If $+_a(x) = +_a(x') = y$, then

$$\langle x_0 + a_0, x_1 + a_1\rangle \sim^{\mathbb{Z}} \langle x'_0 + a_0, x'_1 + a_1\rangle .$$

Thus, $x_0 + a_0 + x'_1 + a_1 = x'_0 + a_0 + x_1 + a_1$, and hence $x \sim^{\mathbb{Z}} x'$ which implies $x = x'$.

□

Proposition 3.56. *Let $f : A \to B, g : B \to C, h : C \to D$ be functions. Then,*

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Proof. $(h \circ g) \circ f$ is the set consisting of tuples $\langle x, w\rangle \in A \times D$ such that there are $y \in B$ and $z \in C$ for which

$$\begin{aligned} &\langle x, y\rangle \in f \text{ and } \langle y, w\rangle \in (h \circ g) \\ \iff &\langle x, y\rangle \in f \text{ and } \langle y, z\rangle \in g \text{ and } \langle z, w\rangle \in h.\end{aligned}$$

These are exactly the tuples for which there exists $z \in C$ such that

$$\langle x, z\rangle \in (g \circ f) \text{ and } \langle z, w\rangle \in h,$$

i.e. the elements of $h \circ (g \circ f)$. Therefore, $\circ$ is associative. □

Definition 3.57. Let $f : A \to B$ be a function. For any subset $A' \subseteq A$, the **restriction of f to A'** is defined by

$$f\,|_{A'} := f \cap (A' \times B).$$

Proposition 3.58. *The restriction of a bijective function $f : A \to B$ to $A' \subseteq A$ is a bijective function from A' to $f[A']$.*

Proof. Since $f : A \hookrightarrow\!\!\!\rightarrow B$ is a function, it is clear by definition that $f\,|_{A'} : A' \to f[A']$ is a function, too. By the bijectivity of f, for every $y \in B$ there exists exactly one $x \in A$ such that $\langle x, y \rangle \in f$. Since $f[A']$ is defined as the set of images of elements of A' under f, the existence of an $x \in A'$ for each $y \in f[A']$ such that $\langle x, y \rangle \in f\,|_{A'}$ is clear, whereas its uniqueness follows from its uniqueness in f. □

Definition 3.59. A set A is called **finite** if there exists $n \in \omega$ and a bijective function $f : A \hookrightarrow\!\!\!\rightarrow n$. We then say that A has n elements. If A is not finite, we say that A is **infinite**.

Proposition 3.60. *Every subset of an element of ω is finite.*

Proof. To prove this, we use induction on $n \in \omega$. Let $n = 0$. Since $0 = \emptyset$ has no subsets, every subset of 0 is finite.
Now assume that every subset of n is finite. If y is a subset of $\mathrm{s}(n) = n \cup \{n\}$, we can write y as

$$y = x \cup x',$$

where $x = y \cap n$ and $x' = y \backslash x$. We then have $x \subseteq n$ and either $x' = \emptyset$ or $x' = \{n\}$. By assumption, there exists $k \in \omega$ such that there is a bijective function $f : x \hookrightarrow\!\!\!\rightarrow k$. If $x' = \emptyset$, we are already finished since $y = x$ and therefore with f we have a bijective function from y to $k \in \omega$. On the other hand, if $x' = \{n\}$, define

$$g := f \cup \{\langle n, k \rangle\}.$$

g is a bijective function from y to $\mathrm{s}(k) \in \omega$ and therefore, y is finite. □

Corollary 3.61. *Every subset of a finite set is finite.*

Proof. Let A be a finite set. Therefore, there exist a natural number $n \in \omega$ and a bijective function $f : A \hookrightarrow\!\!\!\rightarrow n \in \omega$. Now consider an arbitrary subset $A' \subseteq A$. The restriction

$$f\,|_{A'} : A' \hookrightarrow\!\!\!\rightarrow f[A']$$

is bijective by Proposition 3.58. Also, $f[A'] \subseteq n$ is a subset of an element of ω, from which we have a bijective function g to a natural number $k \in \omega$ by Proposition 3.60. Composing $g \circ f$ yields, by Proposition 3.54, a bijective function from A' to $k \in \omega$, implying that A' is finite. □

Proposition 3.62. *The union of two finite sets is finite.*

Proof. Let A and B be two finite sets, i.e. there are two bijective functions

$$f : A \hookrightarrow\!\!\!\!\rightarrow n \in \omega$$
$$g : B \hookrightarrow\!\!\!\!\rightarrow m \in \omega.$$

We show the finiteness of $A \cup B$ by induction on m. First, let $m = 0$. Then,

$$B = g^{-1}[0] = \emptyset$$
$$\implies A \cup B = A \cup \emptyset = A,$$

which is finite by assumption. Now, assume that the union of A with B', for B' an arbitrary set such that there is a bijection $g' : B' \hookrightarrow\!\!\!\!\rightarrow m$, is finite and let B be a finite set such that there exists a bijection $g : B \hookrightarrow\!\!\!\!\rightarrow \mathrm{s}(m)$. Since $m \subseteq \mathrm{s}(m)$, we can define

$$B' := g^{-1}[m] \subseteq B = g^{-1}[\mathrm{s}(m)]$$

and thus we have a bijective function

$$g' := g \mid_{B'} : B' \hookrightarrow\!\!\!\!\rightarrow m.$$

By our assumption, $A \cup B'$ is finite, giving us a bijection

$$h' : A \cup B' \hookrightarrow\!\!\!\!\rightarrow k' \in \omega.$$

We want to find a bijection from $A \cup B$ to a natural number k. There are two possible cases. Either, $g^{-1}(m) \in A$. In this case, $A \cup B = A \cup B'$ and we are finished ($k = k'$). The other case, $g^{-1}(m) \notin A$, is less trivial. Since $A \cup B = (A \cup B') \cup \{g^{-1}(m)\}$, setting

$$h := h' \cup \left\{ \left\langle g^{-1}(m), k' \right\rangle \right\},$$

yields a bijective function from $A \cup B$ to $\mathrm{s}(k') =: k$. Hence, $A \cup B$ is finite. □

Proposition 3.63. *Let $f : A \to B$ be a function. If A is finite, then $f[A]$ is finite, too.*

Proof. The set A being finite implies the existence of a bijective function

$$g : A \hookrightarrow\!\!\!\!\rightarrow n \in \omega.$$

Again, we can prove this by induction on n, the number of elements in A. If $n = 0$, then

$$\begin{aligned} A &= g^{-1}[0] = \emptyset \\ \implies f[A] &= \emptyset, \end{aligned}$$

which is obviously finite. So, assume that for any set A' which has n elements, $f'[A']$ is finite for every function $f' : A' \to B'$. Consider a set A with $\mathrm{s}(n)$ elements. Then, since $n \subseteq \mathrm{s}(n)$, we can define

$$A' := g^{-1}[n] \subseteq A = g^{-1}[\mathrm{s}(n)].$$

Since now $A = A' \cup \{g^{-1}(n)\}$, we have that

$$f[A] = f[A'] \cup f[\{g^{-1}(n)\}].$$

And since $f[A']$ is finite by assumption, it follows from Proposition 3.62 that $f[A]$ is also finite. □

Lemma 3.64. *If two sets $A, B \subseteq \mathbb{Z}$ are finite, then the set*

$$A \oplus B := \{a + b : \ a \in A \wedge b \in B\}$$

is finite, as well.

Proof. Once again this will be proven by induction on n, the number of elements in A. If $n = 0$, $A \oplus B = \emptyset$, and we are finished. So assume that $A' \oplus B$ is finite if A' has n elements. As we have done before, we can decompose a set A of $\mathrm{s}(n)$ elements into

$$A = A' \cup \{a_0\}$$

where A' has n elements. Thus,

$$\begin{aligned} A \oplus B &= \{a + b : \ a \in A \wedge b \in B\} \\ &= \{a + b : \ (a \in A' \vee a = a_0) \wedge b \in B\} \\ &= (A' \oplus B) \cup +_{a_0}[B]. \end{aligned}$$

We have already seen that $+_{a_0}$ is a function from $\mathbb{Z}$ to $\mathbb{Z}$. Restricting it to $B \subseteq \mathbb{Z}$ yields, by Proposition 3.63, that $+_{a_0}[B]$ is finite, $A' \oplus B$ is finite by assumption, and therefore, $A \oplus B$ is, as their union, finite by Proposition 3.62. □

We have reached the point at which the concept of "slopes" can be introduced, later leading us to the definition of the real numbers. The ideas in the rest of this chapter and in most of Chapter 4 are adopted from [A'C03].

3.3.2 Slopes

Definition 3.65. A **slope** is a function $\lambda : \mathbb{Z} \to \mathbb{Z}$ for which the set

$$D_\lambda := \{\lambda(x+y) - \lambda(x) - \lambda(y) : \, x, y \in \mathbb{Z}\}$$

is finite. The set of all slopes will be denoted by $\mathcal{S}$.

One can imagine a slope as a function that is "almost" linear, in the sense that if for a function λ the set D_λ contains only 0, (hence is finite), then λ is linear. If D_λ has more or other elements than 0 but is still finite, it can be shown that λ has some properties that are generalizations of properties of linear functions. We will see some of these properties later, in Section 4.2.

Definition 3.66.

$$\begin{aligned} D_{\lambda,\mu} &:= \{\lambda(x) - \mu(x) : \, x \in \mathbb{Z}\} \\ \sim^{\mathbb{R}} &:= \{\langle \lambda, \mu \rangle \in \mathcal{S} \times \mathcal{S} : \, D_{\lambda,\mu} \text{ is finite}\} \\ \lambda \sim^{\mathbb{R}} \mu &:\iff \langle \lambda, \mu \rangle \in \sim^{\mathbb{R}} . \end{aligned}$$

Proposition 3.67. $\sim^{\mathbb{R}}$ *is an equivalence relation.*

Proof. We show reflexivity, symmetry and transitivity of $\sim^{\mathbb{R}}$.

reflexivity: $D_{\lambda,\lambda} = \{\lambda(x) - \lambda(x) : \, x \in \mathbb{Z}\} = \{0\}$ by Proposition 3.39. The function

$$f : D_\lambda \hookrightarrow\!\!\!\!\rightarrow \{0^{\mathbb{N}}\} = \mathrm{s}(0^{\mathbb{N}}) \in \omega$$

defined by $f = \left\{\left\langle 0^{\mathbb{Z}}, 0^{\mathbb{N}} \right\rangle\right\}$ is bijective. Hence, $\lambda \sim^{\mathbb{R}} \lambda$.

symmetry: Assume $\lambda \sim^{\mathbb{R}} \mu$. This means there is $n \in \omega$ and a bijective function

$$f : D_{\lambda,\mu} = \{\lambda(x) - \mu(x) : \, x \in \mathbb{Z}\} \hookrightarrow\!\!\!\!\rightarrow n.$$

Define

$$g := \{\langle \mu(x) - \lambda(x), \lambda(x) - \mu(x) \rangle : \, x \in \mathbb{Z}\}.$$

Now, g is a bijective function from $D_{\mu,\lambda}$ to $D_{\lambda,\mu}$, since it is just a subset of $-^{\mathbb{Z}}$. Therefore we can consider the function $f \circ g : D_{\mu,\lambda} \hookrightarrow\!\!\!\!\rightarrow n$, which by Proposition 3.54 is bijective. Hence, $\mu \sim^{\mathbb{R}} \lambda$, too.

transitivity: Suppose $\lambda \sim^{\mathbb{R}} \mu$ and $\mu \sim^{\mathbb{R}} \nu$. This means that both $D_{\lambda,\mu}$ and $D_{\mu,\nu}$ are finite.

Now notice that

$$\begin{aligned} D_{\lambda,\nu} &= \{\lambda(x) - \nu(x) : \; x \in \mathbb{Z}\} \\ &= \{\underbrace{(\lambda(x) - \mu(x))}_{\in D_{\lambda,\mu}} + \underbrace{(\mu(x) - \nu(x))}_{\in D_{\mu,\nu}} : \; x \in \mathbb{Z}\} \\ &\subseteq D_{\lambda,\mu} \oplus D_{\mu,\nu}, \end{aligned}$$

which is finite by Lemma 3.64. Hence, by Corollary 3.61, $D_{\lambda,\nu}$ is finite as a subset of a finite set.

□

We have now reached the point at which we are able to define the real numbers.

Definition 3.68.

$$\begin{aligned} \mathbb{R} &:= \{x \in \mathscr{P}(\mathcal{S}) : \; \exists \lambda \in \mathcal{S}(\lambda \in x \wedge \forall \mu \in \mathcal{S}(\mu \in x \leftrightarrow \mu \sim^{\mathbb{R}} \lambda))\} \\ [\lambda]^{\mathbb{R}} = x &:\Longleftrightarrow x \in \mathbb{R} \wedge \lambda \in x. \end{aligned}$$

We call $[\lambda]^{\mathbb{R}}$ the **equivalence class of** λ.

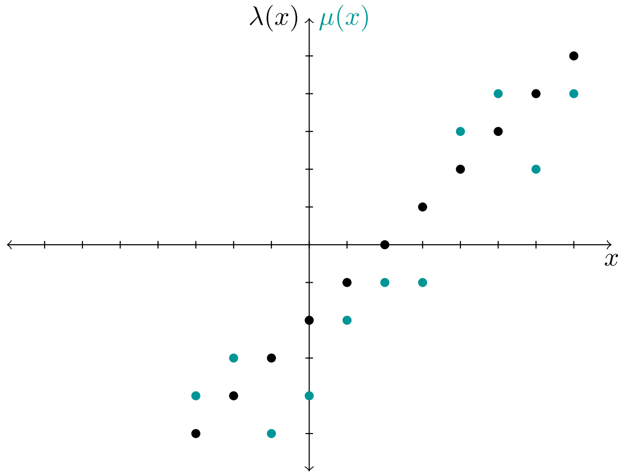

Figure 1: Two slopes, λ and μ, where $D_\lambda = \{2\}$ and $\lambda \sim^{\mathbb{R}} \mu$.

4 The Axioms of $\mathbb{R}$

Now that we have a set we call the "real numbers", we want to check whether it actually behaves as we expect it to. For this purpose we need to make sure that the ZF-transformations of the axioms of $\mathbb{R}$ really hold. The axioms can be divided into three parts:

- the field axioms
- the axioms concerning the order relation
- the axiom of Dedekind completeness.

Before we approach this objective, it is useful to define the constant symbols.

Definition 4.1.

$$\begin{aligned} \mathbf{0}^{\mathcal{S}} &:= \{\langle x, 0\rangle : x \in \mathbb{Z}\} \\ \mathbf{1}^{\mathcal{S}} &:= \{\langle x, x\rangle : x \in \mathbb{Z}\} \end{aligned}$$

Proposition 4.2. *$\mathbf{0}^{\mathcal{S}}$ and $\mathbf{1}^{\mathcal{S}}$ are slopes.*

Proof. What we have to show is that both $D_{\mathbf{0}^{\mathcal{S}}}$ and $D_{\mathbf{1}^{\mathcal{S}}}$ are finite.

$$\begin{aligned} D_{\mathbf{0}^{\mathcal{S}}} &= \{\mathbf{0}^{\mathcal{S}}(x+y) - \mathbf{0}^{\mathcal{S}}(x) - \mathbf{0}^{\mathcal{S}}(y) : x, y \in \mathbb{Z}\} \\ &= \{0 - 0 - 0 : x, y \in \mathbb{Z}\} \\ &= \{0\} \end{aligned}$$

is clearly finite. For $\mathbf{1}^{\mathcal{S}}$, consider

$$\begin{aligned} D_{\mathbf{1}^{\mathcal{S}}} &= \{\mathbf{1}^{\mathcal{S}}(x+y) - \mathbf{1}^{\mathcal{S}}(x) - \mathbf{1}^{\mathcal{S}}(y) : x, y \in \mathbb{Z}\} \\ &= \{x + y - x - y : x, y \in \mathbb{Z}\} \\ &= \{0 : x, y \in \mathbb{Z}\} \\ &= \{0\}, \end{aligned}$$

which is still finite. Hence, $\mathbf{0}^{\mathcal{S}}$ and $\mathbf{1}^{\mathcal{S}}$ are slopes. □

This allows us to define $\mathbf{0}^{\mathbb{R}}$ and $\mathbf{1}^{\mathbb{R}}$ as the equivalence classes with respect to $\sim^{\mathbb{R}}$ of $\mathbf{0}^{\mathcal{S}}$ and $\mathbf{1}^{\mathcal{S}}$, respectively.

Definition 4.3.

$$\begin{aligned} \mathbf{0}^{\mathbb{R}} &:= [\mathbf{0}^{\mathcal{S}}]^{\mathbb{R}} = [\{\langle x, 0\rangle : x \in \mathbb{Z}\}]^{\mathbb{R}} \\ \mathbf{1}^{\mathbb{R}} &:= [\mathbf{1}^{\mathcal{S}}]^{\mathbb{R}} = [\{\langle x, x\rangle : x \in \mathbb{Z}\}]^{\mathbb{R}} \end{aligned}$$

By the definition of equivalence classes of slopes, $\mathbf{0}^{\mathbb{R}}$ and $\mathbf{1}^{\mathbb{R}}$ are both elements of $\mathbb{R}$, allowing us to define $0^{\mathbb{R}}$ and $1^{\mathbb{R}}$ as follows.

Definition 4.4.

$$0^{\mathbb{R}} := \mathbf{0}^{\mathbb{R}} = [\{\langle x, 0\rangle :\ x \in \mathbb{Z}\}]^{\mathbb{R}}$$
$$1^{\mathbb{R}} := \mathbf{1}^{\mathbb{R}} = [\{\langle x, x\rangle :\ x \in \mathbb{Z}\}]^{\mathbb{R}}$$

4.1 The Field Axioms

In order to verify that the axioms of $\mathbb{R}$ hold in our construction, it is essential to define addition, multiplication and an order relation. All of them inevitably have to be defined via corresponding operations or relations on slopes, which is the reason for the following definition.

Definition 4.5. The **sum** of two slopes λ and μ is defined as the function

$$\lambda +^{\mathcal{S}} \mu := \{\langle x, \lambda(x) + \mu(x)\rangle :\ x \in \mathbb{Z}\}.$$

Definition 4.6.

$$+^{\mathbb{R}} := \{\langle\langle x, y\rangle, z\rangle \in (\mathbb{R} \times \mathbb{R}) \times \mathbb{R} :\ \forall\lambda \in x \forall\mu \in y \forall\nu \in z(\nu \sim^{\mathbb{R}} \lambda +^{\mathcal{S}} \mu)\}$$

Proposition 4.7. *The sum of two slopes is still a slope.*

Proof. We have

$$\begin{aligned} D_{\lambda +^{\mathcal{S}} \mu} &= \{(\lambda +^{\mathcal{S}} \mu)(x+y) - (\lambda +^{\mathcal{S}} \mu)(x) - (\lambda +^{\mathcal{S}} \mu)(y) :\ x, y \in \mathbb{Z}\} \\ &= \{\lambda(x+y) + \mu(x+y) - \lambda(x) - \mu(x) - \lambda(y) - \mu(y) :\ x, y \in \mathbb{Z}\} \\ &= \{\underbrace{(\lambda(x+y) - \lambda(x) - \lambda(y))}_{\in D_\lambda} + \underbrace{(\mu(x+y) - \mu(x) - \mu(y))}_{\in D_\mu} :\ x, y \in \mathbb{Z}\} \\ &\subseteq D_\lambda \oplus D_\mu. \end{aligned}$$

Again, we have that $D_{\lambda +^{\mathcal{S}} \mu}$ is a subset of a finite set, since D_λ and D_μ are finite. This yields that $D_{\lambda +^{\mathcal{S}} \mu}$ is finite and hence, $\lambda +^{\mathcal{S}} \mu$ is a slope. □

We are from now on writing $+$ instead of $+^{\mathcal{S}}$.

Proposition 4.8. *If $\lambda, \lambda', \mu, \mu'$ are slopes with $\lambda \sim^{\mathbb{R}} \lambda'$ and $\mu \sim^{\mathbb{R}} \mu'$, then*

$$\lambda + \mu \sim^{\mathbb{R}} \lambda' + \mu'.$$

Proof. Suppose $\lambda \sim^{\mathbb{R}} \lambda'$ and $\mu \sim^{\mathbb{R}} \mu'$ and consider

$$\begin{aligned} D_{\lambda+\mu, \lambda'+\mu'} &= \{(\lambda + \mu)(x) - (\lambda' + \mu')(x) :\ x \in \mathbb{Z}\} \\ &= \{\lambda(x) + \mu(x) - (\lambda'(x) + \mu'(x)) :\ x \in \mathbb{Z}\} \\ &= \{\underbrace{(\lambda(x) - \lambda'(x))}_{\in D_{\lambda,\lambda'}} + \underbrace{(\mu(x) - \mu'(x))}_{\in D_{\mu,\mu'}} :\ x \in \mathbb{Z}\} \\ &\subseteq D_{\lambda,\lambda'} \oplus D_{\mu,\mu'}. \end{aligned}$$

Hence, with the same arguments as above, $\lambda + \mu \sim^{\mathbb{R}} \lambda' + \mu'$. $\square$

Propositions 4.7 and 4.8 verify that $+^{\mathbb{R}}$ as we defined it is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, i.e. for each pair of real numbers $x, y \in \mathbb{R}$ there exists a real number z, the equivalence class of the sum of two arbitrary slopes in x and y, such that $\langle \langle x, y \rangle, z \rangle \in +^{\mathbb{R}}$, and this real number z is unique. Hence we can now use the following notation.

