BOUNDARY REGULARITY FOR FULLY NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

We study fine boundary regularity properties of solutions to fully nonlinear elliptic integro-differential equations of order $2s$*, with* $s \in (0,1)$*. We consider the class of nonlocal operators* $\mathcal{L}_* \subset \mathcal{L}_0$*, which consists of infinitesimal generators of stable Lévy processes belonging to the class* $\mathcal{L}_0$ *of Caffarelli–Silvestre. For fully nonlinear operators* I *elliptic with respect to* $\mathcal{L}_*$*, we prove that solutions to* $\mathrm{I}u = f$ *in* Ω*,* $u = 0$ *in* $\mathbb{R}^n \setminus \Omega$*, satisfy* $u/d^s \in C^{s+\gamma}(\bar{\Omega})$*, where* d *is the distance to* $\partial\Omega$ *and* $f \in C^\gamma$*. We expect the class* $\mathcal{L}_*$ *to be the largest scale-invariant subclass of* $\mathcal{L}_0$ *for which this result is true. In this direction, we show that the class* $\mathcal{L}_0$ *is too large for all solutions to behave as* d^s*. The constants in all the estimates in this article remain bounded as the order of the equation approaches* 2*. Thus, in the limit* $s \uparrow 1$*, we recover the celebrated boundary regularity result due to Krylov for fully nonlinear elliptic equations.*

Contents

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1. Introduction and results

This article is concerned with boundary regularity for fully nonlinear elliptic integro-differential equations.

Since the foundational work of Caffarelli and Silvestre [14], ellipticity for a nonlinear integro-differential operator is defined relatively to a given set $\mathcal{L}$ of linear translation-invariant elliptic operators. This set $\mathcal{L}$ is called the *ellipticity class*. The reference ellipticity class from [14] is the class $\mathcal{L}_0 = \mathcal{L}_0(s)$ containing all operators L of the form

$$Lu(x) = \int_{\mathbb{R}^n} \Big(\frac{u(x+y)+u(x-y)}{2} - u(x) \Big) K(y)\,dy \tag{1.1}$$

with even kernels $K(y)$ bounded between two positive multiples of $(1-s)|y|^{-n-2s}$, which is the kernel of the fractional Laplacian $(-\Delta)^s$.

In their articles [14], [15], and [16], Caffarelli and Silvestre studied the interior regularity for solutions u to

$$\begin{cases} \mathrm{I}u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.2}$$

where I is a translation-invariant fully nonlinear integro-differential operator of order $2s$ (see the definition later in this Introduction). Caffarelli and Silvestre proved existence of viscosity solutions, established $C^{1+\alpha}$ interior regularity of solutions in [14] and $C^{2s+\alpha}$ regularity in the case of convex equations in [15], and developed a perturbative theory for nontranslation-invariant equations in [16]. Thus, the interior regularity for these equations is well understood.

However, very little is known about the boundary regularity for fully nonlinear nonlocal problems.

When I is the fractional Laplacian $-(-\Delta)^s$, the boundary regularity of solutions u to (1.2) is now quite well understood. The first result in this direction was obtained by Bogdan, who established the boundary Harnack principle for s-harmonic functions in [6] (i.e., for solutions to $(-\Delta)^s u = 0$). More recently, the present authors proved in [42] that if $f \in L^\infty$, $g \equiv 0$, and Ω is $C^{1,1}$, then $u \in C^s(\mathbb{R}^n)$ and $u/d^s \in C^\alpha(\bar{\Omega})$ for some small $\alpha > 0$, where d is the distance to the boundary $\partial\Omega$. Moreover, the limit of $u(x)/d^s(x)$ as $x \to \partial\Omega$ is typically nonzero (in fact, it is positive if $f < 0$), and thus the C^s regularity of u is optimal. After this, Grubb [21] showed that when $f \in C^\gamma$ with $\gamma > 0$ (resp., $f \in L^\infty$), $g \equiv 0$, and Ω is smooth, then $u/d^s \in C^{\gamma+s-\epsilon}(\bar{\Omega})$ (resp., $u/d^s \in C^{s-\epsilon}(\bar{\Omega})$) for all $\epsilon > 0$. In particular, $f \in C^\infty$ leads to $u/d^s \in C^\infty(\bar{\Omega})$. Thus, the correct notion of boundary regularity for equations of order $2s$ is the Hölder regularity of the quotient u/d^s. In a new work [20], Grubb removes the ϵ in the previous estimates, obtaining that u/d^s is $C^{s+\gamma}$ whenever $f \in C^\gamma$ and neither $\gamma + s$ nor $\gamma + 2s$ are integers.

