Integro-differential equations:

Regularity theory and Pohozaev identities

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PhD Thesis

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Structure of the thesis

- **PART I**: Integro-differential equations

$$Lu(x) = PV \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big) K(y)\,dy$$

- **PART II**: Regularity of stable solutions to elliptic equations

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega \subset \mathbb{R}^n$$

- **PART III**: Isoperimetric inequalities with densities

$$\frac{|\partial\Omega|}{|\Omega|^{\frac{n-1}{n}}} \geq \frac{|\partial B_1|}{|B_1|^{\frac{n-1}{n}}}$$

PART I

1. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, [*J. Math. Pures Appl.* '14]
2. The Pohozaev identity for the fractional Laplacian, [*ARMA* '14]
3. Nonexistence results for nonlocal equations with critical and supercritical nonlinearities, [*Comm. PDE* '14]
4. Boundary regularity for fully nonlinear integro-differential equations, *Preprint*.

PART II

5. Regularity of stable solutions up to dimension 7 in domains of double revolution, [*Comm. PDE* '13]
6. The extremal solution for the fractional Laplacian, [*Calc. Var. PDE* '14]
7. Regularity for the fractional Gelfand problem up to dimension 7, [*J. Math. Anal. Appl.* '14]

PART III

8. Sobolev and isoperimetric inequalities with monomial weights, [*J. Differential Equations* '13]
9. Sharp isoperimetric inequalities via the ABP method, *Preprint*.

PART I:

Integro-differential equations

Nonlocal equations

Linear **elliptic** integro-differential operators:

$$Lu(x) = PV \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big) K(y) dy,$$

with $K \geq 0$, $K(y) = K(-y)$, and

$$\int_{\mathbb{R}^n} \min\big(1, |y|^2\big) K(y) dy < \infty.$$

Brownian motion	$\longrightarrow$	2nd order PDEs
Lévy processes	$\longrightarrow$	Integro-Differential Equations

Expected payoff

Brownian motion

$$\boxed{\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}}$$

$$\begin{aligned} u(x) &= \mathbb{E}\big(\phi(X_\tau)\big) \quad \text{(expected payoff)} \\ X_t &= \text{Random process}, \quad X_0 = x \\ \tau &= \text{first time } X_t \text{ exits } \Omega \end{aligned}$$

Expected payoff

Brownian motion

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

Lévy processes

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = \phi & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

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More equations from Probability

Distribution of the process X_t	Fractional heat equation $\partial_t u + Lu = 0$
Expected hitting time / running cost	
Controlled diffusion	
Optimal stopping time	

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Controlled diffusion	Fully nonlinear equations $\sup_{\alpha \in \mathcal{A}} L_\alpha u = 0$
Optimal stopping time	

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Controlled diffusion	Fully nonlinear equations $\sup_{\alpha \in \mathcal{A}} L_\alpha u = 0$
Optimal stopping time	Obstacle problem

The fractional Laplacian

- Most canonical example of elliptic integro-differential operator:

$$(-\Delta)^s u(x) = c_{n,s} PV \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}}\, dy, \qquad s \in (0,1).$$

- Notation justified by

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\, \widehat{u}(\xi), \qquad \rightarrow \qquad (-\Delta)^s \circ (-\Delta)^t = (-\Delta)^{s+t}.$$

- It corresponds to *stable* and radially symmetric Lévy process.

Stable Lévy processes

Special class of Lévy processes: stable processes

$$Lu(x) = PV \int_{\mathbb{R}^n} (u(x) - u(x+y)) \frac{a(y/|y|)}{|y|^{n+2s}}\, dy$$

- Very important and well studied in Probability
- These are processes with self-similarity properties ($X_t \approx t^{-1/\alpha} X_1$)
- Central Limit Theorems $\longleftrightarrow$ stable Lévy processes
- $a(\theta)$ is called the *spectral measure* (defined on S^{n-1}).

Why studying nonlocal equations?

Nonlocal equations are used to model (among others):

- Prices in Finance (since the 1990's)
- Anomalous diffusions (Physics, Ecology, Biology): $u_t + Lu = f(x, u)$

Also, they arise naturally when long-range interactions occur:

- Image Processing
- Relativistic Quantum Mechanics $\sqrt{-\Delta + m}$
- Boltzmann equation

Why studying nonlocal equations?