Definition 4.9.

$$x +^{\mathbb{R}} y = z :\Longleftrightarrow \langle \langle x, y \rangle, z \rangle \in +^{\mathbb{R}}$$

Proposition 4.10. $\widetilde{\mathrm{RA}_1} - \widetilde{\mathrm{RA}_3}$ *hold, i.e.*

(i) $\forall x, y, z, \in \mathbb{R}((x +^{\mathbb{R}} y) +^{\mathbb{R}} z = x +^{\mathbb{R}} (y +^{\mathbb{R}} z))$

(ii) $\forall x \in \mathbb{R}(x +^{\mathbb{R}} 0^{\mathbb{R}} = 0^{\mathbb{R}} +^{\mathbb{R}} x = x)$

(iii) $\forall x, y \in \mathbb{R}(x +^{\mathbb{R}} y = y +^{\mathbb{R}} x)$.

Proof. Consider arbitrary slopes $\lambda \in x, \mu \in y, \nu \in z$.

(i)

$$\begin{aligned}
(x +^{\mathbb{R}} y) +^{\mathbb{R}} z &= [(\lambda + \mu)]^{\mathbb{R}} +^{\mathbb{R}} z \\
&= [(\lambda + \mu) + \nu]^{\mathbb{R}} \\
&= [\{\langle x, (\lambda(x) + \mu(x)) + \nu(x) \rangle : x \in \mathbb{Z}\}]^{\mathbb{R}} \\
&\overset{3.35(i)}{=} [\{\langle x, \lambda(x) + (\mu(x) + \nu(x)) \rangle : x \in \mathbb{Z}\}]^{\mathbb{R}} \\
&= \dots = x +^{\mathbb{R}} (y +^{\mathbb{R}} z).
\end{aligned}$$

(ii)

$$\begin{aligned}
x +^{\mathbb{R}} 0^{\mathbb{R}} &= [\lambda + \{\langle x, 0 \rangle : x \in \mathbb{Z}\}]^{\mathbb{R}} \\
&= [\{\langle x, \lambda(x) + 0 \rangle : x \in \mathbb{Z}\}]^{\mathbb{R}} \\
&\overset{3.35(ii)}{=} [\{\langle x, 0 + \lambda(x) \rangle\} : x \in \mathbb{Z}]^{\mathbb{R}} = \dots && = 0^{\mathbb{R}} +^{\mathbb{R}} x \\
&\overset{3.35(ii)}{=} [\{\langle x, \lambda(x) \rangle : x \in \mathbb{Z}\}]^{\mathbb{R}} = \dots && = x.
\end{aligned}$$

(iii)

$$\begin{aligned}
x +^{\mathbb{R}} y &= [\lambda + \mu]^{\mathbb{R}} \\
&= [\{\langle x, \lambda(x) + \mu(x) \rangle : x \in \mathbb{Z}\}]^{\mathbb{R}} \\
&\overset{3.35(iii)}{=} [\{\langle x, \mu(x) + \lambda(x) \rangle : x \in \mathbb{Z}\}]^{\mathbb{R}} \\
&= \dots = y +^{\mathbb{R}} x.
\end{aligned}$$

$\square$

Like in $\mathbb{Z}$, we need an inverse with respect to addition in $\mathbb{R}$, as well.

Definition 4.11. For a slope $\lambda \in \mathcal{S}$ we define

$$-^{\mathcal{S}}\lambda := \{\langle x, -\lambda(x)\rangle : x \in \mathbb{Z}\}$$

as the **additive inverse of** λ.

Proposition 4.12. *The additive inverse of a slope is still a slope.*

Proof.

$$\begin{aligned} D_{-^{\mathcal{S}}\lambda} &= \{-^{\mathcal{S}}\lambda(x+y) - -^{\mathcal{S}}\lambda(x) - -^{\mathcal{S}}\lambda(y) : x, y \in \mathbb{Z}\} \\ &= \{-\lambda(x+y) - (-\lambda(x)) - (-\lambda(y)) : x, y \in \mathbb{Z}\} \\ &= -[D_\lambda], \end{aligned}$$

which is the image under $-^{\mathbb{Z}}$ of a finite set, and therefore finite. □

Proposition 4.13. *If two slopes λ, μ are equivalent, then so are their additive inverses.*

Proof.

$$\begin{aligned} D_{-^{\mathcal{S}}\lambda, -^{\mathcal{S}}\mu} &= \{-^{\mathcal{S}}\lambda(x) - -^{\mathcal{S}}\mu(x) : x \in \mathbb{Z}\} \\ &= -[D_{\lambda,\mu}]. \end{aligned}$$

With the same reasoning as before, $-^{\mathcal{S}}\lambda \sim^{\mathbb{R}} -^{\mathcal{S}}\mu$ follows from $\lambda \sim^{\mathbb{R}} \mu$. □

Definition 4.14.

$$-^{\mathbb{R}} := \{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} : \forall \lambda \in x \forall \mu \in y(\mu \sim^{\mathbb{R}} -^{\mathcal{S}}\lambda)\}$$

With Propositions 4.12 and 4.13 it follows that $-^{\mathbb{R}}$ is a function from $\mathbb{R}$ to $\mathbb{R}$ and we can immediately define

$$\begin{aligned} -^{\mathbb{R}}x = y &:\Longleftrightarrow \langle x, y\rangle \in -^{\mathbb{R}} \\ x -^{\mathbb{R}} x = z &:\Longleftrightarrow x + -^{\mathbb{R}}y = z. \end{aligned}$$

Proposition 4.15.

$$-^{\mathbb{R}}x + x = x -^{\mathbb{R}} x = 0^{\mathbb{R}}$$

Proof. Let $\lambda \in x$. Then,

$$\begin{aligned} -^{\mathbb{R}}x + x &= [-^{\mathcal{S}}\lambda + \lambda]^{\mathbb{R}} \\ &= \left\{\left\langle x, -^{\mathcal{S}}\lambda(x) + \lambda(x)\right\rangle : x \in \mathbb{Z}\right\} \\ &= \{\langle x, -\lambda(x) + \lambda(x)\rangle : x \in \mathbb{Z}\} \\ &\overset{3.39}{=} \cdots = 0^{\mathbb{R}} \end{aligned}$$

□

Corollary 4.16. *As a simple consequence of Proposition 4.15, we have*

$$\widetilde{\mathrm{RA}_4} \equiv \forall x \in \mathbb{R} \exists y \in \mathbb{R}(x + y = 0^{\mathbb{R}}).$$

The properties of $+^{\mathbb{R}}$ we wanted have been shown now and we can proceed with further definitions, writing $-$ instead of $-^{\mathcal{S}}$ and $+, -$ instead of $+^{\mathbb{R}}, -^{\mathbb{R}}$.

Definition 4.17. We define the **product** of two slopes λ, μ as the function

$$\lambda \cdot^{\mathcal{S}} \mu := \lambda \circ \mu$$

Definition 4.18.

$$\cdot^{\mathbb{R}} := \{\langle\langle x, y\rangle, z\rangle \in (\mathbb{R} \times \mathbb{R}) \times \mathbb{R} : \forall \lambda \in x \forall \mu \in y \forall \nu \in z(\nu \sim^{\mathbb{R}} \lambda \cdot^{\mathcal{S}} \mu)\}.$$

Proposition 4.19. *The product of two slopes is a slope.*

Proof. For $x, y \in \mathbb{Z}$ define

$$\begin{aligned}
\mu(x+y) - \mu(x) - \mu(y) &=: r_{x,y} \in D_\mu \\
\lambda(\mu(x) + \mu(y)) - \lambda(\mu(x)) - \lambda(\mu(y)) &=: s_{x,y} \in D_\lambda \\
\lambda(r_{x,y} + \mu(x) + \mu(y)) - \lambda(r_{x,y}) - \lambda(\mu(x) + \mu(y)) &=: t_{x,y} \in D_\lambda.
\end{aligned}$$

Then we have that

$$\begin{aligned}
D_{\lambda \cdot^{\mathcal{S}} \mu} &= \{(\lambda \cdot^{\mathcal{S}} \mu)(x+y) - (\lambda \cdot^{\mathcal{S}} \mu)(x) - (\lambda \cdot^{\mathcal{S}} \mu)(y) : x, y \in \mathbb{Z}\} \\
&= \{\lambda(\mu(x+y)) - \lambda(\mu(x)) - \lambda(\mu(y)) : x, y \in \mathbb{Z}\} \\
&= \{\lambda(r_{x,y} + \mu(x) + \mu(y)) - (\lambda(\mu(x) + \mu(y)) - s_{x,y}) : x, y \in \mathbb{Z}\} \\
&= \{t_{x,y} + \lambda(\mu(x) + \mu(y)) + \lambda(r_{x,y}) - (\lambda(\mu(x) + \mu(y)) - s_{x,y}) : x, y \in \mathbb{Z}\} \\
&= \{\underbrace{t_{x,y}}_{\in D_\lambda} + \underbrace{\lambda(r_{x,y})}_{\in \lambda[D_\mu]} + \underbrace{s_{x,y}}_{\in D_\lambda} : x, y \in \mathbb{Z}\} \\
&\subseteq D_\lambda \oplus \lambda[D_\mu] \oplus D_\lambda.
\end{aligned}$$

With Proposition 3.63, Lemma 3.64 and Corollary 3.61, we get that $D_{\lambda \cdot^{\mathcal{S}} \mu}$ is finite and hence, $\lambda \cdot^{\mathcal{S}} \mu$ is a slope. □

Proposition 4.20. *If $\lambda, \lambda', \mu, \mu'$ are slopes with $\lambda \sim^{\mathbb{R}} \lambda'$ and $\mu \sim^{\mathbb{R}} \mu'$, then*

$$\lambda \cdot^{\mathcal{S}} \mu \sim^{\mathbb{R}} \lambda' \cdot^{\mathcal{S}} \mu'.$$

Proof. For $x \in \mathbb{Z}$ define

$$\begin{aligned}
\mu(x) - \mu'(x) &=: r_x \in D_{\mu,\mu'} \\
\lambda(\mu'(x)) - \lambda'(\mu'(x)) &=: s_x \in D_{\lambda,\lambda'} \\
\lambda(r_x + \mu'(x)) - \lambda(r_x) - \lambda(\mu'(x)) &=: t_x \in D_\lambda.
\end{aligned}$$

Then we have

$$\begin{aligned}
D_{\lambda \cdot^{\mathcal{S}} \mu, \lambda' \cdot^{\mathcal{S}} \mu'} &= \{\lambda \cdot^{\mathcal{S}} \mu(x) - \lambda' \cdot^{\mathcal{S}} \mu'(x) : x \in \mathbb{Z}\} \\
&= \{\lambda(r_x + \mu'(x)) - \lambda'(\mu'(x)) : x \in \mathbb{Z}\} \\
&= \{\lambda(r_x + \mu'(x)) - (\lambda(\mu'(x)) - s_x) : x \in \mathbb{Z}\} \\
&= \{\underbrace{t_x}_{\in D_\lambda} + \underbrace{\lambda(r_x)}_{\in \lambda[D_{\mu,\mu'}]} + \underbrace{s_x}_{\in D_{\lambda,\lambda'}} : x \in \mathbb{Z}\} \\
&\subseteq D_\lambda \oplus \lambda[D_{\mu,\mu'}] \oplus D_{\lambda,\lambda'}.
\end{aligned}$$

This implies that, as we desired,

$$\lambda \cdot^{\mathcal{S}} \mu \sim^{\mathbb{R}} \lambda' \cdot^{\mathcal{S}} \mu'.$$

□

We now know that the set $\boldsymbol{\cdot}^{\mathbb{R}}$ is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ and hence we introduce the corresponding notation. Propositions 4.19 and 4.20 correspond to Lemma 1 of [A'C03].

Definition 4.21.

$$x \cdot^{\mathbb{R}} y = z :\Longleftrightarrow \langle\langle x, y\rangle, z\rangle \in \boldsymbol{\cdot}^{\mathbb{R}}.$$

Proposition 4.22.

(i) $\forall x, y, z \in \mathbb{R}((x \cdot^{\mathbb{R}} y) \cdot^{\mathbb{R}} z = x \cdot^{\mathbb{R}} (y \cdot^{\mathbb{R}} z))$

(ii) $\forall x \in \mathbb{R}(x \cdot^{\mathbb{R}} 1^{\mathbb{R}} = 1^{\mathbb{R}} \cdot^{\mathbb{R}} x = x)$,

meaning that both $\widetilde{\mathrm{RM}_1}$ *and* $\widetilde{\mathrm{RM}_2}$ *hold.*

Proof. Let $\lambda \in x, \mu \in y$ and $\nu \in z$.

(i) We have already seen in Proposition 3.56 that the composition of functions is associative. This transfers to $\cdot^{\mathbb{R}}$ as follows.

$$\begin{aligned}
(x \cdot^{\mathbb{R}} y) \cdot^{\mathbb{R}} z &= [\lambda \cdot^{\mathcal{S}} \mu]^{\mathbb{R}} \cdot^{\mathbb{R}} \nu \\
&= [(\lambda \cdot^{\mathcal{S}} \mu) \cdot^{\mathcal{S}} \nu]^{\mathbb{R}} \\
&= [(\lambda \circ \mu) \circ \nu]^{\mathbb{R}} \\
&\overset{3.56}{=} [\lambda \circ (\mu \circ \nu)]^{\mathbb{R}} \\
&= \cdots = x \cdot^{\mathbb{R}} (y \cdot^{\mathbb{R}} z).
\end{aligned}$$

(ii) By the definition of $1^{\mathbb{R}}$ it is clear that $\lambda \circ \mathbf{1}^{\mathcal{S}} = \lambda$, as well as $\mathbf{1}^{\mathcal{S}} \circ \lambda = \lambda$. This already proves the statement.

□

Proposition 4.23. $\widetilde{\mathrm{RD}_1}$ *holds, i.e.*

$$\forall x, y, z \in \mathbb{R}(x \cdot^{\mathbb{R}} (y + z) = (x \cdot^{\mathbb{R}} y) + (x \cdot^{\mathbb{R}} z)).$$

Proof. Let $\lambda \in x, \mu \in y, \nu \in z$.

$$\begin{aligned} D_{\lambda \cdot^{\mathcal{S}} (\mu +^{\mathcal{S}} \nu), (\lambda \cdot^{\mathcal{S}} \mu) +^{\mathcal{S}} (\lambda \cdot^{\mathcal{S}} \nu)} &= \{\lambda(\mu(x) + \nu(x)) - (\lambda(\mu(x)) + \lambda(\nu(x))) : x \in \mathbb{Z}\} \\ &= \{\lambda(\mu(x) + \nu(x)) - \lambda(\mu(x)) - \lambda(\nu(x)) : x \in \mathbb{Z}\} \\ &\subseteq D_\lambda, \end{aligned}$$

which is finite. Thus,

$$\lambda \cdot^{\mathcal{S}} (\mu +^{\mathcal{S}} \nu) \sim^{\mathbb{R}} (\lambda \cdot^{\mathcal{S}} \mu) +^{\mathcal{S}} (\lambda \cdot^{\mathcal{S}} \nu).$$

This directly translates to

$$\begin{aligned} x \cdot^{\mathbb{R}} (y +^{\mathbb{R}} z) &= x \cdot^{\mathbb{R}} [\mu +^{\mathcal{S}} \nu]^{\mathbb{R}} \\ &= [\lambda \cdot^{\mathcal{S}} (\mu +^{\mathcal{S}} \nu)]^{\mathbb{R}} \\ &= [(\lambda \cdot^{\mathcal{S}} \mu) +^{\mathcal{S}} (\lambda \cdot^{\mathcal{S}} \nu)]^{\mathbb{R}} \\ &= \cdots = (x \cdot^{\mathbb{R}} y) +^{\mathbb{R}} (x \cdot^{\mathbb{R}} z). \end{aligned}$$

□

For $(\mathbb{R}, +, \cdot^{\mathbb{R}})$ to be a field we still need commutativity of $\cdot^{\mathbb{R}}$ and the existence of a multiplicative inverse. Before proving that the remaining field axioms hold as well, we will show some interesting properties of the real numbers, which will later be of use while showing the rest of the axioms.

4.2 Well Adjusted Slopes

Definition 4.24. A function $f : \mathbb{Z} \to \mathbb{Z}$ is called **odd**, if

$$f(x) = -f(-x)$$

holds for all $x \in \mathbb{Z}$.

Definition 4.25. Let us introduce some notation. For $k \in \mathbb{Z}$ define

$$\begin{aligned} \mathbb{Z}_{<k} &:= \{x \in \mathbb{Z} : x < k\} \\ \mathbb{Z}_{\leq k} &:= \{x \in \mathbb{Z} : x \leq k\} \\ \mathbb{Z}_{>k} &:= \{x \in \mathbb{Z} : x > k\} \\ \mathbb{Z}_{\geq k} &:= \{x \in \mathbb{Z} : x \geq k\}. \end{aligned}$$

Every odd function f is already determined by its restriction $f|_{\mathbb{Z}_{>0}}$, since $f(0) = 0$ follows from the definition of oddity. Analogously, the equivalence class of a slope depends only on the slope's restriction to $\mathbb{Z}_{>0}$, as the following proposition shows.

Proposition 4.26. *For a slope $\lambda \in \mathcal{S}$, the function*

$$\tilde{\lambda}(x) := \begin{cases} 0 & \textit{if } x = 0 \\ \lambda(x) & \textit{if } x \in \mathbb{Z}_{>0} \\ -\lambda(-(x)) & \textit{if } x \in \mathbb{Z}_{<0} \end{cases}$$

is an odd slope which is equivalent to λ.

Proof. First we show that $\tilde{\lambda}$ is a slope.

$$\begin{aligned} D_{\tilde{\lambda}} &= \{\widetilde{\lambda}(x+y) - \widetilde{\lambda}(x) - \widetilde{\lambda}(y) : \ x, y \in \mathbb{Z}\} \\ &= \{\widetilde{\lambda}(x+y) - \widetilde{\lambda}(x) - \widetilde{\lambda}(y) : \ x, y \in \mathbb{Z}_{>0}\} \\ &\quad \cup \{\widetilde{\lambda}(x+y) - \widetilde{\lambda}(x) - \widetilde{\lambda}(y) : \ x, y \in \mathbb{Z}_{\leq 0}\} \\ &\quad \cup \{\widetilde{\lambda}(x+y) - \widetilde{\lambda}(x) - \widetilde{\lambda}(y) : \ x \in \mathbb{Z}_{>0}, y \in \mathbb{Z}_{\leq 0}\} \\ &\subseteq D_\lambda \cup -[D_\lambda] \cup \{0\} \cup \underbrace{\{\widetilde{\lambda}(x+y) - \widetilde{\lambda}(x) - \widetilde{\lambda}(y) : \ x \in \mathbb{Z}_{>0}, y \in \mathbb{Z}_{\leq 0}\}}_{=:A}. \end{aligned}$$

Now consider the set A. If $y = 0$, then $\widetilde{\lambda}(x+y) - \widetilde{\lambda}(x) - \widetilde{\lambda}(y) = 0$. Otherwise, there are two possible cases:

$$0 < x + y \text{ or } x + y \leq 0.$$

In the first case we have

$$\begin{aligned} \widetilde{\lambda}(x+y) - \widetilde{\lambda}(x) - \widetilde{\lambda}(y) &= \lambda(x+y) - \lambda(x) + \lambda(-y) \\ &= -\big(\lambda((x+y) - y) - \lambda(x+y) - \lambda(-y)\big) \\ &\in -[D_\lambda]. \end{aligned}$$

In the other case, if $x + y = 0$ we have $x = -y$ and the result is 0 again, whereas if $x + y < 0$,

$$\begin{aligned} \widetilde{\lambda}(x+y) - \widetilde{\lambda}(x) - \widetilde{\lambda}(y) &= -\lambda(-x-y) - \lambda(x) + \lambda(-y) \\ &= \lambda((-x-y) + x) - \lambda(-x-y) - \lambda(x) \\ &\in D_\lambda. \end{aligned}$$

Hence,

$$A \subseteq D_\lambda \cup -[D_\lambda] \cup \{0\}$$

and therefore $D_{\widetilde{\lambda}}$ is finite, making $\widetilde{\lambda}$ a slope as well.
To see that $\lambda \sim^{\mathbb{R}} \widetilde{\lambda}$, consider

$$\begin{aligned} D_{\lambda, \widetilde{\lambda}} &= \{\lambda(x) - \widetilde{\lambda}(x) : \ x \in \mathbb{Z}\} \\ &= \{\lambda(x) - 0 : x = 0\} \cup \{\lambda(x) - \lambda(x) : \ x \in \mathbb{Z}_{>0}\} \\ &\quad \cup \{\lambda(x) + \lambda(-x) : \ x \in \mathbb{Z}_{<0}\}. \end{aligned}$$

If we set

$$\lambda(0) - \lambda(x) - \lambda(-x) =: r_x \in D_\lambda,$$

this yields

$$\begin{aligned} D_{\lambda,\widetilde{\lambda}} &\subseteq \{\lambda(0)\} \cup \{0\} \cup \{\lambda(0) - r_x : r_x \in D_\lambda\} \\ &= \{\lambda(0), 0\} \cup +_{\lambda(0)}[-[D_\lambda]]. \end{aligned}$$

It follows that $D_{\lambda,\widetilde{\lambda}}$ is finite and thereby,

$$\lambda \sim^{\mathbb{R}} \widetilde{\lambda}.$$

□

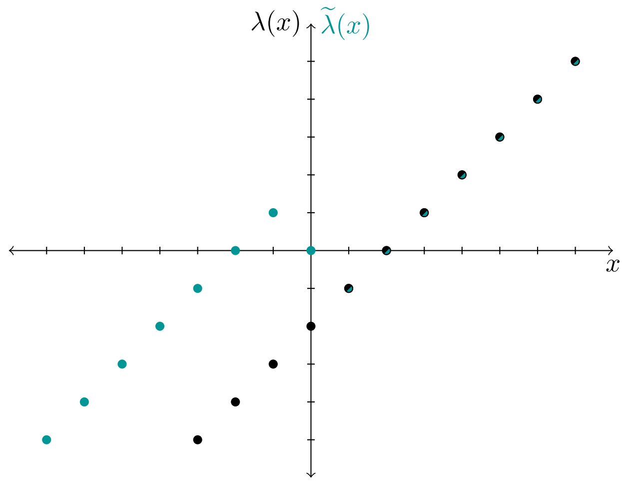

Figure 2: A slope λ together with $\widetilde{\lambda}$, which is odd and equivalent to λ.

