Big Intersections in Projective Space
Masterarbeit

Judith Keller

Januar 2015

Betreut durch:

Prof. Dr. Elisa Gorla, Universität Neuchatel

Prof. Dr. Joachim Rosenthal, Universität Zürich

Contents

1 Commutative Algebra **3**
1.1 Localization, Dimension, Cohen-Macaulay rings 3
1.1.1 Localization 3
1.1.2 The Krull dimension 5
1.1.3 Transcendence degree 6
1.1.4 Depth of a module 7
1.1.5 The ext functor 8
1.2 Graded rings, homogeneous ideals, saturation of an ideal 12
1.2.1 Graded rings 12
1.2.2 Saturated ideals 13
1.2.3 Hilbert polynomial 14

2 Curves in projective space, some notions from algebraic geometry **17**
2.1 The projective space and the Zariski topology 17
2.1.1 The projective space 17
2.1.2 Zariski closed subsets 17
2.1.3 Dimension of Zariski closed subsets 21
2.2 Curves in $\mathbb{P}^n$: Cohen-Macaulay and saturation 22
2.3 Intersection of hypersurfaces in $\mathbb{P}^n$ 23
2.4 Curves in $\mathbb{P}^2$: Bézout theorem 26
2.5 Curves in $\mathbb{P}^3$ 27
2.5.1 Big intersection 27
2.5.2 Cohen-Macaulay implies big intersection 27

3 The theorems of Hilbert-Burch and Buchsbaum-Eisenbud **30**
3.1 Free minimal resolutions 30
3.2 Determinantial ideals 35
3.3 The Hilbert-Burch Theorem 39
3.4 Multilinear algebra 41
3.4.1 Exterior powers 41
3.4.2 Pfaffians and Pfaffian ideals 44
3.5 The Buchsbaum-Eisenbud Theorem 45
3.5.1 Ext, Tor and minimal resolutions 45
3.5.2 Gorenstein rings 49
3.5.3 Algebra structures on free resolutions 50
3.5.4 Proof of the Eisenbud-Buchsbaum Theorem 53
3.6 Application: union of complete intersections of codimension two 56

1 Commutative Algebra

In this section we will collect some facts and techniques that we will use later.

1.1 Localization, Dimension, Cohen-Macaulay rings

Let R be a commutative ring, with unit, and let M be an R-module.

1.1.1 Localization

Definition 1.1 *We denote* $\mathrm{Spec}(R)$ *the set of all prime ideals of* R*.*

Definition 1.2 *A subset* $S \subset R$ *is called* multiplicative*, if* $1 \in S$ *and if for all elements* $a, b \in S$ *the product* $a \cdot b$ *is also in* S*.*

Given such a multiplicative set $S \subset R$*, we define a relation on* $S \times R$ *by:*

$$(s, x) \sim_S (t, y) \Leftrightarrow \exists u \in S : utx = usy.$$

The set $S \times R / \sim_S$ *is denoted by* $S^{-1}R$*; it is a ring. The class of* $(s, x) \in S \times R$ *is denoted by* $\frac{x}{s}$*. By replacing* R *by* M *we get similarly* $S^{-1}M$ *which is an* $S^{-1}R$*-module.*

The ring $S^{-1}R$ *and the module* $S^{-1}M$ *are called the* localizations *of* R *and* M *with respect to* S*.*

Lemma 1.1 *There is a canonical homomorphism of rings*

$$\begin{aligned} \eta_S : R &\to S^{-1}R \\ x &\mapsto \frac{x}{1}, \end{aligned}$$

and a canonical homomorphism of R*-modules*

$$\begin{aligned} \eta_S : M &\to S^{-1}M \\ m &\mapsto \frac{m}{1}. \end{aligned}$$

Moreover

$$\mathrm{Ker}(\eta_S) = \{m \in M \mid \exists u \in S : u \cdot m = 0\}.$$

Proof: The map η_S is a homomorphism, since

$$\eta_S(a \cdot x + b \cdot y) = \frac{a \cdot x + b \cdot y}{1} = \frac{a \cdot x}{1} + \frac{b \cdot y}{1} = a \cdot \eta_S(x) + b \cdot \eta_S(y)$$

for $a, b \in R$ and $x, y \in R$ or M. Clearly, $m \in \mathrm{Ker}(\eta_S)$ if and only if

$$(1, m) \sim_S (1, 0).$$

By definition, this is equivalent to the existence of $u \in S$ such that $u \cdot m = 0$. □

Remark 1.1 *For a homomorphism $h: M \to N$ of R-modules and a multiplicative set $S \subset R$ there is an induced homomorphism*

$$\begin{array}{rcl} S^{-1}h : S^{-1}M & \to & S^{-1}N \\ \dfrac{m}{s} & \mapsto & \dfrac{h(m)}{s} \end{array}$$

of $S^{-1}R$-modules, which makes the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{h} & N \\ \downarrow{\scriptstyle \eta_S} & & \downarrow{\scriptstyle \eta_S} \\ S^{-1}M & \xrightarrow{S^{-1}h} & S^{-1}N. \end{array}$$

This defines a functor $\{R - modules\} \to \{S^{-1}R - modules\}$ (see Definition 1.14). In particular we have $S^{-1}h \circ S^{-1}h' = S^{-1}(h \circ h')$, and $S^{-1}0 = 0$.

Lemma 1.2 *For an exact sequence of R-modules*

$$M \xrightarrow{h} N \xrightarrow{g} L$$

and a multiplicative set $S \subset R$, the induced sequence of $S^{-1}R$-modules

$$S^{-1}M \xrightarrow{S^{-1}h} S^{-1}N \xrightarrow{S^{-1}g} S^{-1}L$$

is also exact.

Proof: Clearly $S^{-1}g \circ S^{-1}h = 0$. Let $\frac{n}{s} \in \mathrm{Ker}(S^{-1}g)$. This means that $\frac{g(n)}{s} = 0$. So, there exists an element $u \in S$ such that $g(un) = ug(n) = 0$. It follows, that $un \in \mathrm{Ker}(g) = \mathrm{Im}(h)$. So there must be an element $m \in M$ such that $h(m) = un$. It follows that

$$S^{-1}h(\frac{m}{us}) = \frac{h(m)}{us} = \frac{un}{us} = \frac{n}{s}.$$

This proves that $\frac{n}{s} \in \mathrm{Im}(S^{-1}h)$. □

Remark 1.2 *The previous lemma says that localization is an exact functor, i.e. it takes a short exact sequence to a short exact sequence.*

Example 1.1 *Consider $p \in \mathrm{Spec}(R)$, a prime ideal. Then $R \setminus p$ is a multiplicative set and*

$$(R \setminus p)^{-1}R = \left\{ \frac{a}{b} \mid a \in R,\ b \in (R \setminus p) \right\}$$

is called the localization of R at p; it is denoted by R_p. This is a local ring, with maximal ideal

$$pR_p = \left\{ \frac{a}{b} \mid a \in p,\ b \in R \setminus p \right\}.$$

The R_p-module

$$M_p := (R \setminus p)^{-1}M$$

is also called the localization of M at p.

1.1.2 The Krull dimension

Definition 1.3 *The* support *of an R-module M is the subset of the spectrum of the ring R, defined as follows:*

$$\mathrm{Supp}(M) := \{p \in \mathrm{Spec}(R) \mid M_p \neq 0\}.$$

To stress the dependence on R, we sometimes write $\mathrm{Supp}_R(M)$ for this subset.

Example 1.2 *For R considered as an R-module, we have $\mathrm{Supp}(R) = \mathrm{Spec}(R)$.*

Lemma 1.3 *For an exact sequence of R-modules*

$$0 \to M \xrightarrow{h} N \xrightarrow{g} L \to 0,$$

it holds that $\mathrm{Supp}(N) = \mathrm{Supp}(M) \cup \mathrm{Supp}(L)$.

Proof: For every prime $p \in \mathrm{Spec}(R)$ we have an exact sequence

$$0 \to M_p \xrightarrow{h} N_p \xrightarrow{g} L_p \to 0$$

by Lemma 1.2. Therefore, $N_p = 0$ if and only if $M_p = 0$ and $L_p = 0$. So $p \in \mathrm{Supp}(N)$ if and only if $p \in \mathrm{Supp}(M)$ or $p \in \mathrm{Supp}(L)$. $\square$

Definition 1.4 *For a subset $U \subset R$ we denote by $\mathrm{Z}(U)$ the set:*

$$\mathrm{Z}(U) := \{p \in \mathrm{Spec}(R) | U \subset p\}.$$

Remark 1.3 *If $I \subset R$ is an ideal, there is a canonical bijection*

$$\begin{array}{ccc} \mathrm{Z}(I) & \xrightarrow{\sim} & \mathrm{Spec}(R/I) \\ p & \mapsto & p/I \end{array}$$

Definition 1.5 *Let $P \subset M$ be any subset. We define the* annihilator *of P to be the ideal:*

$$\mathrm{Ann}(P) = \{r \in R \mid \forall m \in P : r \cdot m = 0\}.$$

Proposition 1.1

(1) $\mathrm{Supp}(M) \subset \mathrm{Z}(\mathrm{Ann}(M))$.

(2) If M is finitely generated, then $\mathrm{Supp}(M) = \mathrm{Z}(\mathrm{Ann}(M))$.

Proof: Consider $p \in \mathrm{Supp}(M)$. This means, that $M_p \neq 0$. So there must be an element $m \in M$ such that $\frac{m}{1}$ is not in the same class as $\frac{0}{1}$. Thus for every $u \in R \setminus p$ we have $um \neq 0$. This means, that $\mathrm{Ann}(M) \cap R \setminus p = \emptyset$ which is equivalent to say that $\mathrm{Ann}(M) \subset p$. This proves that $p \in \mathrm{Z}(\mathrm{Ann}(M))$ and hence:

$$\mathrm{Supp}(M) \subset \mathrm{Z}(\mathrm{Ann}(M)).$$

Now assume that M is finitely generated and suppose that $M_p = 0$ for a $p \in \mathrm{Spec}(R)$ (i.e., $p \notin \mathrm{Supp}(M)$). We will show that $p \notin \mathrm{Z}(\mathrm{Ann}(M))$ which will finish the proof. Let $m_1, m_2, \ldots, m_r$ be a generating set for M. As

$$\frac{m_1}{1} = \frac{m_2}{1} = \ldots = \frac{m_r}{1} = \frac{0}{1}$$

in M_p, there must be elements $u_1, u_2, \ldots, u_r \in R \setminus p$ such that $m_1 \cdot u_1 = m_2 \cdot u_2 = \ldots = m_r \cdot u_r = 0$. But then, the product $u = u_1 \cdot u_2 \cdot \ldots \cdot u_r$ is in the annihilator of M and as p is prime we also have $u \notin p$. This implies, that $\mathrm{Ann}(M) \not\subset p$ and hence $p \notin \mathrm{Z}(\mathrm{Ann}(M))$. □

Definition 1.6 *The* Krull dimension $\dim_R(M)$ *of an R-module M is defined as:*

$$\dim_R(M) := sup\{l \in \mathbb{N} \mid \exists p_0, p_1, \ldots, p_l \in \mathrm{Supp}(M) : p_0 \subsetneq p_1 \subsetneq \ldots \subsetneq p_l\}.$$

The Krull dimension $\dim_R(R)$ *of the ring R is denoted by* $\dim(R)$.

Example 1.3 *If k is a field,* $\dim(k[x_1, \ldots, x_n]) = n$. *The proof is difficult and will not be given here. It can be found for example in [6, Theorem 13.1].*

Lemma 1.4 *For an ideal $I \subset R$ with $IM = 0$, so that M can be considered as an R/I-module, we have:*

$$\dim_R(M) = \dim_{R/I}(M).$$

Proof: First we show, that $\dim_R(M) \leq \dim_{R/I}(M)$. Note, that $I \subset p$ for all $p \in \mathrm{Supp}_R(M)$. Indeed, if there is an element $a \in I \setminus p$, we have that $aM_p = 0$ since $IM = 0$. But, as $\frac{a}{1}$ is invertible in R_p, we also have $aM_p = M_p$. This shows that $M_p = 0$ and p cannot be in $\mathrm{Supp}_R(M)$. Because of this, given a chain of prime ideals $p_0 \subsetneq p_1 \subsetneq \ldots \subsetneq p_l$ in $\mathrm{Supp}_R(M)$, we can construct a chain of the same length in $\mathrm{Supp}_{R/I}(M)$ by taking the ideals $p_0/I \subsetneq p_1/I \subsetneq \ldots \subsetneq p_l/I$.

On the other hand $\dim_R(M) \geq \dim_{R/I}(M)$. Indeed, a chain of prime ideals $q_0 \subsetneq q_1 \subsetneq \ldots \subsetneq q_l$ in $\mathrm{Supp}_{R/I}(M)$ induces a chain of the same length in $\mathrm{Supp}_R(M)$ given by the inverse images of the q_i's under the canonical projection $\bar{\cdot} : R \to R/I$. This proves the lemma. □

1.1.3 Transcendence degree

Definition 1.7 *Let k be a field and let A be a k-algebra. We say that a family $a_1, \ldots, a_n \in A$ is* algebraically independent *over k if there is no polynomial f in the ring $k[x_1, \ldots x_n] \setminus \{0\}$ such that $f(a_1, \ldots, a_n) = 0$. Otherwise we say that the family is* algebraically dependent.

Definition 1.8 *Let $l \supset k$ be a field extension. We define the* transcendence degree $\mathrm{trdeg}_k(l)$ *of l over k to be the maximal cardinality of an algebraically independent family in l over k:*

$$\mathrm{trdeg}_k(l) := sup\{n \in \mathbb{N} \mid \exists\, a_1, \ldots, a_n \in l \text{ algebraically independent over } k\}.$$

If A is an integral domain we denote by $\mathrm{Frac}(A)$ its field of fractions.

Proposition 1.2 *Let k be a field and let A be a finitely generated k-algebra. Then*

$$\dim(A) = \sup_{p \in \mathrm{Spec}(A)} \mathrm{trdeg}_k(\mathrm{Frac}(A/p)).$$

In the formula above, one can restrict to minimal prime ideals of A. In particular, if A is an integral domain, then

$$\dim(A) = \mathrm{trdeg}_k(\mathrm{Frac}(A)).$$

Proof: The proof is difficult and will be omitted. It can be found for example in [6, p. 290, Theorem A]. □

1.1.4 Depth of a module

Definition 1.9 *An element $x \in R \setminus \{0\}$ is called a* zero divisor *on the R-module M if there is an element $y \in M \setminus \{0\}$, such that:*

$$x \cdot y = 0.$$

Definition 1.10 *A sequence of elements $x_1, \dots, x_r \in R$ is called a* regular sequence *for M if*

(1) $\forall i \in \{1, \dots, r\}$, x_i is not a zero divisor on $M/(x_1, \dots, x_{i-1})M$, and

(2) $M \neq (x_1, \dots, x_r)M$.

Remark 1.4 *If $M \neq 0$, then the empty sequence is a regular sequence for M. If $M = 0$, there are no regular sequences for M.*

Definition 1.11 *Let $I \subset R$ be an ideal. The supremum of the length of regular sequences $x_1, \dots, x_r \in I$ for M is called the depth of M with respect to I:*

$$\mathrm{depth}_I(M) := \sup\{r \in \mathbb{N} \mid \exists x_1, \dots, x_r \in I \text{ a regular sequence for } M\}.$$

(The length of the empty sequence is 0; if $M = 0$, then $\mathrm{depth}_I(M) = -\infty$.)

If R is a local ring with maximal ideal m, we write $\mathrm{depth}(M)$ for $\mathrm{depth}_m(M)$.

Lemma 1.5 *For an ideal $I \subset R$ and a second ideal J with $JM = 0$, we have:*

$$\mathrm{depth}_I(M) = \mathrm{depth}_{I(R/J)}(M).$$

Proof: Indeed, the morphism $R \to R/J$ takes a regular sequence in I for M (as an R-module) to a regular sequence in $I(R/J)$ for M (as a R/J-module). Conversely, given a regular sequence in $I(R/J)$ for M (as an R/J-module), arbitrary lifts in I provide a regular sequence for M (as an R-module). □

Lemma 1.6 *For a local ring R with maximal ideal m and a finitely generated R-module $M \neq 0$ we have:*

$$\mathrm{depth}(M) \leq \dim_R(M)$$

Proof: We argue by induction on the Krull dimension of M.

First assume that $\dim_R(M) = 0$. Since R is local, this means, that $\mathrm{Supp}(M) = \{m\}$. By Proposition 1.1, m is the unique prime ideal that contains $\mathrm{Ann}(M)$. Therefore, we have $\sqrt{\mathrm{Ann}(M)} = m$. In other words, for every $a \in m$ there is an integer $N \in \mathbb{N}$ such that $a^N \in \mathrm{Ann}(M)$. But this means, that there are no non zero divisors in m for M. Hence, $\mathrm{depth}(M) = 0$.

Consider now the case $\dim_R(M) > 0$. Suppose, there is a maximal regular sequence $(a_0, a_1, \dots a_r)$ of length $r+1$ in m. Then $(a_1, \dots a_r)$ is a maximal regular sequence of length r in M for $M/(a_0)M$. On the other hand, we have by the principal ideal theorem of Krull that $\dim_R(M/(a_0)M) = \dim_R(M) - 1$. Using induction we thus get

$$\dim_R(M) - 1 = \dim_R(M/(a_0)M) \geq \mathrm{depth}(M/(a_0)M) = \mathrm{depth}(M) - 1.$$

This finishes the proof. □

Definition 1.12 *Let R be a local ring and M an R-module. Then M is said to be* Cohen-Macaulay *if*

$$\mathrm{depth}(M) = \dim_R(M).$$

Lemma 1.7 *Let R be a local ring and $J \subset R$ an ideal such that $JM = 0$. Then M is Cohen-Macaulay as an R-module if and only if M is Cohen-Macaulay as an R/J-module.*

Proof: This is a direct consequence of Lemmas 1.4 and 1.5. □

Definition 1.13 *For a ring R, a prime ideal $p \subset R$ and an R-module M, we say that M is* Cohen-Macaulay *at p, if M_p is Cohen-Macaulay as an R_p-module.*

1.1.5 The ext functor

Definition 1.14 *Let R and R' be commutative rings and M an R-module. An assignment*

$$F : (M \xrightarrow{g} N) \rightsquigarrow (F(M) \xrightarrow{F(g)} F(N)),$$

*which assigns to each R-module M an R'-module $F(M)$ and to each homomorphism of R-modules $h : M \to N$ a homomorphism of R'-modules $F(h) : F(M) \to F(N)$, is called an R-*linear functor *from the category of R-modules to the category of R'-modules if the following properties hold:*

(1) $F(id_M) = id_{F(M)}$ for each R-module M.

(2) $F(h \circ l) = F(h) \circ F(l)$ if $h : N \to P$ and $l : M \to N$ are homomorphisms of R-modules.

(3) $F(h + l) = F(h) + F(l)$ if $h, l : M \to N$ are homomorphisms of R-modules.

(4) There exists a homomorphism of rings $f : R \to R'$ such that $F(ah) = f(a)F(h)$ for each $a \in R$ and each homomorphism of R-modules $h : M \to N$.

Example 1.4 *The assignment $\mathrm{Hom}_R(M, \bullet)$ is an R-linear functor from the category of R-modules to itself. It assigns to each R-module N the R-module $\mathrm{Hom}_R(M, N)$ and to each homomorphism $h : N \to N'$ the homomorphism $\mathrm{Hom}_R(M, h)$ defined as follows:*

$$\begin{array}{rcl} \mathrm{Hom}_R(M,h) : \mathrm{Hom}_R(M,N) & \to & \mathrm{Hom}_R(M,N') \\ l & \mapsto & h \circ l. \end{array}$$

The functor $\mathrm{Hom}_R(M, \bullet)$ is left exact. This means that for an exact sequence of R-modules

$$0 \to N' \xrightarrow{g} N \xrightarrow{h} N'',$$

the sequence

$$0 \to \mathrm{Hom}_R(M, N') \xrightarrow{\mathrm{Hom}_R(M,g)} \mathrm{Hom}_R(M, N) \xrightarrow{\mathrm{Hom}_R(M,h)} \mathrm{Hom}_R(M, N'')$$

is also exact.

Definition 1.15 *By a complex of R-modules we mean a sequence of R-modules and homomorphisms of R-modules*

$$\ldots \xrightarrow{g^{n-1}} N^n \xrightarrow{g^n} N^{n+1} \xrightarrow{g^{n+1}} N^{n+2} \xrightarrow{g^{n+2}} \ldots$$

such that, for all $n \in \mathbb{Z}$, we have $\mathrm{Im}(g^{n-1}) \subset \mathrm{Ker}(g^n)$; this is equivalent to saying, that $g^n \circ g^{n-1} = 0$. Such a complex will be denoted by $(N^\bullet, g^\bullet)$.

A homomorphism of complexes $h^\bullet : (N^\bullet, g^\bullet) \to (M^\bullet, l^\bullet)$ is a family $(h^n)_{n\in\mathbb{Z}}$ of homomorphisms $h^n : N^n \to M^n$ such that for all $n \in \mathbb{Z}$ the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & N^{n-1} & \xrightarrow{g^{n-1}} & N^n & \xrightarrow{g^n} & N^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle h^{n-1}} & & \downarrow{\scriptstyle h^n} & & \downarrow{\scriptstyle h^{n+1}} & & \\ \cdots & \longrightarrow & M^{n-1} & \xrightarrow{l^{n-1}} & M^n & \xrightarrow{l^n} & M^{n+1} & \longrightarrow & \cdots \end{array}$$

commutes.

Definition 1.16 *An R-module I is said to be injective if for each injection $i : N \hookrightarrow M$ of R-modules and each homomorphism $h : N \to I$, there exists a homomorphism $l : M \to I$ such that $h = l \circ i$.*

Remark 1.5 *By the Lemma of Eckmann-Schöpf [6, Corollary A3.9], we know that for each R-module M there is an injective homomorphism $M \hookrightarrow I$ into an injective R-module I.*

Definition 1.17 *An injective resolution* $(I^\bullet, d^\bullet)$ *of an R-module M is a complex of R-modules* $I^\bullet$*, such that* $I^i = 0$ *when* $i < 0$*, together with a homomorphism* $b : M \hookrightarrow I^0$ *such that the sequence*

$$0 \to M \xrightarrow{b} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \xrightarrow{d^3} \dots$$

is exact.

