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Unitary Matrices with Maximal or near Maximal Diversity Product*

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Abstract

Fully diversified constellations with large diversity product are playing an important role in improving the data rate of systems with multiple antennas. In this paper we study some optimal and some near optimal constellations. For constellations with exactly 3 elements several methods are provided to estimate the performance of the constellation.

In the case of diagonal constellations, tools from linear programming are used to describe an upper bound and a lower bound for the best diversity product of a constellation with a fixed number of elements.

For 2-dimensional constellations we provide two construction methods and we prove that they have better performance than orthogonal constellations.

Index Terms –wireless communications, diversity product, constellation design, space-time coding.

1 Introduction

Multiple-antennas can enhance the data rate for wireless communication systems without increasing the error probability. At this point we still know little about how to design so called space-time codes for multiple-antennas. Mathematically, we are facing the following problem:

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Let $U(n)$ denote the set of $n \times n$ unitary matrices. Given a subset (also called constellation) $\mathcal{V} \subset U(n)$ one defines the diversity product of $\mathcal{V}$ as:

$$\zeta\mathcal{V} = \frac{1}{2} \min_{A,B \in \mathcal{V},\, A \neq B} |\det(A-B)|^{\frac{1}{n}} \tag{1.1}$$

Our goal is to find a constellation $\mathcal{V}$ of cardinality m such that the diversity product $\zeta\mathcal{V}$ is as large as possible. A lot of research has already been devoted to this question.

Constellations involving cyclic unitary groups have been studied by Hochwald and Sweldens [5]. Orthogonal constellations as in [9] provide solutions to 2-dimensional constellation design. Hamkins and Zeger analyzed in [1] Hamilton constellations which offer higher performance as the constellation gets larger. Hassibi, Hochwald, Shokrollahi, and Sweldens [3] studied a method of nongroup constellation construction which allows to construct any dimensional constellation with any number of elements. Meanwhile, many authors are addressing their efforts to group constellation construction. In [3], all the finite fully diversified groups are classified. Babak Hassibi, Mohammad Khorrami [4] considered constellations of fully diversified Lie groups.

In this paper we are studying the following basic question: For a fixed number of elements in a finite constellation of unitary (respectively diagonal matrices) what is the maximal possible diversity product. Our major results are concerned with n-dimensional diagonal constellations of arbitrary cardinality and with arbitrary constellations of three elements.

2 Some basic estimates for the maximal diversity product

Let $U(n)$ denote the set of all $n \times n$ unitary matrices. A constellation is simply a subset $\mathcal{V} \subset U(n)$. $U(n)$ has the structure of a smooth compact manifold of real dimension n^2. Together with the usual matrix multiplication $U(n)$ forms also a Lie group. Elements of $\mathcal{V}$ represent code symbols. If $A \in \mathcal{V}$ is sent one uses the jth antenna to send the jth column of A.

In this paper we are going to concentrate on constellations with only a few elements and we will try obtain estimates for the best possible diversity product in these situations. For this let $D(n)$ be the set of $n \times n$ diagonal matrices. If $\mathcal{V}$ is a constellation, $|\mathcal{V}|$ denotes the number of elements in $\mathcal{V}$. Using the definition (1.1) for the diversity product of $\mathcal{V}$ we define:

$$\begin{aligned}
\zeta(m,n) &:= \sup\{\zeta\mathcal{V} \mid \mathcal{V} \subset U(n) \text{ and } |\mathcal{V}| = m\} \\
\zeta_C(m,n) &:= \sup\{\zeta\mathcal{V} \mid \mathcal{V} \subset U(n) \text{ a cyclic subgroup and } |\mathcal{V}| = m\} \\
\zeta_D(m,n) &:= \sup\{\zeta\mathcal{V} \mid \mathcal{V} \subset U(n) \cap D(n) \text{ and } |\mathcal{V}| = m\} \\
\zeta_G(m,n) &:= \sup\{\zeta\mathcal{V} \mid \mathcal{V} \subset U(n) \text{ an arbitrary subgroup and } |\mathcal{V}| = m\}
\end{aligned}$$

Since $U(n)$ is compact, the supremum in each of the above definitions is well defined. Our main concern in this paper are estimates for above quantities. The first result shows how the different quantities relate.