Remark 4.27. In the previous proof it becomes clear that for any slope λ we always have

$$D_{\widetilde{\lambda}} \subseteq D_\lambda \cup -[D_\lambda] \cup \{0\}.$$

If λ is an odd function, the finiteness of the set

$$D_\lambda^+ := \{\lambda(x+y) - \lambda(x) - \lambda(y) : x, y \in \mathbb{Z}_{>0}\}$$

therefore already implies that $\lambda \in \mathcal{S}$.

Definition 4.28. An odd slope $\lambda \in \mathcal{S}$ is called **well adjusted** if

$$-1 \leq \lambda(x+y) - \lambda(x) - \lambda(y) \leq 1$$

holds for all $x, y \in \mathbb{Z}$.

Remark 4.29. If λ is a well adjusted slope, one can always write

$$\lambda(x+y) = \lambda(x) + \lambda(y) + r,$$

where $-1 \leq r \leq 1$.

The main goal of this section is to show that every slope is equivalent to a well adjusted slope. For this reason we define some functions on $\mathbb{Z}$ and $\mathscr{P}(\mathbb{Z})$ that will be needed to prove this.

Proposition 4.30. *Every nonempty finite subset $\emptyset \neq A \subseteq \mathbb{Z}$ has an element x_m for which*

$$\forall x \in A(x \leq x_m)$$

holds. Furthermore, this element is unique.

Proof. We show this by induction on n, the number of elements of A. Let $n = \mathrm{s}(0)$. Then, A has exactly one element x, and for this x we have of course that $x \leq x$, since $x = x$. So if we define $x_m := x$, we have that $x \leq x_m$ holds for every element out of A. Clearly, x_m is unique since it is the only element in A.
Now assume that every set with n elements has such a unique maximal element. We can decompose a set A with $\mathrm{s}(n)$ elements into two sets as we have already done earlier, for example in the proof of Proposition 3.63, and get

$$A = A' \cup \{a_0\},$$

where A' is a set of n elements. By our assumption, A' has a unique maximal element we will call x'_m. From our construction of A' we know that $a_0 \notin A'$, implying that $x'_m \neq a_0$. With Proposition 3.47 (iii) we can conclude that either

$$x'_m < a_0 \text{ or } a_0 < x'_m.$$

In the first case, we choose $x_m := a_0$, otherwise $x_m := x'_m$. If $x_m = a_0$, we have by the transitivity of $<$ that

$$\forall x \in A(x \leq x_m)$$

holds. On the other hand if $x_m = x'_m$, then by assumption all elements of A' are already less than or equal to x_m, and since we defined x_m to be x'_m, a_0 is less than x_m, too. Again, the statement follows.
Uniqueness of this maximal element is the result of the following argument. If there were two elements with this property, let's call them x_{m_0} and x_{m_1}, then we would have $x_{m_0} \leq x_{m_1}$ as well as $x_{m_1} \leq x_{m_0}$, i.e.

$$(x_{m_0} = x_{m_1} \vee x_{m_0} < x_{m_1}) \wedge (x_{m_0} = x_{m_1} \vee x_{m_1} < x_{m_0}),$$

which is equivalent to

$$x_{m_0} = x_{m_1} \vee (x_{m_0} < x_{m_1} \wedge x_{m_1} < x_{m_0}).$$

But the latter yields a contradiction since by transitivity, $x_{m_0} < x_{m_0}$ would follow. Hence, $x_{m_0} = x_{m_1}$. □

Definition 4.31. For nonempty finite sets $A \subseteq \mathbb{Z}$ we define

$$\max(A) = x :\Longleftrightarrow x \in A \land \forall y \in A(y \leq x)$$
$$\min(A) = x :\Longleftrightarrow x \in A \land \forall y \in A(y \geq x).$$

$\max(A)$ and $\min(A)$ will be called the **maximum of** A and **minimum of** A, respectively. By Proposition 4.30, max is a function from the nonempty finite subsets of $\mathbb{Z}$ to $\mathbb{Z}$ and in the same way one can show that min is so, as well.

Definition 4.32. For $x, y \in \mathbb{Z}$ define

$$|x| = y :\Longleftrightarrow y = \max(\{x, -x\})$$

as the **absolute value of** x.

The absolute value is certainly a function from $\mathbb{Z}$ to $\mathbb{Z}$ since the set $\{x, -x\}$ is clearly nonempty and finite.

Remark 4.33. One can easily show that the absolute value has the following properties

(i) $\forall x(|x| = 0 \leftrightarrow x = 0)$

(ii) $\forall x, y \in \mathbb{Z}(|x \cdot y| = |x| \cdot |y|)$

(iii) $\forall x, y \in \mathbb{Z}(|x + y| \leq |x| + |y|)$ (Triangle inequality).

Remark 4.34. We can make use of the Peano Axiom of Induction when working on any of the sets introduced in Definition 4.25, since there exist bijective functions from each of them to ω. For example, one can define a function from $\mathbb{Z}_{\leq k}$ to ω similarly to how we defined addition and multiplication on ω. We set $f(k) := 0$ and for $z < k$ demand that $f(z) = x$ implies that $f(z+1) = x'$, where x' is the predecessor of x. Then, proofs by induction need to start with showing the formula holds for k, and after that, one has to prove that if the formula holds for z, then it also holds for $z - 1$.

Definition 4.35. We define the **optimal euclidean division** as

$$:= \{\langle\langle x, y\rangle, z\rangle \in (\mathbb{Z} \times (\mathbb{Z}\backslash\{0\})) \times \mathbb{Z} : \ 2x - |y| \leq 2yz < 2x + |y|\},$$

where 2 is an abbreviation for $\langle \mathrm{s}(\mathrm{s}(0)), 0\rangle$. In the sequel we will sometimes use the commonly known symbols $1, 2, 3, \ldots$ as abbreviations for the according integer numbers.

Proposition 4.36. *For each pair of integers $x \in \mathbb{Z}, y \in \mathbb{Z}\backslash\{0\}$ there exists a unique $z \in \mathbb{Z}$ such that the inequalities*

$$2x - |y| \leq 2yz < 2x + |y|$$

are satisfied.

Proof. The usual euclidean division, which is a well known consequence of the axioms of $\mathbb{Z}$, gives us unique $q, r \in \mathbb{Z}$ for each pair of integers x, y, with $y \neq 0$ such that

$$x = q \cdot y + r, \text{ where } 0 \leq r < |y|.$$

Now consider $0 < y$, that is $|y| = y$. The inequalities then read

$$2x - y \leq 2yz < 2x + y.$$

We claim that for $z = q$ or $z = q + 1$ from the euclidean division, they are satisfied. Distinguishing the two cases

$$0 \leq 2r \leq y \text{ (I) and } y < 2r < 2y \text{ (II)}$$

we have

$$\begin{aligned} \underset{\sim\sim\sim}{2x - y} &= 2(qy + r) - y = (2q - 1)y + 2r \\ &\overset{\text{(I)}}{\leq} (2q - 1)y + y = \underset{\sim\sim}{2qy} < 2qy + 2r + y = \underset{\sim\sim\sim}{2x + y} \end{aligned}$$

in the first case and

$$\begin{aligned} \underset{\sim\sim\sim}{2x - y} &= (2q - 1)y + 2r \overset{\text{(II)}}{<} (2q - 1)y + 2y = (2q + 1)y \\ &< \underset{\sim\sim\sim\sim}{2(q + 1)y} = 2qy + 2y \overset{\text{(II)}}{<} 2qy + y + 2r = \underset{\sim\sim\sim}{2x + y} \end{aligned}$$

in the second case. Analogously, the existence of a z satisfying these inequalities can be shown for the case where $y < 0$. For the uniqueness of this z, notice that the values of $2yz_0$ and $2yz_1$ have a difference of at least $2y$ as soon as $z_0 \neq z_1$. But there are only $2|y|$ integers lying between $2x - |y|$ and $2x + |y|$ (including $2x - |y|$ and excluding $2x + |y|$). Thus, only one of the numbers z_0 and z_1 can satisfy both inequalities. This yields that the optimal euclidean division is indeed a function from $\mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ to $\mathbb{Z}$. □

Definition 4.37. For two integers $x \in \mathbb{Z}, y \in \mathbb{Z}\backslash\{0\}$ the notation $x : y = z$ can now be introduced as

$$x : y = z :\Longleftrightarrow 2x - |y| \leq 2yz < 2x + |y|.$$

The optimal euclidean division will be very important when constructing well adjusted slopes out of slopes. It is more or less what is usually called division with rounding. In the sequel, properties of this division will be shown that are needed for the proof of the Concentration Lemma, which states that every slope has an equivalent well adjusted slope, and for other important proofs later in the thesis.

Corollary 4.38. *For any integers $x, y \in \mathbb{Z}$ with $y \neq 0$, we have*

$$2 \cdot |y \cdot (x : y) - x| \leq |y|.$$

Proof. In the proof of Proposition 4.36 we have seen that if we write $x = qy + r$ with $0 \leq r < |y|$, the value of $x : y$ turns out to be either q or $q+1$ if $0 < y$ and q or $q-1$ if $y < 0$. We consider those three cases separately:

$x : y = q$: This happens only when $2r \leq |y|$ is true. Hence, we can write

$$2 \cdot |y \cdot (x : y) - x| = 2 \cdot |yq - (qy + r)| = 2 \cdot |r| = |2r| \leq |y|.$$

$x : y = q + 1$: In this case, y is positive and $y < 2r < 2y$ holds. We have

$$2 \cdot |y \cdot (x : y) - x| = 2 \cdot |yq + y - (qy + r)| = 2 \cdot |y - r| = |2y - 2r| \leq |y|.$$

$x : y = q - 1$: Here, y has to be negative and $-y < 2r < -2y$.

$$2 \cdot |y \cdot (x : y) - x| = 2 \cdot |yq - y - (qy + r)| = 2 \cdot | - y - r| = | - 2y - 2r| \leq |y|.$$

Thus, we have proven the inequality. □

Corollary 4.39. *For any $a, b \in \mathbb{Z}$ we have*

$$(ay + b) : y = a + b : y.$$

Proof. From the definition of the optimal euclidean division we have that

$$2b - |y| \leq 2y(b : y) < 2b + |y|. \tag{17}$$

It follows immediately that

$$\begin{aligned} 2(ay + b) - |y| &= 2ay + 2b - |y| \\ &\overset{(17)}{\leq} 2ay + 2y(b : y) = 2y(a + (b : y)) \\ &\overset{(17)}{<} 2ay + 2b + |y| = 2(ay + b) + |y|. \end{aligned}$$

□

Corollary 4.40. *For $x \in \mathbb{Z}, y \in \mathbb{Z}_{>0}$,*

$$(x : y) + (-x : y) \in \{-1, 0\},$$

and otherwise if $y \in \mathbb{Z}_{<0}$,

$$(x : y) + (-x : y) \in \{0, 1\}.$$

Proof. The optimal euclidean quotients z_+ and z_- are defined to satisfy

$$\begin{aligned} 2x - |y| &\leq 2yz_+ < 2x + |y| \\ -2x - |y| &\leq 2yz_- < -2x + |y|. \end{aligned}$$

By adding these inequalities we obtain by Proposition 3.48 (i)

$$-2|y| \leq 2y(z_+ + z_-) < 2|y|.$$

Now if $y \in \mathbb{Z}_{>0}$, "dividing" by $2y$ (i.e. applying the monotony of $\cdot^{\mathbb{Z}}$) yields

$$-1 \leq (z_+ + z_-) < 1,$$

while for $y \in \mathbb{Z}_{<0}$ it gives us

$$1 \geq (z_+ + z_-) > -1.$$

□

Lemma 4.41 (Lemma 2 of [A'C03])**.** *Let $y \in \mathbb{Z}_{>0}$ and $a, b, c \in \mathbb{Z}$. If $-y \leq a - b - c \leq y$, then*

$$-1 \leq a : 3y - b : 3y - c : 3y \leq 1.$$

Proof. First we consider

$$\begin{aligned} 2 \cdot 3y \cdot |a : 3y - b : 3y - c : 3y| &= 2 \cdot |3y \cdot (a : 3y) + 3y \cdot (-b : 3y) + 3y \cdot (-c : 3y)| \\ &= 2 \cdot |3y \cdot (a : 3y) - a + a + 3y \cdot (-b : 3y) + b - b \\ &\qquad + 3y \cdot (-c : 3y) + c - c| \\ &\leq 2 \cdot |3y \cdot (a : 3y) - a| + 2 \cdot |3y \cdot (-b : 3y) + b| \\ &\qquad + 2 \cdot |3y \cdot (-c : 3b) + c| + 2 \cdot |a - b - c| \\ &\overset{4.38}{\leq} 3y + 3y + 3y + 2 \cdot |a - b - c| \leq 11y. \end{aligned}$$

Now since both $6y$ and $|a : 3y - b : 3y - c : 3y|$ are non-negative integer numbers, $|a : 3y - b : 3y - c : 3y|$ can only be 0 or 1, otherwise it would contradict the inequality stated above. □

Remark 4.42. In contrast to the proof in [A'C03] we have multiplied the inequalities by $6y$ in order to avoid using fractions; otherwise we would have needed to develop the entire theory of rational numbers within Set Theory in addition to the natural, integer and real numbers. It is however one of the main aspects making this construction especially beautiful that one can directly pass on to the real numbers from the integers without having to introduce fractions.

The analogous procedure will be used in following proofs, without changing the general ideas of A'Campo's proofs.

Lemma 4.43 (Lemma 3 of [A'C03])**.** *Let $x, y \in \mathbb{Z}_{>0}$ and $c \in \mathbb{Z}$. Then,*

$$-1 \leq c : (x \cdot y) - c : \big(x \cdot (x + y)\big) - c : \big(y \cdot (x + y)\big) \leq 1.$$

Proof. Consider

$$\begin{aligned}
&2xy(x{+}y)\cdot\big|c:(x\cdot y)-c:\big(x\cdot(x+y)\big)-c:\big(y\cdot(x+y)\big)\big|\\
&\quad=2\cdot\big|(x+y)\cdot\big(xy\cdot(c:xy)-c+c\big)\\
&\qquad+y\cdot\big(x(x+y)\cdot(-c:(x(x+y))+c-c)\big)\\
&\qquad+x\cdot\big(y(x+y)\cdot(-c:(y(x+y))+c-c)\big)\big|\\
&\quad\le 2(x+y)\cdot|xy\cdot(c:xy)-c|+2y\cdot\big|x(x+y)\cdot\big(-c:(x(x+y))\big)+c\big|\\
&\qquad+2x\cdot\big|y(x+y)\cdot\big(-c:(y(x+y))\big)+c\big|+\underbrace{2\cdot|(x+y)c-yc-xc|}_{=0}\\
&\quad\overset{4.38}{\le}(x+y)xy+yx(x+y)+xy(x+y)=3xy(x+y).
\end{aligned}$$

Again, $\big|c:(x\cdot y)-c:\big(x\cdot(x+y)\big)-c:\big(y\cdot(x+y)\big)\big|$ can only take the values 0 or 1. ☐

At this point we are able to prove the main result of this section: for every slope there is an equivalent slope which is well adjusted. In the following lemma, a construction of a well adjusted slope out of an arbitrary slope is presented. The basic idea underlying the construction is to take advantage of the fact that slopes are generalizations of linear functions.

Consider a linear function $f:\mathbb{Z}\to\mathbb{Z}$. By definition, it certainly satisfies $f(ax)=a\cdot f(x)$ for all integers $a,x\in\mathbb{Z}$. Defining a new function f' through $f'(x)=f(ax):a$ will therefore yield $f'(x)=f(x)$ for all $x\in\mathbb{Z}$, hence $f'=f$. Now for slopes λ that are not linear, $\lambda'(x)=\lambda(ax):a$ will not necessarily turn out to be equal to $\lambda(x)$, but the following lemma shows that their difference is bounded and, if a is chosen big enough, λ' is well adjusted.

Lemma 4.44 (Concentration Lemma, Lemma 4 of [A'C03])**.** *Let $\lambda\in\mathcal{S}$ be a slope and $s\in\mathbb{Z}_{>0}$ such that*

$$-s\le\lambda(x+y)-\lambda(x)-\lambda(y)\le s$$

for all $x,y\in\mathbb{Z}$. Then, $\widetilde{\lambda}'$, the odd function corresponding to

$$\lambda'(x):=\lambda(3sx):3s$$

is a well adjusted slope which is equivalent to λ.

Proof. First we show that $D_{\lambda'}$ is contained in $\{-1,0,1\}$. By our assumption we have that

$$-s\le\lambda(3sx+3sy)-\lambda(3sx)-\lambda(3sy)\le s.$$

With Lemma 4.41 it follows that

$$\begin{aligned}
-1\le\lambda(3s(x+y)):3s-\lambda(3sx):3s-\lambda(3sy):3s\le 1\\
\Longrightarrow -1\le\lambda'(x+y)-\lambda'(x)-\lambda'(y)\le 1
\end{aligned}$$

for all $x,y\in\mathbb{Z}$. It therefore follows from Remark 4.27 that for $\widetilde{\lambda}'$, too,

$$-1\le\widetilde{\lambda}'(x+y)-\widetilde{\lambda}'(x)-\widetilde{\lambda}'(y)\le 1$$

holds and since by definition $\widetilde{\lambda}'$ is odd, it is well adjusted.

Next we prove the equivalence of λ and $\widetilde{\lambda}'$. Without loss of generality we can assume that λ is odd, otherwise we take $\widetilde{\lambda}$ which by Proposition 4.26 is equivalent to λ. By induction on $t \in \mathbb{Z}_{>0}$, we can prove that

$$-s(t-1) \leq \lambda(tx) - t\lambda(x) \leq s(t-1). \tag{18}$$

First let $t = 1$. Since λ is odd, $\lambda(0) = 0$ and hence,

$$0 \leq \lambda(0) - 0 \cdot \lambda(x) \leq 0$$

holds. If we now assume these inequalities hold for $t \in \mathbb{Z}_{>0}$, we have that

$$\begin{aligned} -st = -s(t-1) - s &\leq \lambda(tx) - t\lambda(x) + \big(\lambda(tx + x) - \lambda(tx) - \lambda(x)\big) \\ &= \lambda((t+1)x) - (t+1)\lambda(x) \\ &\leq s(t-1) + s = st, \end{aligned}$$

and thus, (18) is proven. Setting $t = 3s$, this yields

$$-s \cdot 3s \leq -s(3s-1) \leq \lambda(3sx) - 3s\lambda(x) \leq s(3s-1) \leq s \cdot 3s. \tag{19}$$

By Corollary 4.39 it is clear that $(x \cdot y) : y = x$ for all $x, y \in \mathbb{Z}$ with $y \neq 0$. Also, it follows quite easily that $x_0 : y \leq x_1 : z$ for $x_0 \leq x_1$ and $y \neq 0$. Therefore, we have

$$-s \leq \lambda'(x) - \lambda(x) \leq s,$$

by applying ": $3s$" to each side of (19), proving that $\lambda \sim^{\mathbb{R}} \lambda'$ and since we have seen in Proposition 4.26 that $\lambda' \sim^{\mathbb{R}} \widetilde{\lambda}'$, we deduce $\lambda \sim^{\mathbb{R}} \widetilde{\lambda}'$. □

This is a very interesting result, since it tells us that every real number can be represented by a function from $\mathbb{Z}$ to $\mathbb{Z}$ which is — although not necessarily linear — very close to being linear.