Remark 1.6 *By a recursive use of the Lemma of Eckmann-Schöpf, every R-module has an injective resolution.*

Definition 1.18 *For a complex of R-modules* $(N^\bullet, d^\bullet)$ *and an integer* $n \in \mathbb{Z}$*, the n-th cohomology of* $(N^\bullet, d^\bullet)$ *is defined by*

$$H^n((N^\bullet, d^\bullet)) = H^n(N) := \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1}).$$

Definition 1.19 *For* $n \in \mathbb{Z}$*, the n-th derived functor of* Hom_R*, denoted* Ext^n_R*, is defined as follows. Let M and N be R-modules. Choose an injective resolution* $(I^\bullet, d^\bullet)$ *of N. Then*

$$\mathrm{Ext}^n_R(M, N) := H^n(\mathrm{Hom}_R(M, I^\bullet)).$$

By construction, $\mathrm{Ext}^n_R(M, N) = 0$ *for* $n < 0$.

If there is no risk of confusion, we write Ext^n *instead of* Ext^n_R.

Remark 1.7 *It can be shown that* $\mathrm{Ext}^n(M, N)$ *is independent, up to a canoical isomorphism, of the choice of an injective resolution of N. See [6, Corollary A3.14] for a proof.*

Definition 1.20 *A short exact sequence of complexes*

$$0 \to (N'^\bullet, d'^\bullet) \xrightarrow{g^\bullet} (N^\bullet, d^\bullet) \xrightarrow{h^\bullet} (N''^\bullet, d''^\bullet) \to 0$$

consists of morphisms of complexes $g^\bullet$ *and* $h^\bullet$ *such that for every* $i \in \mathbb{Z}$,

$$0 \to N'^i \xrightarrow{g^i} N^i \xrightarrow{h^i} N''^i \to 0$$

is a short exact sequence.

Lemma 1.8 *Keep the notation as in the previous definition. Then, there are natural homomorphisms* $H^i(N'') \to H^{i+1}(N')$ *for* $i \in \mathbb{Z}$ *yielding a long exact sequence*

$$\dots \to H^i(N') \to H^i(N) \to H^i(N'') \to H^{i+1}(N') \to \dots$$

Proof: See [6, Proposition A3.15]. □

Definition 1.21 *An injective resolution of an exact sequence of R-modules*

$$0 \to N' \to N \to N'' \to 0$$

is an exact sequence of complexes of R-modules

$$0 \to (I'^{\bullet}, d'^{\bullet}) \to (I^{\bullet}, d^{\bullet}) \to (I''^{\bullet}, d''^{\bullet}) \to 0$$

together with a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I'^0 & \longrightarrow & I^0 & \longrightarrow & I''^0 & \longrightarrow & 0 \end{array}$$

such that $(I'^{\bullet})$, $(I^{\bullet})$ *and* $(I''^{\bullet})$ *are injective resolutions of* N', N *and* N'' *respectively.*

Remark 1.8 *It can be shown, using the Lemma of Eckmann-Schöpf, that every exact sequence of R-modules has an exact sequence of injective resolutions.*

Proposition 1.3 *Let M be an R-module and let*

$$0 \to N' \to N \to N'' \to 0$$

be an exact sequence of R-modules. Then there is a long exact sequence of R-modules

$$0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'')$$

$$\to \mathrm{Ext}^1(M, N') \to \mathrm{Ext}^1(M, N) \to \mathrm{Ext}^1(M, N'') \to \mathrm{Ext}^2(M, N') \to \dots$$
$$\dots \to \mathrm{Ext}^i(M, N') \to \mathrm{Ext}^i(M, N) \to \mathrm{Ext}^i(M, N'') \to \mathrm{Ext}^{i+1}(M, N') \to \dots$$

Proof: Choose an injective resolution of the exact sequence in the statement:

$$0 \to (I'^{\bullet}, d'^{\bullet}) \to (I^{\bullet}, d^{\bullet}) \to (I''^{\bullet}, d''^{\bullet}) \to 0.$$

This gives an exact sequence of complexes

$$0 \to \mathrm{Hom}(M, I'^{\bullet}) \to \mathrm{Hom}(M, I^{\bullet}) \to \mathrm{Hom}(M, I''^{\bullet}) \to 0.$$

The long exact sequence we want is then obtained by applying Lemma 1.8. □

Proposition 1.4 *For two finitely generated R-modules M and N, it holds that either* $\mathrm{Ann}(M) + \mathrm{Ann}(N) = R$ *or*

$$\mathrm{depth}_{\mathrm{Ann}(M)}(N) = \min\{r \in \mathbb{N} \mid \mathrm{Ext}^r(M, N) \neq 0)\}.$$

Proof: See [6, prop. 18.4]. □

1.2 Graded rings, homogeneous ideals, saturation of an ideal

1.2.1 Graded rings

Definitions 1.1 *Let R be a ring and Γ a commutative monoid. A* Γ-grading *on R is a decomposition*

$$R = \bigoplus_{i \in \Gamma} R_i$$

where R_i are additive abelian subgroups such that $R_i \cdot R_j \subset R_{i+j}$ for each $(i,j) \in \Gamma^2$. A Γ-graded *ring is a ring endowed with a Γ-grading.*

Given a Γ-graded ring R, a Γ-graded R-module M is an R-module M with a decomposition (called a Γ-grading)

$$M = \bigoplus_{i \in \Gamma} M_i$$

where M_i are additive abelian subgroups such that $R_i \cdot M_j \subset M_{i+j}$ for each $(i,j) \in \Gamma^2$.

- *The elements of R_i and M_i are called* homogeneous *elements of degree i. The set of homogeneous elements in R and M are denoted by R^{Hom} and M^{Hom}. We have $R^{Hom} = \bigcup_{i \in \Gamma} R_i$ and $M^{Hom} = \bigcup_{i \in \Gamma} M_i$.*

- *For an element r of R or M, there is a unique decomposition $r = \sum_{i \in \Gamma} r_i$ where the r_i are homogeneous of degree i and zero except for finitely many of them. The r_i's are called the* homogeneous components *of r.*

- *An ideal $I \subset R$ is called* homogeneous *if it admits a system of homogenous generators. This is equivalent to saying that $I = \bigoplus_{i \in \Gamma} R_i \cap I$.*

Remark 1.9 *We are mainly interested in $\mathbb{N}$-graded rings and modules. Since $\mathbb{N}$ is an ordered monoid, we also have a natural order on the grading. When $\Gamma = \mathbb{N}$, we simply say graded instead of $\mathbb{N}$-graded.*

Definitions 1.2 *Let R be an $\mathbb{N}$-graded ring. We denote:*

- *$R_+ := \bigoplus_{i \in \mathbb{N}_+} R_i$ the graded ideal generated by the elements of strictly positive degrees.*

- $\mathrm{Sph}(R)$ *the set of homogenous prime ideals in R which do not contain R_+; this is a subset of* $\mathrm{Spec}(R)$.

Example 1.5 *The ring of polynomials $k[x_0, \ldots, x_m]$ has a natural $\mathbb{N}$-grading given by the total degree of monomials.*

1.2.2 Saturated ideals

Definition 1.22 *Let k be a ring and $R := k[x_0, \ldots, x_m]$. Let $I \subset R$ be a homogeneous ideal. Then the* saturation *of I is defined by*

$$I^{Sat} := \{r \in R \mid \forall i \in \{1, \ldots, m\}, \exists n \in \mathbb{N} : x_i^n r \in I\}.$$

The ideal I is called saturated *if $I = I^{Sat}$.*

Lemma 1.9 *Let I be a homogenous ideal. Then, I^{Sat} is also a homogeneous ideal.*

Proof: This is clear. □

Lemma 1.10 *Let I be a homogenous ideal. Then*

$$I^{Sat} = (I : R_+^\infty) := \bigcup_{n \in \mathbb{N}} (I : R_+^n) = \bigcup_{n \in \mathbb{N}} \{r \in R \mid R_+^n r \subset I\}.$$

Proof: For $r \in (I : R_+^\infty)$ there is an $n \in \mathbb{N}$, such that $R_+^n r \subset I$. In particular $x_i^n r \in I$, and this means that r is in the saturation of I.

On the other hand, let $r \in R$ such that for all $i \in \{1, \ldots, m\}$ there exists $n_i \in \mathbb{N}$ with $x_i^{n_i} r \in I$. If n is greater than all the n_i's, we have $x_i^n r \in I$. It follows, that $R_+^{mn} r \subset I$. This proves the lemma. □

Example 1.6 *The ideal $(x, y) \subset k[x, y]$ is not saturated. Its saturation is equal to $k[x, y]$. More generally, the saturation of R_+ is equal to R.*

Example 1.7 *For an ideal $I \subset R$, IR_+ is not saturated. Indeed, I is contained in $(IR_+)^{Sat}$.*

Lemma 1.11 *A homogenous prime ideal $p \subset R$ which does not contain R_+ is saturated.*

Proof: Let $r \in p^{Sat}$. Thus, for every $i \in \{1, \ldots, m\}$ there is an integer $n_i \in \mathbb{N}$ such that $x_i^{n_i} r \in p$. Since p is prime, this means that either $r \in p$ or $x_i \in p$ for all i. Since p does not contain R_+, the second alternative is excluded. So r must be in p. □

Example 1.8 *Let k be a field. The ideal $(x) \subset k[x, y]$ is saturated since it is a prime ideal not containing y.*

Lemma 1.12 *Let J be a graded ideal of R. Then J is saturated if and only if*

$$\mathrm{Hom}_R(R/R_+, R/J) = 0.$$

Proof: Let $f : R/R_+ \to R/J$ be a homomorphism and let $f(1 + R_+) = a + J$. Then, for $0 \leq i \leq m$, we have $0 = f(x_i(1 + R_+)) = x_i f(1 + R_+) = x_i a + J$. This means that $x_i a \in J$. As J is saturated, $a \in J$ and f is the zero map.

Conversely, assume that $\mathrm{Hom}_R(R/R_+, R/J) = 0$. Let $a \in R$ and assume that $x_i^n a \in J$. We want to show that $a \in J$. By induction on n we may assume that $n = 1$. Then we have a map $f : R/R_+ \to R/J$ given by $f(u + R_+) = ua + J$. As this map must be zero, we see that $a \in J$. □

Lemma 1.13 *Let $f_1, \ldots, f_r \in R = k[x_0, \ldots, x_m]$ be a regular sequence consisting of homogenous polynomial. Assume that $r \leq m$. Then $(f_1, \ldots, f_r)$ is a saturated ideal.*

Proof: As $r \leq m$, there must be an element $l \in R_+$ such that $(f_1, \ldots, f_r, l)$ is a regular sequence. This is equivalent to saying, that the class of l is a nonzero divisor in $R/(f_1, \ldots, f_r)$. Take now an element $a \in (f_1, \ldots, f_r)^{Sat}$. There is an $n \in \mathbb{N}$ such that $R_+^n a \subset (f_1, \ldots, f_r)$. In particular,

$$l^n a \in (f_1, \ldots, f_r).$$

This means that the class of $l^n a$ is zero in $R/(f_1, \ldots, f_r)$. But this is only possible if the class of a is zero since the class of l is a non zero divisor. This shows that a is in $(f_1, \ldots, f_r)$ as needed. $\square$

1.2.3 Hilbert polynomial

Let k be a field. We set $R = k[x_0, \ldots, x_m]$ with its natural grading.

Proposition 1.5 *Let M be a finitely generated graded R-module. There exists a unique polynomial $P_M(t) \in \mathbb{Q}[t]$ such that $P_M(d) = \dim_k(M_d)$ for d big enough. $P_M(z)$ is called the* Hilbert Polynomial *of M.*

Proof: We show the existence of P_M by induction on m. For $m = -1$, the module M is a finite dimensional graded k-vectorspace. Therefore, we have $\dim_k(M_d) = 0$ for d big enough.

Suppose now that $m \geq 0$ and that the claim is true for $m - 1$. We define the submodules N^s for $s \in \mathbb{N}$ to be the kernel of the multiplication by x_m^s. We then have a chain of graded submodules

$$0 = N^0 \subset N^1 \subset \ldots \subset N^s \subset \ldots,$$

which is stationary, since M is a noetherian module. This means, that for s big enough $N^s/N^{s-1} = 0$. Now, for $s \geq 1$, N^s/N^{s-1} is a finitely generated graded $k[x_0, \ldots, x_m]$-module on which x_m acts as zero. Therefore, it may be considered as a finitely generated graded $k[x_0, \ldots, x_m]/(x_m) \cong k[x_0, \ldots, x_{m-1}]$-module. By the induction hypothesis, there are polynomials $P_{N^s/N^{s-1}}$ such that

$$\dim_k(N_d^s/N_d^{s-1}) = P_{N^s/N^{s-1}}(d)$$

for d big enough. Consider now $N := \bigcup_{s \in \mathbb{N}} N^s$, the so called x_m-torsion submodule of M. From the previous discussion, we see that

$$\dim_k(N_d) = \sum_{s>1} \dim_k(N_d^s/N_d^{s-1}) = \sum_{s>1} P_{N^s/N^{s-1}}(d),$$

where the second equality holds for d big enough. This shows, that the theorem is true for N.

Next, we consider the short exact sequence

$$0 \to N \to M \to M/N \to 0.$$

We clearly have $\dim_k(M_d) = \dim_k(N_d) + \dim_k(M_d/N_d)$. Therefore, it is enough to prove the claim for the R-module M/N. By construction, the multiplication by x_m is injective on M/N and shifts the grading by one. We form the exact sequence:

$$0 \to M/N \stackrel{\cdot x_m}{\to} M/N[+1] \to K \to 0,$$

where $M/N[+1]$ has the same underlying module as M/N but with a shifted grading: $(M/N[+1])_{d-1} = (M/N)_d$. Also, K is a finitely generated $k[x_0, \ldots, x_m]$-module on which x_m acts as zero. Hence, it can be considered as a $k[x_0, \ldots, x_{m-1}]$-module and the induction hypothesis gives a polynomial $P_K(z) \in \mathbb{Q}[z]$, such that $\dim_k(K_d) = P_K(d)$ for d big enough. Using the above exact sequence we get:

$$\dim_k(M_d/N_d) - \dim_k(M_{d-1}/N_{d-1}) = P_K(d-1)$$

for d big enough. We conclude using the lemma below. □

Lemma 1.14 *Let $f : \mathbb{N} \to \mathbb{Q}$ be a function. Assume that there exists a polynomial $Q \in \mathbb{Q}[t]$ such that $f(d) - f(d-1) = Q(d-1)$ for d big enough. Then, there exists a polynomial $P \in \mathbb{Q}[t]$ such that $f(d) = P(d)$ for d big enough.*

Proof: Let n be the degree of Q and at^n its leading monomial. Define a function $g : \mathbb{N} \to \mathbb{Q}$ by

$$g(d) = f(d) - \frac{a \cdot d^{n+1}}{n+1}.$$

Then $g(d) - g(d-1)$ is given by

$$\begin{aligned}
& f(d) - \frac{a \cdot d^{n+1}}{n+1} - f(d-1) + \frac{a \cdot (d-1)^{n+1}}{n+1} \\
= \quad & Q(d) - \frac{a}{n+1}(d^{n+1} - (d^{n+1} - \binom{n+1}{1} d^n + \binom{n+1}{2} d^{n-1} - \ldots + (-1)^{n+1})) \\
= \quad & Q(d) - \frac{a}{n+1}(\binom{n+1}{1} d^n - \binom{n+1}{2} d^{n-1} + \ldots - (-1)^{n+1}) \\
= \quad & Q(d) - a \cdot d^n + \frac{a}{n+1}(\binom{n+1}{2} d^{n-1} - \ldots + (-1)^{n+1})
\end{aligned}$$

Hence there is a polynomial $R \in \mathbb{Q}[t]$ of degree $n-1$ such that $g(d) - g(d-1) = R(d)$ for d big enough. We now use induction on the degree of Q to deduce that there is a polynomial $S \in \mathbb{Q}[t]$ such that $g(d) = S(d)$ for d big enough. We can now take $P = S + \frac{a \cdot t^{n+1}}{n+1}$. □

The following proposition was implicitely used in the proof of Proposition 1.5.

Proposition 1.6 *Let*

$$0 \to N \to M \to L \to 0$$

be a short exact sequence of graded finitely generated R-modules. Then, we have the equality $P_M = P_N + P_L$.

Proof: This is clear. □

Lemma 1.15 *Let $I \subset R$ be a graded ideal. For d big enough we have $(I^{Sat})_d = I_d$.*

Proof: The inclusion "$\supset$" is clear. For the other inclusion, let $f_1, \ldots, f_r \in I^{Sat}$ be a set of homogeneous generators of degree $d_1, \ldots, d_r$. For every $j \in \{1, \ldots, r\}$ there exists an integer $e_j \in \mathbb{N}$ such that $pf_j \in I$ for every homogenous polynomial $p \in R$ of degree greater than e_j.

Let n be an integer greater than $\max\{d_1 + e_1, \ldots, d_r + e_r\}$. Every $g \in (I^{Sat})_n$ can be written as a sum $g = \sum_{j=1}^{r} q_j f_j$ where $\deg(q_j) = n - d_j \geq e_j$. Therefore $q_j f_j \in I_n$ and $g \in I_n$. □

Corollary 1.1 *$P_I = P_{I^{Sat}}$ and $P_{R/I} = P_{R/I^{Sat}}$.*

2 Curves in projective space, some notions from algebraic geometry

In this section, we fix once and for all an algebraically closed field k.

2.1 The projective space and the Zariski topology

2.1.1 The projective space

Definition 2.1 *Consider the $n+1$-dimensional vectorspace k^{n+1}. We define an equivalence relation $\sim$ on $k^{n+1} \setminus \{0\}$ by $P \sim Q$ if and only if there is an element $r \in k \setminus \{0\}$ such that $P = rQ$. The set*

$$\mathbb{P}^n := (k^{n+1} \setminus \{0\})/\sim$$

is the n-dimensional projective space. *The class in $\mathbb{P}^n$ of a point $Q = (x_0, \ldots, x_n) \in k^{n+1} \setminus \{0\}$ is denoted by $\bar{Q} = [x_0 : \ldots : x_n]$.*

Remark 2.1 *There is a injective map from k^n to $\mathbb{P}^n$:*

$$\begin{aligned} i_0 : k^n &\to \mathbb{P}^n \\ (x_1, x_2, \ldots, x_n) &\mapsto [1 : x_1 : x_2 : \ldots : x_n]. \end{aligned}$$

This map reaches all the points in $\mathbb{P}^n$ except those with $x_0 = 0$. For the latter we have another injective map:

$$\begin{aligned} j_0 : \mathbb{P}^{n-1} &\to \mathbb{P}^n \\ [x_0 : x_1 : \ldots : x_{n-1}] &\mapsto [0 : x_0 : x_1 : \ldots : x_{n-1}]. \end{aligned}$$

Therefore, we have a bijection between the set $k^n \cup \mathbb{P}^{n-1}$ and the projective space $\mathbb{P}^n$. By induction, one gets also a bijection between $\mathbb{P}^n$ and $k^n \cup \ldots \cup k^1 \cup k^0$.

2.1.2 Zariski closed subsets

Remark 2.2 *For a homogenous polynomial in $n+1$ variables $p \in k[x_0, \ldots, x_n]$, the condition that $p(x_0, \ldots, x_n) = 0$ for $(x_0, \ldots, x_n) \in k^{n+1} \setminus \{0\}$ depends only on its class $[x_0 : \ldots : x_n]$ in $\mathbb{P}^n$.*

Definition 2.2 *Consider a set of homogenous polynomials in $n+1$ variables $S \subset k[x_0, \ldots, x_n]$. It defines a subset $\mathrm{V}(S)$ in $\mathbb{P}^n$:*

$$\mathrm{V}(S) := \{\bar{Q} \in \mathbb{P}^n \mid p(Q) = 0 \; \forall p \in S\}.$$

These subsets are the Zariski closed *subsets of $\mathbb{P}^n$.*

Definition 2.3 *For a homogenous ideal $I \subset k[x_0, \ldots, x_n]$, we define I^{Hom} to be the set of homogenous elements of I.*

Definition 2.4 *For a homogenous ideal $I \subset k[x_0, \ldots, x_n]$, we define:*

$$\mathrm{V}(I) := \{\bar{Q} \in \mathbb{P}^n \mid p(Q) = 0 \ \forall p \in I^{Hom}\} = \mathrm{V}(I^{Hom}).$$

Lemma 2.1 *Let $p_1, ..., p_r$ be homogenous generators of a homogenous ideal I. Then, we have:*

$$\mathrm{V}(p_1, ..., p_r) = \mathrm{V}(I).$$

Proof: For $\bar{Q} \in \mathrm{V}(p_1, ..., p_r)$ we have $p_i(Q) = 0$ for all $i \in \{1, \ldots, r\}$. Therefore, $g \cdot p_i(Q) = 0$ for all $g \in k[x_0, \ldots, x_n]^{Hom}$ and the same is true for homogeneous linear combinations of the p_i's. Therefore $\bar{Q} \in \mathrm{V}(I)$.

On the other hand, for $\bar{Q} \in \mathrm{V}(I)$, all polynomials of I^{Hom} vanish at Q. In particular, $p_i(Q) = 0$ for all $i \in \{1, \ldots, r\}$. Hence, $Q \in \mathrm{V}(p_1, ..., p_r)$. □

Proposition 2.1 *For two homogenous ideals $I, J \subset k[x_0, \ldots, x_n]$, it holds that*

a) $\mathrm{V}(I) \cup \mathrm{V}(J) = \mathrm{V}(I \cap J) = \mathrm{V}(IJ)$*;*

b) $\mathrm{V}(I) \cap \mathrm{V}(J) = \mathrm{V}(I + J)$*;*

c) $I \subset J \Rightarrow \mathrm{V}(J) \subset \mathrm{V}(I)$*;*

d) $\mathrm{V}(I) = \mathrm{V}(\sqrt{I}) = \mathrm{V}(I^{Sat})$*.*

Proof:

c) Consider a point $\bar{P} \in \mathrm{V}(J)$. For all $f \in J^{Hom}$ we have that $f(P) = 0$. Since $I \subset J$ this is also true for all $f \in I^{Hom}$. This means that $\bar{P}$ is also a point in $\mathrm{V}(I)$.

a) We have seen, that $I \subset J \Rightarrow \mathrm{V}(J) \subset \mathrm{V}(I)$, and we know that $I \cdot J \subset I \cap J \subset I$ and $I \cdot J \subset I \cap J \subset J$. It follows that $\mathrm{V}(I) \subset \mathrm{V}(I \cap J) \subset \mathrm{V}(I \cdot J)$ and $\mathrm{V}(J) \subset \mathrm{V}(I \cap J) \subset \mathrm{V}(I \cdot J)$. Hence, $\mathrm{V}(I) \cup \mathrm{V}(J) \subset \mathrm{V}(I \cap J) \subset \mathrm{V}(I \cdot J)$.