Still, these operators appear in:

- Fluid Mechanics (surface quasi-geostrophic equation)
- Conformal Geometry

Finally, all PDEs are limits of nonlocal equations (as $s \uparrow 1$).

Important works

- Works in Probability 1950-2014 (Kac, Getoor, Bogdan, Bass, Chen,...)
- Fully nonlinear equations: Caffarelli-Silvestre '07-10 [CPAM, Annals, ARMA]
- Reaction-diffusion equations $u_t + Lu = f(x, u)$
- Obstacle problem, free boundaries
- Nonlocal minimal surfaces, fractional perimeters
- Math. Physics: (Lieb, Frank,...) [JAMS'08], [Acta Math.'13]
- Fluid Mech.: Caffarelli-Vasseur [Annals'10], [JAMS'11]

The classical Pohozaev identity

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Theorem (Pohozaev, 1965)

$$\int_\Omega \left\{ n\,F(u) - \frac{n-2}{2}\,u\,f(u) \right\} = \frac{1}{2}\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 (x\cdot\nu)d\sigma$$

Follows from: For any function u with $u = 0$ on $\partial\Omega$,

$$\boxed{\int_\Omega (x\cdot\nabla u)\,\Delta u = \frac{2-n}{2}\int_\Omega u\,\Delta u + \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial \nu}\right)^2 (x\cdot\nu)d\sigma}$$

And this follows from the divergence theorem.

The classical Pohozaev identity

Applications of the identity:

- Nonexistence of solutions: critical exponent $-\Delta u = u^{\frac{n+2}{n-2}}$
- Unique continuation "from the boundary"
- Monotonicity formulas
- Concentration-compactness phenomena
- Radial symmetry
- Stable solutions: uniqueness results, H^1 interior regularity
- Other: Geometry, control theory, wave equation, harmonic maps, etc.

Pohozaev identities for $(-\Delta)^s$

Assume

$$\begin{aligned} \left|(-\Delta)^s u\right| &\leq C \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{aligned}$$

(+ some interior regularity on u)

Theorem (R-Serra'12; ARMA)

If Ω is $C^{1,1}$,

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u = \frac{2s-n}{2} \int_\Omega u\,(-\Delta)^s u - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{d^s}(x)\right)^2 (x \cdot \nu)$$

Here, Γ is the gamma function.

Remark

$$\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2(x\cdot\nu) \qquad \rightsquigarrow \qquad \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{d^s}\right)^2(x\cdot\nu)$$

$$\boxed{\left.\frac{u}{d^s}\right|_{\partial\Omega} \quad \text{plays the role that} \quad \frac{\partial u}{\partial\nu} \quad \text{plays in 2nd order PDEs}}$$

Pohozaev identities for $(-\Delta)^s$

Changing the origin in our identity, we find

$$\boxed{\int_\Omega u_{x_i}(-\Delta)^s u = \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{d^s}\right)^2 \nu_i}$$

Thus,

Corollary

Under the same hypotheses as before

$$\int_\Omega (-\Delta)^s u \; v_{x_i} = -\int_\Omega u_{x_i}(-\Delta)^s v + \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{d^s}\frac{v}{d^s}\,\nu_i$$

Note the contrast with the nonlocal flux in the formula $\int_\Omega (-\Delta)^s w = \int_{\mathbb{R}^n\setminus\Omega}\int_\Omega \cdots$

Ideas of the proof

1. $\boxed{u_\lambda(x) = u(\lambda x)}$, $\lambda > 1$, $\Rightarrow$

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u = \frac{d}{d\lambda}\Big|_{\lambda=1^+} \int_\Omega u_\lambda (-\Delta)^s u$$

2. Ω star-shaped $\Rightarrow$

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u + \frac{1}{2}\frac{d}{d\lambda}\Big|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda},$$

$\boxed{w = (-\Delta)^{\frac{s}{2}} u}$

3. Analyze very precisely the singularity of $(-\Delta)^{\frac{s}{2}} u$ along $\partial\Omega$, and compute.

4. Deduce the result for general $C^{1,1}$ domains.

Fractional Laplacian: Two explicit solutions

1. $u(x) = (x_+)^s$ satisfies $(-\Delta)^s u = 0$ in $(0, +\infty)$.