Here, we obtain boundary regularity for *fully nonlinear* integro-differential problems of the form (1.2) which are elliptic with respect to the class $\mathcal{L}_* \subset \mathcal{L}_0$ as defined in the following. We have that $\mathcal{L}_*$ consists of all linear operators of the form

$$Lu(x) = (1-s)\int_{\mathbb{R}^n}\Big(\frac{u(x+y)+u(x-y)}{2} - u(x)\Big)\frac{\mu(y/|y|)}{|y|^{n+2s}}\,dy, \tag{1.3}$$

with

$$\mu \in L^\infty(S^{n-1}) \quad \text{satisfying } \mu(\theta) = \mu(-\theta) \text{ and } \lambda \le \mu \le \Lambda, \tag{1.4}$$

where $0 < \lambda \le \Lambda$ are called *ellipticity constants*. The class $\mathcal{L}_*$ consists of all infinitesimal generators of *stable* Lévy processes belonging to $\mathcal{L}_0$. Our main result essentially establishes that when $f \in C^\gamma$, $g \equiv 0$, and Ω is $C^{2,\gamma}$, then the viscosity solutions u satisfy

$$u/d^s \in C^{s+\gamma}(\bar{\Omega}). \tag{1.5}$$

In case of flat boundary, we assume $\mu \in C^\gamma(S^{n-1})$. In general $C^{2,\gamma}$ domains Ω, we need to assume $\mu \in C^{1,\gamma}(S^{n-1})$.

We expect the class $\mathcal{L}_*$ to be the largest scale-invariant subclass of $\mathcal{L}_0$ for which this result is true.

For general elliptic equations with respect to $\mathcal{L}_0$, no fine boundary regularity results like (1.5) hold. In fact, the class $\mathcal{L}_0$ is too large for all solutions to be comparable to d^s near the boundary. Indeed, we show in Section 2 that there are powers $0 < \beta_1 < s < \beta_2$ for which the functions $(x_n)_+^{\beta_1}$ and $(x_n)_+^{\beta_2}$ satisfy

$$M^+_{\mathcal{L}_0}(x_n)_+^{\beta_1} = 0 \qquad \text{and} \qquad M^-_{\mathcal{L}_0}(x_n)_+^{\beta_2} = 0 \quad \text{in } \{x_n > 0\},$$

where $M^+_{\mathcal{L}_0}$ and $M^-_{\mathcal{L}_0}$ are the extremal operators for the class $\mathcal{L}_0$ (see their definition in Section 2). Hence, since $(-\Delta)^s(x_n)_+^s = 0$ in $\{x_n > 0\}$, we have at least three functions which solve fully nonlinear elliptic equations with respect to $\mathcal{L}_0$ but which are not even comparable near the boundary $\{x_n = 0\}$. As we show in Section 2, the same happens for the subclasses $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}_0$, which have more regular kernels and were considered in [14]–[16].

The constants in our estimates remain bounded as $s \uparrow 1$. Thus, in the limit we recover the celebrated boundary regularity estimate of Krylov for second-order fully nonlinear elliptic equations [29].

1.1. *The class $\mathcal{L}_*$*

The class $\mathcal{L}_*$ consists of all infinitesimal generators of stable Lévy processes belonging to $\mathcal{L}_0$. This type of Lévy process has been well studied in probability, as is explained next. In that context, the function $\mu \in L^\infty(S^{n-1})$ is called the *spectral measure*.

Stable processes are for several reasons a natural extension of Gaussian processes. For instance, the *generalized central limit theorem* states that the distribution of a sum of independent, identically distributed random variables with heavy tails converges to a stable distribution (see [31], [43], or [3] for a precise statement of this result). Thus, stable processes are often used to model sums of many random independent perturbations with heavy-tailed distributions (i.e., when large outcomes are not unlikely). In particular, they arise frequently in the fields of financial mathematics, internet traffic statistics, or signal processing (see, e.g., [1], [23], [27], [32]–[35], [37], [39], [40] and the books [36], [43]).