Example 4.45. Consider the slope $\lambda(x) := x + k$ with $k \in \mathbb{Z}$, $k \neq 0$. (If $k = 0$, λ is already linear and hence well adjusted.) Clearly,

$$\lambda(x + y) - \lambda(x) - \lambda(y) = -k$$

for all $x, y \in \mathbb{Z}$ and hence we can use $s := |k|$ to construct a well adjusted slope equivalent to λ. For $x \in \mathbb{Z}$, λ' is given by

$$\lambda'(x) = \lambda(3sx) : 3s = (3 \cdot |k| \cdot x - k) : (3 \cdot |k|).$$

Thus we have

$$6|k|x \underbrace{-2k - 3|k|}_{\leq 0} \leq 6|k| \cdot \lambda'(x) < 6|k|x \underbrace{-2k + 3|k|}_{>0}$$
$$\implies \lambda'(x) = x.$$

Since this is already odd, $\widetilde{\lambda}' = \lambda'$, and therefore a linear function in the usual sense is our well adjusted slope equivalent to λ.

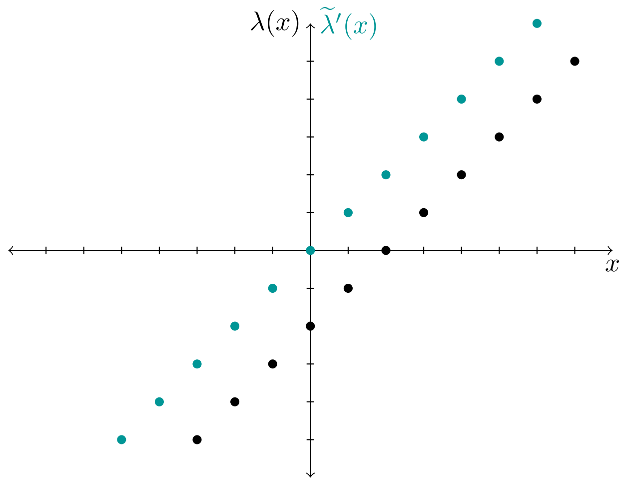

Figure 3: The well adjusted slope $\widetilde{\lambda}'$ which is equivalent to λ.

To become more familiar with well adjusted slopes and slopes in general, we will point out some of their properties.

Definition 4.46. For a slope $\lambda \in \mathcal{S}$ define

$$d_\lambda := \max(|[D_\lambda]|) = \max(\{|x| : x \in D_\lambda\}).$$

Remark 4.47. If λ is an odd slope with $d_\lambda \leq 1$, then λ is well adjusted, and if $d_\lambda = 0$, then λ is linear.

Proposition 4.48. *For $\lambda, \mu \in \mathcal{S}$,*

(i) $d_{\lambda \pm \mu} \leq d_\lambda + d_\mu$

(ii) $d_{\lambda \cdot \mu} \leq \mu(d_\lambda) + 2d_\mu$.

Proof.

(i) Consider

$$\begin{aligned} D_{\mu \pm \lambda} &= \{(\mu \pm \lambda)(x+y) - (\mu \pm \lambda)(x) - (\mu \pm \lambda)(y) : x, y \in \mathbb{Z}\} \\ &= \{\mu(x+y) - \mu(x) - \mu(y) \pm (\lambda(x+y) - \lambda(x) - \lambda(y)) : x, y \in \mathbb{Z}\} \\ &\subseteq D_\mu \oplus \pm[D_\lambda]. \end{aligned}$$

Thus, $d_{\mu\pm\lambda} \leq d_\lambda + d_\mu$.

(ii) Here we proceed similarly. Consider

$$\begin{aligned} \mu(\lambda(x)+\lambda(y)) - \mu(\lambda(x)) - \mu(\lambda(y)) &=: r_{x,y} \in D_\mu \\ \mu(\lambda(x+y)-\lambda(x)-\lambda(y)) - \mu(\lambda(x+y)) + \mu(\lambda(x)+\lambda(y)) &=: s_{x,y} \in D_\mu \\ \lambda(x+y)-\lambda(x)-\lambda(y) &=: t_{x,y} \in D_\lambda. \end{aligned}$$

Then we have

$$\begin{aligned} D_{\mu\cdot\lambda} &= \{\mu\cdot\lambda(x+y) - \mu\cdot\lambda(x) - \mu\cdot\lambda(y) :\ x,y\in\mathbb{Z}\} \\ &= \{\mu(\lambda(x+y)) - \mu(\lambda(x)) - \mu(\lambda(y)) :\ x,y\in\mathbb{Z}\} \\ &= \{\mu(\lambda(x+y)) - \mu(\lambda(x)+\lambda(y)) + r_{x,y} :\ x,y\in\mathbb{Z}\} \\ &= \{\mu(\lambda(x+y)-\lambda(x)-\lambda(y)) + r_{x,y} - s_{x,y} :\ x,y\in\mathbb{Z}\} \\ &= \{\mu(t_{x,y}) + r_{x,y} - s_{x,y} :\ x,y\in\mathbb{Z}\} \\ &\subseteq \mu[D_\lambda] \oplus D_\mu \oplus -[D_\mu]. \end{aligned}$$

This proves that $d_{\mu\cdot\lambda} \leq \mu(d_\lambda) + 2d_\mu$.

□

Proposition 4.49. *For every slope λ and $x\in\mathbb{Z}$ with $n>0$; or λ odd and $n\neq 0$,*

$$|\lambda(nx) - n\lambda(x)| \leq (|n|-1)\cdot d_\lambda$$

holds.

Proof. We prove this by induction on $|n|\in\mathbb{Z}_{>0}$. Let $|n|=1$, that is $n=1$ or $n=-1$. Then,

$$|\lambda(x) - 1\cdot\lambda(x)| = |0| \overset{\text{if } \lambda \text{ odd}}{=} |\lambda(-x)+\lambda(x)|,$$

and $|0|\leq 0\cdot d_\lambda$ of course. Assuming n is positive and $|\lambda(nx)-n\lambda(x)|\leq(|n|-1)\cdot d_\lambda$, we get

$$\begin{aligned} &|\lambda((n+1)x) - (n+1)\lambda(x)| \\ &\quad = |\lambda((n+1)x) - \lambda(nx) - \lambda(x) + \lambda(nx) + \lambda(x) - (n+1)\lambda(x)| \\ &\quad \leq \underbrace{|\lambda((n+1)x) - \lambda(nx) - \lambda(x)|}_{\leq d_\lambda} + \underbrace{|\lambda(nx) - n\lambda(x)|}_{\leq(|n|-1)\cdot d_\lambda} + |\lambda(x)-\lambda(x)| \\ &\quad \leq |n|\cdot d_\lambda. \end{aligned}$$

And for negative n and λ odd,

$$|\lambda((n-1)x) - (n-1)\lambda(x)| \leq |n|\cdot d_\lambda$$

follows analogously. □

Proposition 4.50. *Let $\lambda \in \mathcal{S}$ be a slope. Then,*

(i) for any $n \in \mathbb{Z}$, $|\lambda(n+1) - \lambda(n)| \leq |\lambda(1)| + d_\lambda$.

If additionally $d_\lambda < \lambda(m)$ holds for some $m \in \mathbb{Z}_{>0}$, the following holds.

(ii) For any $n \in \mathbb{Z}$, $\lambda(n) < \lambda(n+m)$.

(iii) For any $n \in \mathbb{Z}_{>0}$, $-d_\lambda + (n : m) \leq \lambda(n)$.

(iv) For any $v \in \mathbb{Z}$, the set $\lambda^{-1}(v) := \{x \in \mathbb{Z} : \lambda(x) = v\}$ has m or fewer elements. That is, there exists a bijection from the set $\lambda^{-1}(v)$ to an element $m^* \in \omega$ for which $m \geq \langle m^*, 0\rangle$ holds.

(v) For any $v \in \mathbb{Z}$ there exists $n \in \mathbb{Z}$ such that $|v - \lambda(n)| \leq |\lambda(1)| + d_\lambda$.

Proof.

(i) The reverse triangle inequality ($|x| - |y| \leq |x-y|$), which is a consequence of the triangle inequality stated in Remark 4.33 (iii), induces that

$$|\lambda(n+1) - \lambda(n)| - |\lambda(1)| \leq |\lambda(n+1) - \lambda(n) - \lambda(1)| \leq d_\lambda,$$

implying

$$|\lambda(n+1) - \lambda(n)| \leq |\lambda(1)| + d_\lambda.$$

(ii) First we observe that

$$-d_\lambda \leq \lambda(n+m) - \lambda(n) - \lambda(m). \tag{20}$$

Together with $d_\lambda < \lambda(m)$ this yields

$$\lambda(n) = -d_\lambda + \lambda(n) + d_\lambda < -d_\lambda + \lambda(n) + \lambda(m) \overset{(20)}{\leq} \lambda(n+m).$$

(iii) Thanks to euclidean division we can always write $n = am + b$ with $0 \leq b < m$. Statement (iii) will now be shown by induction on a. In the case $a = 0$, we have that $0 \leq n < m$. Therefore, the optimal euclidean quotient $n : m$ is either 0 or 1. Hence, showing $\lambda(n) \geq -d_\lambda + 1$ is sufficient for knowing $\lambda(n) \geq -d_\lambda + (n : m)$. So suppose the opposite, i.e. that $\lambda(n) \leq -d_\lambda$.

By Proposition 4.49 we now have that

$$\begin{aligned} \lambda(kn) &\leq k \cdot \lambda(n) + (k-1) \cdot d_\lambda \\ &\leq k \cdot (-d_\lambda) + (k-1) \cdot d_\lambda \\ &= -d_\lambda \end{aligned} \tag{21}$$

holds for any $k \in \mathbb{Z}_{>0}$. Similarly, again for $k \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned} \lambda(km) &\geq k \cdot \lambda(m) - (k-1) \cdot d_\lambda \\ &\geq k \cdot (d_\lambda + 1) - (k-1) \cdot d_\lambda \\ &= k + d_\lambda. \end{aligned} \tag{22}$$

Since both $m, n \in \mathbb{Z}_{>0}$, these two inequalities together imply

$$0 < n + d_\lambda \overset{(22)}{\leq} \lambda(nm) \overset{(21)}{\leq} -d_\lambda \leq 0,$$

yielding a contradiction. Hence, the assumption $\lambda(n) \leq -d_\lambda$ must have been false, showing $\lambda(n) \geq -d_\lambda + (n : m)$ in the case $a = 0$.

Assuming that $\lambda(n) \geq -d_\lambda + (n : m)$ holds if $n = am + b$, $0 \leq b < m$, we now show that it also holds if a is increased by one.

$$\begin{aligned} \lambda(n) &= \lambda((a+1)m + b) = \lambda((am + b) + m) \\ &\overset{\text{(ii)}}{\geq} \lambda(am + b) + 1 \\ &\overset{\text{ass.}}{\geq} -d_\lambda + ((am + b) : m) + 1 \\ &= -d_\lambda + \big(((a+1)m + b) : m\big) = -d_\lambda + (n : m). \end{aligned}$$

This proves the inequality for every $n \in \mathbb{Z}_{>0}$.

(iv) We divide $\mathbb{Z}$ into m sets as follows.

$$\mathbb{Z}_{k,m} := \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}(x = ym + k)\}.$$

It is a consequence of the existence of the euclidean division that $\mathbb{Z}$ is the union of all $\mathbb{Z}_{k,m}$ with $0 \leq k < m$. Furthermore, there can be at most one element $x \in \mathbb{Z}_{k,m}$ for which $\lambda(x) = v$, since if there were more, say $x_0 = y_0 m + k$ and $x_1 = y_1 m + k$ where without loss of generality $y_0 < y_1$, this would imply (with induction on the difference $d := y_1 - y_0$ between y_0 and y_1)

$$\lambda(x_0) = \lambda(y_0 m + k) \overset{\text{(ii)}}{<} \lambda((y_0 + 1)m + k) \overset{\text{(ii)}}{<} \ldots \overset{\text{(ii)}}{<} \lambda(y_1 m + k) = \lambda(x_1).$$

Thus, we can now write

$$\begin{aligned} \{x \in \mathbb{Z} : \lambda(x) = v\} &= \bigcup \big\{\mathbb{Z}_{k,m} : 0 \leq k < m\big\} \cap \{x \in \mathbb{Z} : \lambda(x) = v\} \\ &= \bigcup \big\{\mathbb{Z}_{k,m} \cap \{x \in \mathbb{Z} : \lambda(x) = v\} : 0 \leq k < m\big\}, \end{aligned}$$

which is, by induction on m, a finite set containing at most m elements.

(v) Using (ii), one can show by induction that $\lambda(zm) \geq z$ for any $z \in \mathbb{Z}_{>0}$ and moreover, $\lambda(-zm) \leq \lambda(0) - z$ for any $z \in \mathbb{Z}_{>0}$. Therefore, we can surely find $z_0, z_1 \in \mathbb{Z}$ for which

$$\lambda(z_0) < v < \lambda(z_1)$$

holds. Now we can prove the statement by induction on the difference between z_0 and

z_1. For let $d := z_0 - z_1$ first be equal to 1. Then,

$$|\lambda(z_1) - v| = \lambda(z_1) - v < \lambda(z_1) - \lambda(z_0) \\ = |\lambda(z_1) - \lambda(z_1 + 1)| \overset{(i)}{\leq} |\lambda(1)| + d_\lambda.$$

So now, assume that for a distance d, we can always find a z between z_0 and z_1 where the value of $\lambda(z)$ is near v. For $d + 1$, it follows that the distance between $z_0 + 1$ and z_1 is d. There are now three possible cases.

$\lambda(z_0 + 1) = v$: In this case we are already finished.

$\lambda(z_0 + 1) < v$: Here we can reduce the problem to the one where the distance was d, since we have

$$\lambda(z_0 + 1) < v < \lambda(z_1).$$

$v < \lambda(z_0 + 1)$: In this case it is even easier. We are in the problem with distance 1, since we have found

$$\lambda(z_0) < v < \lambda(z_0 + 1).$$

□

4.3 The Field Axioms Revisited

In Section 4.1 we have already seen that most of the field axioms hold in our model of $\mathbb{R}$. But it is only now that we are sufficiently prepared to show the remaining ones.

Proposition 4.51. $\widetilde{\mathrm{RM}_3}$ *holds. That is,*

$$\forall x, y \in \mathbb{R}(x \cdot^{\mathbb{R}} y = y \cdot^{\mathbb{R}} x).$$

Proof. We will show that for two slopes $\lambda, \mu \in \mathcal{S}$,

$$\lambda \cdot \mu \sim^{\mathbb{R}} \mu \cdot \lambda$$

holds. From this, commutativity of $\cdot^{\mathbb{R}}$ follows automatically. Without loss of generality, assume λ and μ are well adjusted, i.e. $d_\lambda \leq 1$ and $d_\mu \leq 1$. Thus, $\lambda \cdot \mu(0) = 0 = \mu \cdot \lambda(0)$. For $n \in \mathbb{Z}\backslash\{0\}$ we have by Proposition 4.49

$$n(\lambda \cdot \mu(n)) = n\lambda(\mu(n)) = \lambda(n\mu(n)) + r_0 = \lambda(\mu(n)n) + r_0 = \mu(n) \cdot \lambda(n) + s_0 + r_0$$
$$n(\mu \cdot \lambda(n)) = n\mu(\lambda(n)) = \mu(n\lambda(n)) + r_1 = \mu(\lambda(n)n) + r_1 = \lambda(n) \cdot \mu(n) + s_1 + r_1,$$

where

$$\begin{aligned}
|r_0| &\le |n| \cdot d_\lambda \le |n| \\
|s_0| &\le |\mu(n)| \cdot d_\lambda \le |\mu(n)| \le |n| \cdot d_\mu + |n||\mu(1)| = |n| \cdot (d_\mu + |\mu(1)|) \\
&\le |n| \cdot (1 + |\mu(1)|) \\
|r_1| &\le |n| \cdot d_\mu \le |n| \\
|s_1| &\le |\lambda(n)| \cdot d_\mu \le |\lambda(n)| \le |n| \cdot d_\lambda + |n||\lambda(1)| = |n| \cdot (d_\lambda + |\lambda(1)|) \\
&\le |n| \cdot (1 + |\lambda(1)|).
\end{aligned}$$

Hence we can write

$$\begin{aligned}
|n||\lambda \cdot \mu(n) - \mu \cdot \lambda(n)| &= |n \cdot (\lambda \cdot \mu(n)) - n \cdot (\mu \cdot \lambda(n))| \\
&= |\mu(n) \cdot \lambda(n) + s_0 + r_0 - (\lambda(n) \cdot \mu(n) + s_1 + r_1)| \\
&= |s_0 + r_0 - s_1 - r_1| \le |s_0| + |r_0| + |s_1| + |r_1| \\
&\le |n|\big(4 + \mu(1) + \lambda(1)\big).
\end{aligned}$$

Now,

$$|\lambda \cdot \mu(n) - \mu \cdot \lambda(n)| \le 4 + \mu(1) + \lambda(1)$$

follows for all $n \in \mathbb{Z}$, including 0, hence the set $D_{\lambda\cdot\mu,\mu\cdot\lambda}$ has finitely many elements. The equivalence of $\lambda \cdot \mu$ and $\mu \cdot \lambda$ follows, and with it the commutativity of our multiplication on $\mathbb{R}$. □

Proposition 4.52.

$$\forall x \in \mathbb{R}\backslash\{0^{\mathbb{R}}\} \exists y \in \mathbb{R}(x \cdot^{\mathbb{R}} y = y \cdot^{\mathbb{R}} x = 1^{\mathbb{R}}),$$

which means that $\widetilde{\mathrm{RM}_4}$ *is true.*

Proof. Let $\lambda \in x$ be a well adjusted slope. We are now going to construct a slope that is a $\cdot^{\mathcal{S}}$-right inverse of λ. Since x is nonzero, $\lambda[\mathbb{Z}]$ must be an infinite set and hence there exists $m \in \mathbb{Z}_{>0}$ with $1 < |\lambda(m)|$. As a consequence of Proposition 4.50 (v) we know that for each $v \in \mathbb{Z}$ there is $n_v \in \mathbb{Z}$ for which

$$|v - \lambda(n_v)| \le |\lambda(1)| + 1.$$

Since there are only finitely many values $\lambda(n_v)$ could possibly take, we know that the set

$$P_\lambda(v) := \{x \in \mathbb{Z} : \ |v - \lambda(x)| \le |\lambda(1)| + 1\},$$

being the union of finitely many sets of the form $\lambda^{-1}(n_v^*)$ which by Proposition 4.50 (iv) are all finite, is itself a finite set. Therefore, we are allowed to define the following function from $\mathbb{Z}$ to $\mathbb{Z}$.

$$\mu(v) := \min(P_\lambda(v)).$$

We now want to show that μ is a slope. Consider

$$\begin{aligned}|\lambda(\mu(x+y)-\mu(x)-\mu(y))| &= |\lambda(z_{x+y}-z_x-z_y)| \\ &\leq |\lambda(z_{x+y})+\lambda(-z_x-z_y)|+1 \\ &= |\lambda(z_{x+y})-\lambda(z_x+z_y)|+1 \\ &\leq |\lambda(z_{x+y})-\lambda(z_x)-\lambda(z_y)|+2 \\ &\leq |(x+y)-x-y|+3\cdot\big(|\lambda(1)|+1\big)+2 \\ &= 3\cdot|\lambda(1)|+5,\end{aligned}$$

where $x, y \in \mathbb{Z}, z_{x+y} \in P_\lambda(x+y), z_x \in P_\lambda(x)$ and $z_y \in P_\lambda(y)$. This is a finite number, implying that $\lambda[D_\mu]$ is finite. Again, since λ takes each value only finitely many times, this induces that also D_μ is finite and hence, $\mu \in \mathcal{S}$. It remains to show that $\lambda \cdot \mu \sim^{\mathbb{R}} \mathbf{1}^{\mathcal{S}}$. By the definition of $P_\lambda(x)$, we have

$$|\lambda\cdot\mu(x)-x| = |\lambda(\mu(x))-x| = |\lambda(\min(P_\lambda(x)))-x| \leq |\lambda(1)|+1.$$

This implies that $D_{\lambda\cdot\mu,\mathbf{1}^{\mathcal{S}}}$ is finite, showing $\lambda\cdot\mu \sim^{\mathbb{R}} \mathbf{1}^{\mathcal{S}}$. Setting $y := [\mu]^{\mathbb{R}}$, this yields

$$x \cdot^{\mathbb{R}} y = [\lambda\cdot\mu]^{\mathbb{R}} = [\mathbf{1}^{\mathcal{S}}]^{\mathbb{R}} = 1^{\mathbb{R}}.$$

$\square$

We have now shown that in our construction of $\mathbb{R}$, all field axioms hold.