   On the other hand, let $\bar{P} \in \mathrm{V}(I \cdot J)$ and suppose that $\bar{P} \notin \mathrm{V}(J)$. This means, that we have at least one element $f \in J^{Hom}$ such that $f(P) \neq 0$. Consider now an element $g \in I^{Hom}$. The product $f \cdot g$ is in $I \cdot J$ and therefore we have that $f \cdot g(P) = f(P) \cdot g(P) = 0$. It follows, that $g(P) = 0$ and therefore $\bar{P} \in \mathrm{V}(I)$.

b) Since $I \subset (I + J)$ and $J \subset (I + J)$ we know by c), that $\mathrm{V}(I) \supset \mathrm{V}(I + J)$ and $\mathrm{V}(J) \supset \mathrm{V}(I + J)$. It follows that $\mathrm{V}(I) \cap \mathrm{V}(J) \supset \mathrm{V}(I + J)$.

   For the other inclusion, we consider a point $\bar{P} \in V(I) \cap V(J)$ and a polynomial $f + g \in I + J$. We have:

$$(f + g)(P) = f(P) + g(P) = 0 + 0 = 0.$$

   It follows that $\bar{P}$ is also an element of $\mathrm{V}(I + J)$.

d) Again the inclusions $\mathrm{V}(I) \supset \mathrm{V}(\sqrt{I})$ follow from c), since $I \subset \sqrt{I}$. On the other hand, let $\bar{P} \in \mathrm{V}(I)$ and $f \in \sqrt{I}$. There is an integer $n \in \mathbb{N}$ such that $f^n \in I$. Therefore $f^n(P) = (f(P))^n = 0$. This implies that $f(P) = 0$. Therefore, we have $\bar{P} \in \mathrm{V}(\sqrt{I})$.

   We now turn to the equality $\mathrm{V}(I) = \mathrm{V}(I^{Sat})$. The inclusion $\mathrm{V}(I) \supset \mathrm{V}(I^{Sat})$ is given by c) since $I \subset I^{Sat}$. For the other inclusion, consider $\bar{P} \in \mathrm{V}(I)$ and $f \in I^{Sat}$. By the definition of the saturation of an ideal there exists an $l_i \in \mathbb{N}$ for all variables x_i with $i \in \{0, \ldots, n\}$ such that $x_i^{l_i} \cdot f \in I$. It follows that

$$x_i^{l_i} \cdot f(P) = x_i^{l_i}(P) \cdot f(P) = p_i^{l_i} \cdot f(P) = 0.$$

   (In the formula above, the p_i's are the coordinates of the point P.) Since $P \in k^{n+1} \setminus \{0\}$, at least one of the p_i's is not equal to zero. This shows that $f(P) = 0$. Therefore, we have $\bar{P} \in \mathrm{V}(I^{Sat})$.

□

Corollary 2.1 *The Zariski closed subsets of $\mathbb{P}^n$ define a topology on $\mathbb{P}^n$ called the Zariski topology.*

Definition 2.5 *For a subset $Z \subset \mathbb{P}^n$ we define:*

$$\mathrm{I}(Z) := (\{f \in (k[x_0, \ldots, x_n])^{Hom} \mid f(P) = 0, \ \forall \bar{P} \in Z\}).$$

Remark 2.3 *By construction, $\mathrm{I}(Z)$ is a homogeneous ideal. The homogenous part of degree d is equal to $\{f \in (k[x_0, \ldots, x_n])_d \mid f(P) = 0, \ \forall \bar{P} \in Z\}$. Therefore, we have*

$$I(Z) = \bigoplus_{d \in \mathbb{N}} \{f \in (k[x_0, \ldots, x_n])_d \mid f(P) = 0, \ \forall \bar{P} \in Z\}.$$

Lemma 2.2 *For a subset $Z \subset \mathbb{P}^n$, $\mathrm{I}(Z)$ is a radical and saturated ideal.*

Proof: Consider a homogenous polynomial $f \in k[x_0, \ldots, x_n]$ with $f^n \in \mathrm{I}(Z)$ for some integer $n \in \mathbb{N}$. This means, that $f^n(P) = (f(P))^n = 0$ for all $\bar{P} \in Z$. It follows, that $f(P) = 0$ for all $\bar{P} \in Z$. Hence $f \in \mathrm{I}(Z)$.

Now consider a homogenous polynomial $f \in k[x_0, \ldots, x_n]$ for which there exist integers $l_i \in \mathbb{N}$ such that $x_i^{l_i} \cdot f \in \mathrm{I}(Z)$ for each $i \in \{0, \ldots, n\}$. This means, that $(x_i^{l_i} \cdot f)(P) = p_i^{l_i} f(P) = 0$ for all $\bar{P} = [p_0 : \ldots : p_n] \in Z$. Since at least one of the p_i's is non zero, we have $f(P) = 0$. Hence $f \in \mathrm{I}(Z)$. □

Proposition 2.2 *For a subset $Z \subset \mathbb{P}^n$ and for a homogenous ideal $J \subset k[x_0, \ldots, x_n]$ it holds that*

a) $\mathrm{V}(\mathrm{I}(Z)) = \overline{Z}$ *(where $\overline{Z}$ is the Zariski closure of Z) and*

b) $\mathrm{I}(\mathrm{V}(J)) = (\sqrt{J})^{Sat}$.

Proof:

a) For a point $\bar{Q} \in Z$ it holds that $f(Q) = 0$ for all $f \in \mathrm{I}(Z)$ since this is the property of the elements of $\mathrm{I}(Z)$. This shows that $Z \subset \mathrm{V}(\mathrm{I}(Z))$. As $\mathrm{V}(\mathrm{I}(Z))$ is Zariski closed, we also get that $\overline{Z} \subset \mathrm{V}(\mathrm{I}(Z))$.

   Conversely, let $\bar{Q} \in \mathrm{V}(\mathrm{I}(Z))$. To show that $\bar{Q} \in \overline{Z}$ it is enough to show that $\bar{Q} \in T$ for every Zariski closed subset $T = \mathrm{V}(I)$ containing Z. But $Z \subset \mathrm{V}(I)$ implies that $I \subset \mathrm{I}(Z)$. Hence, $f(Q) = 0$ for every $f \in I$ and thus $\bar{Q} \in T$.

b) When $\mathrm{V}(J) = \emptyset$, we have $\sqrt{J} = k[x_0, \ldots, x_n]$ or $\sqrt{J} = k[x_0, \ldots, x_n]_+$ by [7, p. 11, Exercise 2.2]. Therefore, in this case we have $\mathrm{I}(\mathrm{V}(J)) = k[x_0, \ldots, x_n]$ and $(\sqrt{J})^{Sat} = k[x_0, \ldots, x_n]$.

   We now assume that $\mathrm{V}(J) \neq \emptyset$. By [7, p. 11, Exercise 2.3], we have $\mathrm{I}(\mathrm{V}(J)) = \sqrt{J}$. Moreover, $\mathrm{I}(\mathrm{V}(J))$ is saturated being the intersection of saturated ideals. It follows that $(\sqrt{J})^{Sat} = \sqrt{J}$. This finishes the proof.

□

Corollary 2.2 *For two homogenous ideals I and J in $k[x_0, \ldots, x_n]$ we have the following equivalence:*

$$\mathrm{V}(I) \subset \mathrm{V}(J) \Leftrightarrow J \subset (\sqrt{I})^{Sat}.$$

Proof: If $J \subset (\sqrt{I})^{Sat}$, we have $\mathrm{V}((\sqrt{I})^{Sat}) \subset \mathrm{V}(J)$. But $\mathrm{V}((\sqrt{I})^{Sat}) = \mathrm{V}(I)$ by Proposition 2.1. This gives the implication "$\Leftarrow$".

For the reverse implication, we remark that $\mathrm{V}(I) \subset \mathrm{V}(J)$ implies that $\mathrm{I}(\mathrm{V}(J)) \subset \mathrm{I}(\mathrm{V}(I))$. By Proposition 2.2, we get $(\sqrt{J})^{Sat} \subset (\sqrt{I})^{Sat}$. This gives the result since $J \subset (\sqrt{J})^{Sat}$. □

Corollary 2.3 *There is a bijection:*

$$\begin{array}{rcl} \{Z \subset \mathbb{P}^n \mid Z \text{ Zariski closed }\} & \tilde{\to} & \{I \subset k[x_0, \ldots, x_n] \mid I \text{ saturated radical ideal}\} \\ Z & \mapsto & \mathrm{I}(Z). \end{array}$$

Definition 2.6 *A Zariski closed subset $Z \subset \mathbb{P}^n$ is called* irreducible *if there aren't two Zariski closed subsets*

$$\emptyset \neq Z_1, Z_2 \subsetneq Z$$

such that $Z = Z_1 \cup Z_2$.

Proposition 2.3 *A Zariski closed subset $Z \subset \mathbb{P}^n$ is irreducible if and only if* $\mathrm{I}(Z)$ *is a prime ideal in $k[x_0, \ldots, x_n]$. This gives another bijection:*

$$\begin{array}{rcl} \{Z \subset \mathbb{P}^n \mid Z \text{ Zariski closed and irreducible}\} & \tilde{\to} & Sph(k[x_0, \ldots, x_n]) \\ Z & \mapsto & \mathrm{I}(Z). \end{array}$$

Proof: Let $Z \subset \mathbb{P}^n$ be an irreducible Zariski closed subset. Consider two homogenous polynomials $a, b \in k[x_0, \ldots, x_n]$ such that $a \cdot b \in \mathrm{I}(Z)$. For each $P \in Z$ we have:

$$a \cdot b(P) = a(P) \cdot b(P) = 0.$$

But this means that either $a(P) = 0$ or $b(P) = 0$. This divides Z in two closed subsets $Z_1 := \mathrm{V}(a) \cap Z$ and $Z_2 := \mathrm{V}(b) \cap Z$ and every $P \in Z$ has to be in one of them. Since Z is irreducible, we must have $Z = Z_1$ or $Z = Z_2$. Therefore, a or b has to be in $\mathrm{I}(Z)$. This proves that $\mathrm{I}(Z)$ is a prime ideal.

For the other implication consider a closed subset $Z \subset \mathbb{P}^n$ such that $p = \mathrm{I}(Z)$ is prime. Suppose that there are two closet subsets $Z_1,\ Z_2 \subset \mathbb{P}^n$ with $Z_1 \cup Z_2 = Z$. So there must be two homogenous ideals $J_1,\ J_2 \subset k[x_0, \ldots, x_n]$ such that $\mathrm{V}(J_1) = Z_1$ and $\mathrm{V}(J_2) = Z_2$. So $\mathrm{V}(p) = \mathrm{V}(J_1 \cap J_2)$. By Corollary 2.2, it follows that $J_1 \cap J_2 \subset (\sqrt{p})^{Sat} = p$. We now want to show, that either J_1 or J_2 is contained in p. Suppose that $J_1 \not\subset p$ and consider an element $a \in J_1 \setminus p$. Let b be an element of J_2. Then $a \cdot b$ is in $J_1 \cap J_2$ and hence it is in p. Since p is prime and $a \notin p$, b must be in p. Because this is true for all $b \in J_2$ we have $J_2 \subset p$. □

Definition 2.7 *Let $Z \subset \mathbb{P}^n$ be a closed subset. A subset $T \subset Z$ is called an* irreducible component *of Z if T is irreducible and maximal for this property. Here the maximality means that there is no irreducible subset T' contained in Z and strictly containing T.*

Proposition 2.4 *Every closed subset $Z \subset \mathbb{P}^n$ has finitely many irreducible components. Moreover, if $Z_1, \ldots, Z_r$ are the irreducible components of Z, it holds that:*

$$Z = \bigcup_{i=1}^{r} Z_i.$$

Proof: Let $I \subset k[x_0, \ldots, x_n]$ be a homogenous ideal such that $Z = \mathrm{V}(I)$. There exist finitely many minimal prime ideals $q_1, \ldots, q_r$ in $k[x_0, \ldots, x_n]/I$ by [2, p. 102, Corollary 3]. Let $p_1, \ldots, p_r$ be their inverse images in $k[x_0, \ldots, x_n]$. Those are the minimal prime ideals containing I. Moreover, we have $\sqrt{I} = p_1 \cap \ldots \cap p_r$ by [2, p. 63, Corollary 1]. Also, the prime ideals $p_1, \ldots, p_r$ are graded. We now conclude using Proposition 2.1. □

2.1.3 Dimension of Zariski closed subsets

Definition 2.8 *For a Zariski closed subset Z in $\mathbb{P}^n$, we define its* dimension *as the maximal length of a chain of irreducible subsets:*

$$\dim(Z) := \sup\{n \in \mathbb{N} \mid \exists\, Z_0 \subsetneq \ldots \subsetneq Z_n \subset Z,\ Z_i \text{ irreducible } \forall i \in \{0, \ldots, n\}\}.$$

Example 2.1 *$\mathbb{P}^n$ has dimension n.*

Remark 2.4 *A closed subset $Z \subset \mathbb{P}^n$ of dimension 0 is a finite set of points.*

Lemma 2.3 *If $Z \subset \mathbb{P}^n$ is a Zariski closed subset and $Z_1, \ldots, Z_r$ its irreducible components, then we have:*

$$\dim(Z) = \max_{i=1,\ldots,r} \dim(Z_i).$$

Proof: It is clear, that $\dim(Z) \geq \max_{i=1,\ldots,r} \dim(Z_i)$. Suppose that n is the dimension of Z. There is a chain of inclusions of irreducible closed subsets

$$T_0 \subsetneq \ldots \subsetneq T_n.$$

But then, T_0 is contained in one of the Z_i's, say Z_{i_0}. It follows that the above chain is also a chain of irreducible closed subsets of Z_{i_0}. This proves that $\dim(Z) \leq \dim(Z_{i_0}) \leq \max_{i=1,\ldots,r} \dim(Z_i)$. □

Definition 2.9 *We say that a Zariski closed subset $Z \subset \mathbb{P}^n$ has* pure dimension d *if all its irreducible components are of dimension d.*

Definition 2.10 *For a Zariski closed subset $Z \subset \mathbb{P}^n$, we set*

$$k[Z] := k[x_0, \ldots, x_n]/\mathrm{I}(Z).$$

This ring is called the k-algebra associated to Z.

Proposition 2.5 *Let $Z \subset \mathbb{P}^n$ be a Zariski closed subset. Then*

$$\dim(Z) = \dim(k[Z]) - 1.$$

If Z is also irreducible, then

$$\dim(Z) = \dim(k[Z]) - 1 = \mathrm{trdeg}_k(\mathrm{Frac}(k[Z])) - 1.$$

Proof: The proof is omitted. See [7, Exercise 2.6]. □

Definition 2.11 *A Zariski closed subset $C \subset \mathbb{P}^n$ of pure dimension 1 is called a* curve*.*

Definition 2.12 *A curve $C \subset \mathbb{P}^n$ is called a* complete intersection, *if $\mathrm{I}(C) = (f_1, \ldots, f_{n-1})$ for some homogenous polynomials $f_1, \ldots, f_{n-1} \in k[x_0, \ldots, x_n]$.*

Remark 2.5 *If C is a curve in $\mathbb{P}^2$, then $\mathrm{I}(C)$ is a principal graded ideal, i.e., there exists a homogenous polynomial $f \in k[x_0, x_1, x_2]$ such that $\mathrm{I}(C) = (f)$. In particular, every curve in $\mathbb{P}^2$ is a complete intersection.*

Note that if $n \geq 3$, there are curves in $\mathbb{P}^n$ that are not complete intersections.

2.2 Curves in $\mathbb{P}^n$: Cohen-Macaulay and saturation

In this section, we set $R = k[x_0, \ldots, x_n]$. To ease notation, given a curve C in $\mathbb{P}^n$, we will denote by I_C the homogenous ideal $\mathrm{I}(C)$ associated to C.

Definition 2.13 *A curve $C \subset \mathbb{P}^n$ is called* Cohen-Macaulay *if $k[C] := R/\mathrm{I}_C$ is Cohen-Macaulay at the maximal ideal $k[C]_+$.*

Proposition 2.6 *Let C and D be two Cohen-Macaulay curves in $\mathbb{P}^n$. Then the curve $C \cup D$ is Cohen-Macaulay if and only if the ideal $\mathrm{I}_C + \mathrm{I}_D$ is saturated.*

Proof: The graded ring associated to $C \cup D$ is $R/\mathrm{I}_C \cap \mathrm{I}_D$. By Proposition 2.5, its Krull dimension is equal to 2. By Proposition 1.4, the depth of $R/\mathrm{I}_C \cap \mathrm{I}_D$ (at R_+) is

$$\min\{r \in \mathbb{N} \mid \mathrm{Ext}^r_R(R/R_+, R/\mathrm{I}_C \cap \mathrm{I}_D) \neq 0\}.$$

As the ideal $\mathrm{I}_C \cap \mathrm{I}_D$ is saturated, the vanishing of $\mathrm{Hom}_R(R/R_+, R/\mathrm{I}_C \cap \mathrm{I}_D)$ is granted by Lemma 1.12. Hence $R/\mathrm{I}_C \cap \mathrm{I}_D$ is Cohen-Macaulay if and only if

$$\mathrm{Ext}^1_R(R/R_+, R/\mathrm{I}_C \cap \mathrm{I}_D) = 0.$$

There is an exact sequence

$$0 \to R/\mathrm{I}_C \cap \mathrm{I}_D \to R/\mathrm{I}_C \oplus R/\mathrm{I}_D \to R/(\mathrm{I}_C + \mathrm{I}_D) \to 0,$$

Where the first application maps $r + \mathrm{I}_C \cap \mathrm{I}_D$ to $(r + \mathrm{I}_C, r + \mathrm{I}_D)$ and the second $(r + \mathrm{I}_C, s + \mathrm{I}_D)$ to $r - s + (\mathrm{I}_C + \mathrm{I}_D)$. By Proposition 1.3, this gives a long exact sequence whose first terms are:

$$0 \to \mathrm{Hom}(R/R_+, R/\mathrm{I}_C \cap \mathrm{I}_D) \to \mathrm{Hom}(R/R_+, R/\mathrm{I}_C \oplus R/\mathrm{I}_D) \to \mathrm{Hom}(R/R_+, R/\mathrm{I}_C + \mathrm{I}_D)$$

$$\to \mathrm{Ext}^1(R/R_+, R/\mathrm{I}_C \cap \mathrm{I}_D) \to \mathrm{Ext}^1(R/R_+, R/\mathrm{I}_C \oplus R/\mathrm{I}_D) \to \mathrm{Ext}^1(R/R_+, R/\mathrm{I}_C + \mathrm{I}_D).$$

As R/I_C and R/I_D are Cohen-Macaulay rings of Krull dimension 2, their depth as R-modules is 2. Therefore, the depth of $R/\mathrm{I}_C \oplus R/\mathrm{I}_D$ is also 2 and we have

$$\mathrm{Hom}_R(R/R_+, R/\mathrm{I}_C \oplus R/\mathrm{I}_D) = \mathrm{Ext}^1_R(R/R_+, R/\mathrm{I}_C \oplus R/\mathrm{I}_D) = 0.$$

This gives the isomorphism

$$\mathrm{Hom}(R/R_+, R/\mathrm{I}_C + \mathrm{I}_D) \cong \mathrm{Ext}^1(R/R_+, R/\mathrm{I}_C \cap \mathrm{I}_D).$$

By Lemma 1.12, $\mathrm{Hom}(R/R_+, R/\mathrm{I}_C + \mathrm{I}_D)$ is zero if and only if $\mathrm{I}_C + \mathrm{I}_D$ is saturated. This finishes the proof of the proposition. □

2.3 Intersection of hypersurfaces in $\mathbb{P}^n$

In this section, we set $R = k[x_0, \ldots, x_n]$.

Lemma 2.4 *Let C be a field of characteristic 0. Let $Q \in C[t]$ be a polynomial of degree $d \geq 1$ and leading coefficient $a \in C \setminus 0$. Let $c \in C \setminus \{0\}$. Then the polynomial $Q(t) - Q(t-c)$ is of degree $d-1$ and has leading coefficient acd.*

Proof: Let b be such that $Q(t) - (a \cdot t^d + b \cdot t^{d-1})$ has degree less than or equal to $d-2$, and set $\tilde{Q}(t) = a \cdot t^d + b \cdot t^{d-1}$ and $P = Q(t) - \tilde{Q}(t)$. Then $P(t) - P(t-c)$ has degree at most $d-3$. Therefore, it suffices to show that $\tilde{Q}(t) - \tilde{Q}(t-c)$ has degree $d-1$ and leading coefficient acd. A simple computation gives:

$$\begin{aligned}
\tilde{Q}(t) - \tilde{Q}(t-c) &= a \cdot t^d + b \cdot t^{d-1} - a \cdot (t-c)^d - b \cdot (t-c)^{d-1} \\
&= a \cdot t^d + b \cdot t^{d-1} - a \cdot \sum_{i=0}^{d} \binom{d}{i} (-c)^i t^{d-i} - b \cdot \sum_{i=0}^{d-1} \binom{d-1}{i} (-c)^i t^{d-1-i} \\
&= acd \cdot t^{d-1} + \ldots.
\end{aligned}$$

□

Lemma 2.5 *Let $f_1, \dots, f_r \in R$ be a regular sequence consisting of homogeneous polynomials. Consider the graded R-module $R/(f_1, \dots, f_r)$. Then, its Hilbert polynomial $P_{R/(f_1,\dots,f_r)}$ has degree $n-r$ and leading coefficient*

$$\frac{\deg(f_1)\cdot \ldots \cdot \deg(f_r)}{(n-r)!}.$$

Moreover, the equality

$$\dim((R/(f_1, \dots, f_r))_d) = P_{R/(f_1,\dots,f_r)}(d)$$

holds for $d \geq \deg(f_1) + \ldots + \deg(f_r)$.