Linear equations $Lu = f$ with L in the class $\mathcal{L}_*$ have already been studied, specially by Sztonyk and Bogdan (see, e.g., [7], [41], [49], [50]). When the spectral measure μ in (1.3) belongs to $C^\infty(S^{n-1})$, the regularity up to the boundary of u/d^s in C^∞ domains follows from the recent results of Grubb [21] for linear pseudodifferential operators.

Notice that all second-order linear uniformly elliptic operators are recovered as limits of operators in $\mathcal{L}_* = \mathcal{L}_*(s)$ as $s \to 1$. In particular, all second-order fully nonlinear equations $F(D^2u) = f(x)$ are recovered as limits of the fully nonlinear integro-differential equations that we consider. Furthermore, when $s < 1$, the class of translation-invariant linear operators $\mathcal{L}_*(s)$ is much richer than the one of second-order uniformly elliptic operators. Indeed, while any operator in the latter class is determined by a positive definite $n \times n$ matrix, a function $\mu : S^{n-1} \to \mathbb{R}^+$ is needed to determine an operator in $\mathcal{L}_*(s)$.

A key feature of the class $\mathcal{L}_*$ for boundary regularity issues is that

$$L(x_n)_+^s = 0 \quad \text{in } \{x_n > 0\} \text{ for all } L \in \mathcal{L}_*.$$

This is essential first to construct barriers which are comparable to d^s and later to prove finer boundary regularity.

1.2. Equations with bounded measurable coefficients

The first result of this paper, on which all the other results rely, is Proposition 1.1.

Here and throughout the article, we use the definition of viscosity solutions and inequalities from [14]. Moreover, for $r > 0$, we denote

$$B_r^+ = B_r \cap \{x_n > 0\} \qquad \text{and} \qquad B_r^- = B_r \cap \{x_n < 0\},$$

and the constants λ and Λ in (1.4) are ellipticity constants. The extremal operators associated to the class $\mathcal{L}_*$ are denoted by $M^+_{\mathcal{L}_*}$ and $M^-_{\mathcal{L}_*}$:

$$M^+_{\mathcal{L}_*}u = \sup_{L\in\mathcal{L}_*} Lu \qquad \text{and} \qquad M^-_{\mathcal{L}_*}u = \inf_{L\in\mathcal{L}_*} Lu.$$

Note that, since $\mathcal{L}_* \subset \mathcal{L}_0$, then $M^-_{\mathcal{L}_0} \le M^-_{\mathcal{L}_*} \le M^+_{\mathcal{L}_*} \le M^+_{\mathcal{L}_0}$.

PROPOSITION 1.1
Let $s_0 \in (0,1)$ and $s \in [s_0,1)$. Assume that $u \in C(B_1) \cap L^\infty(\mathbb{R}^n)$ is a viscosity solution of

$$\begin{cases} M^+_{\mathcal{L}_*}u \ge -C_0 & \text{in } B_1^+, \\ M^-_{\mathcal{L}_*}u \le C_0 & \text{in } B_1^+, \\ u = 0 & \text{in } B_1^-, \end{cases} \tag{1.6}$$

for some nonnegative constant C_0. Then, u/x_n^s is $C^{\bar\alpha}(\overline{B^+_{1/2}})$ for some $\bar\alpha > 0$, with the estimate

$$\|u/x_n^s\|_{C^{\bar\alpha}(B^+_{1/2})} \le C\bigl(C_0 + \|u\|_{L^\infty(\mathbb{R}^n)}\bigr). \tag{1.7}$$

The constants $\bar\alpha$ and C depend only on n, s_0, and the ellipticity constants.

It is important to note that the constants in our estimate remain bounded as $s \to 1$. This means that from Proposition 1.1 we can recover the classical boundary Harnack inequality of Krylov from [29].

The estimate of Proposition 1.1 is only a first step toward our results. It is obtained via a nonlocal version of the method of Caffarelli [26, p. 39] for second-order equations with bounded measurable coefficients (see also Section 9.2 in [10]). This method has been adapted to nonlocal equations by the authors in [42],† where we proved estimate (1.7) for the fractional Laplacian $(-\Delta)^s$ in $C^{1,1}$ domains.