Theorem 2.1 *For all $m \geq 2$ and $n \geq 1$ one has:*

$$\sin\left(\frac{\pi}{m}\right) \leq \zeta_C(m,n) \leq \zeta_D(m,n), \zeta_G(m,n) \leq \zeta(m,n) \leq 1. \tag{2.1}$$

Proof: Assume $\mathcal{V} \subset U(n)$ forms the cyclic subgroup generated by the diagonal matrix $\operatorname{diag}(e^{2\pi i/m}, \ldots, e^{2\pi i/m})$. Then $\zeta\mathcal{V} = \sin\left(\frac{\pi}{m}\right)$ and this shows the first inequality. Assume now that $\mathcal{V} \subset U(n)$ forms an arbitrary cyclic subgroup generated by some matrix A. If SAS^{-1} is diagonal then $S\mathcal{V}S^{-1}$ is a diagonal constellation and also a group and $\zeta\mathcal{V} = \zeta(S\mathcal{V}S^{-1})$ and this shows the second inequality. Clearly $\zeta(m,n)$ is larger than either $\zeta_G(m,n)$ or $\zeta_D(m,n)$ and this shows the third inequality. For the last inequality, observe that if $A, B \in U(n)$ are two arbitrary elements, then

$$|\det(A-B)| = |\det(I_n - A^{-1}B)| = |\det(I_n - SA^{-1}BS^{-1})| = \prod_{i=1}^{n} |1 - e^{i\varphi_i}|,$$

where $e^{i\varphi_i}$ are the eigenvalues of $A^{-1}B$. But the last quantity is clearly less than 2^n, i.e. $\zeta(m,n) \leq 1$. □

In many simple cases the inequalities are sharp. E.g. if the constellation contains only two elements:

Lemma 2.2 *For all $n \geq 1$ one has:*

$$\zeta_C(2,n) = \zeta_D(2,n) = \zeta_G(2,n) = \zeta(2,n) = 1. \tag{2.2}$$

The constellations involving 3 elements constitute the first non-trivial case:

Theorem 2.3 *For all $n \geq 1$ one has:*

$$\frac{\sqrt{3}}{2} = \zeta_C(3,n) = \zeta_D(3,n) = \zeta_G(3,n) \leq \zeta(3,n) \leq 1. \tag{2.3}$$

For the proof of this theorem we will need the following technical lemma: Let $S(m,n)$ be the set of $n \times m$ matrices of the form $\Phi = (\Phi_{ij})$ where $0 \leq \Phi_{ij} \leq \pi$ and $\sum_{i=1}^{n} \Phi_{ij} = \pi$. Let $d_i(\Phi) := \prod_{j=1}^{m} \sin \Phi_{ij}$, then we have:

Lemma 2.4

$$\max_{\Phi \in S(m,n)} \min_{i=1,2,\cdots,n} d_i(\Phi) = \left(\sin \frac{\pi}{n}\right)^m$$

and equality holds if and only if $\Phi_{ij} = \frac{\pi}{n}$, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$.

Proof: Let d be the left hand side in above formula. Then

$$\begin{aligned} d^n \leq \prod_{i=1}^{n} d_i(\Phi) = \prod_{i=1}^{n}\prod_{j=1}^{m} \sin \Phi_{ij} &= \prod_{j=1}^{m}\prod_{i=1}^{n} \sin \Phi_{ij} \\ &\leq \prod_{j=1}^{m} \left(\frac{\sum_{i=1}^{n} \sin \Phi_{ij}}{n}\right)^n \leq \prod_{j=1}^{m} \left(\sin \frac{\pi}{n}\right)^n = \left(\sin \frac{\pi}{n}\right)^{nm} \end{aligned}$$

that means

$$d \leq \left(\sin \frac{\pi}{n}\right)^m$$

On the other hand if one chooses $\Phi_{ij} = \frac{\pi}{n}$ one sees that

$$d \geq \left(\sin \frac{\pi}{n}\right)^m$$

which establishes the claimed equality. We leave it to the reader to verify that the maximal value is achieved in a unique manner. □