Proof: We argue by induction on r. For $r = 0$, we are considering the ring R. Its Hilbert polynomial is

$$\binom{t+n}{n} = \frac{(t+1)\cdot \ldots \cdot (t+n)}{n!}.$$

This is a polynomial of degree n, with leading coefficient $\frac{1}{n!}$.

Now we suppose that the thesis is true for $r-1$. Consider the following exact sequence:

$$0 \to (R/(f_1, \dots, f_{r-1}))\cdot f_r \to R/(f_1, \dots, f_{r-1}) \to R/(f_1, \dots, f_r) \to 0$$

As f_r is a non zero divisor of $R/(f_1, \dots, f_{r-1})$, this gives the relation

$$P_{R/(f_1,\dots,f_r)}(t) = P_{R/(f_1,\dots,f_{r-1})}(t) - P_{R/(f_1,\dots,f_{r-1})}(t-\deg(f_r)).$$

By Lemma 2.4 and the induction hypothesis, we deduce that $P_{R/(f_1,\dots,f_r)}(t)$ has degree $n-r$ and leading coefficient

$$\begin{aligned} & \frac{\deg(f_1)\cdot \ldots \cdot \deg(f_{r-1})}{(n-r+1)!}\cdot \deg(f_r)\cdot (n-r+1) \\ = \quad & \frac{\deg(f_1)\cdot \ldots \cdot \deg(f_r)}{(n-r)!}. \end{aligned}$$

Moreover, by induction, the equality

$$\dim((R/(f_1, \dots, f_{r-1}))_d) = P_{R/(f_1,\dots,f_{r-1})}(d)$$

holds for $d \geq \deg(f_1) + \ldots + \deg(f_{r-1})$. It follows that the equality

$$\dim(((R/(f_1, \dots, f_{r-1}))\cdot f_r)_d) = P_{R/(f_1,\dots,f_{r-1})}(d-\deg(f_r))$$

holds for $d \geq \deg(f_1) + \ldots + \deg(f_r)$. This gives that the equality

$$\dim((R/(f_1, \dots, f_r))_d) = P_{R/(f_1,\dots,f_r)}(d)$$

holds for $d \geq \deg(f_1) + \ldots + \deg(f_r)$. □

Lemma 2.6 *Let $P_1, \ldots, P_m \in \mathbb{P}^n$ be m distinct points. Then, the Hilbert polynomial $P_{R/\mathrm{I}(P_1)\cap\ldots\cap\mathrm{I}(P_m)}$ of the R-module $R/\mathrm{I}(P_1)\cap\ldots\cap\mathrm{I}(P_m)$ is constant and equals m.*

Proof: We will do this by induction on the number of points. First assume that $m=1$. Without loss of generality, we may assume that $P_1 = [1 : x_1 : \ldots : x_n]$ (i.e., the first coordinate of P_1 is non zero). In this case, $\mathrm{I}(P_1) = (x_1 - a_1x_0, \ldots, x_n - a_nx_0)$ and there is an isomorphism

$$R/\mathrm{I}(P_1) \xrightarrow{\sim} k[x_0], \quad x_i \mapsto a_ix_0.$$

This shows that $P_{R/\mathrm{I}(P_1)} = 1$.

Now, we assume that the claim is true for some $m \geq 1$ and we show that it is still true for $m+1$. Using the short exact sequence:

$$\begin{aligned} 0 \to R/\mathrm{I}(P_1)\cap\ldots\cap I(P_{m+1}) &\to R/\mathrm{I}(P_1)\cap\ldots\cap\mathrm{I}(P_m)\oplus R/\mathrm{I}(P_{m+1}) \\ &\to R/(\mathrm{I}(P_1)\cap\ldots\cap\mathrm{I}(P_m)+\mathrm{I}(P_{m+1})) \to 0, \end{aligned}$$

it is enough to show that the Hilbert polynomial of the module

$$N = R/(\mathrm{I}(P_1)\cap\ldots\cap\mathrm{I}(P_m)+\mathrm{I}(P_{m+1}))$$

is zero. But N is a non trivial graded quotient of $R/\mathrm{I}(P_{m+1})$ since $\mathrm{I}(P_1)\cap\ldots\cap\mathrm{I}(P_m) \not\subset \mathrm{I}(P_{m+1})$ (which is equivalent to $P_{m+1} \notin \{P_1, \ldots, P_m\}$). Assuming that the first coordinate of P_{m+1} is non zero, we have an isomorphism $R/\mathrm{I}(P_{m+1}) \cong k[x_0]$ (see the case $m=1$). Hence, N is a non trivial graded quotient of $k[x_0]$ and must be of the form $k[x_0]/(x_0^s)$ for some $s \in \mathbb{N}$. As

$$k[x_0]/(x_0^s) \cong k \oplus kx_0 \oplus kx_0^2 \oplus \ldots \oplus kx_0^{s-1},$$

the Hilbert polynomial of N is indeed zero. □

Proposition 2.7 *Let $f_1, \ldots, f_n \in R$ be a regular sequence consisting of homogeneous polynomials. Then, $\mathrm{V}(f_1, \ldots, f_n)$ is a finite set and we have*

$$|\mathrm{V}(f_1, \ldots, f_n)| \leq \deg(f_1)\cdot\ldots\cdot\deg(f_n).$$

If moreover, the ideal $(f_1, \ldots, f_n)$ is radical, then we have

$$|\mathrm{V}(f_1, \ldots, f_n)| = \deg(f_1)\cdot\ldots\cdot\deg(f_n).$$

Proof: As $f_1, \ldots, f_n \in R$ is a regular sequence, the set $\mathrm{V}(f_1, \ldots, f_n)$ has dimension 0 and hence consists of finitely many points, say $P_1, \ldots, P_m$. Using Corollary 2.2, it follows that

$$\sqrt{(f_1, \ldots, f_n)} = \mathrm{I}(P_1)\cap\ldots\cap\mathrm{I}(P_m).$$

Using Lemmas 2.5 and 2.6, we get that

$$m = P_{R/\mathrm{I}(P_1)\cap\ldots\cap\mathrm{I}(P_m)} = P_{R/\sqrt{(f_1,\ldots,f_n)}} \leq P_{R/(f_1,\ldots,f_n)} = \deg(f_1)\cdot\ldots\cdot\deg(f_n)$$

with equality when $(f_1, \ldots, f_n)$ is radical. □

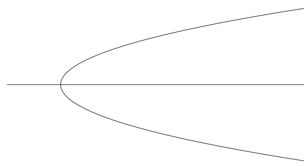
2.4 Curves in $\mathbb{P}^2$: Bézout theorem

Definition 2.14 *Let C and D be two curves in $\mathbb{P}^2$ without a common irreducible component. We say that C and D* intersect without multiplicities *if the ideal* $\mathrm{I}_C + \mathrm{I}_D$ *is radical.*

Recall from Remark 2.5 that curves in $\mathbb{P}^2$ are defined by a single equation, i.e., $\mathrm{I}_C = (f)$ and $\mathrm{I}_D = (g)$. Therefore, one has $\mathrm{I}_C + \mathrm{I}_D = (f, g)$.

The previous definition is motivated by the following two examples.

Example 2.2 *Consider the curves C and D defined by the equations $zx - y^2 = 0$ and $y = 0$. Said differently, we have $C = \mathrm{V}(xz - y^2)$ and $D = \mathrm{V}(y)$. Setting $z = 1$, and taking points with real coordinates, we get the following picture:*

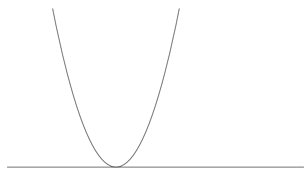


showing that the intersection point has multiplicity one. On the other hand, we have

$$\mathrm{I}_C + \mathrm{I}_D = (xz - y^2, y) = (xz, y)$$

which is a radical ideal.

Example 2.3 *Consider the curves E and F defined by the equations $zy - x^2 = 0$ and $y = 0$. Said differently, we have $E = \mathrm{V}(zy - x^2)$ and $E = \mathrm{V}(y)$. Setting $z = 1$, and taking points with real coordinates, we get the following picture:*



showing that the intersection point has multiplicity two. On the other hand, we have

$$\mathrm{I}_E + \mathrm{I}_F = (yz - x^2, y) = (x^2, y)$$

which is not a radical ideal.

Proposition 2.8 *Let C and D be two curves in $\mathbb{P}^2$ without a common irreducible component. Assume that $\mathrm{I}_C = (f)$ and $\mathrm{I}_D = (g)$. Then, $|C \cap D| \leq \deg(f) \cdot \deg(g)$. If moreover, C and D intersect without multiplicities, then*

$$|C \cap D| = \deg(f) \cdot \deg(g).$$

Proof: Note that $C \cap D = \mathrm{V}(f, g)$. Hence, the proposition is a particular case of Proposition 2.7. □

2.5 Curves in $\mathbb{P}^3$

2.5.1 Big intersection

Definition 2.15 *Let C and D be two curves in $\mathbb{P}^3$ and assume that C and D are complete intersections with $\mathrm{I}_C = (f_1, f_2)$ and $\mathrm{I}_D = (f_3, f_4)$. We say C and D have* big intersection *if the following conditions are satisfied:*

(1) C and D have no common irreducible components,

(2) there is $i_0 \in \{1, 2, 3, 4\}$ such that f_{i_0} is in the ideal generated by the f_j's for $j \in \{1, 2, 3, 4\} \setminus \{i_0\}$.

Lemma 2.7 *Let C and D be two curves in $\mathbb{P}^3$ with $\mathrm{I}_C = (f_1, f_2)$ and $\mathrm{I}_D = (f_3, f_4)$. Assume that $f_4 \in (f_1, f_2, f_3)$ (so that C and D have big intersection). Then*

$$|C \cap D| \leq \deg(f_1) \cdot \deg(f_2) \cdot \deg(f_3)$$

with equality if $\mathrm{I}_C + \mathrm{I}_D$ is radical.

Proof: Note that $\mathrm{I}_C + \mathrm{I}_D = (f_1, f_2, f_3)$. This is then a special case of Proposition 2.7 □

Proposition 2.9 *Keep the notation as in Definition 2.15 and assume that C and D have big intersection. Then, the ideal $\mathrm{I}_C + \mathrm{I}_D$ is saturated and the curve $C \cup D$ is Cohen-Macaulay.*

Proof: By Proposition 2.6, $\mathrm{I}_C + \mathrm{I}_D$ is saturated if and only if $C \cup D$ is Cohen-Macaulay. Therefore, it suffices to show that $\mathrm{I}_C + \mathrm{I}_D$ is saturated.

Without loss of generality, we may assume that f_4 is contained in (f_1, f_2, f_3). Then $\mathrm{I}_C + \mathrm{I}_D = (f_1, f_2, f_3)$ and f_1, f_2, f_3 is a regular sequence in $R := k[x_0, x_1, x_2, x_3]$ since $C \cap D$ consists of finitely many points. Now (f_1, f_2, f_3) is saturated by Lemma 1.13. □

2.5.2 Cohen-Macaulay implies big intersection

Here we investigate when the condition that $C \cup D$ is Cohen-Macaulay implies that C and D have big intersection. Our main result is the following partial converse to Proposition 2.9.

Proposition 2.10 *Let C and D be two curves in $\mathbb{P}^3$, such that $\mathrm{I}_C = (f_1, f_2)$ and $\mathrm{I}_D = (f_3, f_4)$. Let $d_i = \deg(f_i)$ for $i \in \{1, 2, 3, 4\}$. Assume that f_1, f_2, f_3 is a regular sequence and that $d_4 > d_1 + d_2 + d_3$. Then the following conditions are equivalent:*

a) C and D have big intersection;

b) $f_4 \in (f_1, f_2, f_3)$;

c) $C \cup D$ is Cohen-Macaulay;

d) $\mathrm{I}_C + \mathrm{I}_D$ *is saturated.*

Proof: Note that C and D do not have any common irreducible component. Indeed, $C \cap D \subset \mathrm{V}(f_1, f_2, f_3)$ which consists of finitely many points as f_1, f_2, f_3 is a regular sequence.

By Proposition 2.6, we have c) $\Leftrightarrow$ d). By Proposition 2.9, we know that a) $\Rightarrow$ c). Also, we clearly have b) $\Rightarrow$ a). Therefore, to finish the proof, it remains to check that d) $\Rightarrow$ b).

From now on, we assume that $\mathrm{I}_C + \mathrm{I}_D = (f_1, f_2, f_3, f_4)$ is saturated. Our goal is to prove that $f_4 \in (f_1, f_2, f_3)$. Let $J = (f_1, f_2, f_3)$; this is a saturated ideal by Lemma 1.13. Also, let $Z = \mathrm{V}(J)$; it consists of finitely many points. We split the proof in three steps.

Step 1: Fix an element $l \in R_1$ such that $l(Q) \neq 0$ for all $\bar{Q} \in Z$. It follows that $\mathrm{V}(f_1, f_2, f_3, l) = \emptyset$. Using Proposition 2.2, we get $\sqrt{(f_1, f_2, f_3, l)^{Sat}} = R$ and hence $(f_1, f_2, f_3, l)^{Sat} = R$. By Lemma 1.15, it follows that $(f_1, f_2, f_3, l)_d = R_d$ for d big enough. This shows that the map

$$\begin{array}{rcl} \bar{l} \cdot : (R/J)_{d-1} & \to & (R/J)_d \\ \bar{r} & \mapsto & \bar{l} \cdot \bar{r}, \end{array}$$

is surjective for d big enough. On the other hand, by Lemma 2.5, $\dim(R_d/J_d) = d_1 d_2 d_3$ for $d \geq d_1 + d_2 + d_3$. This proves that the above map is actually an isomorphism for d big enough.

Step 2: Given $b \in R$, we denote by $\bar{b}$ its class in R/J. In this step, we consider the localized ring $R/J[\bar{l}^{-1}]$. This is a $\mathbb{Z}$-graded ring. Its homogeneous elements of degree d are of the form $\dfrac{\bar{b}}{\bar{l}^r}$ where $b \in R_{r+d}$ and $r \in \mathbb{N}$. It follows from Step 1 that the map

$$\varphi_d : (R/J)_d \to (R/J[\bar{l}^{-1}])_d$$

is an isomorphism for d big enough. Indeed, for d big enough, every element of $(R/J)_{r+d}$ is a multiple of $\bar{l}^r$.

We claim that the map $R/J \to R/J[\bar{l}^{-1}]$ is injective. To show this, let N be its kernel. This is a graded R-submodule of R/J such that $N_d = 0$ for d big enough. Assume by contradiction that $N \neq 0$. Then, there exists $e \in \mathbb{N}$ such that $N_e \neq 0$ and $N_{e+1} = 0$. But then, every element $s \in N_e$ defines a map $R/R_+ \to N$ sending $1 + R_+$ to s. By Lemma 1.12, this contradicts the property that J is saturated.

As φ_d is injective for all d and because the dimension of $(R/J)_d$ is the same for $d \geq d_1 + d_2 + d_3$, we deduce that φ_d is an isomorphism for $d \geq d_1 + d_2 + d_3$.

Step 3: We are now ready to end the proof. Recall that our goal is to show that $f_4 \in J$ assuming that (f_1, f_2, f_3, f_4) is saturated.

As $d_4 > d_1 + d_2 + d_3$, one can find by the previous step an element $h \in R_{d_4 - 1}$ such that

$$\varphi_{d_4-1}(\bar{h}) = \frac{\overline{f_4}}{\bar{l}}.$$

Equivalently, we have $\overline{h} \cdot \bar{l} = \overline{f_4}$ in R/J. This proves that

$$(f_1, f_2, f_3, l) \cdot h \subset (f_1, f_2, f_3, f_4).$$

As $(f_1, f_2, f_3, l)^{Sat} = R$, we deduce that $R_+^m \subset (f_1, f_2, f_3, l)$ for m big enough. This gives that

$$R_+^m \cdot h \subset (f_1, f_2, f_3, f_4).$$

As (f_1, f_2, f_3, f_4) is assumed to be saturated, we actually get that $h \in (f_1, f_2, f_3, f_4)$. Therefore, there exists $g \in R$ such that $\overline{h} = \overline{g}\overline{f_4}$. But, by degree considerations this cannot happen unless $\overline{h} = 0$. This implies that $\overline{f_4} = 0$ and thus $f_4 \in (f_1, f_2, f_3)$. $\square$

3 The theorems of Hilbert-Burch and Buchsbaum-Eisenbud

3.1 Free minimal resolutions

Proposition 3.1 *For an R-module M there is an exact sequence*

$$\ldots \to F_{n+1} \stackrel{\varphi_{n+1}}{\to} F_n \stackrel{\varphi_n}{\to} \ldots \to F_1 \stackrel{\varphi_1}{\to} F_0 \to M \to 0 \qquad (1)$$

of R-modules such that F_i is free for all $i \in \mathbb{N}$. If R is noetherian and M is finitely generated, then it is possible to choose the F_i's to be of finite rank.

Furthermore, if R is a graded ring, then it is possible to choose the F_i's to be graded free modules.

Proof: We treat the case where M is a finitely generated graded R-module as this is the case we need later. We construct a graded free resolution as follows. For M, we choose homogeneous generators $m_1, \ldots, m_n$ of degrees $a_1, \ldots, a_n \in \mathbb{Z}$. Define F_0 to be $\oplus_{i=1}^n R[-a_i]$, where $R[-a_i]$ is the same as R considered as an R-module, but whit a shifted grading: $(R[-a_i])_n = R_{n-a_i}$. Denote by e_j the generator $1 \in R[-a_j]$ for $1 \le j \le n$. By construction, e_j has degree a_j. We consider the map

$$\begin{aligned} \varphi_0 : F_0 = \oplus_{i=1}^n R[-a_i] &\to M \\ e_i &\mapsto m_i. \end{aligned}$$

This a graded surjective homomorphism of R-modules.

Suppose now $i > 0$, and that F_j and φ_j are constructed for $0 \le j \le i-1$. We set $M_i := \mathrm{Ker}(\varphi_{i-1})$. As R is noetherian and F_{i-1} is free of finite rank, M_i is a finitely generated graded R-module. Let $g_1, \ldots, g_k$ be generators of M_i of homogenous degree $c_1, \ldots, c_k \in \mathbb{Z}$. Let $F_i = \oplus_{s=1}^k R[-c_s]$ and denote by h_s the generator $1 \in R[-c_s]$ for $1 \le s \le k$. Again we define the morphism:

$$\begin{aligned} \varphi_i : F_i = \oplus_{s=1}^k R[-c_s] &\to M_i \\ f_s &\mapsto m_s. \end{aligned}$$

This finishes the induction step. By construction it holds that $\mathrm{Im}(\varphi_{i-1}) = \mathrm{Ker}(\varphi_{i-1})$. Therefore, we indeed have an exact sequence. □

Definition 3.1 *An exact sequence as in Proposition 3.1 is called a* free resolution *of M.*

Definition 3.2 *Let M be a finitely generated R-module. A family $(m_i)_{i \in I}$ in M is a* minimal system of generators *of M if $M = \sum_{i \in I} R \cdot m_i$ but $M \neq \sum_{i \neq j} R \cdot m_i$ for all $j \in I$.*

Proposition 3.2 *Let R be a $\mathbb{N}$-graded noetherian ring and M a finitely generated graded R-module. A family of homogeneous elements $(m_i)_{1 \le i \le r}$ generates M if and only if the family $(m_i + R_+M)_{1 \le i \le r}$ generates the R/R_+-module M/R_+M.*

Proof: If $(m_i)_{1\leq i\leq r}$ generates M, then clearly $(m_i+R_+M)_{1\leq i\leq r}$ generated M/R_+M.

Conversely, suppose that $(m_i+R_+M)_{1\leq i\leq r}$ generates M/R_+M and consider the submodule $N\subset M$ generated by the family $(m_i)_{1\leq i\leq r}$. As the m_i's are homogeneous, N is a graded submodule of M and so is the quotient $P:=M/N$. We need to show that $P=0$.

We argue by contradiction assuming that $P\neq 0$. As P is finitely generated and R is $\mathbb{N}$-graded, there is an integer $s\in\mathbb{Z}$ such that $P_s\neq 0$ and $P_t=0$ for all $t\leq s-1$. But then, the projection map $P\to P/R_+P$ induces an isomorphism $P_s\simeq (P/R_+P)_s$. This is impossible as $P/R_+P\simeq 0$. Indeed, one has an exact sequence

$$N/R_+N\to M/R_+M\to P/R_+P\to 0$$

and the left morphism is surjective as M/R_+M is generated by the (m_i+R_+M)'s which are the images of the (m_i+R_+N)'s. □

Corollary 3.1 *Let R be an $\mathbb{N}$-graded noetherian ring such that R/R_+ is a field. Let M be a finitely generated graded R-module. Then the cardinality of a minimal system of homogeneous generators is independent of the choice of the family. In fact, this cardinality is equal to the dimension of the R/R_+-vector space M/R_+M.*

Proof: By the previous proposition, $(m_i)_{i\in I}$ is a minimal system of homogeneous generators of M if and only if $(m_i+R_+M)_{i\in I}$ is a minimal system of generators of the R/R_+-vector space M/R_+M. The last condition means that $(m_i+R_+M)_{i\in I}$ is a basis of M/R_+M. □

Lemma 3.1 *Let R be a noetherian $\mathbb{N}$-graded ring with R/R_+ a field. Let M be a finitely generated graded R-module and $h:M\to N$ a homomorphism of graded R-modules. Then the following conditions are equivalent.*

1. *For every minimal system of homogeneous generators $(m_i)_{i\in I}$ of M, $(h(m_i))_{i\in I}$ is a minimal system of homogeneous generators of $\mathrm{Im}(h)$.*

2. *There exists a minimal system of homogeneous generators $(m_i)_{i\in I}$ of M such that $(h(m_i))_{i\in I}$ is a minimal system of homogeneous generators of $\mathrm{Im}(h)$.*

3. $\dim(M/R_+M)=\dim(h(M)/R_+h(M))$, *where M/R_+M and $h(M)/R_+h(M)$ are viewed as R/R_+-vector spaces.*

4. *The homomorphism of R/R_+-vector spaces:*

$$\begin{aligned}\bar{h}:M/R_+M &\to h(M)/R_+h(M)\\ m+R_+M &\mapsto h(m)+R_+h(M)\end{aligned}$$

   is an isomorphism.

5. $\mathrm{Ker}(h)\subset R_+M$.