Remark 4.53. The multiplicative inverse, whose existence for all real numbers $x \neq 0^{\mathbb{R}}$ we have just shown above, is unique. This is a consequence of the field axioms and makes the following a definition of a function from $\mathbb{R}\backslash\{0^{\mathbb{R}}\}$ to itself.

Definition 4.54.

$$\begin{aligned}{}^{\mathbf{-1}} &:= \{\langle x,y\rangle \in (\mathbb{R}\backslash\{0^{\mathbb{R}}\})\times(\mathbb{R}\backslash\{0^{\mathbb{R}}\}) :\ x\cdot^{\mathbb{R}} y = 1^{\mathbb{R}}\} \\ x^{-1} = y &:\Longleftrightarrow \langle x,y\rangle \in^{\mathbf{-1}} .\end{aligned}$$

We say that y is the **multiplicative inverse of** x and sometimes will also write $\frac{1}{x}$ instead of x^{-1}.

4.4 The Order Relation

Now we can advance to the axioms of order, $\widetilde{\mathrm{RO}_1} - \widetilde{\mathrm{RO}_5}$. For this purpose we continue with the definition of an order relation on $\mathbb{R}$.

Definition 4.55. For two slopes λ, μ we define

$$\begin{aligned}G_{\lambda,\mu} &:= \{\mu(x)-\lambda(x) :\ x \in \mathbb{Z}_{\geq 0} \wedge \mu(x)-\lambda(x) \leq 0\} \\ \lambda <^{\mathcal{S}} \mu &:\Longleftrightarrow G_{\lambda,\mu} \text{ is finite and } D_{\lambda,\mu} \text{ is infinite.}\end{aligned}$$

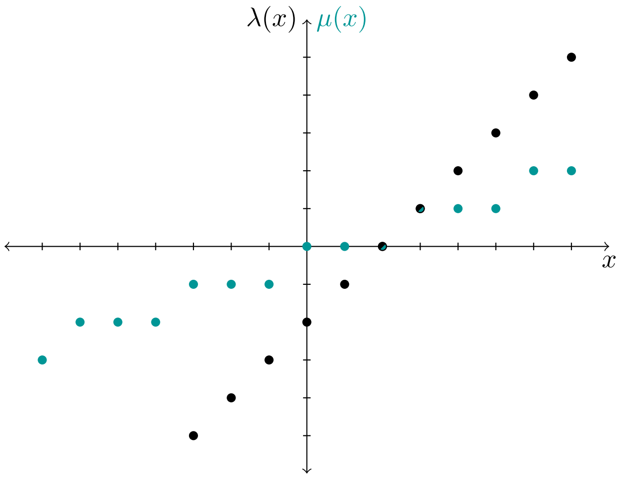

Figure 4: Two slopes, λ and μ, with $\mu <^{\mathcal{S}} \lambda$.

Proposition 4.56. *Let $\lambda, \lambda', \mu, \mu' \in \mathcal{S}$ be such that $\lambda \sim^{\mathbb{R}} \lambda'$ and $\mu \sim^{\mathbb{R}} \mu'$. Then,*

$$\lambda <^{\mathcal{S}} \mu \Longleftrightarrow \lambda' <^{\mathcal{S}} \mu'.$$

Proof. Assuming that $\lambda <^{\mathcal{S}} \mu$, we know that $G_{\lambda,\mu}$ is finite, whereas $D_{\lambda,\mu}$ is infinite. So, consider

$$G_{\lambda',\mu'} = \{\mu'(x) - \lambda'(x) : \ x \in \mathbb{Z}_{\geq 0} \wedge \mu'(x) - \lambda'(x) \leq 0\}.$$

With

$$\begin{aligned} \mu(x) - \mu'(x) &=: r_x \in D_{\mu,\mu'} \\ \lambda(x) - \lambda'(x) &=: s_x \in D_{\lambda,\lambda'} \end{aligned}$$

we can rewrite the condition:

$$\begin{aligned} \mu'(x) - \lambda'(x) \leq 0 &\Longleftrightarrow \mu'(x) - \mu(x) + \mu(x) - \lambda(x) + \lambda(x) - \lambda'(x) \leq 0 \\ &\Longleftrightarrow -r_x + \mu(x) - \lambda(x) + s_x \leq 0 \\ &\Longleftrightarrow \mu(x) - \lambda(x) \leq r_x - s_x. \end{aligned}$$

Moreover, since $D_{\lambda,\lambda'}$ and $D_{\mu,\mu'}$ are both finite, $D_{\mu,\mu'} \oplus -[D_{\lambda,\lambda'}]$ is a finite nonempty set and therefore has a maximal element we will call m. We now know that

$$\begin{aligned} G_{\lambda',\mu'} &= \{\mu'(x) - \lambda'(x) : \ x \in \mathbb{Z}_{\geq 0} \wedge \mu(x) - \lambda(x) \leq r_x - s_x\} \\ &\subseteq \{\mu'(x) - \lambda'(x) : \ x \in \mathbb{Z}_{\geq 0} \wedge \mu(x) - \lambda(x) \leq m\}. \end{aligned}$$

If $m \leq 0$, then $\mu(x) - \lambda(x) \leq m$ implies that $\mu(x) - \lambda(x) \leq 0$. Hence,

$$\begin{aligned} G_{\lambda',\mu'} &\subseteq \{\mu'(x) - \lambda'(x) : \, x \in \mathbb{Z}_{\geq 0} \wedge \mu(x) - \lambda(x) \leq 0\} \\ &= \{\mu'(x) - \mu(x) + \mu(x) - \lambda(x) + \lambda(x) - \lambda'(x) : \\ &\qquad x \in \mathbb{Z}_{\geq 0} \wedge \mu(x) - \lambda(x) \leq 0\} \\ &\subseteq \{\mu(x) - \lambda(x) : \, x \in \mathbb{Z}_{\geq 0} \wedge \mu(x) - \lambda(x) \leq 0\} \oplus -[D_{\mu,\mu'}] \oplus D_{\lambda,\lambda'} \\ &= G_{\lambda,\mu} \oplus -[D_{\mu,\mu'}] \oplus D_{\lambda,\lambda'}, \end{aligned}$$

which is finite and thus, $G_{\lambda',\mu'}$ is finite.
On the other hand if $m \leq 0$ doesn't hold, i.e. $0 < m$, with the same reasoning as above we get that

$$\begin{aligned} G_{\lambda',\mu'} &\subseteq \{\mu(x) - \lambda(x) : \, x \in \mathbb{Z}_{\geq 0} \wedge \mu(x) - \lambda(x) \leq m\} \oplus -[D_{\mu,\mu'}] \oplus D_{\lambda,\lambda'} \\ &= \big(\{\mu(x) - \lambda(x) : \, x \in \mathbb{Z}_{\geq 0} \wedge \mu(x) - \lambda(x) \leq 0\} \\ &\quad \cup \{\mu(x) - \lambda(x) : \, x \in \mathbb{Z}_{\geq 0} \wedge \mu(x) - \lambda(x) \in M\}\big) \oplus -[D_{\mu,\mu'}] \oplus D_{\lambda,\lambda'} \\ &\subseteq \big(G_{\lambda,\mu} \cup M\big) \oplus -[D_{\mu,\mu'}] \oplus D_{\lambda,\lambda'}, \end{aligned}$$

where M is the set containing the numbers $0, 1, \ldots, m$, i.e.

$$M = \{x \in \mathbb{Z}_{\geq 0} : \, x \leq m\}.$$

It is clear that M is finite, hence $G_{\lambda,\mu} \cup M$ is finite and thereby $G_{\lambda',\mu'}$ is finite, as well.

The second thing we need to show is that $D_{\lambda',\mu'}$ is infinite. This will be proven by contradiction. Suppose $D_{\lambda',\mu'}$ is finite. Then,

$$\begin{aligned} D_{\lambda,\mu} &= \{\lambda(x) - \mu(x) : \, x \in \mathbb{Z}\} \\ &= \{\underbrace{\lambda(x) - \lambda'(x)}_{\in D_{\lambda,\lambda'}} + \lambda'(x) - \mu'(x) + \underbrace{\mu'(x) - \mu(x)}_{\in -[D_{\mu,\mu'}]} : \, x \in \mathbb{Z}\} \\ &\subseteq D_{\lambda,\lambda'} \oplus -[D_{\mu,\mu'}] \oplus D_{\lambda',\mu'}. \end{aligned}$$

This is of course finite, contradicting our assumption that $D_{\lambda,\mu}$ is infinite. Therefore, $D_{\lambda',\mu'}$ cannot be finite, implying that it is infinite and thus,

$$\lambda' <^{\mathcal{S}} \mu'$$

holds. The other implication follows in the exact same way. □

Since we now know that the $<^{\mathcal{S}}$ relation is independent of the choice of a representative in an equivalence class of slopes, it makes sense to transfer its definition to $\mathbb{R}$.

Definition 4.57. For $x, y \in \mathbb{R}$ define

$$x <^{\mathbb{R}} y :\Longleftrightarrow \forall \lambda \in x \forall \mu \in y(\lambda <^{\mathcal{S}} \mu).$$

Lemma 4.58 (Order Lemma)**.** *Let $x, y \in \mathbb{R}$ and $\lambda \in x, \mu \in y$ be odd slopes representing them. Then,*

$$x <^{\mathbb{R}} y \Longleftrightarrow \exists n \in \mathbb{Z}_{>0}(\mu(n) - \lambda(n) > d_\lambda + d_\mu).$$

Proof. λ and μ being odd implies that $\mu - \lambda$ is odd and we know from Proposition 4.48 that $d_{\lambda-\mu} \leq d_\lambda + d_\mu$. Now for the first direction, assume that there doesn't exist $n \in \mathbb{Z}_{>0}$ for which $\mu(n) - \lambda(n) > d_\lambda + d_\mu$. Hence, either $G_{\lambda,\mu}$ is infinite, in which case $x \not<^{\mathbb{R}} y$ would follow, or $G_{\lambda,\mu}$ is finite. In that case, we observe the following.

$$\begin{aligned}
\{\mu(x) - \lambda(x) : x \in \mathbb{Z}_{>0}\} &= \{\mu(x) - \lambda(x) : x \in \mathbb{Z}_{>0} \wedge \mu(x) - \lambda(x) \leq 0\} \\
&\quad \cup \{\mu(x) - \lambda(x) : x \in \mathbb{Z}_{>0} \wedge \mu(x) - \lambda(x) > 0\} \\
&= G_{\lambda,\mu} \cup \{\mu(x) - \lambda(x) : x \in \mathbb{Z}_{>0} \wedge 0 < \mu(x) - \lambda(x) \leq d_\lambda + d_\mu\} \\
&\subseteq G_{\lambda,\mu} \cup \{1, 2, \ldots, d_\lambda + d_\mu\}.
\end{aligned}$$

So $\{\mu(x) - \lambda(x) : x \in \mathbb{Z}_{>0}\}$ is still finite. But this implies that also $D_{\mu,\lambda}$ is finite, since

$$\begin{aligned}
D_{\mu,\lambda} &= \{\mu(x) - \lambda(x) : x \in \mathbb{Z}\} \\
&= \{\mu(x) - \lambda(x) : x \in \mathbb{Z}_{>0}\} \cup \{\mu(x) - \lambda(x) : x \in \mathbb{Z}_{\leq 0}\} \\
&= \{\mu(x) - \lambda(x) : x \in \mathbb{Z}_{>0}\} \cup \{0\} \cup \{-(\mu - \lambda)(-x) : -x \in \mathbb{Z}_{>0}\} \\
&= \{\mu(x) - \lambda(x) : x \in \mathbb{Z}_{>0}\} \cup \{0\} \cup -[\{(\mu - \lambda)(x) : x \in \mathbb{Z}_{>0}\}].
\end{aligned}$$

Therefore, $x \not<^{\mathbb{R}} y$ follows in the second case as well and thus by contraposition, the first direction is proven.
Now for the other direction, assume there is $n \in \mathbb{Z}_{>0}$ for which $\mu(n) - \lambda(n) > d_\lambda + d_\mu$. Then, with Proposition 4.50 (iii) applied to the slope $\mu - \lambda$, we get that

$$-(d_\lambda + d_\mu) + (k : n) \leq (\mu - \lambda)(k)$$

for all $k \in \mathbb{Z}_{>0}$. Hence from a certain point on, $(\mu - \lambda)(k)$ will always be positive and consequently $G_{\lambda,\mu}$ is finite. From Proposition 4.50 (ii) we can deduce that $\mu(x) - \lambda(x)$ can get arbitrarily small and therefore, $D_{\mu,\lambda}$ is infinite, implying $x <^{\mathbb{R}} y$. □

Remark 4.59. The statement of the Order Lemma also holds for any number greater than $d_\lambda + d_\mu$.

Proposition 4.60. *The* ZF*-transformed order axioms $\widetilde{\mathrm{RO}_1} - \widetilde{\mathrm{RO}_3}$ are true:*

(i) $\forall x \in \mathbb{R} \neg(x <^{\mathbb{R}} x)$

(ii) $\forall x, y, z \in \mathbb{R}(x <^{\mathbb{R}} y \wedge y <^{\mathbb{R}} z \to x <^{\mathbb{R}} z)$

(iii) $\forall x, y \in \mathbb{R}(x <^{\mathbb{R}} y \vee y <^{\mathbb{R}} x \vee x = y)$.

Proof. For real numbers x, y, z let $\lambda \in x, \mu \in y, \nu \in z$ be well adjusted slopes.

(i) Consider the set

$$\begin{aligned} D_{\lambda,\lambda} &= \{\lambda(x) - \lambda(x) : \ x \in \mathbb{Z}\} \\ &= \{0\}. \end{aligned}$$

It is clearly finite and therefore not infinite, which implies that $\lambda \not<^{\mathcal{S}} \lambda$ and accordingly, $x \not<^{\mathbb{R}} x$.

Remark 4.61. The condition that $D_{\lambda,\mu}$ must be infinite for $\lambda <^{\mathcal{S}} \mu$ to hold induces that λ and μ are not $\sim^{\mathbb{R}}$-equivalent slopes.

(ii) With the Order Lemma, $x <^{\mathbb{R}} y$ and $y <^{\mathbb{R}} z$ implies that there are $n, m \in \mathbb{Z}_{>0}$ such that

$$\mu(n) - \lambda(n) > 2 \text{ and } \nu(m) - \mu(m) > 2.$$

Moreover, $d_{\mu-\lambda}$ and $d_{\nu-\mu}$ are both less than or equal to 2 by Proposition 4.48 (i). Therefore,

$$\begin{aligned} \nu(nm) - \lambda(nm) &= \nu(nm) - \mu(nm) + \mu(nm) - \lambda(nm) \\ &= (\nu - \mu)(nm) + (\mu - \lambda)(mn) \\ &\overset{4.49}{=} n \cdot (\nu - \mu)(m) + r + m \cdot (\mu - \lambda)(n) + s \\ &\geq (n + m) \cdot 3 + r + s, \end{aligned}$$

where $|r| \leq (n-1) \cdot 2$ and $|s| \leq (m-1) \cdot 2$. This means

$$\nu(nm) - \lambda(nm) \geq n + m + 4.$$

Hence we are done since with the Order Lemma, $x <^{\mathbb{R}} z$ follows.

(iii) If $D_{\mu,\lambda}$ is finite, we have that $\lambda \sim^{\mathbb{R}} \mu$ and thus $x = y$. So assume $D_{\mu,\lambda}$ is infinite, i.e.

$$\{\mu(x) - \lambda(x) : \ x \in \mathbb{Z}\}$$

is infinite. Since both λ and μ are odd, $\mu - \lambda$ is odd as well and therefore there must exist $n \in \mathbb{Z}_{>0}$ for which

$$|\mu(n) - \lambda(n)| > 2.$$

If $|\mu(n) - \lambda(n)| = \mu(n) - \lambda(n) > 2$, it follows from the Order Lemma that $x <^{\mathbb{R}} y$. The other possibility, $|\mu(n) - \lambda(n)| = -(\mu(n) - \lambda(n)) = \lambda(n) - \mu(n) > 2$ analogously implies $y <^{\mathbb{R}} x$, proving the statement.

□

Corollary 4.62. *If λ and μ are two well adjusted slopes such that $\lambda \sim^{\mathbb{R}} \mu$, then*

$$D_{\lambda,\mu} \subseteq \{-2, -1, 0, 1, 2\}.$$

Proof. Suppose $|\lambda(x_0) - \mu(x_0)| > 2$ for some $x_0 \in \mathbb{Z}$. This x_0 must be different from 0 since both λ and μ are odd and hence $\lambda(0) = \mu(0) = 0$. So now, either $x_0 \in \mathbb{Z}_{>0}$ or $x_0 \in \mathbb{Z}_{<0}$. But we only need to consider the first case, since again by the oddity of λ and μ,

$$|\lambda(x_0) - \mu(x_0)| = | - \lambda(-x_0) + \mu(-x_0)| = |\lambda(-x_0) - \mu(-x_0)|.$$

Without loss of generality we have that $\lambda(x_0) - \mu(x_0) > 2$ with $x_0 \in \mathbb{Z}_{>0}$ and thus, by the Order Lemma, $\mu <^{\mathcal{S}} \lambda$. But since $\lambda \sim^{\mathbb{R}} \mu$, this contradicts Proposition 4.60 (i). □

Proposition 4.63. $\widetilde{\mathrm{RO}_4}$ *and* $\widetilde{\mathrm{RO}_5}$ *hold. I.e.*

(i) $\forall x, y, z \in \mathbb{R}(x <^{\mathbb{R}} y \to (x + z <^{\mathbb{R}} y + z))$

(ii) $\forall x, y, z \in \mathbb{R}(x <^{\mathbb{R}} y \wedge 0^{\mathbb{R}} <^{\mathbb{R}} z \to (x \cdot^{\mathbb{R}} z <^{\mathbb{R}} y \cdot^{\mathbb{R}} z))$.

Proof. Let $\lambda \in x, \mu \in y, \nu \in z$ be well adjusted slopes. If $x <^{\mathbb{R}} y$, we have $\lambda <^{\mathcal{S}} \mu$, i.e. there is $m_0 \in \mathbb{Z}_{>0}$ such that $\mu(m_0) - \lambda(m_0) > 2$, which is equivalent to

$$\lambda(m_0) \leq \mu(m_0) - 3.$$

(i) For $n \in \mathbb{Z}_{>0}$ we therefore have

$$\begin{aligned} \lambda(nm_0) &\overset{4.49}{\leq} n \cdot \lambda(m_0) + n - 1 \\ &\leq n \cdot (\mu(m_0) - 3) + n - 1 \\ &= n \cdot \mu(m_0) - 2n - 1 \\ &\overset{4.49}{\leq} \mu(nm_0) + n - 1 - 2n - 1 \\ &= \mu(nm_0) - n - 2. \end{aligned} \tag{23}$$

This implies

$$\lambda(nm_0) + \nu(nm_0) \leq \mu(nm_0) + \nu(nm_0) - n - 2,$$

which, after setting $n = 3$, yields

$$\mu(3m_0) + \nu(3m_0) - (\lambda(3m_0) + \nu(3m_0)) \geq 5 > 4 \geq d_{\lambda+\nu} + d_{\mu+\nu}.$$

Hence, $\lambda + \nu <^{\mathcal{S}} \mu + \nu$ and therefore $x + z <^{\mathbb{R}} y + z$.

(ii) If additionally $0^{\mathbb{R}} <^{\mathbb{R}} z$, there exists $k_0 \in \mathbb{Z}_{>0}$ such that $\nu(k_0) > 2$, implying

$$\nu(nk_0) \geq n \cdot \nu(k_0) - (n - 1) \geq 3n - n + 1 = 2n + 1 \geq 2n$$

for $n \in \mathbb{Z}_{>0}$. We want to make use of the Order Lemma, thus our aim is to find an element x_0 of $\mathbb{Z}_{>0}$ for which

$$\nu \cdot \mu(x_0) - \nu \cdot \lambda(x_0) > \lambda(1) + \mu(1) + 4 \overset{4.48}{\geq} d_{\nu \cdot \lambda} + d_{\nu \cdot \mu}$$

holds. This will then imply $\nu \cdot \lambda <^{\mathcal{S}} \nu \cdot \mu$ and with it,

$$x \cdot^{\mathbb{R}} z = z \cdot^{\mathbb{R}} x <^{\mathbb{R}} z \cdot^{\mathbb{R}} y = y \cdot^{\mathbb{R}} z.$$

We have

$$\begin{aligned}\nu \cdot \lambda(m_0 k_0 n) = \nu(\lambda(n k_0 m_0)) &\overset{(23)}{=} \nu(\mu(n k_0 m_0) - n k_0 - 2 - r_+) \\ &\leq \nu(\mu(n k_0 m_0)) - \underbrace{\nu(n k_0)}_{\geq 2n} - \underbrace{\nu(2 + r_+)}_{\geq -1} + 2 \\ &\leq \nu \cdot \mu(n k_0 m_0) - 2n + 3,\end{aligned}$$

where r_+ is some non-negative integer and therefore by Proposition 4.50 (iii) $\nu(2+r_+) \geq -1$ holds. This implies

$$\nu \cdot \mu(n k_0 m_0) - \nu \cdot \lambda(n k_0 m_0) \geq 2n - 3 > 2n.$$

Since $\lambda(1) + \mu(1) + 4$ is a fixed value it becomes clear that there is certainly $n_0 \in \mathbb{Z}_{>0}$ such that $2n_0 \geq \lambda(1) + \mu(1) + 4$ and in consequence,

$$\nu \cdot \lambda(n_0 k_0 m_0) - \nu \cdot \mu(n_0 k_0 m_0) > 2n_0 \geq \lambda(1) + \mu(1) + 4,$$

which was to be proven.