Proof of Theorem 2.3: Since $\sin(\pi/3) = \sqrt{3}/2$ it follows from Theorem 2.1 that $\sqrt{3}/2 \leq \zeta_C(3,n) \leq \zeta_D(3,n), \zeta_G(3,n)$. We will show that $\zeta_D(3,n), \zeta_G(3,n) \leq \sqrt{3}/2$ and this will establish the claim. If $\mathcal{V} \subset U(n)$ forms a subgroup of three elements then after possible change of basis we can assume that $\mathcal{V} \subset U(n)$ forms a diagonal constellation. So we assume that $\mathcal{V} = \{V_1, V_2, V_3\}$. Let

$$V_k := \mathrm{diag}\left(e^{i\varphi_{k1}}, e^{i\varphi_{k2}}, \cdots, e^{i\varphi_{kn}}\right)$$

for $k = 1, 2, 3$. Note that $|\det(V_k - V_l)| = 2^n |\prod_{j=1}^n \sin\left(\frac{\varphi_{kj} - \varphi_{lj}}{2}\right)|$. Let

$$\Phi_{1j} := \frac{\varphi_{2j} - \varphi_{1j}}{2} \text{ and } \Phi_{2j} := \frac{\varphi_{3j} - \varphi_{2j}}{2} \text{ and } \Phi_{3j} := \frac{\varphi_{1j} - \varphi_{3j}}{2}.$$

Since $|\sin x| = |\sin(-x)|$ and $|\sin(\pi - x)| = |\sin x|$, we can assume with loss of generality that

$$0 \leq \Phi_{kj} \leq \pi \text{ and } \Phi_{1j} + \Phi_{2j} + \Phi_{3j} = \pi \quad \text{for all} \quad k, j.$$

According to Lemma 2.4, $\zeta_D(3,n) \leq \frac{\sqrt{3}}{2}$ and this completes the proof. □

For n-dimensional diagonal constellation with m elements, we can sharpen the bounds. The following theorem gives an upper bound and a lower bound for the diversity product.

Let $T = \{1, 2, \cdots, m\}$, $I = \{1, 2, \cdots, n\}$ and S be the set of all the map from $T \times T \times I$ to $\{0, 1\}$, for a fixed $\delta \in S$, consider the following linear programming problem

$$-x \leq \frac{\theta_{ti} - \theta_{si}}{2} + (-1)^{\delta(t,s,i)} \frac{\pi}{2} \leq x$$

where $t, s = 1, 2, \cdots, m$ and $i = 1, 2, \cdots, n$ and $0 \leq x \leq \pi$. Let ε_δ denote the least possible x satisfying the inequality above and

$$\varepsilon = \min_\delta \varepsilon_\delta.$$

Then we have:

Theorem 2.5

$$\cos \varepsilon \leq \zeta_D(m,n) \leq (\cos \varepsilon)^{\frac{1}{n}}. \tag{2.4}$$

Remark 2.6 The complexity of calculating ε is on the order of $O((m-1)!^{n-1})$.

3 Estimation of diversity product of three element constellation

Let $A = (a_{ij})_{n\times n}$ be a $n \times n$ matrix and denote with S_n the symmetric group in n elements. Let

$$F(n) := \sup_{A\in U(n)} \sum_{\sigma\in S_n} |\prod_{i=1}^{n} a_{i\sigma(i)}|$$

Using $F(n)$, we can give an upper bound for the diversity product of an arbitrary constellation with exactly three elements:

Theorem 3.1

$$\zeta(3,n) \leq \sqrt[n]{F(n)}\frac{\sqrt{3}}{2} \tag{3.1}$$

Proof: Consider a constellation $\mathcal{V} \in C^n$ of cardinality $|\mathcal{V}| = 3$. Without loss of generality we can assume that $\mathcal{V}$ has the form: $\mathcal{V} = \{I_n, D, A\}$, where I_n is the $n\times n$ identity matrix and D is a diagonal matrix of the form

$$D = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n})$$

and A is an arbitrary fixed unitary matrix. Assume A has eigenvalues $e^{i\varphi_1}, \ldots, e^{i\varphi_n}$, i.e. there is a unitary matrix $U = (u_{ij})_{n\times n}$ such that

$$U^{-1}AU = \operatorname{diag}(e^{i\varphi_1}, e^{i\varphi_2}, \cdots, e^{i\varphi_n}).$$