Proof: The implication 1) $\Rightarrow$ 2) is trivial and the implication 2) $\Rightarrow$ 3) follows from Corollary 3.1. Now, assume that 3) holds and let $(m_i)_{1\leq i\leq r}$ be a minimal system of generators of M. By Corollary 3.1, we have $r = \dim(M/R_+M)$. Clearly $(h(m_i))_{1\leq i\leq r}$ generates $h(M)$. Applying Corollary 3.1 again, we see that a minimal generating sub-family of $(h(m_i))_{1\leq i\leq r}$ has cardinality r and hence $(h(m_i))_{1\leq i\leq r}$ is itself a minimal system of generators of $h(M)$. This proves that 3) $\Rightarrow$ 1).

The morphism $\bar{h}$ is surjective by construction. Therefore $\bar{h}$ is an isomorphism if and only if $\dim(M/R_+M) = \dim(h(M)/R_+h(M))$. This proves that 3) $\Leftrightarrow$ 4).

It remains to show that 4) $\Leftrightarrow$ 5). First, assume we that 4) holds. To prove 5), we argue by contradiction, i.e., we assume that $\mathrm{Ker}(h) \not\subset R_+M$. Choose $m \in R_+M\backslash\mathrm{Ker}(h)$. Then $\bar{h}(\bar{m}) = h(m)+R_+h(M) = 0+R_+h(M) = 0 \in h(M)/R_+h(M)$. This means that $\bar{h}$ is not injective, contradiction 4).

Finally, assume that 5) holds. It suffices to prove that $\bar{h}$ is injective. Let $m \in M$ with $\bar{h}(\bar{m}) = 0$ in $h(M)/R_+h(M)$. This means that $h(m) \in R_+h(M)$. Therefore, we can find homogeneous elements $m_1, \ldots, m_r \in M$ and $x_1, \ldots, x_r \in R_+$ such that

$$h(m) = x_1h(m_1) + \ldots + x_rh(m_r) = h(x_1m_1 + \ldots + x_rm_r).$$

It follows that $m - \sum_{i=1}^{r} x_im_i \in \mathrm{Ker}(h)$. The assumption $\mathrm{Ker}(h) \subset R_+M$ implies now that $m - \sum_{i=1}^{r} x_im_i \in R_+M$. As the x_i's are in R_+, we deduce that $m \in R_+M$. This shows that $\bar{m} = 0$ in M/R_+M and that $\bar{h}$ is injective. This finishes the proof as $\bar{h}$ is surjective by construction. □

Definition 3.3 *Let R be an $\mathbb{N}$-graded noetherian ring with R/R_+ a field. Let M be a finitely generated graded R-module. A homomorphism of graded R-modules $h : M \to N$ is called* minimal *if one (and hence all) of the conditions in Lemma 3.1 is fulfilled.*

Proposition 3.3 *Let R be a noetherian $\mathbb{N}$-graded ring with R/R_+ a field. Let M be a graded finitely generated R-module. Then M has a free resolution*

$$\ldots \to F_{n+1} \stackrel{\varphi_{n+1}}{\to} F_n \stackrel{\varphi_n}{\to} \ldots \to F_1 \stackrel{\varphi_1}{\to} F_0 \stackrel{\varphi_0}{\to} M \to 0$$

such that F_i has finite rank and φ_i is a minimal homomorphism for all $i \in \mathbb{N}$.

Proof: Let $r = \dim(M/R_+M)$ as an R/R_+-vector space. Let $(m_i)_{1\leq i\leq r}$ be a minimal system of homogeneous generators of M. The map

$$\begin{array}{rcl} \varphi_0 : F_0 = R^{\oplus r} & \to & M \\ (a_1, \ldots, a_r) & \mapsto & \displaystyle\sum_{i=1}^{r} a_i \cdot m_i \end{array}$$

defines a surjective minimal morphism of graded R-modules. Suppose now that F_i and φ_i are defined for $0 \leq i \leq n$, i.e., that we have an exact sequence of minimal morphisms form free graded R-modules of finite rank

$$F_n \stackrel{\varphi_n}{\to} \ldots \to F_1 \stackrel{\varphi_1}{\to} F_0 \stackrel{\varphi_0}{\to} M \to 0.$$

Since F_n is finitely generated and R is noetherian, $N := \operatorname{Ker}(\varphi_n)$ is a finitely generated graded R-module. Let $(n_i)_{1\leq i\leq s}$ be a minimal system of homogeneous generators of N. We consider the surjective morphism of graded R-modules

$$\begin{array}{rcl} F_{n+1} = R^{\oplus s} & \to & N \\ (a_1, \dots, a_s) & \mapsto & \sum_{i=1}^{s} a_i \cdot m_i. \end{array}$$

If we compose this morphism with the inclusion $N \hookrightarrow F_n$, we get a minimal morphism $\varphi_{n+1} : F_{n+1} \to F_n$ such that $\operatorname{Im}(\varphi_{n+1}) = \operatorname{Ker}(\varphi_n)$. This complete the induction step and the proof of the proposition. □

Definition 3.4 *Let R be a noetherian $\mathbb{N}$-graded ring with R/R_+ a field. Let M be a finitely generated graded R-module. An exact sequence as in Proposition 3.3 is called a* minimal free resolution *of M.*

Theorem 3.1 *Let R be a noetherian $\mathbb{N}$-graded ring with R/R_+ a field. Let $f : M' \to M$ be a morphism of finitely generated graded R-modules and let $F'_\bullet \to M'$ and $F_\bullet \to M$ be free resolutions of finite rank. Then, there exists a morphism of complexes $g_\bullet : F'_\bullet \to F_\bullet$ such that the following diagram is commutative*

$$\begin{array}{ccccccccccccccc} \cdots \longrightarrow & F'_{n+1} & \xrightarrow{\varphi'_{n+1}} & F'_n & \xrightarrow{\varphi'_n} & \cdots & \xrightarrow{\varphi'_2} & F'_1 & \xrightarrow{\varphi'_1} & F'_0 & \xrightarrow{\varphi'_0} & M' & \longrightarrow & 0 \\ & \downarrow{\scriptstyle g_{n+1}} & & \downarrow{\scriptstyle g_n} & & & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle f} & & \\ \cdots \longrightarrow & F_{n+1} & \xrightarrow{\varphi_{n+1}} & F_n & \xrightarrow{\varphi_n} & \cdots & \xrightarrow{\varphi_2} & F_1 & \xrightarrow{\varphi_1} & F_0 & \xrightarrow{\varphi_0} & M & \longrightarrow & 0. \end{array}$$

Moreover, if f has a section and $F_\bullet \to M$ is a minimal free resolution, then necessarily the g_n's are surjective for all $n \in \mathbb{N}$.

Proof: We construct g_0 as follows. Let $e'_1, \dots, e'_r$ be a homogeneous basis of the R-module F'_0. As φ_0 is surjective, we may find elements $f_1, \dots, f_r \in F_0$ such that $\varphi_0(f_i) = f(\varphi'_0(e'_i))$ for all $1 \leq i \leq r$. We define $g_0 : F'_0 \to F_0$ by $g_0(\sum_{i=1}^r a_i \cdot e'_i) = \sum_{i=1}^r a_i \cdot f_i$. It is clear that $\varphi_0 \circ g_0 = f \circ \varphi'_0$.

Now, assume that g_n is constructed. We construct g_{n+1} as before. We fix a homogeneous basis $e'_1, \dots, e'_r$ of F'_{n+1}. The elements $g_n(\varphi'_n(e'_i))$ belongs to $\operatorname{Ker}(\varphi_{n-1})$ because $\varphi_{n-1} \circ g_n \circ \varphi'_n = g_{n-1}\varphi'_{n-1} \circ \varphi'_n = 0$. As $\operatorname{Im}(\varphi_{n+1}) = \operatorname{Ker}(\varphi_{n-1})$, me may find elements $f_1, \dots, f_r \in F_{n+1}$ such that $\varphi_{n+1}(f_i) = g_n(\varphi'_n(e'_i))$ for all $1 \leq i \leq r$. We define $g_{n+1} : F'_{n+1} \to F_{n+1}$ by $g_{n+1}(\sum_{i=1}^r a_i \cdot e'_i) = \sum_{i=1}^r a_i \cdot f_i$. It is clear that $\varphi_{n+1} \circ g_{n+1} = g_n \circ \varphi'_{n+1}$. This finishes the construction of the g_i's.

Now, assume that f is has a section and that $F_\bullet \to M$ is a minimal free resolution. We need to show that the g_n's are surjective. In fact, we will prove a stronger statement: we will prove by induction on $n \in \mathbb{N}$ that $g_n : F'_n \to F_n$ is surjective and that $g_n : \operatorname{Ker}(\varphi'_n) \to \operatorname{Ker}(\varphi_n)$ has a section. Assume that the result is known for $n \geq 0$ and let us prove it for $n + 1$. By hypothesis, the surjection $F_{n+1} \to \operatorname{Ker}(\varphi_n)$ is minimal. Let $e_1, \dots, e_r$ be a homogeneous basis of F_{n+1} such that $\varphi_{n+1}(e_1), \dots, \varphi_{n+1}(e_r)$ is a minimal system of homogeneous generators of

$\mathrm{Ker}(\varphi_n)$. Fix a section $s : \mathrm{Ker}(\varphi_n) \to \mathrm{Ker}(\varphi'_n)$ and choose homogeneous elements $e'_1, \ldots, e'_r \in F'_{n+1}$ such that $\varphi'_{n+1}(e'_i) = s(\varphi_{n+1}(e_i))$ for $1 \leq i \leq r$. Hence, we have $\varphi_{n+1}(e_i) = \varphi_{n+1}(g_{n+1}(e'_i))$. Therefore, $e_i - g_{n+1}(e'_i) \in \mathrm{Ker}(\varphi_{n+1})$. By Lemma 3.1, we deduce that $e_i - g_{n+1}(e_i) \in R_+F_{n+1}$. This proves that $(g_{n+1}(e_i) + R_+F_{n+1})_{1\leq i\leq r}$ is a basis of F_{n+1}/RF_{n+1} and hence $(g_{n+1}(e_i))_{1\leq i\leq r}$ generated F_{n+1}. This proves that g_{n+1} is surjective.

It remains to show that $g_{n+1} : \mathrm{Ker}(\varphi'_{n+1}) \to \mathrm{Ker}(\varphi_{n+1})$ admits a section. Let $N \subset \mathrm{Ker}(\varphi'_n)$ be the kernel of $g_{n+1} : \mathrm{Ker}(\varphi'_n) \to \mathrm{Ker}(\varphi_n)$. Using the section $s : \mathrm{Ker}(\varphi_n) \to \mathrm{Ker}(\varphi'_n)$ fixed above, one obtains an isomorphism

$$\mathrm{Ker}(\varphi'_n) \simeq N \oplus \mathrm{Ker}(\varphi_n).$$

Let $H \subset F'_{n+1}$ be the submodule generated by $e'_1, \ldots, e'_r$. As argued above, H is free of rank r and the map g_{n+1} induces an isomorphism $H \simeq F_{n+1}$. Let $G \subset F'_{n+1}$ be the kernel of g_{n+1}. We then have a direct sum decomposition $F'_{n+1} = G \oplus H$. By construction, the map $\varphi'_{n+1} : F'_{n+1} \to \mathrm{Ker}(\varphi'_n)$ takes G to N and H to $\mathrm{Ker}(\varphi_n)$. This gives a decomposition

$$\mathrm{Ker}(\varphi'_{n+1}) \simeq \mathrm{Ker}(G \to N) \oplus \mathrm{Ker}(\varphi_{n+1} : F_{n+1} \to \mathrm{Ker}(\varphi_n))$$

and modulo this decomposition $g_{n+1} : \mathrm{Ker}(\varphi'_{n+1}) \to \mathrm{Ker}(\varphi_{n+1})$ is the projection to the second factor. □

Corollary 3.2 *Let R be a noetherian $\mathbb{N}$-graded ring with R/R_+ a field. Let M be a finitely generated graded R-module and let $F'_\bullet \to M$ and $F_\bullet \to M$ be two minimal free resolutions of M. Then, there is an isomorphism of resolutions $g_\bullet : F'_\bullet \to F_\bullet$.*

Proof: We apply Theorem 3.1 to the identity of M. This gives two surjections $F'_\bullet \twoheadrightarrow F_\bullet$ and $F_\bullet \twoheadrightarrow F'_\bullet$. This shows that F_n and F'_n have the same rank for every $n \in \mathbb{N}$ and that the above surjections are in fact isomorphisms. □

Definition 3.5 *Let R be a noetherian $\mathbb{N}$-graded ring with R/R_+ a field. Let M be a finitely generated graded R-module. We define the* projective dimension *of M to be the supremum of the set $\{n \in \mathbb{N}; F_n \neq 0\}$ for a free minimal resolution $F_\bullet \to M$. Thanks to Corollary 3.2, this is independent of the choice of the minimal free resolution. The projective dimension of M is denoted by* $\mathrm{pdim}(M)$*. It is an element of* $\mathbb{N} \cup \{\infty\}$*.*

Proposition 3.4 *If k is a field, then every finitely generated graded module over $k[x_1, \ldots, x_n]$ has projective dimension $\leq n$ and in particular has a finite free resolution.*

Proof: See [6, Corollary 19.7] □

Recall from Definition 1.11 the depth of an R-module M with respect to an ideal I.

Proposition 3.5 (The Auslander-Buchsbaum formula) *Let R be a noetherian $\mathbb{N}$-graded ring with R/R_+ a field. Let M be an non-zero finitely generated graded R-module of finite projective dimension. Then it holds:*

$$\operatorname{pdim}(M) + \operatorname{depth}(M) = \operatorname{depth}(R)$$

where the depth is taken with respect to the maximal ideal R_+.

Proof: See [6, Theorem 19.9]. □

3.2 Determinantial ideals

Definition 3.6 *Given an $m \times n$ matrix*

$$\varphi = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

over a ring R, we define the determinantial ideal $I_k(\varphi)$ *as the ideal generated by determinants of the $k \times k$ submatrices of φ. These determinants are called the k-minors of φ:*

$$I_k(\varphi) := (k\text{-minors of } \varphi).$$

The main goal of this section is to prove Theorem 3.2 that gives a lower bound on the codimension of a determinantial ideal. To do this, we need some preliminary results.

Remark 3.1 *Keep the notation as in Definition 3.6. For a prime ideal $\mathfrak{p} \subset R$, we denote by $\varphi(\mathfrak{p})$ the $m \times n$ matrix with coefficients in the field* $\operatorname{Frac}(R/\mathfrak{p})$ *whose entries are*

$$\frac{a_{ij} + \mathfrak{p}}{1}.$$

The matrix φ determines a morphism of R-modules $\varphi : R^n \to R^m$. Similarly, $\varphi(\mathfrak{p})$ determines a morphism of $\operatorname{Frac}(R/\mathfrak{p})$*-vectorspaces $\varphi(\mathfrak{p}) : \operatorname{Frac}(R/\mathfrak{p})^n \to \operatorname{Frac}(R/\mathfrak{p})^m$. In particular, we may speak of the rank of $\varphi(\mathfrak{p})$.*

Lemma 3.2 *For an $m \times n$ matrix φ over a ring R it holds:*

$$\sqrt{I_k(\varphi)} = \bigcap_{\operatorname{rank}(\varphi(\mathfrak{p})) \leq k-1} \mathfrak{p}.$$

In words, $\sqrt{I_k(\varphi)}$ is the intersection of the prime ideals $\mathfrak{p} \subset R$ such that the matrix $\varphi(\mathfrak{p})$ has rank at most $k-1$.

Proof: It is well known that the radical of an ideal I is the intersection of all prime ideals containing I. In particular, we have:

$$\sqrt{I_k(\varphi)} = \bigcap_{I_k(\varphi) \subset \mathfrak{p}} \mathfrak{p}.$$

Therefore, it is enough to show that $I_k(\varphi) \subset \mathfrak{p}$ if and only if $\varphi(\mathfrak{p})$ has rank at most $k-1$.

Fix a prime ideal $\mathfrak{p} \subset R$. Clearly, $I_k(\varphi) \cdot \mathrm{Frac}(R/\mathfrak{p}) = I_k(\varphi(\mathfrak{p}))$ and, as $\mathrm{Frac}(R/\mathfrak{p})$ is a field,

$$I_k(\varphi(\mathfrak{p})) = \begin{cases} \mathrm{Frac}(R/\mathfrak{p}) & \text{if} \quad \mathrm{rank}(\varphi(\mathfrak{p})) \geq k, \\ 0 & \text{if} \quad \mathrm{rank}(\varphi(\mathfrak{p})) \leq k-1. \end{cases}$$

Now, if $I_k(\varphi) \subset \mathfrak{p}$, then

$$I_k(\varphi(\mathfrak{p})) = I_k(\varphi) \cdot \mathrm{Frac}(R/\mathfrak{p}) \subset \mathfrak{p} \cdot \mathrm{Frac}(R/\mathfrak{p}) = 0.$$

This shows that $I_k(\varphi(\mathfrak{p})) = 0$. Thus we have $\mathrm{rank}(\varphi(\mathfrak{p})) \leq k-1$ as needed. Conversely, if $\mathrm{rank}(\varphi(\mathfrak{p})) \leq k-1$, then $I_k(\varphi) \cdot \mathrm{Frac}(R/\mathfrak{p}) = I_k(\varphi(\mathfrak{p})) = 0$. This shows that $I_k(\varphi)$ is contained in the kernel of the ring homomorphism $R \to \mathrm{Frac}(R/\mathfrak{p})$ which is equal to $\mathfrak{p}$. This gives $I_k(\varphi) \subset \mathfrak{p}$ as needed. □

Remark 3.2 *Let R be a ring and let $\varphi = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be an $m \times n$ matrix. Assume that a_{11} is invertible, i.e., $a_{11} \in R^\times$. Then $\varphi(e_1) = a_{11}e_1 + \ldots + a_{m1}e_m$ can be completed to a basis of R^m. More precisely, $(\varphi(e_1), e_2, \ldots, e_m)$ is a basis of R^m.*

Now, the morphism $\varphi : R^n \to R^m$ induces a morphism of R-modules

$$\bar{\varphi} : R^n/R \cdot e_1 \to R^m/R \cdot \varphi(e_1).$$

Clearly, $R^n/R \cdot e_1 \simeq R^{n-1}$ and, by the previous discussion, $R^m/R \cdot \varphi(e_1) \simeq R^{m-1}$. Using these identifications, we obtain a morphism of R-modules $R^{n-1} \to R^{m-1}$ that we still denote by $\bar{\varphi}$. Thus, $\bar{\varphi}$ is an $(m-1) \times (n-1)$ matrix.

Lemma 3.3 *Keep the notation and assumption as in Remark 3.2. Then we have*

$$\sqrt{I_k(\varphi)} = \sqrt{I_{k-1}(\bar{\varphi})}.$$

Proof: By Lemma 3.2, it is enough to show that for all prime ideals $\mathfrak{p} \subset R$ we have $\mathrm{rank}(\varphi(\mathfrak{p})) \leq k-1$ if and only if $\mathrm{rank}(\bar{\varphi}(\mathfrak{p})) \leq k-2$. More precisely, we will show that

$$\mathrm{rank}(\bar{\varphi}(\mathfrak{p})) = \mathrm{rank}(\varphi(\mathfrak{p})) - 1.$$

As $\bar{\varphi}(\mathfrak{p}) = \overline{\varphi(\mathfrak{p})}$ it is enough to show that for an $m \times n$ matrix $\beta = (b_{ij})$ over a field k with $b_{11} \neq 0$, we have

$$\mathrm{rank}(\bar{\beta}) = \mathrm{rank}(\beta) - 1.$$

But, by construction, we have

$$\mathrm{Im}(\bar{\beta}) \simeq \mathrm{Im}(\beta)/k \cdot \beta(e_1).$$

Hence $\dim(\mathrm{Im}(\bar{\beta})) = \dim(\mathrm{Im}(\beta)) - 1$ which gives the equality on ranks. □

Definition 3.7 *The* codimension *of a prime ideal $\mathfrak{p} \subset R$ is defined to be the dimension of the local ring $R_{\mathfrak{p}}$. For a general ideal $I \subset R$, the codimension of I is defined to be the minimum of the codimensions of the prime ideals containing I. We write* codim(I) *for the codimension of I. Sometimes, the codimension of an ideal is also called its* height.

Remark 3.3 *Let R be a ring and let $I \subset R$ be an ideal. As a prime ideal contains I if and only if it contains $\sqrt{I}$, we get:*

$$\operatorname{codim}(\sqrt{I}) = \operatorname{codim}(I).$$

Lemma 3.4 *Let R be a local ring with maximal ideal $\mathfrak{m}$, and let $I \subset \mathfrak{m}$ be an ideal of R such that $\sqrt{I} = \mathfrak{m}$ (this is equivalent to saying that $\mathfrak{m}$ is the unique prime ideal containing I).*

Let T be an indeterminate over R. Let $I' \subset \mathfrak{m} \cdot R[T]$ be an ideal of $R[T]$ such that $I' + (T) = I \cdot R[T] + (T)$. Then $\mathfrak{m}R[T]$ is a minimal prime ideal containing I'.

Proof: We have $R[T]/(I' + (T)) \simeq R/I$ and $\mathfrak{m}/I$ is a minimal prime ideal of R/I. It follows that $\mathfrak{m}R[T] + (T)$ is a minimal prime ideal containing $I' + (T)$.

Consider a prime ideal $\mathfrak{n} \subset R[T]$ between I' and $\mathfrak{m}R[T]$, i.e., such that

$$I' \subset \mathfrak{n} \subset \mathfrak{m}R[T].$$

We will show that the Krull dimension of $R[T]/\mathfrak{n}$ is 1. As $R[T]/\mathfrak{m}R[T] \simeq (R/\mathfrak{m})[T]$ has also Krull dimension 1 and as both rings $R[T]/\mathfrak{n}$ and $(R/\mathfrak{m})[T]$ are intergral domains, this will proves that $\mathfrak{n} = \mathfrak{m}R[T]$ showing that $\mathfrak{m}R[T]$ is indeed a minimal prime ideal containing I'.

Consider the ideal (T) in $R[T]/\mathfrak{n}$. It is equal to $(\mathfrak{n}+(T))/\mathfrak{n}$ which is contained in $(\mathfrak{m}R[T]+(T))/\mathfrak{n}$. This inclusion makes $(\mathfrak{m}R[T]+(T))/\mathfrak{n}$ into a minimal prime ideal containing $(\mathfrak{n}+(T))/\mathfrak{n}$. Indeed, by the previous dicussion we know that $\mathfrak{m}R[T]+(T)$ is a minimal prime ideal containing $I' + (T)$ which itself is contained in $\mathfrak{n} + (T)$.