□

4.5 Dedekind Completeness

The only axiom remaining is the least-upper-bound property characteristic for the reals. Without it, we could just as well be dealing with the rational numbers since they satisfy all the other axioms as well. For this purpose it is of course essential to define what a least upper bound is.

Definition 4.64. Let A be a subset of $\omega, \mathbb{Z}$ or $\mathbb{R}$. An **upper bound of** A is a natural, integer or real number m for which

$$\forall x \in A (x \leq m),$$

where for $x, y \in \mathbb{R}$ we define

$$x \leq^{\mathbb{R}} y :\Longleftrightarrow x <^{\mathbb{R}} y \vee x = y$$

and from now on write $<$ and $\leq$ instead of $<^{\mathbb{R}}$ and $\leq^{\mathbb{R}}$. m' is said to be the **least upper bound of** A if for all upper bounds m of A

$$m' \leq m$$

holds.

Remark 4.65. A subset of $\mathbb{R}$ can have several upper bounds, however never more than one least upper bound.

Proposition 4.66. *Every nonempty subset A of $\mathbb{Z}$ which has an upper bound also has a maximum, which simultaneously is its least upper bound.*

Proof. We can prove this by induction. Since A is nonempty, we are given an element $x \in A$ and $m \in \mathbb{Z}$ an upper bound of A. Consider $d := m - x$. Certainly, $d \in \mathbb{Z}_{\geq 0}$ and we can use induction on d. If $d = 0$, this means that $x = m$ and hence we have already found an element of A satisfying $y \leq x$ for all $y \in A$, which makes x the maximum of A.
Assuming that for any given difference d we find that A has a maximum, we want to show it for $d + 1$, also. We differentiate between two cases. If $m - 1$ is still an upper bound of A, we can reduce this to the case of d and find the maximum of A. Otherwise we know that there exists $y \in A$ such that $y \leq m$ but $y \nleq m-1$, i.e. $y \geq m$. Thus, $m = y \in A$ is the maximum of A.

Now of course the maximum m of any subset of $\mathbb{Z}$ is automatically its least upper bound since it satisfies $x \leq m$ for all $x \in A$ and any $m^* < m$ obviously doesn't. □

Proposition 4.67. *Every nonempty subset of* $\mathbb{R}$ *which has an upper bound also has a least upper bound. Or in other words,* RC_1 *holds.*

Proof. Let D be an arbitrary nonempty subset of $\mathbb{R}$ with an upper bound m. Let $\mu \in m$ be a well adjusted slope and consider

$$\Delta := \{\delta \in \mathcal{S} : \delta \text{ is well adjusted and } \exists d \in D(\delta \in d)\}.$$

Since m is an upper bound of D, $\delta <^{\mathcal{S}} \mu \vee \delta \sim^{\mathbb{R}} \mu$ holds for every $\delta \in \Delta$. Therefore, $\mu \not<^{\mathcal{S}} \delta$ which implies that $\delta(n) - \mu(n) \leq 2$ for all $n \in \mathbb{Z}_{>0}, \delta \in \Delta$, or equivalently,

$$\delta(n) \leq \mu(n) + 2.$$

That is, $\mu(n) + 2$ is an upper bound of $\{\delta(n) : \delta \in \Delta\}$. By Proposition 4.66, the set $\{\delta(n) : \delta \in \Delta\}$ has a maximum, making the odd extension of

$$\sigma(n) := \max(\{\delta(n) : \delta \in \Delta\})$$

a valid definition of a function from $\mathbb{Z}$ to $\mathbb{Z}$. Our aim is to show that σ represents a real number, i.e. σ is a slope, and that the real number containing σ is the least upper bound of D.
For each $z \in \mathbb{Z}$ we denote by Δ_z the set of all slopes δ_z in Δ for which

$$\delta_z(z) = \sigma(z)$$

holds. Notice that Δ_z is nonempty for every $z \in \mathbb{Z}$. Consider $p, k \in \mathbb{Z}_{>0}$ and $q := pk$. Now for arbitrary elements $\delta_p \in \Delta_p$ and $\delta_q \in \Delta_q$ we know that δ_p and δ_q are well adjusted slopes, hence we have

$$\begin{aligned}
2k \cdot |\delta_q(q) : k - \delta_q(p)| &= 2|k \cdot (\delta_q(q) : k) - k \cdot \delta_q(p)| \\
&\overset{4.38}{\leq} 2|\delta_q(q) - k \cdot \delta_q(p)| + k \\
&= 2|\delta_q(pk) - k \cdot \delta_q(p)| + k \\
&\overset{4.49}{\leq} 3k - 2 < 4k, \text{ hence} \\
|\delta_q(q) : k - \delta_q(p)| \leq 1.
\end{aligned}$$

Furthermore, $\delta_q(p) \leq \delta_p(p)$ by the definition of δ_p. Together this implies

$$\delta_q(q) : k \leq \delta_q(p) + 1 \leq \delta_p(p) + 1. \tag{24}$$

Besides that, we have

$$k \cdot \delta_p(p) \leq \delta_p(kp) + k - 1 \leq \delta_q(q) + k - 1,$$

from which we deduce

$$\begin{aligned}
(k \cdot \delta_p(p) - \delta_q(q)) : k &\leq (k-1) : k \leq 1 \\
\overset{4.39}{\Longrightarrow} \delta_p(p) + (-\delta_q(q) : k) &\leq 1 \\
\Longrightarrow \delta_p(p) + \underbrace{(-\delta_q(q) : k) + \delta_q(q) : k}_{\overset{4.40}{\geq} -1} - \delta_q(q) : k &\leq 1 \\
\Longrightarrow \delta_p(p) - \delta_q(q) : k &\leq 2.
\end{aligned} \tag{25}$$

Inequalities (24) and (25) together imply

$$|\delta_p(p) - \delta_q(q) : k| \leq 2. \tag{26}$$

Since $\delta_p \in \Delta_p$ and $\delta_q \in \Delta_q$ were arbitrary, (26) holds for all elements of Δ_p and Δ_q respectively. Now consider $x, y \in \mathbb{Z}_{>0}$ and define $c := \max(\{\delta(xy(x+y)) : \delta \in \Delta\}) = \sigma(xy(x+y))$. Choosing different values among x, y and $x+y$ for p and k and applying (26) we get

$$\begin{aligned}
|\sigma(x) - c : (y(x+y))| &\leq 2 \\
|\sigma(y) - c : (x(x+y))| &\leq 2 \\
|\sigma(x+y) - c : xy| &\leq 2.
\end{aligned}$$

Now, finally, the time has come to make use of Lemma 4.43.

$$\begin{aligned}|\sigma(x+y)-\sigma(x)-\sigma(y)| =& |\sigma(x+y)-c:xy+c:xy \\ & -\sigma(x)+c:(y(x+y))-c:(y(x+y)) \\ & -\sigma(y)+c:(x(x+y))-c:(x(x+y))| \\ \leq & 2+2+2+|c:xy-c:(y(x+y))-c:(x(x+y))| \\ \overset{4.43}{\leq} & 7.\end{aligned}$$

Thus, D_σ is finite with $d_\sigma \leq 7$, implying that σ, as desired, is indeed a slope. It remains to show that the real number $s := [\sigma]^{\mathbb{R}}$ is the least upper bound of D. First of all, s is an upper bound of D because for any $d \in D$, let $\delta \in \Delta$ be a well adjusted slope representing d. Furthermore,

$$\delta(x) \leq \sigma(x)$$

holds for every $x \in \mathbb{Z}_{>0}$ and thus, $\sigma <^{\mathcal{S}} \delta$ can't be satisfied, yielding $d \leq s$.
Now consider $s' < s$. Then there is $x \in \mathbb{Z}_{>0}$ for which

$$\sigma(x) - \sigma'(x) > 8.$$

Since for every $\delta_x \in \Delta_x$ we have that $\delta_x(x) = \sigma(x)$, we now know that $\sigma' <^{\mathcal{S}} \delta_x$ and hence,

$$s' < [\delta_x]^{\mathbb{R}} \in D.$$

This implies that any real number less than s is not an upper bound of D and thus, s is its least upper bound. □

Remark 4.68. In A'Campo's proof, Δ is defined to be a set of well adjusted representatives of the elements of D. But since we wanted to avoid choosing representatives of these real numbers, we included all well adjusted slopes contained in elements of D in Δ. As we have seen, the proof still works and additionally, we didn't make use of the Axiom of Choice. However, there is a constructive way to "choose" a unique slope as the odd function corresponding to $\rho(x) := \max(\{\lambda(x) : \lambda \in r\})$ for $x \in \mathbb{Z}_{>0}$ out of any real number r, inspired by the proof we have just seen.

From this, the Archimedean Property follows as a corollary. However, it is nice to know that it can also be shown independently. But for being able to even state it, we need to introduce what natural numbers are within ℝ.

Definition 4.69. We define

$$\begin{aligned}\omega_{\mathbb{R}} &:= \{x \in \mathbb{R} : \exists \lambda \in x \exists n \in \mathbb{Z}_{\geq 0} \forall z \in \mathbb{Z} (\lambda(z) = n \cdot z)\} \\ \mathbb{Z}_{\mathbb{R}} &:= \{x \in \mathbb{R} : \exists \lambda \in x \exists n \in \mathbb{Z} \forall z \in \mathbb{Z} (\lambda(z) = n \cdot z)\}\end{aligned}$$

to be the sets of **natural numbers in** ℝ and **integer numbers in** ℝ, respectively.

One can easily check that all properties we have shown for ω and $\mathbb{Z}$ also hold within $\omega_{\mathbb{R}}$ and $\mathbb{Z}_{\mathbb{R}}$ by identifying n with the equivalence class of the slope λ defined by $\lambda(x) = \langle n, 0\rangle \cdot x$ or $\lambda(x) = n \cdot x$ for an element n of ω or $\mathbb{Z}$ respectively.

Definition 4.70. We define $\widetilde{n} := [\lambda]^{\mathbb{R}}$, where $\lambda(x) = k \cdot x$ for all $x \in \mathbb{Z}$ and

$$k = \begin{cases} n & \text{if } n \in \mathbb{Z} \\ \langle n, 0\rangle & \text{if } n \in \omega. \end{cases}$$

Proposition 4.71 (Archimedean Property)**.** *Let $x \in \mathbb{R}$ be a real number greater than $0^{\mathbb{R}}$ and $x < y$. Then there exists a number $n \in \omega_{\mathbb{R}}$ such that*

$$y < n \cdot^{\mathbb{R}} x.$$

Proof. Let $\lambda \in x$ and $\mu \in y$ be well adjusted slopes. Since both x and y are greater than $0^{\mathbb{R}}$, there must be $z_0 \in \mathbb{Z}_{>0}$ such that $\lambda(z_0) > 1$. Then,

$$\lambda(2z_0) \geq 2\underbrace{\lambda(z_0)}_{\geq 2} - 1 \geq 3$$

and hence, $\lambda(2z_0) > 2$. Now define $k := 1 + \max(\{\mu(2z_0), 0\})$ and let $k\lambda$ be the slope defined by

$$k\lambda(x) = k \cdot \lambda(x).$$

As a consequence of Proposition 4.48, $d_{k\lambda} \leq k$ holds and thus,

$$k\lambda(2z_0) \geq k \cdot 3 = k + 2k > \mu(2z_0) + 2k \geq \mu(2z_0) + k + 1$$

implies, by the Order Lemma, that $\mu <^{\mathcal{S}} k\lambda$. We now notice that $k\lambda = \kappa \cdot \lambda$, where $\kappa(x) = k \cdot x$ represents an element of $\omega_{\mathbb{R}}$. It turns out we have found a natural number, $n := [\kappa]^{\mathbb{R}}$, for which

$$y < n \cdot^{\mathbb{R}} x$$

holds. ∎

We have now completely shown that the set $\mathbb{R}$ constructed as the set of equivalence classes of slopes, together with addition ($+^{\mathbb{R}}$), multiplication ($\cdot^{\mathbb{R}}$) and the order relation ($<^{\mathbb{R}}$) satisfies all the desired axioms and therefore yields a model of the real numbers. An interesting observation to make is that during the entire process of defining sets and proving properties we only made use of the axioms ZF_0-ZF_6 (ZF_{0-6}). In particular, the construction does not depend on the Axiom of Choice.
Another important thing to remark is that the existence of this model of the reals is only granted if we assume the existence of a model of ZF_{0-6}. But as we have seen in Section 3.1, Peano Arithmetic can be formalized within this theory and therefore, by Gödel's Second Incompleteness Theorem, it cannot prove its own consistency. Hence, by his Completeness Theorem, we do not know whether or not there is a model of ZF_{0-6} and with it, a model based on A'Campo's construction.

5 Nonstandard Models

After having constructed a model of $\mathbb{R}$ based on a model of Set Theory, assuming there is one, the question arises what the model of the reals looks like when considering a nonstandard model of Set Theory. For example by the Löwenheim-Skolem Theorem we know that ZF has a countable model and since each real number is an object in this model, the existence of a countable model of $\mathbb{R}$ follows. Other approaches to obtain nonstandard models of $\mathbb{R}$ are considering the ultrapower of a standard model or, what will be done below, using the Compactness Theorem. There are various reasons for which one can be interested in nonstandard models of the reals, one of which is nonstandard analysis. It was discovered by Abraham Robinson that many proofs in analysis can be strongly simplified with the help of nonstandard models of the real numbers [Rob96]. In this chapter we illustrate a nonstandard model of $\mathbb{R}$ based on a nonstandard model of Set Theory inducing a nonstandard model of ω. The picture of nonstandard real numbers which is obtained when considering them as equivalence classes of slopes remains very concrete.

5.1 Introduction

For the rest of the chapter, we assume consistency of ZF and and although it is an abuse of notation, denote by $\omega, \mathbb{Z}, \mathbb{R}$ the corresponding objects in the domain of a model of ZF_{0-6} in which the set ω together with the relevant operations and relations is isomorphic to the standard model of PA defined in [Hal14, pp. 67-70]. We call the elements of $\omega, \mathbb{Z}, \mathbb{R}$ **standard natural, integer** and **real numbers** respectively and denote the model itself by V. Moreover, addition, subtraction, multiplication and order relation will be denoted by $+, -, \cdot, <$ respectively, regardless of the set on which they are defined. $\mathscr{L}$ is defined to be the the language one obtains after adding all usual operation and relation symbols and those we have introduced in this thesis so far to $\mathscr{L}_{\mathrm{ZF}} = \{\in\}$; by Ψ we mean the collection of all axioms of ZF_{0-6} together with the formulae needed to define the symbols other than $\in$, and $\overline{\Psi}$ denotes the set of all sentences that hold in V.

Definition 5.1. We define $\mathscr{L}^* := \mathscr{L} \cup \{\mathrm{c}\}$ for a constant symbol c which isn't already contained in $\mathscr{L}$. Consider

$$\overline{\Psi}^* := \overline{\Psi} \cup \{\mathrm{c} \in \omega\} \cup \{n < c :\ \varphi(n)\}, \text{ where}$$
$$\varphi(n) \equiv \text{“}n \text{ is an } \mathscr{L}\text{-term of the form } \mathrm{s}\ldots\mathrm{s}(0) \text{ with a finite chain of } \mathrm{s}\text{”}.$$

Theorem 5.2. $\overline{\Psi}^*$ *has a model.*

Proof. Remember that we are assuming that ZF is consistent and with it is Ψ. Every finite subset of $\overline{\Psi}^*$ is consistent, since it contains only finitely many sentences of $\overline{\Psi}^* \backslash \overline{\Psi}$ and for any finite set of natural numbers we can always find an element of ω greater than all of them. Hence by the Compactness Theorem, $\overline{\Psi}^*$ is consistent, too. By the Completeness Theorem, $\overline{\Psi}^*$ has a model. □

Let V^* be a model of $\overline{\Psi}^*$. From now on, we denote by $\omega^*, \mathbb{Z}^*, \mathbb{R}^*, \mathrm{c}$ the corresponding objects in the domain of V^*. We call $\omega^*, \mathbb{Z}^*$ and $\mathbb{R}^*$ the **nonstandard natural, integer** and **real numbers**, respectively.

Remark 5.3.

(i) Notice that every sentence which is true in V is automatically true in V^*, as well. In particular, all properties of number systems, slopes, and sets in general that we have shown in the previous chapters, such as the Concentration Lemma, still hold in V^*.

(ii) For the construction of the real numbers out of the natural numbers, we have only needed the axioms $ZF_0 - ZF_6$. But in the proof of Theorem 5.2 we have made use of the Compactness Theorem, which, according to [Hal11, pp. 131-134], follows from the Axiom of Choice and is even equivalent to a choice principle which is only slightly weaker than AC. Also, the Ultrafilter Theorem, which is needed for the other main possibility to obtain nonstandard models, is equivalent to the Compactness Theorem.

Definition 5.4. According to [Rob96, p. 49], every object a in the domain of V has a corresponding object a^* in the domain of V^* with the same $\mathscr{L}$-properties (properties that can be described using $\mathscr{L}$-sentences). a^* will be called the **nonstandard** or V^***-version of** a.

Like this, we can interpret the sets $\omega, \mathbb{Z}, \mathbb{R}$ as "subsets" of $\omega^*, \mathbb{Z}^*, \mathbb{R}^*$. They are not actual subsets because there are no objects in V^* containing just the nonstandard versions of natural numbers of V for example. Being the nonstandard version of an object of V is not something that can be described with $\mathscr{L}^*$-formulae.

We refer to elements of nonstandard number systems that are also contained in their corresponding standard version as **standard***, otherwise they will be called **nonstandard***.

Similarly, the set of slopes $\mathcal{S}^*$ in V^* contains $\mathcal{S}$, where the latter again is called the set of standard* slopes. Notice that even though we call them standard*, they are functions from $\mathbb{Z}^*$ to $\mathbb{Z}^*$, which is the reason we don't call them standard.

Let us now point out some properties of these sets.

Proposition 5.5. *There exists an integer number in $\mathbb{Z}^*$ which is greater (smaller) than all standard* integer numbers, i.e. $\mathbb{Z}^*$ has nonstandard* elements.*

Proof. By the definition of V^*, both $\mathrm{c} \in \omega$ and $n <^{\mathbb{N}} \mathrm{c}$ are true for every standard* natural number n. Consider $\langle \mathrm{c}, 0 \rangle$. It is an element of $\mathbb{Z}^*$ and by the definition of the order relation on the integers,

$$\langle n, 0 \rangle <^{\mathbb{Z}} \langle \mathrm{c}, 0 \rangle$$

holds for every standard* natural number n, hence $\langle \mathrm{c}, 0 \rangle$ is greater than all standard* non-negative integers. That the negative standard* integers are smaller as well follows from the non-negativity of c. For a number smaller than all standard* integers, consider $\langle 0, \mathrm{c} \rangle$ analogously. □

We denote the integer $\langle \mathrm{c}, 0\rangle$ by c as well from now on. It should be clear from the context whether the natural or the integer version is meant.

Proposition 5.6. *In $\mathbb{R}^*$ there exist real numbers greater than all standard* real numbers, as well as real numbers that are greater than $0^{\mathbb{R}}$ but smaller than any positive standard* real number.*

Proof. Consider $\widetilde{\mathrm{c}} \in \mathbb{R}^*$. For a standard* real number r we can always find a standard* integer number $\widetilde{n}$ greater than r. To show this we consider a well adjusted slope $\lambda \in r$ and define $n := \lambda(1) + 2$. Then, $\nu(x) = n \cdot x$ defines a linear and hence well adjusted slope ν which is contained in $\widetilde{n}$. Furthermore,

$$\nu(1) - \lambda(1) = 3 > 2 \geq d_\nu + d_\lambda,$$

implying $\lambda <^{\mathcal{S}} \nu$ and therefore $r < \widetilde{n}$.

n is a standard* integer since we have not used c for its definition. Therefore,

$$(n+2) \cdot x < \mathrm{c} \cdot x$$

holds for all positive x, in particular for $x = 1$, which implies that $\widetilde{n} <^{\mathbb{R}} \widetilde{\mathrm{c}}$ and thus, $r <^{\mathbb{R}} \widetilde{\mathrm{c}}$.