2. Explicit solution by [Getoor, 1961]:

$$\left.\begin{array}{rcll} (-\Delta)^s u & = & 1 & \text{in } B_1 \\ u & = & 0 & \text{in } \mathbb{R}^n \setminus B_1 \end{array}\right\} \Longrightarrow \quad u(x) = c\left(1 - |x|^2\right)^s$$

- They are C^∞ inside Ω, but $C^s(\overline{\Omega})$ and not better!
- In both cases, they are comparable to d^s, where $d(x) = \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega)$.

Boundary regularity: First results

$$\begin{cases} (-\Delta)^s u = g & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

Then, $\|u\|_{C^s(\overline{\Omega})} \leq C\|g\|_{L^\infty}$. Moreover,

Theorem (R-Serra'12; J. Math. Pures Appl.)

Ω bounded and $C^{1,1}$ domain. Then,

- $\|u/d^s\|_{C^\gamma(\overline{\Omega})} \leq C\|g\|_{L^\infty}$ *for some small* $\gamma > 0$,

where d is the distance to $\partial\Omega$.

Proof: Can not do odd reflection! (boundary behavior different from interior!)

Boundary for integro-differential operators?

$$\begin{cases} (-\Delta)^s u = g(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \qquad \Longrightarrow \qquad u/d^s \in C^\gamma(\overline{\Omega}).$$

We answer an open question: What about boundary regularity for more general operators of "order" $2s$?

$$Lu(x) = PV \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big) K(y)dy$$

Is it true that u/d^s is Hölder continuous? At least bounded?

- We answer this for linear and also for fully nonlinear equations

Fully nonlinear integro-differential equations

Let us consider solutions to

$$\begin{cases} Iu = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where I is a fully nonlinear operator like

$$\boxed{Iu(x) = \sup_{\alpha} L_\alpha u(x)} \qquad \text{(controlled diffusion)}$$

Here, all $\boxed{L_\alpha \in \mathcal{L}}$ for some class of linear operators $\mathcal{L}$.

- The class $\mathcal{L}$ is called the ellipticity class.

(When L_α are 2nd order operators, we have $F(D^2u) = f(x)$ in Ω)

Interior regularity:

- Was developed by Caffarelli and Silvestre in 2007-2010 (CPAM, Annals, ARMA)
- They established: Krylov-Safonov, Evans-Krylov, perturbative theory, etc.
- The reference ellipticity class of Caffarelli-Silvestre is $\boxed{\mathcal{L}_0}$, with kernels

$$\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}$$

Boundary regularity:

We establish boundary regularity for the class $\boxed{\mathcal{L}_*}$, with kernels

$$K(y) = \frac{a\left(y/|y|\right)}{|y|^{n+2s}}, \qquad \lambda \leq a(\theta) \leq \Lambda$$

$\mathcal{L}_* =$ Subclass of $\mathcal{L}_0$, corresponding to *stable* Lévy processes

Boundary regularity for fully nonlinear equations

Let $I(u, x)$ be a fully nonlinear operator elliptic w.r.t. $\mathcal{L}_*$, and

$$\begin{cases} I(u,x) &= f(x) \quad \text{in } \Omega \\ u &= 0 \qquad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

Theorem (R-Serra; preprint'14)

If Ω is $C^{1,1}$, then any viscosity solution satisfies

$$\|u/d^s\|_{C^{s-\epsilon}(\overline{\Omega})} \leq C\|f\|_{L^\infty(\Omega)} \quad \textit{for all } \epsilon > 0$$

Even for $(-\Delta)^s$ we improve our previous results!

- Novelty: We obtain higher regularity for u/d^s!
- The exponent $s-\epsilon$ is optimal for $f \in L^\infty$
- Also, it cannot be improved if $a \in L^\infty(S^{n-1})$
- Very important: $\mathcal{L}_*$ is the good class for boundary regularity!