Now, thanks to the principal ideal theorem of Krull we conclude:

$$\operatorname{codim}((\mathfrak{m}R[T] + (T))/\mathfrak{n}) = 1.$$

Since $(\mathfrak{m}R[T] + (T))/\mathfrak{n})$ can not be zero. On the other hand $(\mathfrak{m}R[T] + (T))/\mathfrak{n}$ is a maximal ideal in $R[T]/\mathfrak{n}$. This follows from the isomorphisms:

$$(R[T]/\mathfrak{n})/((\mathfrak{m}R[T] + (T))/\mathfrak{n}) \simeq R[T]/(\mathfrak{m}R[T] + (T)) \simeq R/\mathfrak{m}$$

and the fact that $R/\mathfrak{m}$ is a field. We thus obtained a maximal chain of inclusions of prime ideals $(0) \subset (\mathfrak{m}R[T] + (T))/\mathfrak{n}$ of the ring $R[T]/\mathfrak{n}$. This proves that $R[T]/\mathfrak{n}$ has Krull dimension 1. □

We are now ready to prove the main result of this section.

Theorem 3.2 *Let R be a ring and let φ be an $m \times n$ matrix with coefficients in R. Then, we have:*

$$\operatorname{codim}(I_k(\varphi)) \leq (m - k + 1)(n - k + 1).$$

More precisely, for every minimal prime ideal $I_k(\varphi) \subset \mathfrak{p}$, we have $\operatorname{codim}(\mathfrak{p}) \leq (m - k + 1)(n - k + 1)$.

Proof: We argue by induction on k. When $k = 1$, the inequality $\operatorname{codim}(I_1(\varphi)) \leq m \cdot n$ is a consequence of Krull's principal ideal theorem. From now on, we assume that $k \geq 2$ and that the claim is known for $k - 1$.

We fix a minimal prime ideal $\mathfrak{p}$ containing $I_k(\varphi)$. To prove that $\operatorname{codim}(\mathfrak{p}) \leq (m-k+1)(n-k+1)$, we may replace R, $\mathfrak{p}$ and φ by $R_\mathfrak{p}$, $\mathfrak{p}R_\mathfrak{p}$ and the matrix $\varphi_\mathfrak{p}$ obtained by replacing the entries of φ by their images in $R_\mathfrak{p}$. Thus, we may assume that R is a local ring with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m} = \sqrt{I_k(\varphi)}$ (i.e., $\mathfrak{m}$ is a minimal prime ideal containing $I_k(\varphi)$). If one of the entries of φ is not in $\mathfrak{m}$, this entry is invertible in $R_\mathfrak{p}$. By Lemma 3.3, we then get $\operatorname{codim}(I_k(\varphi_\mathfrak{p})) = \operatorname{codim}(I_{k-1}(\overline{\varphi_\mathfrak{p}}))$ and the induction hypothesis on k yields the desired conclusion.

By the previous discussion, we may thus assume that all the entries of φ belong to $\mathfrak{m}$. Let T be a indeterminate over R and consider a new $m \times n$ matrix:

$$\varphi' = \varphi + \begin{pmatrix} T & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Remark that $I_k(\varphi')$ is an ideal of $R[T]$.

- In a first step we claim that $I_k(\varphi') \subset \mathfrak{m}R[T]$. Indeed, as the cofficients of φ belong to $\mathfrak{m}$, the image of φ' in $R[T]/\mathfrak{m}R[T] \simeq R/\mathfrak{m}[T]$ is the matrix

$$\begin{pmatrix} T & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

  As $k \geq 2$, all the k-minors of this matrix are zero. This shows that the image of $I_k(\varphi')$ in $R[T]/\mathfrak{m}R[T]$ is zero and hence that $I_k(\varphi') \subset \mathfrak{m}R[T]$ as claimed.

- In a second step, we claim that $I_k(\varphi') + (T) = I_k(\varphi)R[T] + (T)$. Indeed, the image of φ' in $R[T]/(T) \simeq R$ is equal to φ. This shows that the image of $I_k(\varphi')$ in $R[T]/(T) \simeq R$ is equal to $I_k(\varphi)$. This gives the desired equality.

We may now use Lemma 3.4 to deduce that $\mathfrak{m}R[T]$ is a minimal prime ideal containing $I_k(\varphi')$. Moreover, by construction, the first entry of φ', namely $a_{11} + T$, does not belong to $\mathfrak{m}R[T]$. Therefore, this coefficient becomes invertible in $R[T]_{\mathfrak{m}R[T]}$. Using Lemma 3.3 and induction again, we thus obtain that

$$\operatorname{codim}_{R[T]}(\mathfrak{m}R[T]) \leq (m - k + 1)(n - k + 1).$$

To finish the proof it remains to see that $\operatorname{codim}_{R[T]}(\mathfrak{m}R[T]) = \operatorname{codim}_R(\mathfrak{m})$. This is a well known fact; it can be proved by remarking that for a maximal chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{m}$ in R, we get a maximal chain of prime ideals in $R[T]_{\mathfrak{m}R[T]}$ by taking $\mathfrak{p}_0 R[T]_{\mathfrak{m}R[T]} \subset \cdots \subset \mathfrak{p}_n R[T]_{\mathfrak{m}R[T]} = \mathfrak{m}R[T]_{\mathfrak{m}R[T]}$. □

3.3 The Hilbert-Burch Theorem

In order to motivate the Hilbert-Burch Theorem, we first establish a property of Cohen-Macaulay rings.

Lemma 3.5 *Let $R = k[x_0, \ldots, x_n]$ and let $I \subset R$ be a graded ideal of codimension 2. If the ring R/I is Cohen Macaulay, then the R-module R/I has projective dimension 2.*

Proof: Since the codimension of $I \subset R$ is 2 and the dimension of R is $n+1$, we deduce that the dimension of R/I is $n-1$. As R/I is Cohen-Macaulay, its depth with respect to R_+/I is equal to its dimension. By Lemma 1.5, this gives $\mathrm{depth}_{R_+}(R/I) = n-1$. We now apply the formula of Auslander-Buchsbaum (see Proposition 3.5) to obtain:

$$\begin{aligned} \mathrm{pdim}(R/I) &= \mathrm{depth}_{R_+}(R) - \mathrm{depth}_{R_+}(R/I) \\ &= n+1-(n-1) \\ &= 2. \end{aligned}$$

□

Corollary 3.3 *With the notations and hypothesis of Lemma 3.5, there exists an exact sequence of graded R-modules*

$$0 \to F_2 \stackrel{\varphi_2}{\to} F_1 \stackrel{\varphi_1}{\to} R \to R/I \to 0$$

where F_2 and F_1 are free of finite rank.

Proof: Take a minimal free resolution of R/I: is should start by the surjection $R \to R/I$ and it has length 2 by Lemma 3.5. □

Exact sequences as in Corollary 3.3 are the subject of the Hilbert-Burch Theorem.

Theorem 3.3 *Let R be an integral domain and $\{0\} \neq I \subset R$ an ideal. For an exact complex*

$$0 \to F_2 \stackrel{\varphi_2}{\to} F_1 \stackrel{\varphi_1}{\to} R \to R/I \to 0,$$

where F_2 and F_1 are free R-modules it holds that:

1. $\mathrm{rank}(F_1) = \mathrm{rank}(F_2) + 1$.

2. *Let $n = \mathrm{rank}(F_2)$. There exists $a \in R \setminus \{0\}$ such that*

$$I = a \cdot I_n(\varphi_2).$$

Proof: The proof of the first statement is easy. Take an element $0 \neq u \in I$ and consider the ring $R[u^{-1}]$. Assume, that $F_2 = R^n$ and $F_1 = R^m$. As localisation is an exact functor (see Lemma 1.2), we get an exact sequence of $R[u^{-1}]$-modules:

$$0 \to R[u^{-1}]^n \overset{\varphi_2}{\to} R[u^{-1}]^m \overset{\varphi_1}{\to} R[u^{-1}] \to R/I[u^{-1}] \to 0.$$

Now, as $u \in R$ acts by the zero map on R/I, we deduce that $R/I[u^{-1}] \simeq 0$. Therefore, the previous exact sequence becomes a short exact sequence of free $R[u^{-1}]$-modules:

$$0 \to R[u^{-1}]^n \overset{\varphi_2}{\to} R[u^{-1}]^m \overset{\varphi_1}{\to} R[u^{-1}] \to 0.$$

As the rank is additive in short exact sequences, we get that $m = n + 1$.

We will not give a complete proof of the second statement (which is the main part of the Hilbert-Burch Theorem). Instead, we will outline some of the main ideas; for a complete proof see [6, Theorem 20.15].

From now on, we assume that $F_2 = R^n$ and $F_1 = R^{n+1}$ endowed with their canonical basis. We define a morphism of R-modules:

$$\tilde{\varphi}_1 : R^{n+1} \to R$$

by the line matrix $(m_1, -m_2, m_3, \ldots, (-1)^n m_{n+1})$ where the m_i's are the n-minors of φ_2. More precisely, m_i is the determinant of the sub-matrix of the $(n+1) \times n$ matrix φ_2 obtained by removing the i-th horizontal line. Clearly, the image of $\tilde{\varphi}_1$ is the determinantial ideal $I_n(\varphi_2)$.

We claim that $\tilde{\varphi}_1 \circ \varphi_2 = 0$. If

$$\varphi_2 = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n+1,1} & \cdots & a_{n+1,n} \end{pmatrix},$$

this amounts to checking that

$$\sum_{i=1}^{n+1} (-1)^i m_i a_{i,j} = 0$$

for all $1 \leq j \leq n$. This can be proved easily by considering the square matrix

$$\begin{pmatrix} a_{1,j} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n+1,j} & a_{n+1,1} & \cdots & a_{n+1,n} \end{pmatrix}$$

and computing its determinant, which is zero, using an expansion with respect to the first line.

Therefore, the composition of the two consecutive maps in the sequence

$$0 \longrightarrow R^n \xrightarrow{\varphi_2} R^{n+1} \xrightarrow{\tilde{\varphi}_1} R \longrightarrow R/I_n(\varphi_2) \longrightarrow 0$$

is always zero. In fact, as proved in [6, Theorem 20.15], this is an exact sequence. Moreover, there are unique maps $R \to R$ and $R/I_n(\varphi_2) \to R/I$ making the following diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & R^n & \xrightarrow{\varphi_2} & R^{n+1} & \xrightarrow{\tilde{\varphi_1}} & R & \longrightarrow & R/I_n(\varphi_2) & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow{\scriptstyle a} & & \downarrow & & \\ 0 & \longrightarrow & R^n & \xrightarrow{\varphi_2} & R^{n+1} & \xrightarrow{\varphi_1} & R & \longrightarrow & R/I & \longrightarrow & 0 \end{array}$$

commutative. This gives an element $a \in R \setminus \{0\}$ such that

$$I = \mathrm{Im}(\varphi_1) = a \cdot \mathrm{Im}(\tilde{\varphi_1}) = a \cdot I_n(\varphi_2).$$

This finishes our sketch of the proof of the Hilbert-Burch Theorem. □

Corollary 3.4 *Let $R = k[x_0, \ldots, x_n]$ and let $I \subset R$ be a graded ideal of codimension 2. If the ring R/I is Cohen-Macaulay, then there exists an $(n+1) \times n$ matrix φ with coefficients in R such that $I = I_n(\varphi)$.*

Proof: By Corollary 3.3, we have an exact sequence of graded R-modules

$$0 \to F_2 \overset{\varphi_2}{\to} F_1 \overset{\varphi_1}{\to} R \to R/I \to 0$$

where F_2 and F_1 are free of finite rank. By the first statement in Theorem 3.3, we may assume that $F_2 = R^n$ and $F_1 = R^{n+1}$, i.e., φ_2 is an $(n+1) \times n$ matrix. By the second statement in Theorem 3.3 there exists $a \in R \setminus \{0\}$ such that $I = a \cdot I_n(\varphi_2)$.

To conclude, we will show that a is invertible; this implies that $I = I_n(\varphi_2)$. We argue by contradiction: assume that a is not invertible and let $\mathfrak{p}$ be a minimal prime ideal containing a. Then $\mathfrak{p}$ contains $a \cdot I_n(\varphi_2) = I$. But, by Krull's principal ideal theorem, $\mathrm{codim}(\mathfrak{p}) \leq 1$. This shows that $\mathrm{codim}(I) \leq 1$ contradicting the assumption that $\mathrm{codim}(I) = 2$. □

3.4 Multilinear algebra

3.4.1 Exterior powers

In this subsection, we fix a ring R. Given two R-modules M and N we denote by $M \otimes_R N$ their tensor product. We also denote by $M^{\otimes n}$ the tensor product of n copies of M, i.e.,

$$M^{\otimes n} = \underbrace{M \otimes_R \cdots \otimes_R M}_{n \text{ times}}.$$

Recall that the map

$$\begin{array}{rcl} \overbrace{M \times \cdots \times M}^{n \text{ times}} & \to & M^{\otimes n} \\ (m_1, \ldots, m_n) & \mapsto & m_1 \otimes \ldots \otimes m_n \end{array}$$

is the universal R-multilinear map, i.e., for any R-multilinear map $\varphi : M\times\cdots\times M \to N$, there exists a unique R-linear map $\tilde{\varphi} : M^{\otimes n} \to N$ making the following triangle

$$\begin{array}{ccc} M \times \cdots \times M & \xrightarrow{\varphi} & N \\ \downarrow & \nearrow_{\tilde{\varphi}} & \\ M^{\otimes n} & & \end{array}$$

commutative.

Definition 3.8 *Let M and N be R-modules. An R-multilinear map*

$$\varphi : \underbrace{M \times \cdots \times M}_{n \text{ times}} \to N$$

is called alternating *if $\varphi(m_1, \ldots, m_n) = 0$ whenever $m_i = m_j$ for some $i \neq j$.*

Remark 3.4 *If 2 is invertible in R, the condition of being alternating can be replaced by the condition of being antisymmetric which means that*

$$\varphi(m_1, \ldots, m_i, \ldots, m_j, \ldots, m_n) = -\varphi(m_1, \ldots, m_j, \ldots, m_i, \ldots, m_n)$$

for all $1 \leq i < j \leq n$.

Definition 3.9 *Let M be an R-module. The* exterior n-th power *of M denoted by $\bigwedge^n M$ is the quotient of $M^{\otimes n}$ by the R-submodule generated by tensors of the form $m_1 \otimes \cdots \otimes m_n$ such that $m_i = m_j$ for some $i \neq j$.*

The image of a tensor $m_1 \otimes \ldots \otimes m_n$ by the quotient map $M^{\otimes n} \to \bigwedge^n M$ is denoted by $m_1 \wedge \cdots \wedge m_n$.

Lemma 3.6 *The morphism*

$$\begin{array}{ccc} \overbrace{M \times \ldots \times M}^{n \text{ times}} & \to & \bigwedge^n M \\ (m_1, \ldots, m_n) & \mapsto & m_1 \wedge \ldots \wedge m_n \end{array}$$

is the universal alternating R-multilinear map. More precisely, for any alternating R-multilinear map $\varphi : M \times \cdots \times M \to N$, there exists a unique R-linear map $\tilde{\varphi} : \bigwedge^n M \to N$ making the following triangle

$$\begin{array}{ccc} M \times \cdots \times M & \xrightarrow{\varphi} & N \\ \downarrow & \nearrow_{\tilde{\varphi}} & \\ \bigwedge^n M & & \end{array}$$

commutative.

Proof: This follows directly from the construction and the universal property of the tensor product. □

Proposition 3.6 *Let $m_1, \dots, m_n \in M$ and $\sigma \in \mathfrak{S}_n$ a permutation. Then, it holds:*

$$m_{\sigma(1)} \wedge \ldots \wedge m_{\sigma(n)} = \operatorname{sgn}(\sigma) \cdot m_1 \wedge \ldots \wedge m_n$$

where $\operatorname{sgn} : \mathfrak{S}_n \to \{\pm 1\}$ *is the signature homomorphism.*

Proof: See [1, chapter III.7]. □

Lemma 3.7 *Let $(m_\beta)_{\beta \in B}$ be a generating family of the R-module M. If B has a total order, then the elements*

$$m_{\beta_1} \wedge \ldots \wedge m_{\beta_n} \text{ with } \beta_1 < \beta_2 < \ldots < \beta_n$$

generate $\bigwedge^n M$.

Proof: See [1, chapter III.7]. □

Theorem 3.4 *If M is a free R-module of rank n, then $\bigwedge^k M$ is a free R-module of rank $\binom{n}{k}$. More precisely, if $m_1, \dots, m_n$ form a basis of M, then the elements*

$$m_{i_1} \wedge \ldots \wedge m_{i_l}, \text{ with } i_1 < \ldots < i_l$$

form a basis of $\bigwedge^k M$.

Proof: See [1, chapter III.7]. □

Corollary 3.5 *If M is a free R-module of rank n, then $\bigwedge^n M \simeq R$.*

Lemma 3.8 *Let M and N be R-modules and let $\theta : M \to N$ be a morphism of R-modules. Then there is a unique morphism of R-modules*

$$\bigwedge^n \theta : \bigwedge^n M \to \bigwedge^n N$$

such that $\bigwedge^n \theta(m_1 \wedge \ldots \wedge m_n) = \theta(m_1) \wedge \ldots \wedge \theta(m_n)$.

Remark 3.5 *Let M be a free R-module of rank n and let $\varphi : M \to M$ be an endomorphism of M. Then $\bigwedge^n \varphi$ is an endomorphism of $\bigwedge^n M$. As $\bigwedge^n M \simeq R$, every endomorphism of $\bigwedge^n M$ is given by a multiplication by an element of R. The corresponding element for $\bigwedge^n \varphi$ is the determinant $\det(\varphi)$. Thus, we have the formula*

$$\varphi(m_1) \wedge \ldots \wedge \varphi(m_n) = \det(\varphi) \cdot m_1 \wedge \ldots \wedge m_n.$$

3.4.2 Pfaffians and Pfaffian ideals

We fix a ring R that we assume, for simplicity, to be of characteristic zero. This implies that for an R-module M, $\bigwedge^n M$ can be identified with the image of the map $\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma$ acting on $M^{\otimes n}$.

Lemma 3.9 *Let M be an R-module and let $\bigwedge M = \bigoplus_{n\geq 0}\bigwedge^n M$. Then $\bigwedge M$ has a natural structure of an associative (non commutative) graded R-algebra. The multiplication is given by the wedge product:*

$$(x_1\wedge\ldots\wedge x_m)\wedge(y_1\wedge\ldots\wedge y_n) = x_1\wedge\ldots\wedge x_m\wedge y_1\wedge\ldots\wedge y_n.$$

Moreover, the R-submodule $\bigoplus_{m\geq 0}\bigwedge^{2m} M$ is a commutative subalgebra of $\bigwedge M$.

Definition 3.10 *A square matrix $\varphi = (a_{ij})_{1\leq i,j\leq n}$ with entries in R is called* anti-symmetric *(or* skew symmetric*) if $a_{ji} = -a_{ij}$ for all $1\leq i, j\leq n$. Note that this implies that $a_{ii} = 0$ for all $1\leq i\leq n$.*

Lemma 3.10 *Denote by $\mathrm{A}_n(R)$ the R-module of antisymmetric $n\times n$ matrices with coefficients in R. There exists an isomorphism of R-modules*

$$\begin{array}{ccc}\mathrm{A}_n(R) & \tilde{\to} & \bigwedge^2 R^n\\ \varphi & \mapsto & \hat{\varphi}.\end{array}$$

If $e_1,\ldots,e_n$ denotes the canonical basis of R^n, then $\hat{\varphi} = \sum_{1\leq i<j\leq n} a_{ij}\cdot e_i\wedge e_j$ for $\varphi = (a_{ij})_{1\leq i,j\leq n}$.

Definition 3.11 *Let $\varphi = (a_{ij})_{1\leq i,j\leq n}$ be an antisymmetric square matrix with entries in R and assume that $n = 2m$ is even. Consider the element $\hat{\varphi}\in\bigwedge^2 R^n$ associated to φ by Lemma 3.10. Then $\hat{\varphi}^m$, the m-th power of $\hat{\varphi}$ in the algebra $\bigwedge R^n$, belongs to $\bigwedge^n R^n$ which is a free R-module of rank 1. We define an element $\mathrm{pf}(\varphi)\in R$ by the condition*

$$\frac{1}{m!}\hat{\varphi}^m = \mathrm{pf}(\varphi)\cdot e_1\wedge\ldots\wedge e_n.$$

The element $\mathrm{pf}(\varphi)$ *is called the* Pfaffian *of φ.*

Proposition 3.7 *Let φ be an antisymetric $n\times n$ matrix with entries in R. The determinant of φ has the following propreties:*

1. *If n is odd, then* $\det(A) = 0$.
2. *If n is even, then* $\det(A) = \mathrm{pf}(A)^2$.

Proof: If A^{T} is the transpose of A, we have:

$$\det(A) = \det(A^{\mathrm{T}}) = \det(-A) = (-1)^n\det(A)$$

where the second equality follows from the fact that A is antisymmetric. This proves the first statement. The second statement is proved in [5, Lemma 2.3]. □

Definition 3.12 *Let φ be an antisymmetric $n \times n$ matrix with entries in R. We define the* Pfaffian ideal $J_k(\varphi)$ *as the ideal generated by the Pfaffians of the $k \times k$ antisymmetric submatrices of φ for k even. Note antisymmetric $k \times k$ submatrices are obtained by retaining the i-th lines and columns for i varying in a subset of $\{1, \dots, n\}$ of cardinality k.*

Theorem 3.5 *Let φ be an antisymmetric $n \times n$ matrix with entries in R. Then we have:*

1. $I_{2m}(\varphi) \subset J_{2m}(\varphi) \subset \sqrt{I_{2m}(\varphi)}$.
2. $I_{2m-1}(\varphi) \subset J_{2m}(\varphi)$.