If either

$$|\det(I - D)| \leq 3^{\frac{n}{2}} \text{ or } |\det(I - A)| \leq 3^{\frac{n}{2}}$$

then automatically, we have

$$\zeta\mathcal{V} \leq \frac{\sqrt[n]{F(n)}\sqrt{3}}{2}.$$

Assume therefore that $|\det(I - D)| > 3^{\frac{n}{2}}$ and $|\det(I - A)| > 3^{\frac{n}{2}}$, that is

$$|\det(I - \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n}))| > 3^{\frac{n}{2}}$$

$$|\det(I - \operatorname{diag}(e^{i\varphi_1}, e^{i\varphi_2}, \cdots, e^{i\varphi_n})| > 3^{\frac{n}{2}}$$

so, according to Theorem 2.3, we have the following inequality:

$$\begin{aligned} |\det(D - A)| &= |\det(D - U\operatorname{diag}(e^{i\varphi_1}, e^{i\varphi_2}, \cdots, e^{i\varphi_n})U^{-1})| \\ &= |\det(DU - U\operatorname{diag}(e^{i\varphi_1}, e^{i\varphi_2}, \cdots, e^{i\varphi_n}))| = |\det(u_{ij}(e^{i\theta_i} - e^{i\varphi_j}))| \\ &\leq \sum_{\sigma\in S_n}\prod_{i=1}^{n} |u_{i\sigma(i)}(e^{i\theta_i} - e^{i\varphi_{\sigma(j)}})| \leq F(n)3^{\frac{n}{2}} \end{aligned}$$

Taking the nth root and dividing the result by 2 shows also in this case that the diversity product is at most the value on the right hand side in (3.1). □

In general we do not know how sharp the estimate is. When $n = 2$ we however have:

Lemma 3.2

$$\zeta(3,2) = \sqrt{3}/2.$$

Remark 3.3 In fact, if $\zeta\mathcal{V} = \sqrt{3}/2$, $\mathcal{V}$ must be one of the following form

$$\{C, CADA^{-1}, CBEB^{-1}\} \text{ or } \{C, CAFA^{-1}, CBGB^{-1}\} \text{ or}$$

$$\{C, ADA^{-1}C, BEB^{-1}C\} \text{ or } \{C, AFA^{-1}C, BGB^{-1}C\}$$

where

$$D = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix} \quad E = \begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & e^{\frac{4\pi i}{3}} \end{pmatrix}$$

$$F = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{4\pi i}{3}} \end{pmatrix} \quad G = \begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}$$

and A, B, C are any unitary matrices.

If $n = 3$ then we computed $F(3) \cong 1.299$, i.e. we have:

Lemma 3.4

$$\zeta(3,3) \leq \sqrt[3]{F(3)}\sqrt{3}/2 \cong 0.95$$

Remark 3.5 When $n \geq 4$, we believe $F(n) \geq (\frac{2}{\sqrt{3}})^n$ and the inequality may be trivial.

Let γ_0 denote the minimal positive root of

$$\cos x(n\cos x - n + 1) - \sin x + \cos x = 0$$

and let γ_1 denote the minimal positive root of

$$\cos 2nx + \cos nx = 0.$$

The following is another diversity product estimation of a constellation with exactly three elements:

Theorem 3.6 *If n is even then*

$$\zeta(3,n) \leq \max\left(\sqrt[n]{\cos\frac{\pi}{2n}}, \sqrt[n]{\cos\frac{\gamma_0}{2}}\right).$$

If n is odd then

$$\zeta(3,n) \leq \max\left(\sqrt[n]{\cos\frac{\pi}{2n}}, \min\left(\sqrt[n]{\cos\frac{\gamma_0}{2}}, \sqrt[n]{\cos\frac{\gamma_1}{2}}\right)\right).$$

4 Construction of 2-dimensional fully diversified constellation

A 2-dimensional orthogonal constellation design as in [9] (See also [3]) is a matrix parameterization given by

$$O(x, y) = \frac{1}{\sqrt{2}} \begin{pmatrix} x & -y^* \\ y & x \end{pmatrix},$$

where $|x|^2 = |y|^2 = 1$; observe that $O(x, y)$ is unitary. Constellation of size n^2 are obtained by letting x and y range over the nth roots of unity $1, e^{2\pi i/n}, \cdots, e^{2\pi i(n-1)/n}$.

The diversity product of this constellation is

$$\zeta O(n) = \frac{\sin(\pi/n)}{\sqrt{2}}.$$

We are going to give two constellations which have better performance than orthogonal constellation.