Let's now turn our attention to the second part of the proposition. For a positive standard* real number r there is always a standard* natural number m such that $\frac{1}{\widetilde{m}} <^{\mathbb{R}} r$. This inequality is equivalent to $\frac{1}{r} <^{\mathbb{R}} \widetilde{m}$ and hence we obtain $\widetilde{m}$ as we just saw in the proof of the first part of the statement. Analogously, $\frac{1}{\widetilde{\mathrm{c}}} <^{\mathbb{R}} \frac{1}{\widetilde{m}}$ is equivalent to $\widetilde{m} <^{\mathbb{R}} \widetilde{\mathrm{c}}$, which is indeed true for every standard* natural number m. Clearly, $0^{\mathbb{R}} <^{\mathbb{R}} \frac{1}{\widetilde{\mathrm{c}}}$, completing our proof. $\square$

Remark 5.7. $-\widetilde{\mathrm{c}}$ is a nonstandard* real number being smaller than all standard* ones; $-\frac{1}{\widetilde{\mathrm{c}}}$ is greater than all negative standard* real numbers but still smaller than $0^{\mathbb{R}}$.

Definition 5.8. We can distinguish two types of nonstandard* real numbers. Those being greater or smaller than *all* standard* real numbers will be called **infinite** while we call the other ones **finite**. Furthermore, we will call those lying between $-r$ and r for *all* standard* real numbers r except $0^{\mathbb{R}}$ **infinitesimal** or **infinitely close to** $0^{\mathbb{R}}$. Slopes in equivalence classes of finite, infinite or infinitesimal real numbers will also be called finite, infinite or infinitesimal slopes, respectively.

While there are no infinitesimal numbers in ω^* or $\mathbb{Z}^*$, we can still distinguish between finite and infinite natural and integer numbers analogously.

Remark 5.9.

(i) Finiteness of numbers as we just defined it above is not to confuse with finiteness of sets defined in Definition 3.59. Since $\mathrm{c} \in \omega^*$, we consider a set with c elements as finite although we call c (or $\widetilde{\mathrm{c}}$) an infinite number. The infiniteness of c is not a property that can be defined with sentences of $\mathscr{L}$ or even $\mathscr{L}^*$.

(ii) The multiplicative inverse of an infinite number is always infinitesimal; and vice versa. This follows immediately from the fact that for positive x and r,

$$x < r \quad \text{for } r \text{ standard}^* \iff \tfrac{1}{r} < \tfrac{1}{x} \quad \text{for } r \text{ standard}^*$$

and that multiplicative inverses of standard* numbers are still standard*.

5.2 Slopes in Nonstandard Models

The primary idea of nonstandard analysis is to consider standard sequences and functions etc. and define properties such as convergence, continuity or differentiability via their nonstandard version. The nonstandard version of a sequence of real numbers $s : \omega \to \mathbb{R}$ is a function $s^* : \omega^* \to \mathbb{R}^*$. Since elements of ω^* are standard* if and only if they are finite, s^* can be viewed as an extension of s.

$$s^*(n) = (s(n))^*$$

holds for all finite natural numbers n. The analogous statement also holds for slopes, i.e. for functions from the integers to themselves instead of from the natural numbers to the reals. In this section, properties of nonstandard slopes will be pointed out, yielding some very nice visual interpretations for what standard* and nonstandard* real numbers look like in our model V^*.

Remark 5.10. Slopes in V^* are defined as functions $\lambda : \mathbb{Z}^* \to \mathbb{Z}^*$ such that $D_\lambda = \{\lambda(x+y) - \lambda(x) - \lambda(y) : x, y \in \mathbb{Z}^*\}$ is finite, i.e. that there is a natural number $n \in \omega^*$ such that a bijection $f : D_\lambda \hookrightarrow\!\!\!\rightarrow n$ exists. This natural number can in particular be a nonstandard* natural number, i.e. an infinite natural number.

In the sequel we will sometimes use the phrase "λ takes the value y at the index x" when talking about slopes λ. With the words "value" and "index" we want to emphasize whether an integer serves as image or as preimage under λ of another integer.

Proposition 5.11. *A function $\lambda : \mathbb{Z}^* \to \mathbb{Z}^*$ for which $\lambda(a) = 0$ for all infinite integers a is equivalent to the zero slope $\mathbf{0}^{\mathcal{S}}(x) = 0$, independent of the values taken at finite indices.*

Proof. We know from Lemma 3.23 that $n \in \mathrm{c}$ for every standard* natural number n. Thus, c is a finite set which contains all the standard* natural numbers. As an easy consequence thereof we have the existence of a finite set of integers containing all standard* integers. We denote this set by Z. Now,

$$\begin{aligned} D_{\lambda,\mathbf{0}^{\mathcal{S}}} &= \{\lambda(x) - \mathbf{0}^{\mathcal{S}}(x) : x \in \mathbb{Z}^*\} \\ &= \{\lambda(x) - 0 : x \in \mathbb{Z}^*\} \\ &= \{\lambda(x) : x \in Z\} \cup \{\underbrace{\lambda(x)}_{=0} : x \in (\mathbb{Z}^* \setminus Z)\} \\ &= \lambda[Z] \cup \{0\}. \end{aligned}$$

which is a finite set since Z is finite. Hence, λ is a slope with $\lambda \sim^{\mathbb{R}} \mathbf{0}^{\mathcal{S}}$.

□

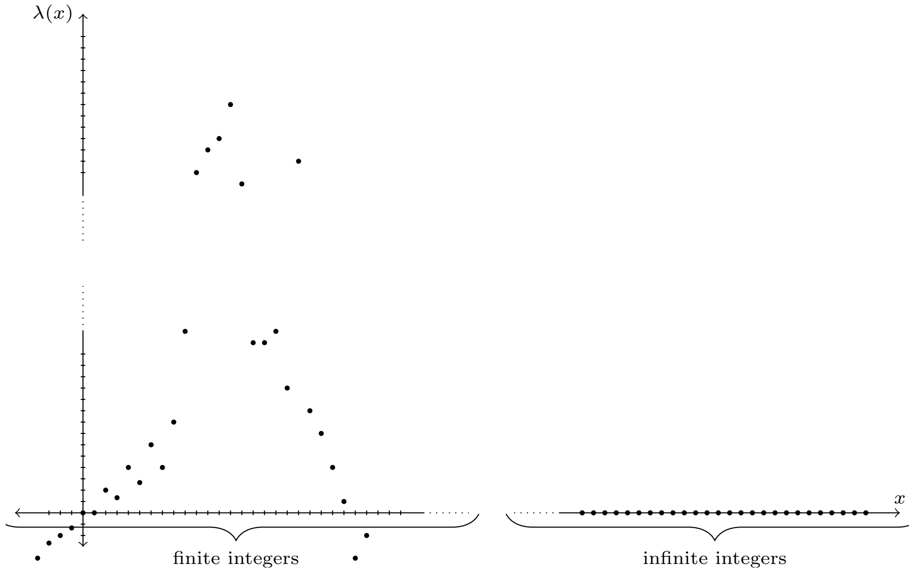

Figure 5: A slope $\lambda \sim^{\mathbb{R}} \mathbf{0}^{\mathcal{S}}$ taking arbitrary values at its finite indices.

Remark 5.12. This result implies that whether or not a function from $\mathbb{Z}^*$ to $\mathbb{Z}^*$ is a slope, and if so, which equivalence class with respect to $\sim^{\mathbb{R}}$ it belongs to, is sufficiently determined by its values taken at infinite indices. To any slope μ, a function λ taking arbitrary values at its finite indices as described in Proposition 5.11 can be added, yielding a slope which is equivalent to μ but only coinciding with it at its infinite indices.

Definition 5.13. It turns out that finite numbers can be uniquely decomposed into a standard* part and an infinitesimal part, that is, it is the sum of a standard* and an infinitesimal real number. We denote the **standard* part of** x by $\mathrm{St}(x)$ and the **infinitesimal part of** x by $\mathrm{Inf}(x)$. Moreover we call the set of numbers sharing their standard* part x the **monad of** x. [Rob96, p. 51]

Proposition 5.14. *If ε is a well adjusted infinitesimal slope, then there is an infinite integer number a_0 such that $\varepsilon(a_0)$ is finite.*

Proof. We show this for positive infinitesimal slopes; the proof is analogous for negative slopes and trivial for 0. Consider the $\mathscr{L}$-sentence

$$\sigma \equiv \forall r \in \mathbb{R} \exists! n \in \omega_{\mathbb{R}} \left(0^{\mathbb{R}} < r < 1^{\mathbb{R}} \rightarrow \tfrac{1}{n+1^{\mathbb{R}}} \leq r < \tfrac{1}{n} \right).$$

It is a consequence of the axioms of $\mathbb{R}$ and hence, it holds in V as well as in V*. Choosing $[\varepsilon]^{\mathbb{R}}$ for r when interpreting σ in V*, we denote the natural number whose existence is guaranteed through σ by $\widetilde{n_\varepsilon}$. We know that $\widetilde{n_\varepsilon}$ must be infinite, since otherwise $[\varepsilon]^{\mathbb{R}}$ would be greater than or equal to $\frac{1}{n+1^{\mathbb{R}}}$ with $n + 1^{\mathbb{R}}$ standard*, contradicting its infinitesimality.

We will now analyse the number $\frac{1}{\widetilde{n_\varepsilon}}$, or rather the slopes contained in it. Surely, the slope ν_ε defined by $\nu_\varepsilon(x) = n_\varepsilon \cdot x$ is an element of $\widetilde{n_\varepsilon}$ and n_ε is an infinite element of $\mathbb{Z}^*$. In the proof of Proposition 4.52 we have seen that if we define

$$\begin{aligned} \mu_\varepsilon(v) &:= \min(P_{\nu_\varepsilon}(v)) \text{ for } v \in \mathbb{Z}, \text{ where} \\ P_{\nu_\varepsilon}(v) &:= \{x \in \mathbb{Z} : \ |v - \nu_\varepsilon(x)| \leq |\nu_\varepsilon(1)| + 1\}, \end{aligned}$$

then $\mu_\varepsilon \in \frac{1}{\widetilde{n_\varepsilon}}$. To estimate $\mu'_\varepsilon(n_\varepsilon)$, where μ'_ε is the well adjusted slope equivalent to μ_ε given by the Concentration Lemma, let s be the minimal number in $\mathbb{Z}_{>0}$ such that

$$-s \leq \mu_\varepsilon(x+y) - \mu_\varepsilon(x) - \mu_\varepsilon(y) \leq s$$

for all $x, y \in \mathbb{Z}$. Then, $\mu'_\varepsilon(n_\varepsilon)$ is given by $\mu_\varepsilon(3sn_\varepsilon) : 3s$, whereas $\mu_\varepsilon(3sn_\varepsilon)$ in turn is given by

$$\begin{aligned} \mu_\varepsilon(3sn_\varepsilon) &= \min\left(P_{\nu_\varepsilon}(3sn_\varepsilon)\right) \\ &= \min\left(\{3s-1, 3s, 3s+1\}\right) \\ &= 3s - 1. \end{aligned}$$

The optimal euclidean quotient $(3s-1) : 3s$ is defined to be the number k for which

$$3s - 2 \leq 6sk < 9s - 2,$$

i.e. $k = 1$ and thereby, $\mu'_\varepsilon(n_\varepsilon) = 1$. Now since $[\varepsilon]^{\mathbb{R}} < \frac{1}{\widetilde{n_\varepsilon}}$, we have that

$$\varepsilon <^{\mathcal{S}} \mu'_\varepsilon,$$

implying that $\varepsilon(n_\varepsilon)$ cannot exceed 3. Also, it can't get below -2, since this would contradict the positivity of ε. Thus, we have shown that at $a_0 := n_\varepsilon$, which is an infinite number, ε takes a finite value. ☐

Next we want to show that the converse of Proposition 5.14 is also true, from which will follow that well adjusted slopes are infinitesimal iff they take a finite value at some infinite index.

Proposition 5.15. *If λ is a well adjusted positive slope which is not infinitesimal, then $\lambda(a)$ is infinite for all infinite indices a.*

Proof. We distinguish the cases $r = [\lambda]^{\mathbb{R}} \geq 1^{\mathbb{R}}$ and $r < 1^{\mathbb{R}}$.
$r \geq 1^{\mathbb{R}}$ implies that $\lambda \geq \mathbf{1}^{\mathcal{S}}$ and hence, for infinite integers a we have $\lambda(a) \geq \mathbf{1}^{\mathcal{S}}(a) - 2 = a - 2$, which is infinite.

The case $r < 1^{\mathbb{R}}$ is more interesting. Again we can squeeze r between two standard* real numbers of the form $\frac{1}{\widetilde{n_\lambda}}$ and $\frac{1}{\widetilde{n_\lambda} - 1^{\mathbb{R}}}$, where $\widetilde{n_\lambda}$ is a finite positive integer greater than 1. (If it was infinite, this would imply that r is infinitesimal, which we assume to be false.) We are now going to show that well adjusted slopes representing real numbers of the form $\frac{1}{\tilde{n}}$ with a finite natural number $n \neq 0$ always take infinite values at infinite indices. From this, it follows immediately that $\lambda(a)$ also must be infinite for all infinite a by the Order Lemma.

So consider an arbitrary finite natural number $n \neq 0$ and let $a \in \mathbb{Z}^*$ be infinite. The slope $\nu(x) = n \cdot x$ is contained in $\widetilde{n}$, and therefore the slope μ defined by

$$\mu(v) := \min(P_\nu(v)) \text{ for } v \in \mathbb{Z}, \text{ where}$$
$$P_\nu(v) := \{x \in \mathbb{Z} : \; |v - \nu(x)| \leq |\nu(1)| + 1\}.$$

is an element of $\frac{1}{\widetilde{n}}$. By the definition of μ, we have that

$$|\nu(\mu(a)) - a| = |n \cdot \mu(a) - a| \leq n + 1.$$

Since n is finite, we know that the difference between a and $n \cdot \mu(a)$ is finite, which is only possible if $\mu(a)$ is infinite. However, μ is not necessarily well adjusted, which is why we need to consider the slope μ' given by the Concentration Lemma which is well adjusted and still contained in $\frac{1}{\widetilde{n}}$.

For this purpose, we want to examine the number s for which

$$-s \leq \mu(x + y) - \mu(x) - \mu(y) \leq s$$

holds, where μ' is then defined through $\mu'(x) = \mu(3sx) : 3s$. In the proof of Proposition 4.52 it becomes clear that

$$\begin{aligned} &|\nu(\mu(x + y) - \mu(x) - \mu(y))| \leq 3 \cdot |\nu(1)| + 5 \\ \Longrightarrow\; &n \cdot |\mu(x + y) - \mu(x) - \mu(y)| \leq 3n + 5 \\ \overset{n\neq 0}{\Longrightarrow}\; &|\mu(x + y) - \mu(x) - \mu(y)| \leq 3n + 5. \end{aligned}$$

Since n is a finite number, this implies that s is finite, as well. Now from the proof of the Concentration Lemma we know that $|\mu'(x) - \mu(x)| \leq s$, which implies that for infinite a, $\mu'(a)$ is still infinite since its difference from $\mu(a)$, which itself has already been shown to be infinite, is finite. □

Corollary 5.16. *A well adjusted slope ε is infinitesimal if and only if there is an infinite integer number a_0 for which $\varepsilon(a_0)$ is finite.*

Remark 5.17. It is comparatively easy to see that a well adjusted slope is infinite iff it takes an infinite value at some finite nonzero index. We can squeeze a well adjusted infinite slope α between two infinite linear slopes representing integers. These take infinite values at all indices except 0 and hence, so does α. Also, a well adjusted slope taking infinite values at finite indices must necessarily be infinite, since every finite real number can be squeezed between two finite natural numbers which also only take finite values at finite indices.

These results permit a characterization of nonstandard well adjusted slopes as follows.

Theorem 5.18. *A well adjusted slope λ is*

- *infinite, if and only if $\lambda(k_0)$ is infinite for some finite k_0.*
- *infinitesimal, if and only if $\lambda(a_0)$ is finite for some infinite a_0.*
- *finite and non-infinitesimal, if and only if $\lambda(x)$ is finite iff x is finite.*

In the sequel we will see that considering only the values taken at finite indices also allows a nice characterization of nonstandard well adjusted slopes.

Proposition 5.19. *If λ is a well adjusted, positive slope which is not infinitesimal, then $\lambda(k_0) \geq 2$ for some finite integer k_0.*

Proof. As in the proof of Proposition 5.15, we distinguish the cases $r = [\lambda]^{\mathbb{R}} \geq 1^{\mathbb{R}}$ and $r < 1^{\mathbb{R}}$.
If $r \geq 1^{\mathbb{R}}$, the proof is easy. The slope $\mathbf{1}^{\mathcal{S}}(x) = x$ is well adjusted and contained in $1^{\mathbb{R}}$. Therefore, $\mathbf{1}^{\mathcal{S}}(4) = 4$ and since r is not less that $1^{\mathbb{R}}$, we know from the Order Lemma that $\lambda(4) \geq 2$. Setting $k_0 := 4$ proves the statement.

Now consider the case $r < 1^{\mathbb{R}}$. The statement can now be proven very similar to Proposition 5.14. As in the proof of Proposition 5.15, we can squeeze a positive, non-infinitesimal real number $r = [\lambda]^{\mathbb{R}} < 1^{\mathbb{R}}$ between two standard* numbers of the form $\frac{1}{\widetilde{n_\lambda}}$ and $\frac{1}{\widetilde{n_\lambda - 1^{\mathbb{R}}}}$, where $\widetilde{n_\lambda}$ is a finite positive integer greater than 1. Now consider the slope ν_λ given by $\nu_\lambda(x) = n_\lambda \cdot x$. Here n_λ is certainly a finite element of $\mathbb{Z}^*$. Define

$$\begin{aligned} \mu_\lambda(v) &:= \min(P_{\nu_\lambda}(v)) \text{ for } v \in \mathbb{Z}, \text{ where} \\ P_{\nu_\lambda}(v) &:= \{x \in \mathbb{Z} : \ |v - \nu_\lambda(x)| \leq |\nu_\lambda(1)| + 1\}. \end{aligned}$$

μ_λ represents the standard* number $\frac{1}{\widetilde{n_\lambda}}$. With the Concentration Lemma, we can again define μ'_λ as we already did in the proof of Proposition 5.14. This time we estimate $\mu'_\lambda(2n_\lambda) = \mu_\lambda(6sn_\lambda) : 3s$, with s defined accordingly.

$$\begin{aligned} \mu_\lambda(6sn_\lambda) &= \min(P_{\nu_\lambda}(6sn_\lambda)) \\ &= \min(\{6s-1, 6s, 6s+1\}) \\ &= 6s - 1. \end{aligned}$$

This gives us

$$\mu'_\lambda(2n_\lambda) = (6s-1) : 3s = 2 > 1 \geq d_{\mu'_\lambda},$$

with $\mu'_\lambda \in \frac{1}{\widetilde{n_\lambda}}$. Now by Proposition 4.50 (ii), $\mu'_\lambda(6n_\lambda) \geq 4$, which, with the Order Lemma, implies that

$$\lambda(6n_\lambda) \geq 2.$$

And since n_λ is a finite integer, the proof is complete if we define $k_0 := 6n_\lambda$. □

Corollary 5.20. *If ε is a well adjusted infinitesimal slope, then $\varepsilon(k) \in \{-1, 0, 1\}$ for all finite integers k.*

Proof. For $k = 0$ this is trivial since we assume ε to be well adjusted. So consider a finite integer $k \neq 0$ such that $|\varepsilon(k)| > 1$ and hence, by the oddity of ε, either $\varepsilon(k) > 1$ or $\varepsilon(-k) > 1$. Without loss of generality, assume that $\varepsilon(k) > 1$ for a positive finite $k \in \mathbb{Z}^*$. As we have seen in Proposition 5.14, there is an infinite number a such that $\varepsilon(a)$ is finite. But by Proposition 4.50 (iii),

$$\varepsilon(a) \geq -1 + (a : k).$$

$(a : k)$ must be an infinite number, since otherwise $a = (a : k) \cdot k + r$ with $|r| \leq k$ would be finite as a composition of finite numbers. Therefore, $-1 + (a : k)$ and hence $\varepsilon(a)$ is infinite as well, yielding a contradiction. □

With this knowledge, we are able to determine whether a well adjusted slope λ is infinitesimal or not, and whether it is infinite or not, by taking into account only the values taken at finite indices.