Proof: See [5, Corollary 2.6] for the full proof. Note that the inclusion $J_{2m}(\varphi) \subset \sqrt{I_{2m}(\varphi)}$ follows from Proposition 3.7. Indeed, by definition $J_{2m}(\varphi)$ is generated by Pfaffians of antisymmetric $2m \times 2m$ submatrices of φ. By Proposition 3.7, the square of such Pfaffian is a $2m$-minor of φ and thus belongs to $I_{2m}(\varphi)$. This shows that the generators of $J_{2m}(\varphi)$ belong to $\sqrt{I_{2m}(\varphi)}$. □

Corollary 3.6 *Let φ be an antisymmetric $n \times n$ matrix with entries in R. Then*

$$\operatorname{codim}(J_{2m}(\varphi)) \leq (n - 2m + 1)^2.$$

More precisely, for every minimal prime ideal $J_{2m}(\varphi) \subset \mathfrak{p}$, we have $\operatorname{codim}(\mathfrak{p}) \leq (n - 2m + 1)^2$.

Proof: By Theorem 3.5 a prime ideal contains $J_{2m}(\varphi)$ if and only if it contains $I_{2m}(\varphi)$. Therefore, if $\mathfrak{p}$ is a minimal prime ideal containing $J_{2m}(\varphi)$ it is also a minimal prime ideal containing $I_{2m}(\varphi)$. We may now use Theorem 3.2 to conclude. □

3.5 The Buchsbaum-Eisenbud Theorem

3.5.1 Ext, Tor and minimal resolutions

We gather in this subsection some results that we will need later.

Remark 3.6 *Let R be a ring and let M be an R-module. Recall that the n-th derived functor of* $\operatorname{Hom}_R(M, -)$, *denoted by* $\operatorname{Ext}^n_R(M, -)$, *is defined as follows. Given an R-module N, we choose an injective resolution $I^\bullet$ of N and we set:*

$$\operatorname{Ext}^n_R(M, N) = H^n(\operatorname{Hom}_R(M, I^\bullet)).$$

There is an alternative definition of the groups $\operatorname{Ext}^n_R(M, N)$ *that we will need later. Indeed, instead of using an injective resolution of N, one can use a projective (or a free) resolution $P_\bullet$ of M and set*

$$\operatorname{Ext}^n_R(M, N) = H^n(\operatorname{Hom}_R(P_\bullet, N)).$$

Remark 3.7 *Let R be a ring and let M be an R-module. The functor $M \otimes_R -$ is right exact, i.e., for an exact sequence of R-modules $N' \to N \to N'' \to 0$ the sequence*

$$M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$

is also exact. The n-th derived functor of $M \otimes_R -$ is denoted by $\mathrm{Tor}_n^R(M, -)$. It is defined as follows. Given an R-module N, we choose a projective (or a free) resolution $Q_\bullet$ of N and set

$$\mathrm{Tor}_n^R(M, N) = H_n(M \otimes_R Q_\bullet).$$

Alternatively, one can use a projective resolution $P_\bullet$ of M and set

$$\mathrm{Tor}_n^R(M, N) = H_n(P_\bullet \otimes_R N).$$

Lemma 3.11 *Let A, B, C be R-modules. There is a canonical morphism*

$$\mathrm{Hom}_R(A, B) \otimes_R C \to \mathrm{Hom}_R(A, B \otimes_R C).$$

Moreover if A is a free R-module of finite rank, then above morphism is an isomorphism.

Proof: The canonical morphism of the statement is given by

$$\begin{array}{rcl} \mathrm{Hom}_R(A, B) \otimes_R C & \to & \mathrm{Hom}_R(A, B \otimes_R C) \\ (\phi : A \to B) \otimes e & \mapsto & \phi \otimes e : A \to B \otimes_R C \end{array}$$

where $\phi \otimes e$ sends m to $\phi(m) \otimes e$.

If $A = R^n$, this morphism is an isomorphism. This can be checked by the following calculation. On the one hand, we have:

$$\mathrm{Hom}(R^n, B) \otimes_R C \simeq B^n \otimes_R C \simeq (B \otimes_R C)^n.$$

On the other hand, we have:

$$\mathrm{Hom}(R^n, B \otimes_R C) = (B \otimes_R C)^n.$$

It can be checked easily that modulo these identifications, the morphism of the statement is the identity. □

Corollary 3.7 *Let M be an R-module admitting a free resolution $P_\bullet$ whose terms have finite rank. Then, for any R-module N, there is an isomorphism of complexes*

$$\mathrm{Hom}_R(P_\bullet, N) \simeq \mathrm{Hom}_R(P_\bullet, R) \otimes_R N.$$

Proof: We apply Lemma 3.11 to each term. □

Lemma 3.12 *Let k be a field and $R = k[x_1, \ldots, x_n]$. Then, there is a free resolution of R/R_+ given by:*

$$0 \to \bigwedge^n R^n \to \ldots \to \bigwedge^2 R^n \to R^n \to R \to R/R_+ \to 0.$$

If $e_1, \ldots, e_n$ is the canonical basis of R^n, the differential $\bigwedge^{s+1} R^n \to \bigwedge^s R^n$ in this resolution is given by

$$e_{i_1} \wedge \ldots \wedge e_{i_s} \mapsto \sum_{t=1}^{s} (-1)^{t+1} x_{i_t} \cdot e_{i_1} \wedge \ldots \wedge \widehat{e_{i_t}} \wedge \ldots \wedge e_{i_s}$$

where the symbol $\widehat{e_{i_t}}$ means that we removed e_{i_t} from the wedge product.

Proof: This is the well known Koszul complex. See [6]. $\square$

Lemma 3.13 *Let k be a field and $R = k[x_1, \ldots, x_n]$. Then, we have:*

$$\mathrm{Ext}^i(R/R_+, R) = \begin{cases} 0 & \textit{if} \quad i \neq n, \\ R/R_+ & \textit{if} \quad i = n. \end{cases}$$

Proof: For $0 \leq s \leq n$, consider the ideal $I_s = (x_1, \ldots, x_s)$. Clearly, $I_0 = 0$ and $I_n = R_+$. To prove the lemma, we will prove that

$$\mathrm{Ext}^i(R/I_s, R) = \begin{cases} 0 & \text{if} \quad i \neq s, \\ R/I_s & \text{if} \quad i = s, \end{cases}$$

by induction on s.

For $s = 0$ there is nothing to prove. Assuming that the result is true for s, we will show it for $s + 1$. Consider the short exact sequence:

$$0 \to R/I_s \xrightarrow{x_{s+1}\cdot} R/I_s \to R/I_{s+1} \to 0.$$

It induces a long exact sequence of Ext groups:

$$\ldots \to \mathrm{Ext}^i_R(R/I_{s+1}, R) \to \mathrm{Ext}^i_R(R/I_s, R) \to \mathrm{Ext}^i_R(R/I_s, R)$$

$$\to \mathrm{Ext}^{i+1}_R(R/I_{s+1}, R) \to \mathrm{Ext}^{i+1}_R(R/I_s, R) \to \ldots.$$

Using the induction hypothesis, we see easily that $\mathrm{Ext}^i_R(R/I_{s+1}, R)$ is zero for $i \notin \{s, s+1\}$. Moreover, using that $\mathrm{Ext}^s_R(R/I_s, R) \simeq R/I_s$, we obtain the following exact sequence:

$$0 \to \mathrm{Ext}^s_R(R/I_{s+1}, R) \to R/I_s \xrightarrow{x_{s+1}\cdot} R/I_s \to \mathrm{Ext}^{s+1}_R(R/I_{s+1}, R) \to 0.$$

As multiplication by x_{s+1} is injective on R/I_s and its cokernel is R/I_{s+1}, this finishes the proof. $\square$

Corollary 3.8 *Let k be a field and let $R = k[x_1, \ldots, x_n]$. Let $F_\bullet$ be a free graded resolution of the R-module R/R_+ of length n and having finite rank in each degree. For example, one can take the resolution in Lemma 3.12. Then the complex* $\mathrm{Hom}_R(F_\bullet, R)[n]$ *is also a free resolution of R/R_+.*

Proof: Recall from Remark 3.6 that

$$\mathrm{Ext}^i(R/R_+, R) = H^i(\mathrm{Hom}_R(F_\bullet, R)).$$

Using Lemma 3.13, we thus get

$$H^i(\mathrm{Hom}_R(F_\bullet, R)) = \begin{cases} 0 & \text{if } \ i \neq n, \\ R/R_+ & \text{if } \ i = n. \end{cases}$$

Hence the complex $\mathrm{Hom}_R(F_\bullet, R)[n]$, which is conentrated in degrees $\{-n, \ldots, 0\}$, has no cohomology except in degree 0 and this cohomology is isomorphic to R/R_+. This proves that $\mathrm{Hom}_R(F_\bullet, R)[n]$ is a resolution of R/R_+. □

Theorem 3.6 *Let k be a field and $R = k[x_1, \ldots, x_n]$. For any R-module M, there is an isomorphism of k-vectorspaces:*

$$\mathrm{Ext}^i_R(R/R_+, M) \simeq \mathrm{Tor}^R_{n-i}(R/R_+, M).$$

Proof: Fix a free resolution $F_\bullet$ as in Corollary 3.8 and denote by $F'_\bullet$ the complex $\mathrm{Hom}_R(F_\bullet, R)[n]$. By Corollary 3.7, we have an isomorphism

$$\mathrm{Hom}_R(F_\bullet, M) \simeq \mathrm{Hom}_R(F_\bullet, R) \otimes_R M \simeq F'_\bullet \otimes_R M[-n].$$

Passing to cohomology, we get

$$H^i(\mathrm{Hom}_R(F_\bullet, M)) \simeq H^{i-n}(F'_\bullet \otimes_R M).$$

The left hand side is $\mathrm{Ext}^i_R(R/R_+, M)$. As $F'_\bullet$ is a free resolution of R/R_+, the right hand side is $\mathrm{Tor}^R_{n-i}(R/R_+, M)$. □

Lemma 3.14 *Let R be a noetherian $\mathbb{N}$-graded ring with R/R_+ a field. Let M be a finitely generated graded R-module and consider a minimal free graded resolution:*

$$\ldots \to F_m \overset{\varphi_m}{\to} F_{m-1} \overset{\varphi_{m-1}}{\to} \ldots \to F_1 \overset{\varphi_1}{\to} F_0 \overset{\varphi_0}{\to} M \to 0.$$

Then, the following formula holds:

$$\mathrm{rank}_R(F_i) = \dim_{R/R_+} \mathrm{Tor}^R_i(R/R_+, M).$$

Proof: By definition, for all $i \in \mathbb{N}$, the morphism φ_i is minimal. This means (see Lemma 3.1) that $\ker(\varphi_i) \subset R_+F_i$. Equivalently, the image of $\varphi_{i+1} : F_{i+1} \to F_i$ is contained in R_+F_i. It follows that

$$R/R_+ \otimes_R \varphi_{i+1} : R/R_+ \otimes_R F_{i+1} \to R/R_+ \otimes_R F_i$$

is the zero map. We thus have shown that the complex $R/R_+ \otimes_R F_\bullet$ has zero differentials and hence

$$\mathrm{Tor}_i^R(R/R_+, M) = H_i(R/R_+ \otimes_R F_\bullet) = R/R_+ \otimes_R F_i \simeq F_i/R_+F_i.$$

The claim follows as the rank of F_i is equal to the dimension of the R/R_+-vectorspace F_i/R_+F_i. □

Later we will only use the following.

Theorem 3.7 *Let k be a field and let $R = k[x_1, \ldots, x_n]$. Let M be a finitely generated graded R-module and consider a minimal free graded resolution:*

$$\ldots \to F_m \overset{\varphi_m}{\to} F_{m-1} \overset{\varphi_{m-1}}{\to} \ldots \to F_1 \overset{\varphi_1}{\to} F_0 \overset{\varphi_0}{\to} M \to 0.$$

Then, the following formula holds:

$$\mathrm{rank}_R(F_i) = \dim_{R/R_+} \mathrm{Ext}_R^{n-i}(R/R_+, M).$$

Proof: By Theorem 3.6, $\mathrm{Ext}_R^{n-i}(R/R_+, M) \simeq \mathrm{Tor}_i^R(R/R_+, M)$. We then use Lemma 3.14 to obtain the result. □

3.5.2 Gorenstein rings

Definition 3.13 *Let R be a noetherian $\mathbb{N}$-graded ring with R/R_+ a field. We say that R is* Gorenstein *if*

1. *R is Cohen-Macaulay, i.e., $\mathrm{Ext}_R^i(R/R_+, R) = 0$ for $0 \le i \le \dim(R) - 1$,*
2. *the R/R_+-vectorspace $\mathrm{Ext}_R^{\dim(R)}(R/R_+, R)$ has dimension 1.*

Example 3.1 *If $\dim(R) = 0$, R is always Cohen-Macaulay and it is Gorenstein if $\mathrm{Hom}_R(R/R_+, R) = \{a \in R \mid R_+ \cdot a = 0\}$ has dimension 1 over R/R_+.*

Proposition 3.8 *Let k be a field and let $R = k[x_1, \ldots, x_n]$. Let $I \subset R$ be a graded ideal of codimension c. If the graded ring R/I is Cohen-Macaulay, then the R-module R/I admits a minimal free resolution of length c:*

$$0 \to F_c \overset{\varphi_c}{\to} F_{c-1} \overset{\varphi_{c-1}}{\to} \ldots \to F_1 \overset{\varphi_1}{\to} R \to R/I \to 0.$$

Moreover, R/I is Gorenstein if and only if F_c has rank 1.

Proof: As I has codimension c, the Krull dimension of R/I is $n-c$. As R/I is Cohen-Macaulay, we have $\mathrm{depth}(R/I) = \dim(R/I) = n - c$. The formula of Auslander-Buchsbaum gives

$$\begin{aligned} \mathrm{pdim}(R/I) &= \mathrm{depth}(R) - \mathrm{depth}(R/I) \\ &= n - (n-c) \\ &= c. \end{aligned}$$

This shows that a minimal free resolution of R/I has length c.

Assuming that R/I Cohen-Macaulay, it is Gorenstein if and only if the k-vectorspace

$$\mathrm{Ext}^{n-c}_{R/I}(R/R_+, R/I) \simeq \mathrm{Ext}^{n-c}_{R}(R/R_+, R/I)$$

has dimension 1. But, thanks to Theorem 3.7, the dimension of $\mathrm{Ext}^{n-c}_{R}(R/R_+, R/I)$ is equal to the rank of F_c. This proves our claim. □

The Eisenbud-Buchsbaum Theorem concerns exact sequences as in Proposition 3.8 for $c = 3$.

Theorem 3.8 (Eisenbud-Buchsbaum) *Let k be a field of characteristic zero and let $R = k[x_1, \ldots, x_n]$. Let $I \subset R$ be a graded ideal of codimension 3 such that R/I is Gorenstein and consider a minimal free resolution of the form:*

$$0 \to R \overset{\varphi_3}{\to} F_2 \overset{\varphi_2}{\to} F_1 \overset{\varphi_1}{\to} R \to R/I \to 0$$

which exists by Proposition 3.8. Then, the following properties hold:

1. $\mathrm{rank}(F_1) = \mathrm{rank}(F_2) = 2m + 1$.
2. *There are basis of F_1 and F_2 for which φ_2 can be given by an antisymmetric matrix.*
3. $I = J_{2m}(\varphi_2)$.

As we did for the Hilbert-Burch Theorem, we will give a sketch of the proof of the Eisenbud-Buchsbaum Theorem. However, in order to do so, we need some preliminaries on algebra structures on resolutions which we gather in the next subsection.

3.5.3 Algebra structures on free resolutions

Definition 3.14 *Let $S_\bullet$ and $T_\bullet$ be complexes of R-modules. Then, $S_\bullet \otimes_R T_\bullet$ is naturally a double complexe:*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle Id\otimes d_{j+2}} & & \downarrow{\scriptstyle Id\otimes d_{j+2}} & & \downarrow{\scriptstyle Id\otimes d_{j+1}} & & \\
\cdots \longrightarrow & S_{i+1}\otimes T_{j+1} & \xrightarrow{d_{i+1}\otimes Id} & S_i\otimes T_{j+1} & \xrightarrow{d_i\otimes Id} & S_{i-1}\otimes T_{j+1} & \xrightarrow{d_{i-1}\otimes Id} & \cdots \\
 & \downarrow{\scriptstyle Id\otimes d_{j+1}} & & \downarrow{\scriptstyle Id\otimes d_{j+1}} & & \downarrow{\scriptstyle Id\otimes d_{j+1}} & & \\
\cdots \longrightarrow & S_{i+1}\otimes T_{j} & \xrightarrow{d_{i+1}\otimes Id} & S_i\otimes T_{j} & \xrightarrow{d_i\otimes Id} & S_{i-1}\otimes T_{j} & \xrightarrow{d_{i-1}\otimes Id} & \cdots \\
 & \downarrow{\scriptstyle Id\otimes d_{j}} & & \downarrow{\scriptstyle Id\otimes d_{j}} & & \downarrow{\scriptstyle Id\otimes d_{j}} & & \\
\cdots \longrightarrow & S_{i+1}\otimes T_{j-1} & \xrightarrow{d_{i+1}\otimes Id} & S_i\otimes T_{j-1} & \xrightarrow{d_i\otimes Id} & S_{i-1}\otimes T_{j-1} & \xrightarrow{d_{i-1}\otimes Id} & \cdots \\
 & \downarrow{\scriptstyle Id\otimes d_{j-1}} & & \downarrow{\scriptstyle Id\otimes d_{j-1}} & & \downarrow{\scriptstyle Id\otimes d_{j-1}} & & \\
 & \vdots & & \vdots & & \vdots & &
\end{array}$$

where every square commutes and the composite of consecutive vertical and horizontal maps is zero.

The complexe $(S \otimes_R T)_\bullet$ is defined to be the simple complex associated to this double complex. More precisely,

$$(S \otimes_R T)_n = \bigoplus_{i+j=n} S_i \otimes T_j.$$

The differential $(S \otimes_R T)_n \to (S \otimes_R T)_{n-1}$ restricted to the direct factor $S_i \otimes S_j$ is given by $d_i \otimes Id + (-1)^i Id \otimes d_j$.

Lemma 3.15 *Let $S_\bullet$, $T_\bullet$ and $U_\bullet$ be complexes of R-modules. Then there is an isomorphism of complexes*

$$((S \otimes_R T) \otimes_R U)_\bullet \simeq (S \otimes_R (T \otimes_R U))_\bullet$$

which is direct sum of the isomorphisms

$$(S_i \otimes_R T_j) \otimes_R U_k \simeq S_i \otimes_R (T_j \otimes_R U_k).$$

As usual, we abuse notation and denote by $(S \otimes_R T \otimes_R U)_\bullet$ these two complexes.

Lemma 3.16 *Let $S_\bullet$ and $T_\bullet$ be complexes of R-modules. There is an isomorphism*

$$\tau : (S \otimes_R T)_\bullet \tilde{\to} (T \otimes_R S)_\bullet$$

which is the direct sum of the isomorphisms

$$(-1)^{ij}\tau : S_i \otimes_R T_j \simeq T_j \otimes_R S_i.$$

Definition 3.15 *Let $S_\bullet$ be a complex of R modules. An* algebra structure *on $S_\bullet$ is a morphism of complexes of R-modules*

$$m : (S \otimes_R S)_\bullet \to S_\bullet$$

which we call multiplication *as usual. We say that the multiplication m is* associative *if the following square*

$$\begin{array}{ccc} (S \otimes_R S \otimes_R S)_\bullet & \xrightarrow{m \otimes Id} & (S \otimes_R S)_\bullet \\ {\scriptstyle Id \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ (S \otimes_R S)_\bullet & \xrightarrow{m} & S \end{array}$$

commutes. We say that the multiplication m is commutative *if the following triangle*

$$\begin{array}{ccc} (S \otimes_R S)_\bullet & \xrightarrow{\tau} & (S \otimes_R S)_\bullet \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & S & \end{array}$$

commutes.

Remark 3.8 *Concretely, an algebra structure on $S_\bullet$ is a family of R-bilinear maps*

$$\begin{array}{rcl} S_i \times S_j & \to & S_{i+j} \\ (x, y) & \mapsto & x \cdot y \end{array}$$

where $x \cdot y = m_{i+j}(x \otimes y)$. The condition that m is a morphism of complexes is equivalent to the Leibniz rule:

$$d_{i+j}(x \cdot y) = d_i(x) \cdot y + (-1)^i x \cdot d_j(y).$$

The associativity of m is equivalent to the familiar identity $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. The commutativity of m is equivalent to the identity $x \cdot y = (-1)^{ij} y \cdot x$.

Proposition 3.9 *Let R be a ring and $I \subset R$ an ideal such that R/I admits a free resolution $F_\bullet$ of the form:*

$$\ldots \to 0 \to F_3 \overset{\varphi_3}{\to} F_2 \overset{\varphi_2}{\to} F_1 \overset{\varphi_1}{\to} R \to R/I \to 0.$$

Then $F_\bullet$ admits an associative and commutative algebra structure.

Proof: The group $\mathfrak{S}_2 = \{1, \tau\}$ acts on the complex $(F \otimes_R F)_\bullet$ and we define the complex $(\mathrm{S}^2 F)_\bullet$ to be the cokernel of $Id - \tau : (F \otimes_R F)_\bullet \to (F \otimes_R F)_\bullet$. This is called the second symmetric power of F.

By construction, we have a surjective morphism of complexes

$$\pi : (F \otimes_R F)_\bullet \twoheadrightarrow (\mathrm{S}^2 F)_\bullet$$

making the following triangle

$$\begin{array}{ccc} (F \otimes_R F)_\bullet & \xrightarrow{\ \tau\ } & (F \otimes_R F)_\bullet \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & (\mathrm{S}^2 F)_\bullet & \end{array}$$

commutative. Moreover, π is universal for this properties, i.e., given a morphism of complexes $p : (F \otimes_R F)_\bullet \to G_\bullet$ such that $p \circ \tau = p$, there exists a unique morphism of complexes $\bar{p} : (\mathrm{S}^2 F)_\bullet \to G_\bullet$ such that $p = \bar{p} \circ \pi$.

Now, the morphism of complexes $F_\bullet \to R/I[0]$ induces a morphism

$$\rho : (F \otimes_R F)_\bullet \to R/I \otimes_R R/I.$$

As $R/I \otimes_R R/I \simeq R/I$, $\tau \in \mathfrak{S}_2$ acts by the identity on $R/I \otimes_R R/I$. Thus, we have $\rho \circ \tau = \rho$. By the previous discussion, there exists a unique morphism $\bar{\rho} : (\mathrm{S}^2 F)_\bullet \to R/I \otimes_R R/I$ such that $\rho = \bar{\rho} \circ \pi$.