Construction 1 Let n be an even number and let $r = \frac{\sqrt{2}}{4\cos\frac{2\pi}{n}}$. Consider the following sets:

$$A_1(n) = \left\{ \frac{\sqrt(2)}{2} e^{i\frac{2\pi}{n}k} | k = 0, 1, \cdots, n-1 \right\},$$

$$A_2(n) = \left\{ r e^{i\frac{4\pi}{n}k} | k = 0, 1, \cdots, \frac{n}{2} - 1 \right\},$$

$$A_3(n) = \left\{ \sqrt{1 - r^2} e^{i\frac{2\pi}{n}k} | k = 0, 1, \cdots, n-1 \right\}.$$

Consider the following subsets of $SU(2)$:

$$C_1(n) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} | a \in A_1(n), b \in A_1(n) \right\}$$

$$C_2(n) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} | a \in A_2(n), b \in A_3(n) \right\},$$

$$C_3(n) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} | a \in A_3(n), b \in A_2(n) \right\}.$$

Let

$$\mathcal{V}_1 := C_1(n) \bigcup C_2(n) \bigcup C_3(n).$$

Theorem 4.1 *$\mathcal{V}_1$ is a fully diversified constellation having $2n^2$ elements with diversity product:*

$$\zeta \mathcal{V}_1 = \min \left\{ \frac{\sin\frac{\pi}{n}}{\sqrt{2}}, \frac{\sqrt{(\frac{\sqrt{2}}{2} - r)^2 + (\frac{\sqrt{2}}{2} - \sqrt{1 - r^2})^2}}{2} \right\}$$

Proof: For this assume that $A, B \in \mathcal{V}_1$, in fact if $A \in C_i$ and $B \in C_j$, we can assume $i \leq j$.

If $A, B \in C_1(n)$, then

$$|\det(A - B)| \geq (\sqrt{2}\sin\frac{\pi}{n})^2 = 2(\sin\frac{\pi}{n})^2.$$

If $A \in C_1(n), B \in C_2(n)$ or $A \in C_1(n), B \in C_3(n)$ then

$$|\det(A - B)| \geq (\frac{\sqrt{2}}{2} - r)^2 + (\frac{\sqrt{(2)}}{2} - \sqrt{1 - r^2})^2 = 2 - \sqrt{2}r - \sqrt{2}\sqrt{1 - r^2}.$$

If $A \in C_2(n)$ and $B \in C_3(n)$ then

$$|\det(A - B)| \geq 2(\sqrt{1 - r^2} - r)^2.$$

If $A, B \in C_3(n)$ or $A, B \in C_2(n)$ then

$$|\det(A - B)| \geq (2r\sin\frac{2\pi}{n})^2.$$

Then the claim is established. □

Based on these calculations, we have the following table:

n	$\frac{\sin\frac{\pi}{n}}{\sqrt{2}} = \zeta O(n)$	$\frac{\sqrt{(\frac{\sqrt{2}}{2} - r)^2 + (\frac{\sqrt{2}}{2} - \sqrt{1-r^2})^2}}{2}$	$\zeta\mathcal{V}_1$
4	0.5	0.131	0.131
6	0.353	0.181	0.181
8	0.271	0.195	0.195
10	0.219	0.201	0.201
12	0.183	0.203	0.183

where $O(n)$ denotes the orthogonal constellation with n^2 elements.

Corollary 4.2 *For* $n \geq 12$,

$$\zeta\mathcal{V}_1 = \frac{\sin\frac{\pi}{n}}{\sqrt{2}} \leq \zeta(2n^2, 2).$$

Remark 4.3 For $n \geq 12$, the diversity product of construction 2 is the same as the orthogonal constellation except that it has twice the number of elements, i.e. it has $2n^2$ elements whereas the orthogonal constellation $O(n)$ has n^2 elements.