Theorem 5.21. *A well adjusted slope λ is*

- *infinite, if and only if there is a finite integer k_0 such that $\lambda(k_0)$ is infinite.*
- *infinitesimal, if and only if $|\lambda(k)| \leq 1$ for all finite integers k.*
- *finite and non-infinitesimal, if and only if $\lambda(k)$ is finite for all finite integers k, but there is a finite integer k_0 such that $|\lambda(k_0)| > 1$.*

Remark 5.22. One could imagine that Corollary 5.20 can be formulated even stronger, i.e. that $\varepsilon(k) = 0$ for all finite k if ε is infinitesimal; however this is not true. Consider the slope

$$\lambda(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

It is obviously odd and for all possible choices of x and y, $\lambda(x + y) - \lambda(x) - \lambda(y)$ will always take its value in $\{-1, 0, 1\}$. Thus, λ is well adjusted and equivalent to 0, making it an infinitesimal slope, although it is not constantly 0 on its finite indices.

What proves true though is the following.

Proposition 5.23. *Every infinitesimal slope $\varepsilon \in \mathcal{S}^*$ is equivalent to a well adjusted slope ε' for which $\varepsilon'(k) = 0$ holds for every finite integer k.*

Proof. Assume without loss of generality that ε is well adjusted. Otherwise we can construct a well adjusted slope equivalent to ε with the Concentration Lemma and continue with it as our new ε. By Corollary 5.20, $|\varepsilon(k)| \leq 1$ for all finite k. We can apply the Concentration Lemma to ε with $s = 1$, which yields

$$\varepsilon'(k) = \underbrace{\varepsilon(3k)}_{|\cdot| \leq 1} : 3 = 0,$$

if k is finite.

□

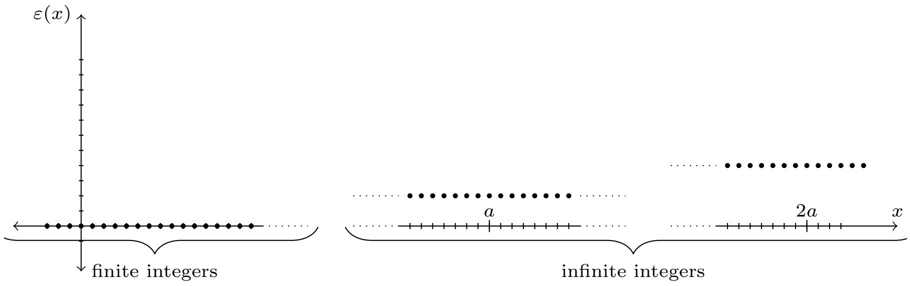

Figure 6: An infinitesimal slope ε taking nonzero values only at infinite indices.

This result is very interesting, since it allows us to picture nonstandard* real numbers nicely. As already mentioned earlier, all finite numbers have a unique standard* part and a unique infinitesimal part. Now that we have the information of Proposition 5.23, this implies that two arbitrary numbers with the same standard* part can be represented by slopes which coincide on the finite indices and only differ on the infinite ones.

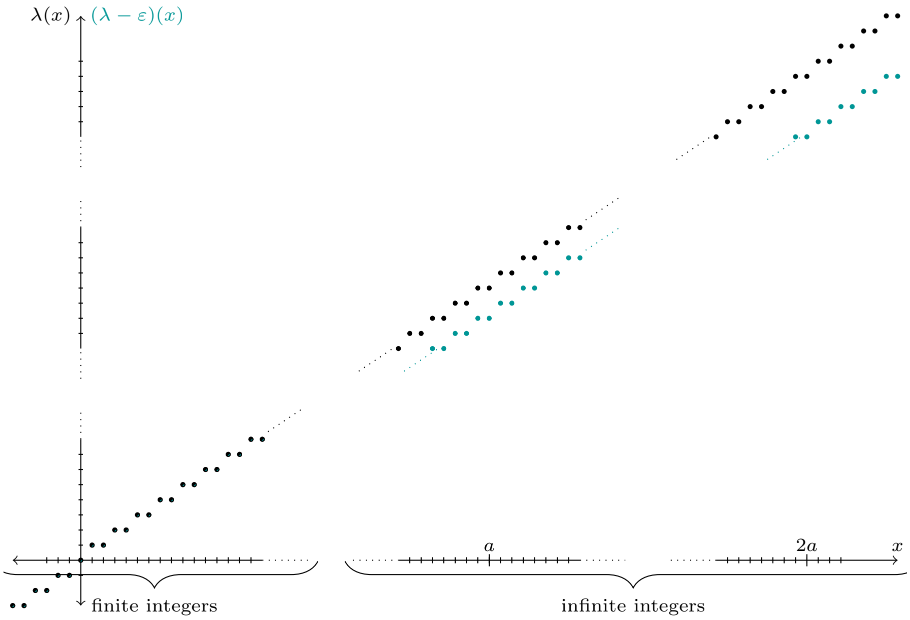

Figure 7: Two slopes representing different real numbers sharing their standard* part.

Corollary 5.24. *A real number is infinitesimal if and only if it contains a well adjusted slope ε for which $\varepsilon(k) = 0$ holds for all finite integers k.*

Proof. Consider an infinitesimal real number e. Proposition 5.23 implies that e contains a well adjusted slope ε which is 0 on all finite indices. What remains to be shown is the other direction. So consider a well adjusted slope ε with $\varepsilon(k) = 0$ for all finite k. If ε was not infinitesimal, there would be a finite integer number k with $|\varepsilon(k)| \geq 2$ by Proposition 5.19, yielding a contradiction. Hence ε is infinitesimal, completing our proof. □

Corollary 5.25.

$$\mathrm{St}(x_0) = \mathrm{St}(x_1)$$

if and only if there are well adjusted slopes $\lambda_0 \in x_0$ and $\lambda_1 \in x_1$ such that

$$\lambda_0(k) = \lambda_1(k)$$

for all finite integers k.

Proof. x_0 and x_1 having the same standard* part is equivalent to their difference $d := x_1 - x_0$ being infinitesimal. Furthermore, by Corollary 5.24, d is infinitesimal iff it contains a well adjusted slope δ with $\delta(k) = 0$ for all finite k, which in turn is equivalent to the existence of well adjusted slopes λ_0 and λ_1 coinciding on their finite indices. To prove the last equivalence, consider a well adjusted slope $\mu_0 \in x_0$ and $\delta \in d$ as described above. Define

$$\lambda_0(x) := \mu_0(6x) : 6$$
$$\lambda_1(x) := (\mu_0 + \delta)(6x) : 6.$$

Since both μ_0 and δ are well adjusted, λ_0 and λ_1 are also well adjusted; furthermore $\lambda_0 \sim^{\mathbb{R}} \mu_0 \in x_0$ and $\lambda_1 \in x_1$ hold. We have

$$\lambda_1(k) = (\mu_0 + \delta)(6k) : 6 = (\mu_0(6k) + \underbrace{\delta(6k)}_{=0}) : 6 = \mu_0(6k) : 6 = \lambda_0(k)$$

for all finite integers k, and therefore we have found well adjusted slopes $\lambda_0 \in x_0, \lambda_1 \in x_1$ which coincide on their finite indices.
For the other direction, consider λ_0, λ_1 as described above. Clearly, $\varepsilon(x) := \lambda_1(x) - \lambda_0(x)$ is a slope contained in $d = x_1 - x_0$ and $\varepsilon(k) = 0$ for all finite k. Now $\delta(x) := \varepsilon(6x) : 6$ is well adjusted, equivalent to ε, and still

$$\delta(k) = \varepsilon(6k) : 6 = 0 : 6 = 0$$

for finite integers k. □

Remark 5.26. With the results obtained by now, we have a nice picture of the standard* part of a real number in $x \in \mathbb{R}^*$. The standard* part of x represented by a well adjusted slope $\lambda \in x \in \mathbb{R}^*$ is fully determined by the values taken at finite indices by Corollary 5.25. If these values are infinite, λ represents an infinite number by Remark 5.17 and therefore x has no standard* part [Rob96, p. 51].

Hence, ignoring every part of the slope λ involving infinite integers and viewing it as a function from "$\mathbb{Z}$" to "$\mathbb{Z}$" yields either nothing, in which case the real number x represented by the slope has no standard* part, or a well adjusted slope of V representing the standard real number $y \in \mathbb{R}$ for which $y^* = \mathrm{St}(x)$ holds.

Notice the contrast to the result obtained at the beginning of this section, where it was shown that if a slope is not assumed to be well adjusted, the values taken at finite indices do not determine the nature of the slope at all (see Remark 5.12).

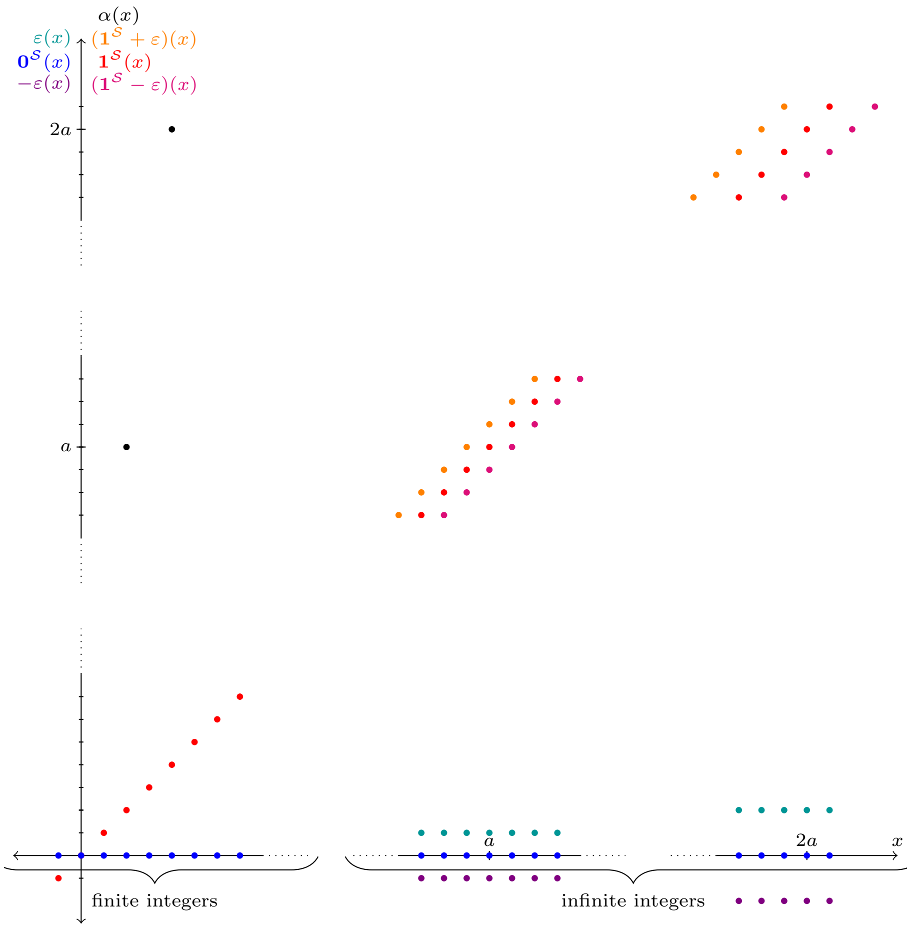

Figure 8: Several well adjusted slopes with or without standard* parts. When only considering the lower left square of the graph, no more than two slopes can be distinguished. α has no standard* part at all, while $-\varepsilon, \mathbf{0}^{\mathcal{S}}, \varepsilon$ and $\mathbf{1}^{\mathcal{S}} - \varepsilon, \mathbf{1}^{\mathcal{S}}, \mathbf{1}^{\mathcal{S}} + \varepsilon$ share their standard* part, respectively.

5.3 Consequences in Analysis

Convergence

[Rob96, Theorem 3.3.7] states that a standard sequence $\{s_n\}$ of real numbers converges to $s \in \mathbb{R}$ if and only if their V*-versions, $\{s_n^*\}$ and s^*, satisfy

$$\mathrm{St}(s_a^*) = s^*$$

for all infinite indices a. Interpreting it in our model, we can translate this theorem as follows.

Theorem 5.27. *A standard sequence $\{s_n\}$ of real numbers converges to $s \in \mathbb{R}$ if and only if its* V*-*version, $\{s_n^*\}$, satisfies*

(Conv.) For every slope $\sigma^* \in s^*$, where s^* is the V*-*version of s, and for every infinite natural number a there is a slope $\sigma_a^* \in s_a^*$ such that σ_a^* coincides with σ^* on its finite indices.*

Proof.

$\Longrightarrow$: Let $\{s_n\}$ be a standard sequence of real numbers converging to s and let a be an infinite natural number. We have just pointed out that then, by [Rob96, Theorem 3.3.7], $\mathrm{St}(s_a^*) = s^*$ holds. Hence,

$$s_a^* = s^* + e$$

for some infinitesimal real number e. By Proposition 5.23, there is an infinitesimal slope $\varepsilon \in e$ such that $\varepsilon(k) = 0$ for all finite integers k. If now $\sigma^* \in s^*$, this means that

$$\sigma_a^* := \sigma^* + \varepsilon \in s_a^*.$$

We have therefore found a slope $\sigma_a^* \in s_a^*$ which coincides with σ^* on its finite indices.

$\Longleftarrow$: Consider a standard sequence of real numbers $\{s_n\}$ and a standard real number s. Suppose that for every slope $\sigma \in s^*$ and every infinite natural number a there is a slope $\sigma_a^* \in s_a^*$ such that σ_a^* and σ coincide on their finite indices. Now consider an arbitrary infinite natural number a_0 and define the slope

$$\varepsilon := \sigma_{a_0}^* - \sigma.$$

$\varepsilon(k) = 0$ for all finite integers k. Therefore, ε must be infinitesimal by Corollary 5.24. This implies that

$$\mathrm{St}(s_{a_0}^*) = s^*$$

and since a_0 was arbitrary, this holds for all infinite natural numbers a. Thus the sequence $\{s_n\}$ converges to s.

□

Continuity

In [Rob96, Theorem 3.4.5] it is shown that a standard function $f : (s,t) \to \mathbb{R}$, where (s,t) denotes the set of all $x \in \mathbb{R}$ for which $s < x < t$ holds for standard real numbers s and t, is continuous at a standard point $x_0 \in (s,t)$ if and only if

$$\mathrm{St}(f^*(x)) = f^*(\mathrm{St}(x))$$

holds for every x with $\mathrm{St}(x) = x_0^*$. The condition can thus be rewritten as

$$\begin{aligned} &\mathrm{St}(f^*(x)) = f^*(x_0^*) \\ \Longleftrightarrow\ &\mathrm{St}(f^*(x)) = \mathrm{St}(f^*(x_0^*)). \end{aligned}$$

Again, we translate this into a condition on the slopes representing relevant numbers.

Theorem 5.28. *A standard function $f : (s,t) \to \mathbb{R}$ is continuous in $x_0 \in (s,t)$ if and only if its* V^**-version $f^* : (s^*, t^*) \to \mathbb{R}^*$ satisfies*

(Cont.) For every $x \in (s^*, t^*)$ it is true that if there are slopes $\lambda \in x$ and $\lambda_0 \in x_0^*$ such that λ and λ_0 coincide on their finite indices, then there also are slopes $\mu \in f^*(x)$ and $\mu_0 \in f^*(x_0^*)$ that coincide on their finite indices.

Proof. Suppose $f : (s,t) \to \mathbb{R}$ is a standard function continuous in $x_0 \in (s,t)$. This is, by [Rob96, Theorem 3.4.5], equivalent to the implication

$$\text{“St}(x) = x_0^* \text{ implies St}(f^*(x)) = \text{St}(f^*(x_0^*)) \text{ for all } x \in (s^*, t^*).\text{”}$$

For x and x_0^* to have the same standard* part is equivalent to the existence of slopes λ and λ_0 as demanded in (Cont.) and analogously, $\mathrm{St}(f^*(x)) = \mathrm{St}(f^*(x_0^*))$ holds if and only if μ and μ_0 as demanded in (Cont.) exist. Hence, the theorem follows. □

A function $f : (s,t) \to \mathbb{R}$ is called continuous if it is continuous in all points $x_0 \in (s,t)$. Informally speaking, one could phrase this as follows. A function $f : (s,t) \to \mathbb{R}$ is continuous if and only if cutting off the infinitely indexed part of a nonstandard slope first and then applying f is equivalent to applying f^* on it and then cutting off the infinitely indexed part. Of course, functions on real numbers are not functions on slopes and thus applying f or f^* on a slope is not something we would be able to do; however it does not chance the picture too much compared to what the actual procedure would be: Cutting off the infinite parts of all slopes in an equivalence class and applying f to the set of their remainders, or applying f^* on the set of initial slopes and after that cutting all of them.

6 Conclusions

Model Theory

The first main result of this thesis is Theorem 2.13. A set of formulae Φ is said to be an insertion of the theory T into ZF if it defines bold versions of the relation, function and constant symbols of the language $\mathscr{L}$ of T as set theoretic relations, functions and constants, respectively, as well as another constant symbol B serving as the "domain" of T. Consider the set Ψ^* of formulae defining the symbols of $\mathscr{L}$, stipulating

(i) $Rv_1 \dots v_n :\Longleftrightarrow \langle v_1, \dots, v_n \rangle \in \boldsymbol{R}$ for relation symbols R of $\mathscr{L}$.

(ii) $Fv_1 \dots v_n = y :\Longleftrightarrow \langle v_1, \dots, v_n, y \rangle \in \boldsymbol{F} \vee (\langle v_1, \dots, v_n \rangle \notin B^n \wedge y = \emptyset)$ for function symbols F of $\mathscr{L}$.

(iii) $c = y :\Longleftrightarrow \boldsymbol{c} = y$ for constant symbols c of $\mathscr{L}$.

If Set Theory together with these definitions is able to prove the ZF-transformations of the non-logical axioms of T, then Φ is called an embedding of T into ZF. (In the ZF-transformation of an $\mathscr{L}$-formula φ, all strings of symbols of the form $\exists \nu$ and $\forall \nu$ are replaced by $\exists \nu \in B$ and $\forall \nu \in B$ respectively, for any variable ν.) Theorem 2.13 now states that if a set of formulae Φ is an embedding of the theory T into ZF and ZF has a model, then T has a model as well. This rather intuitive seeming result can be applied to arbitrary first order theories.

Formalization of A'Campo's Construction

In this thesis, embeddings of Peano Arithmetic as well as the theories of the integer and real numbers were elaborated, leading us the the next important result we have obtained: If ZF_{0-6}, i.e. the subset of ZF containing the axioms $\text{ZF}_0 - \text{ZF}_6$ has a model, then there are models of the natural, integer and real numbers. However, it cannot be known whether or not ZF_{0-6}, has a model, due to Gödel's Completeness and Incompleteness Theorems.

The main challenge was the embedding of the real numbers into ZF; they were defined as equivalence classes of slopes, which in turn are defined as functions $\lambda : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$-s \leq \lambda(x+y) - \lambda(x) - \lambda(y) \leq s$$

for all $x, y \in \mathbb{Z}$ and a fixed natural number s. If $s \leq 1$ and λ is an odd function, then the slope λ is called well adjusted. Following Norbert A'Campo's "A natural construction for the real numbers" [A'C03], various properties of well adjusted slopes and slopes in general were shown in this thesis, one of which is the Concentration Lemma which states that for every slope there is an equivalent well adjusted slope. Other properties of slopes — or more specifically, equivalence classes of slopes — include all axioms of the real numbers. Generally, A'Campo's ideas have been formalized and occasionally changed in order to not using the theory of rational numbers or axioms other than those of ZF_{0-6}.

Slopes in Nonstandard Models

Given the existence of a model of ZF, nonstandard models can be constructed as well. Making use of the Axiom of Choice by applying the Compactness Theorem, we have defined a nonstandard model V^* of Set Theory, inducing nonstandard models of the natural, integer and real numbers. A first important remark is that nonstandard versions of standard slopes can be viewed as extensions of the latter, in the sense that in V^*, where numbers greater than all standard numbers exist, infinitely big numbers are added to the domain and possibly to the image of the slope while leaving it unchanged on its finite part. Several interesting results concerning slopes in nonstandard models have been elaborated. As a consequence of Proposition 5.11, we have that the equivalence class of a slope depends only on the values it takes at infinite indices. On the other hand, Theorem 5.21 characterizes well adjusted slopes relying only on the values they take at finite indices: If infinite values are taken, the slope is infinite, if the values are all between -1 and 1, it is infinitesimal, and if the values are finite but their absolute values exceed 1 at some point, the slope is finite and non-infinitesimal. Led by this idea, it has been shown in Corollary 5.25 that two nonstandard real numbers with the same standard* part must always contain well adjusted slopes which coincide on their finite indices. This suggests that when considering well adjusted slopes, the values taken at finite indices already determine the standard* part of a slope, whereas the values taken at infinite indices are needed only to identify its nonstandard* part. This, in turn, yields nice visualizations of convergence of sequences and continuity of real functions.

Other tasks that could be approached in the future include a further examination of infinitesimal slopes. It would certainly be interesting to know how they behave during addition and multiplication, or what role they play in the differentiation of functions, etc.

List of Figures

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