The class $\mathcal{L}_0$ is too large for fine boundary regularity

There exist positive numbers $0 < \beta_1 < s < \beta_2$ such that

$$L_1(x_+)^{\beta_1} \equiv 0, \qquad L_2(x_+)^{\beta_2} \equiv 0 \qquad \text{in } \{x > 0\}$$

Solutions are not even comparable near the boundary!

Main steps of the proof of $u/d^s \in C^{s-\epsilon}(\overline{\Omega})$:

1. Bounded measurable coefficients $\Longrightarrow$ $u/d^s \in C^\gamma(\overline{\Omega})$
2. Blow up the equation at $x \in \partial\Omega$ + compactness argument.
3. Liouville theorem in half-space

Advantages of the method:

- It allows us to obtain higher regularity of u/d^s, also in the normal direction!
- After blow up, you do not see the geometry of the domain
- Also non translation invariant equations
- Discontinuous kernels $a \in L^\infty(S^{n-1})$

PART II:

Regularity of stable solutions to elliptic equations

Regularity of minimizers

Classical problem in the Calculus of Variations: Regularity of minimizers

Example in Geometry: Regularity of hypersurfaces in $\mathbb{R}^n$ which minimize the area functional.

- These hypersurfaces are smooth if $n \leq 7$
- In $\mathbb{R}^8$ the Simons cone minimizes area and has a singularity at $x = 0$

As we will see, the same happens for other nonlinear PDE in bounded domains.

Regularity of minimizers

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Open problem:

u local minimizer (or stable solution) & $n \le 9 \implies u \in L^\infty$?

- In $\mathbb{R}^{10}$, $u(x) = \log \frac{1}{|x|^2}$ is a stable solution in B_1
- $f(u) = \lambda e^u$ or $f(u) = \lambda(1+u)^p$ & $n \le 9$ [Crandall-Rabinowitz '75]
- $\Omega = B_1$ & $n \le 9$ [Cabré-Capella '06]
- $n \le 4$ [$n \le 3$ Nedev '00; $n \le 4$ Cabré '10]

The extremal solution

If $\boxed{f(u) \rightsquigarrow \lambda f(u)}$, then there is $\lambda^* \in (0, +\infty)$ s.t.

- For $0 < \lambda < \lambda^*$, there is a bounded solution u_λ.
- For $\lambda > \lambda^*$, there is no solution.
- For $\lambda = \lambda^*$,

$$u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_\lambda(x)$$

is a weak solution, called the extremal solution. Moreover, it is stable.

Question: Is the extremal solution bounded? [Brezis-Vázquez '97]

Our work

We have studied the regularity of stable solutions to

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

[*Comm. PDE* '13]

Thm: L^∞ for $n \le 7$ & Ω of double revolution

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

[*Calc. Var. PDE* '13]

[*J. Math. Anal. Appl.* '13]

Thm: L^∞ and H^s bounds in general domains;

Thm: optimal regularity for $f(u) = \lambda e^u$ in x_i-symmetric domains

Sobolev inequalities with weights

When studying $-\Delta u = f(u)$, we needed

$$\int_\Omega \left\{ s^{-\alpha}|u_s|^2 + t^{-\beta}|u_t|^2 \right\} ds\, dt \leq C \qquad \Longrightarrow \qquad u \in L^q(\Omega)\,? \qquad q(\alpha,\beta) = ?$$

After a change of variables, we want

$$\left(\int_{\tilde\Omega} |u|^q\, x_1^a x_2^b\, dx_1\, dx_2 \right)^{1/q} \leq C \left(\int_{\tilde\Omega} |\nabla u|^2\, x_1^a x_2^b\, dx_1\, dx_2 \right)^{1/2}, \qquad q = q(a,b)$$

Thus, we want Sobolev inequalities with weights

$$\left(\int_{\mathbb{R}^n} |u|^q\, w(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla u|^p\, w(x) dx \right)^{1/p}, \qquad \boxed{w(x) = x_1^{A_1} \cdots x_n^{A_n}}$$

PART III:

Isoperimetric inequalities with densities

Sobolev inequalities with weights

Theorem (Cabré-R; J. Differential Equations'13)

Let $A_i \geq 0$,

$$w(x) = x_1^{A_1} \cdots x_n^{A_n}, \qquad D = n + A_1 + \cdots + A_n.$$

Let $1 \leq p < D$. *Then,*

$$\left(\int_{\mathbb{R}^n} |u|^q \, w(x) dx \right)^{1/q} \leq C_{p,A} \left(\int_{\mathbb{R}^n} |\nabla u|^p \, w(x) dx \right)^{1/p},$$

with $q = pD/(D-p)$.