Construction 2 Let $m = 2n$ and consider the following sets

$$A_1(m) = \left\{re^{i\frac{2\pi}{n}j} | j = 0, 1, \cdots, n-1\right\},$$

$$A_2(m) = \left\{\sqrt{1 - r^2}e^{i(\frac{2\pi}{n}j + \frac{\pi}{n})} | j = 0, 1, \cdots, n-1\right\},$$

$$A_3(m) = \left\{\sqrt{1-r^2}e^{i\frac{2\pi}{n}j} | j = 0, 1, \cdots, n-1\right\},$$
$$A_4(m) = \left\{re^{i(\frac{2\pi}{n}j+\frac{\pi}{n})} | j = 0, 1, \cdots, n-1\right\},$$

where

$$r = \frac{1}{\sqrt{2(\sin\frac{\pi}{n})^2 + 2\sqrt{2}\sin\frac{\pi}{n} + 2}}.$$

Consider the following subsets of $SU(2)$

$$C_1(m) = \left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} | a \in A_1(m), b \in A_2(m)\right\}$$

$$C_2(m) = \left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} | a \in A_2(m), b \in A_1(m)\right\}$$

$$C_3(m) = \left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} | a \in A_3(m), b \in A_4(m)\right\}$$

$$C_4(m) = \left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} | a \in A_4(m), b \in A_3(m)\right\}.$$

Let $\mathcal{V}_2 := C_1(m) \bigcup C_2(m) \bigcup C_3(m) \bigcup C_4(m)$.

Theorem 4.4 *$\mathcal{V}_2$ is a fully diversified constellation of m^2 elements with diversity product*

$$\zeta\mathcal{V}_2 = \min\left\{r(\sin\frac{2\pi}{m}), (\sin\frac{\pi}{m})\right\}.$$

Proof: Consider $A, B \in \mathcal{V}_2$, in fact , if $A \in C_i(m), B \in C_j(m)$, then we can assume $i \leq j$.

If $A, B \in C_1(m)$ or $A, B \in C_2(m)$ or $A, B \in C_3(m)$ or $A, B \in C_4(m)$,

$$|\det(A-B)| \geq |re^{i\frac{2\pi}{n}k} - re^{i\frac{2\pi}{n}(k+1)}|^2 = 4r^2(\sin\frac{\pi}{n})^2.$$

If $A \in C_1(m), B \in C_2(m)$ or $A \in C_3(m), B \in C_4(m)$

$$|\det(A-B)| \geq 2|re^{i\frac{2\pi}{n}k} - \sqrt{1-r^2}e^{i(\frac{2\pi}{n}k+\frac{\pi}{n})}| = 2(1 - 2\sqrt{1-r^2}r\cos\frac{\pi}{n}).$$

If $A \in C_1(m)$ and $B \in C_3(m)$ or $A \in C_2(m)$ and $B \in C_4(m)$.

$$|\det(A-B)| \geq 2(\sqrt{1-r^2} - r)^2 = 2(1 - 2\sqrt{1-r^2}r)$$

If $A \in C_1(m)$ and $B \in C_4(m)$ or $A \in C_2(m)$ and $B \in C_3(m)$

$$|\det(A-B)| \geq |re^{i\frac{2\pi}{n}k} - re^{i\frac{2\pi k}{n}+\frac{\pi}{n}}|^2 + |\sqrt{1-r^2}e^{i\frac{2\pi}{n}k} - \sqrt{1-r^2}e^{i\frac{2\pi k}{n}+\frac{\pi}{n}}|^2 = 4(\sin\frac{\pi}{2n})^2.$$

It is easy to check that

$$2(1 - 2\sqrt{1-r^2}r) = 4r^2(\sin\frac{\pi}{n})^2$$

and

$$2(1 - 2\sqrt{1-r^2}r\cos\frac{\pi}{n}) \geq 2(1 - 2\sqrt{1-r^2}r)$$

so the claim is established. □

These arguments lead to the following table:

m	r	$4r^2(\sin\frac{2\pi}{m})^2$	$4(\sin\frac{\pi}{m})^2$	$\zeta\mathcal{V}_2$	$\zeta O(m)$
4	0.383	0.586	2	0.383	0.5
6	0.410	0.504	1	0.355	0.354
8	0.447	0.400	0.586	0.316	0.271
10	0.479	0.317	0.382	0.282	0.219
12	0.505	0.255	0.268	0.253	0.183
14	0.527	0.209	0.198	0.222	0.157

where $O(m)$ is orthogonal constellation with m^2 elements.

Corollary 4.5 *For $m \geq 14$,*

$$\zeta\mathcal{V}_2 = \sin\frac{\pi}{m} \leq \zeta(m^2, 2).$$

Remark 4.6 for $m \geq 14$, the constellation of construction 1 is $\sqrt{2}\zeta O(m)$ with the same elements as in orthogonal constellation.

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