By the universal properties of free resolutions, there exists a morphism of complexes $\bar{m} : (\mathrm{S}^2 F)_\bullet \to F_\bullet$ making the following square

$$\begin{array}{ccc} (\mathrm{S}^2 F)_\bullet & \xrightarrow{\ \bar{\rho}\ } & R/I \otimes_R R/I \\ \downarrow {\scriptstyle \bar{m}} & & \downarrow {\scriptstyle \sim} \\ F_\bullet & \longrightarrow & R/I \end{array}$$

commutes. We define the multiplication $m : (F \otimes_R F)_\bullet \to F_\bullet$ as the composite $\bar{m} \circ \pi$. By construction, m is commutative.

To finish the proof, we will use that our resolution has length 3 to check that associativity is automatic. As usual, we denote by $x \cdot y$ instead of $m(x \otimes y)$. We also denote by d the differential in the complex $F_\bullet$ (instead of $\varphi_\bullet$).

To check associativity, consider three elements $x \in F_i$, $y \in F_j$ and $z \in F_k$. If one of the elements is in $F_0 \simeq R$, we may use the property that the multiplication is R-bilinear to conclude. For example, of $j = 0$, then

$$(x \cdot y) \cdot z = (yx) \cdot z = y(x \cdot z) = x \cdot (yz) = x \cdot (y \cdot z).$$

Thus, we may assume that i, j, $k \geq 1$. But then, because $F_m = 0$ for $m \geq 4$, we only need to consider the case $i = j = k = 1$.

Now, we assume that x, y, $z \in F_1$. As $d : F_3 \to F_2$ is injective, to prove that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, it suffices to check that

$$d((x \cdot y) \cdot z) = d(x \cdot (y \cdot z)).$$

We compute as follows using the Leibniz rule:

$$\begin{aligned} d((x \cdot y) \cdot z) &= d(x \cdot y) \cdot z + (x \cdot y) \cdot d(z) \\ &= (d(x) \cdot y - x \cdot d(y)) \cdot z + (x \cdot y) \cdot d(z) \\ &= (d(x) \cdot y) \cdot z - (x \cdot d(y)) \cdot z + (x \cdot y) \cdot d(z) \end{aligned}$$

$$\begin{aligned} d(x \cdot (y \cdot z)) &= d(x) \cdot (y \cdot z) - x \cdot d(y \cdot z) \\ &= d(x) \cdot (y \cdot z) - x \cdot (d(y) \cdot z - y \cdot d(z)) \\ &= d(x) \cdot (y \cdot z) - x \cdot (d(y) \cdot z) + x \cdot (y \cdot d(z)). \end{aligned}$$

Since dx, dy and dz are elements of $F_0 = R$ and the associativity was established when one of the elements is in F_0, we deduce that the last expressions in the previous calculations are equals. □

3.5.4 Proof of the Eisenbud-Buchsbaum Theorem

We now present a sketch of the proof of Theorem 3.8. We start by fixing some notation. Let k be a field and let $R = k[x_1, \ldots, x_n]$. Let $I \subset R$ be a graded ideal of codimension 3 and assume that R/I is Gorenstein. By Proposition 3.8, we have a free minimal resolution $F_\bullet$ of R/I of the form:

$$0 \to R \overset{\varphi_3}{\to} F_2 \overset{\varphi_2}{\to} F_1 \overset{\varphi_1}{\to} R \to R/I \to 0.$$

We use Proposition 3.9 to endow the complex $F_\bullet$ with an associative and commutative algebra structure. In particular, we have a pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : \quad F_1 \times F_2 &\to R \\ (x, y) &\mapsto x \cdot y \end{aligned}$$

which induced maps $\rho_1 : F_1 \to F_2^\vee$ and $\rho_2 : F_2 \to F_1^\vee$ given by

$$\rho_1(x) = \langle x, - \rangle : F_2 \to R$$

and

$$\rho_2(y) = \langle -, y \rangle : F_1 \to R.$$

We first check the following.

Lemma 3.17 *The following diagram:*

$$\begin{array}{ccccccc} R & \xrightarrow{\varphi_3} & F_2 & \xrightarrow{\varphi_2} & F_1 & \xrightarrow{\varphi_1} & R \\ \downarrow{\scriptstyle -Id} & & \downarrow{\scriptstyle -\rho_2} & & \downarrow{\scriptstyle \rho_1} & & \| \\ R & \xrightarrow{\varphi_1^\vee} & F_1^\vee & \xrightarrow{\varphi_2^\vee} & F_2^\vee & \xrightarrow{\varphi_3^\vee} & R \end{array}$$

is commutative.

Proof: We first check that the middle square commutes. Let $y \in F_2$ and let's show that the linear form $\rho_1(\varphi_2(y)) + \varphi_2^\vee(\rho_2(y))$ is zero. We do this after evaluating on $z \in F_2$. By definition, $[\rho_1(\varphi_2(y))](z) = \langle \varphi_2(y), z \rangle$ and $[\varphi_2^\vee(\rho_2(y))](z) = [\rho_2(y)](\varphi_2(z)) = \langle \varphi_2(z), y \rangle$. Writing d instead of φ_2 for the differential in $F_\bullet$, we are left to show that

$$d(y) \cdot z + d(z) \cdot y = 0.$$

To prove this, we remark that $y \cdot z = 0$ because it belongs to $F_4 \simeq 0$. By the Leibniz rule, we thus get

$$0 = d(y \cdot z) = d(y) \cdot z + y \cdot d(z) = d(y) \cdot z + d(z) \cdot y$$

as needed.

Next, we check that the left square commutes. Let $a \in R$ and let's show that the linear form $\varphi_1^\vee(a) - \rho_2(\varphi_3(a))$ is zero. We do this after evaluating on $x \in F_1$. To avoid confusion, we denote by $a_3 \in F_3$ the element $a \in R$ considered as element of degree 3 in $F_\bullet$. By definition, $[\varphi_1^\vee(a)](x) = a \cdot \varphi_1(x)$ and $[\rho_2(\varphi_3(a_3))](x) = \langle x, \varphi_3(a_3) \rangle$. Writing d for φ_1 and φ_3, we are left to show that

$$d(x) \cdot a_3 - x \cdot d(a_3) = 0.$$

To do this, remark that $x \cdot a_3 = 0$ as it belongs to $F_4 \simeq 0$. By the Leibniz rule, we thus get

$$0 = d(x \cdot a_3) = d(x) \cdot a_3 - x \cdot d(a_3)$$

as needed.

Finally, we check that the right square commutes. Let $x \in F_1$ and let's show that $\varphi_1(x) - \varphi_3^\vee(\rho_1(x))$ is zero. By definition, we have $\varphi_3^\vee(\rho_1(x)) = \langle x, \varphi_3(1) \rangle$. Thus, we are left to check that

$$d(x) + x \cdot d(1_3) = 0.$$

As before, $x \cdot 1_3 = 0$ for degree reasons and by the Leibniz rule, we have

$$0 = d(x \cdot 1_3) = d(x) \cdot 1_3 - x \cdot d(1_3)$$

as needed. $\square$

Corollary 3.9 *The morphisms $\rho_1 : F_1 \to F_2^\vee$ and $\rho_2 : F_2 \to F_1^\vee$ are isomorphisms.*

Proof: As claimed in the proof of [5, Theorem 1.5], $F_\bullet^\vee[3]$ is a free resolution of R/I. This is in fact equivalent to the property that

$$\mathrm{Ext}^i_R(R/I, R) = \begin{cases} 0 & \text{if } i \neq 3, \\ R/I & \text{if } i = 3, \end{cases}$$

which follows from the property that R/I is Gorenstein.

Moreover, $F_\bullet^\vee[3]$ must be a minimal resolution of R/I. Indeed, otherwise, we would have that $\mathrm{rank}(F_2^\vee) > \mathrm{rank}(F_1)$ and $\mathrm{rank}(F_1^\vee) > \mathrm{rank}(F_2)$ which is impossible. Now, by Lemma 3.17, we have a morphism of minimal free resolutions $F_\bullet \to F_\bullet^\vee[3]$. This morphism is necessarily invertible by Theorem 3.1. In particular, ρ_1 and ρ_2 are isomorphisms. □

We now fix a basis $u_1, \dots, u_n$ of F_1. Let $u_1^\vee, \dots, u_n^\vee$ be the dual basis of $F_1^\vee$ and set $v_i = \rho_2^{-1}(u_i^\vee)$. By construction, $v_1, \dots, v_n$ is a basis of F_2 such that $u_i \cdot v_j = \delta_{ij}$ ($\in F_3 = R$).

Lemma 3.18 *In the basis $v_1, \dots, v_n$ and $u_1, \dots, u_n$, the matrix of φ_2 is antisymmetric.*

Proof: Let $(a_{ij})_{1 \leq i,j \leq n}$ be the matrix of φ_2 so that

$$\varphi_2(v_i) = \sum_{j=1}^{n} a_{ji} \cdot u_j.$$

Now, let's write d instead of $\varphi_\bullet$ the differential of the complex $F_\bullet$. For all $1 \leq i, j \leq n$, $v_i \cdot v_j$ is zero because it belongs to $F_4 \simeq 0$. By the Leibniz rule, we get

$$\begin{aligned} 0 &= d(v_i \cdot v_j) \\ &= d(v_i) \cdot v_j + v_i \cdot d(v_j) \\ &= \textstyle\sum_{k=1}^n a_{ki}(u_k \cdot v_j) + \sum_{k=1}^n a_{kj}(v_i \cdot u_k) \\ &= a_{ji} + a_{ij}. \end{aligned}$$

This shows that $a_{ij} = -a_{ji}$ for all $1 \leq i, j \leq n$. □

Proof: (of Theorem 3.8)

At this stage, the minimal resolution of R/I can be written as follows

$$0 \to R \overset{\varphi_3}{\to} R^n \overset{\varphi_2}{\to} R^n \overset{\varphi_1}{\to} R \to R/I \to 0$$

where φ_2 is an antisymmetric matrix and $\varphi_3 = \varphi_1^\vee$.

We first show that n is odd. For this, let $a \in I \setminus \{0\}$ and let's apply $-[a^{-1}]$ to this sequence (as in the proof of Theorem 3.3). We get a short exact sequence of free $R[a^{-1}]$-modules

$$0 \to R[a^{-1}] \to R[a^{-1}]^n \to R[a^{-1}]^n \to R[a^{-1}] \to 0.$$

This shows that the rank $\varphi_2[a^{-1}]$ is $n-1$. But, it is well known that the rank of an antisymmetric matrix is always even. This shows that $n-1$ is even and hence n is odd. Let m be the integer such that $n = 2m+1$.

Now, following [5], we consider the morphism $\tilde{\varphi}_1 : R^{2m+1} \to R$ uniquely determined by the property that

$$\frac{1}{m!} v \wedge (\hat{\varphi}_2)^m = \tilde{\varphi}_1(v) \cdot e_1 \wedge \ldots \wedge e_{2m+1}$$

where $\hat{\varphi}_2 \in \bigwedge^2 R^{2m+1}$ is the element associated to the antisymmetric matrix φ_2 by Lemma 3.10 and $(\hat{\varphi}_2)^m \in \bigwedge^{2m} R^{2m+1}$ its m-th power.

As claimed in [5], the image of $\tilde{\varphi}_1$ is the Pfaffian ideal $J_{2m}(\varphi_2)$. In fact, one has a resolution

$$0 \to R \overset{\tilde{\varphi}_1^\vee}{\to} R^{2m+1} \overset{\varphi_2}{\to} R^{2m+1} \overset{\tilde{\varphi}_1}{\to} R \to R/J_{2m}(\varphi_2) \to 0.$$

In [5] it is also shown that there is a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & R & \xrightarrow{\tilde{\varphi}_1^\vee} & R^{2m+1} & \xrightarrow{\varphi_2} & R^{2m+1} & \xrightarrow{\tilde{\varphi}_1} & R & \longrightarrow & R/J_{2m}(\varphi_2) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \| & & \downarrow a & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{\varphi_1^\vee} & R^{2m+1} & \xrightarrow{\varphi_2} & R^{2m+1} & \xrightarrow{\varphi_1} & R & \longrightarrow & R/I & \longrightarrow & 0.
\end{array}$$

As in the proof of Theorem 3.3, this implies that $I = a \cdot J_{2m}(\varphi_2)$.

To conclude, it thus enough to prove that a must be invertible. We argue by contradiction: assume that a is not invertible and let $\mathfrak{p}$ be a minimal prime ideal containing a. Then $\mathfrak{p}$ contains $a \cdot J_{2m}(\varphi_2) = I$. But, by Krull's principal ideal theorem, $\operatorname{codim}(\mathfrak{p}) \leq 1$. This shows that $\operatorname{codim}(I) \leq 1$ contradicting the assumption that $\operatorname{codim}(I) = 3$. □

3.6 Application: union of complete intersections of codimension two

In this section, we give an application of the Eisenbud-Buchsbaum Theorem following the paper [8] of Ragusa and Zappalà.

We will work in the projective space $\mathbb{P}^n$ over an algebraically closed field k of characteristic zero. The coordinate ring is $R := k[x_0, \ldots, x_n]$. Let X_1 and X_2 be two complete intersections in $\mathbb{P}^n$ of codimension 2. This means, that there are homogeneous polynomials f_1, f_2, g_1, $g_2 \in R$ such that $X_1 = \mathrm{V}(f_1, g_1)$ and $X_2 = \mathrm{V}(f_2, g_2)$. We set $I_1 := (f_1, g_1)$ and $I_2 := (f_2, g_2)$.

Assumptions: We assume that g_2 has minimal degree among the polynomials f_1, f_2, g_1 and g_2. We also assume that X_1 and X_2 have no common irreducible component. Thus, changing if necessary the generators of I_2, we may assume that $\mathrm{V}(f_1, f_2, g_1)$ has codimension 3.

We will be interested in the subvariety $X_1 \cup X_2 = \mathrm{V}(I_1 \cap I_2)$.

Theorem 3.9 *Assume that $X_1 \cup X_2$ is Cohen-Macaulay, i.e., the $\mathbb{N}$-graded ring $R/I_1 \cap I_2$ is Cohen-Macaulay.*

Then, there exists an antisymmetric $(2m+1)\times(2m+1)$ matrix A with coefficients in R such that the following properties hold:

1. *The polynomials f_1, g_1, f_2 belong to the Pfaffian ideal $J_{2m}(A)$. More precisely, for $1 \leq i \leq 2m+1$, let $p_i = (-1)^i \mathrm{pf}(A_i)$ where A_i is the submatrix of A obtained by erasing the i-th line and i-th column. Then, we may find homogeneous polynomials $\alpha_1, \ldots, \alpha_{2m+1}$, $\beta_1, \ldots, \beta_{2m+1}$, and $\gamma_1, \ldots, \gamma_{2m+1}$ such that:*

$$f_1 = \sum_{i=1}^{2m+1} \alpha_i p_i,$$
$$f_2 = \sum_{i=1}^{2m+1} \beta_i p_i,$$
$$g_1 = \sum_{i=1}^{2m+1} \gamma_i p_i.$$

2. *The polynomial g_2 is the pfaffian of the antisymmetric $(2m+4) \times (2m+4)$ matrix $\bar{A}$ given by:*

$$\bar{A} = \begin{pmatrix} 0 & 0 & 0 & \alpha_1 & \ldots & \alpha_{2m+1} \\ 0 & 0 & 0 & \beta_1 & \ldots & \beta_{2m+1} \\ 0 & 0 & 0 & \gamma_1 & \ldots & \gamma_{2m+1} \\ -\alpha_1 & -\beta_1 & -\gamma_1 & & & \\ \vdots & \vdots & \vdots & & A & \\ -\alpha_{2m+1} & -\beta_{2m+1} & -\gamma_{2m+1} & & & \end{pmatrix}.$$

The theorem will be obtained by applying the Eisenbud-Buchsbaum Theorem to a well choosen graded ideal $G \subset R$ of codimension 3 such that R/G is Gorenstein. We first construct the ideal G.

Consider the ideal

$$J = I_1 + I_2 = (f_1, f_2, g_1, g_2).$$

This is the ideal defining $X_1 \cap X_2$. Consider also the ideal

$$N = (f_1, f_2, g_1).$$

By assumption $\mathrm{V}(N)$ is a complete intersection of codimension 3. In particular, the graded ring R/N is Cohen-Macaulay of dimension $n-2$. Clearly, $N \subset J$.

Lemma 3.19 *The graded R-module J/N is cyclic. More precisely, it is generated by $g_2 + N$.*

Let G be the kernel of the morphism

$$\begin{array}{rcl} R & \to & J/N \\ a & \mapsto & ag_2 + N \end{array}$$

so that $R/G \simeq J/N$. By construction, the ideal G is given by

$$G = \{a \in R \mid a \cdot g_2 \in N\}.$$

Therefore, we have $N \subset G$. We have the following result.

Proposition 3.10 *The graded ring R/G is Gorenstein of dimension $n-2$.*

Proof: Since $N \subset G$, we have $\dim(R/G) \leq \dim(R/N) = n-2$. Therefore, it is enough to prove that

$$\mathrm{Ext}^i_R(R/R_+, R/G) = \begin{cases} 0 & \text{if } \ i < n-2 \\ R/R_+ & \text{if } \ i = n-2. \end{cases}$$

Indeed, by Proposition 1.4, this will implies that $\mathrm{depth}(R/G) = n-2$, which by Lemma 1.6, yields that $\dim(R/G) \geq n-2$.

Since $R/G \simeq J/N$, it is enough to calculate $\mathrm{Ext}^i_R(R/R_+, J/N)$.

To do this, we first calculate $\mathrm{Ext}^i_R(R/R_+, N)$. As N has codimension 3 and R/N is Gorenstein, Lemma 3.20 gives:

$$\mathrm{Ext}^i_R(R/R_+, H) = \begin{cases} 0 & \text{if } \ 0 \leq i \leq n-2, \\ R/R_+ & \text{if } \quad\ i = n-1. \end{cases}$$

Next we determine $\mathrm{Ext}^i(R/R_+, J)$. For this, we use the short exact sequence

$$0 \to I_1 \cap I_2 \to I_1 \oplus I_2 \to J \to 0.$$

This gives the long exact sequence of Ext groups:

$$\ldots \mathrm{Ext}^i_R(R/R_+, I_1 \oplus I_2) \to \mathrm{Ext}^i_R(R/R_+, J) \to \mathrm{Ext}^{i+1}_R(R/R_+, I_1 \cap I_2) \to \ldots.$$

Now recall that $R/I_1 \cap I_2$ was assumed to be Cohen-Macaulay. As $I_1 \cap I_2$ has codimension 2, Lemma 3.20 gives:

$$\mathrm{Ext}^{i+1}_R(R/R_+, I_1 \cap I_2) = 0 \quad \text{if} \quad 0 \leq i \leq n-2.$$

Similarly, as R/I_1 and R/I_2 are Cohen-Macaulay, we get

$$\mathrm{Ext}^i_R(R/R_+, I_1 \oplus I_2) = \mathrm{Ext}^i_R(R/R_+, I_1) \oplus \mathrm{Ext}^i_R(R/R_+, I_2) = 0 \quad \text{if} \quad 0 \leq i \leq n-1.$$

Putting these two informations together, we obtain that

$$\mathrm{Ext}^i_R(R/R_+, J) = 0 \quad \text{if} \quad 0 \leq i \leq n-2.$$

To finish the proof, we now consider the short exact sequence:

$$0 \to N \to J \to J/N \to 0.$$

This gives a long exact sequence of Ext groups

$$\ldots \to \mathrm{Ext}^i_R(R/R_+, J) \to \mathrm{Ext}^i_R(R/R_+, J/N) \to \mathrm{Ext}^{i+1}_R(R/R_+, N) \to \ldots.$$

The above computation yields that $\mathrm{Ext}^i_R(R/R_+, J/N) = 0$ for $0 \leq i \leq n-3$ and $\mathrm{Ext}^{n-2}(R/R_+, J/N) \simeq R/R_+$ as needed. □

Lemma 3.20 *Let $H \subset R = k[x_0, \ldots, x_n]$ be a graded ideal of codimension $c \geq 2$ such that R/H is Cohen-Macaulay. Then*

$$\mathrm{Ext}^i_R(R/R_+, H) = 0 \quad \textit{if} \quad 0 \leq i \leq n - c + 1.$$

If moreover, R/H is Gorenstein, then we also have

$$\mathrm{Ext}^{n-c+2}_R(R/R_+, H) \simeq R/R_+.$$

Proof: Using the short exact sequence

$$0 \to H \to R \to R/H \to 0$$

we get a long exact sequence of Ext groups

$$\ldots \to \mathrm{Ext}^{i-1}_R(R/R_+, R/H) \to \mathrm{Ext}^i_R(R/R_+, H) \to \mathrm{Ext}^i_R(R/R_+, R) \to \ldots.$$

As R/H is Cohen-Macaulay of dimension $n-c+1$, we have $\mathrm{Ext}^{i-1}_R(R/R_+, R/N) = 0$ for $i \leq n-c+1$. If moreover R/H is Gorenstein, then $\mathrm{Ext}^{i-1}_R(R/R_+, R/N) \simeq R/R_+$ for $i = n - c + 2$. In both cases, we have $\mathrm{Ext}^i_R(R/R_+, R) = 0$ for $i \leq n$. This gives the result. □

Proof:(of Theorem 3.9)

Thanks to Proposition 3.10, we may apply Theorem 3.8 to the ideal G. This gives an antisymmetric $(2m+1) \times (2m+1)$ matrix A such that $G = J_{2m}(A)$. This proves the first part of Theorem 3.9. The second part of Theorem 3.9 is proved in [9]. □

References

[1] Nicolas Bourbaki: Algebra I Chapter 1-3.

[2] Nicolas Bourbaki: Commutative Algebra Chapter 1-7.

[3] Nicolas Bourbaki: Homological Algebra.

[4] Markus Brodmann: Commutative Algebra, Skript.

[5] David Buchsbaum and David Eisenbud: Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3.

[6] David Eisenbud: Commutative algebra with a view toward algebraic Geometry.

[7] Robert Hartshorne: Introduction to Algebraic Geometry.

[8] Alfio Ragusa and Giuseppe Zappalà: A structure theorem for unions of complete intersections.

[9] Alfio Ragusa and Giuseppe Zappalà: Characterization of the graded Betti numbers for almost complete intersections.