To prove the result, we establish a new weighted isoperimetric inequality.

Isoperimetric inequalities with monomial weights

Theorem (Cabré-R; J. Differential Equations'13)

Let $A_i > 0$, $w(x)$ *and* D *as before, and*

$$\Sigma = \{x \in \mathbb{R}^n \,:\, x_1, ..., x_n > 0\}.$$

Then, for any $E \subset \Sigma$,

$$\frac{P_w(E)}{w(E)^{\frac{D-1}{D}}} \geq \frac{P_w(B_1 \cap \Sigma)}{w(B_1 \cap \Sigma)^{\frac{D-1}{D}}}.$$

We denoted the weighted volume and perimeter

$$w(E) = \int_E w(x)dx \qquad\qquad P_w(E) = \int_{\Sigma \cap \partial E} w(x)dS.$$

Isoperimetric inequalities with weights

This type of isoperimetric inequalities have been widely studied:

- $w(x) = e^{-|x|^2}$ [Borell; Invent. Math.'75]
- Existence and regularity of minimizers (Pratelli, Morgan,...)
- log-convex radial densities $w(|x|)$ [Figalli-Maggi '13], [Chambers '14]
- with $w(x) = e^{|x|^2}$, $w(x) = |x|^{\alpha}$, or other particular weights
- ...

Isoperimetric inequalities in cones

In general cones Σ, a well known result is the following:

Theorem (Lions-Pacella '90)

Let Σ be any open convex cone in $\mathbb{R}^n$. Then, for any $E \subset \Sigma$,

$$\frac{|\Sigma \cap \partial E|}{|E|^{\frac{n}{n-1}}} \geq \frac{|\Sigma \cap \partial B_1|}{|B_1 \cap \Sigma|^{\frac{n}{n-1}}}.$$

Important: Only the perimeter inside Σ is counted

New isoperimetric inequalities weights

Theorem (Cabré-R-Serra; preprint '13)

Let Σ be any convex cone in $\mathbb{R}^n$. Assume

$$w(x) \ \underline{homogeneous} \ of \ degree \ \alpha \geq 0, \qquad \& \qquad w^{1/\alpha} \ \underline{concave} \ in \quad \Sigma.$$

Then, for any $E \subset \Sigma$,

$$\frac{P_w(E)}{w(E)^{\frac{D-1}{D}}} \geq \frac{P_w(B_1 \cap \Sigma)}{w(B_1 \cap \Sigma)^{\frac{D-1}{D}}}.$$

Recall

$$w(E) = \int_E w(x)dx \qquad\qquad P_w(E) = \int_{\Sigma \cap \partial E} w(x)dS.$$

Comments

- Minimizers are radial, while $w(x)$ is not!
- When $w \equiv 1$ we recover the result of Lions-Pacella (with new proof!).
- We can also treat anisotropic perimeters

$$P_{w,H}(E) = \int_{\Sigma \cap \partial E} H(\nu)\, w(x) dS.$$

$w \equiv 1 \quad \Longrightarrow \quad$ new proof of the Wulff theorem.

- Some examples of weights are

$$w(x) = \mathrm{dist}(x, \partial\Sigma)^{\alpha}, \qquad w(x) = \sqrt{x} + \sqrt{y}, \qquad w(x) = \frac{xyz}{x+y+z}, \qquad \ldots$$

The proof

- The proof uses the ABP technique applied to an appropriate PDE
- When $w \equiv 1$, the idea goes back to the work [Cabré '00] (for the classical isoperimetric inequality)
- Here, we need to consider a linear Neumann problem in $E \subset \Sigma$ involving the operator $w^{-1}\mathrm{div}(w\nabla u)$

The end

Thank you!