As explained before, our main result is the $C^{s+\gamma}$ regularity of u/d^s in $C^{2,\gamma}$ domains for solutions u to fully nonlinear integro-differential equations (see Section 1.3). Thus, for solutions to the nonlinear equations, we push the small Hölder exponent $\bar\alpha > 0$ in (1.7) up to the sharp exponent $s + \gamma$ in (1.5). To achieve this, new ideas are needed, and the procedure that we develop differs substantially from that in second-order equations. We use a new compactness method and the "boundary" Liouville-type Theorem 1.4, stated later in the Introduction. This Liouville theorem relies on Proposition 1.1.

†In [42], we incorrectly attributed the method to Krylov. Instead, we were using a method due to Caffarelli, who gave an alternative proof of Krylov's boundary regularity theorem.

1.3. Main result

Before stating our main result, let us recall the definition and motivations of fully nonlinear integro-differential operators.

As defined in [14], a fully nonlinear operator I is said to be elliptic with respect to a subclass $\mathcal{L} \subseteq \mathcal{L}_0$ when

$$M_{\mathcal{L}}^{-}(u-v)(x) \le \mathrm{I}u(x) - \mathrm{I}v(x) \le M_{\mathcal{L}}^{+}(u-v)(x)$$

for all test functions u, v which are C^2 in a neighborhood of x and which have finite integral against $\omega_s(x) = (1-s)(1+|x|^{-n-2s})$. Moreover, if

$$\mathrm{I}\big(u(x_0+\cdot)\big)(x) = (\mathrm{I}u)(x_0+x),$$

then we say that I is *translation-invariant*.

Fully nonlinear elliptic integro-differential equations naturally arise in stochastic control and games. In typical examples, a single player or two players control some parameters (e.g., the volatilities of the assets in a portfolio) affecting the joint distribution of the random increments of n variables $X(t) \in \mathbb{R}^n$. The game ends when $X(t)$ exits for the first time a certain domain Ω (as when having automated orders to sell assets when their prices cross certain limits).

The *value* or *expected payoff* of these games $u(x)$ depends on the starting point $X(0) = x$ (initial prices of all assets in the portfolio). A remarkable fact is that the value $u(x)$ solves an equation of the type $\mathrm{I}u = 0$, where

$$\mathrm{I}u(x) = \sup_a (L_a u + c_a) \qquad \text{or} \qquad \mathrm{I}u(x) = \inf_b \sup_a (L_{ab} u + c_{ab}). \tag{1.8}$$

The first equation, known as the *Bellman equation*, arises in control problems (a single player), while the second one, known as the *Isaacs equations*, arises in zero-sum games (two players). The linear operators L_a and L_{ab} are infinitesimal generators of Lévy processes, standing for all the possible choices of the distribution of time increments of $X(t)$. The constants c_a and c_{ab} are costs associated to the choice of the operators L_a and L_{ab}. More involved equations with 0th-order terms and right-hand sides also have meanings in this context as interest rates or running costs (see [9], [14], [19], [38] and the references therein for more information on these equations).

When all L_a and L_{ab} belong to $\mathcal{L}_*$, then (1.8) are fully nonlinear translation-invariant operators elliptic with respect to $\mathcal{L}_*$, as defined above.

The interior regularity for fully nonlinear integro-differential elliptic equations was mainly established by Caffarelli and Silvestre in their well-known work [14]. More precisely, for some small $\alpha > 0$, they obtain $C^{1+\alpha}$ interior regularity for fully nonlinear elliptic equations with respect to the class $\mathcal{L}_1$ made of kernels in $\mathcal{L}_0$ which are C^1 away from the origin. For $s > 1/2$, the same result in the class $\mathcal{L}_0$ has been recently proved by Kriventsov in [28]. These estimates are uniform as the order of the equations approaches 2, so they can be viewed as a natural extension of the interior regularity for fully nonlinear equations of second order. There were previous interior estimates by Bass and Levin [5] and by Silvestre [46] which are not uniform as the order of the equation approaches 2. An interesting aspect of [46] is that its proof is short and uses only elementary analysis tools, taking advantage of the nonlocal character of the equations. This is why the same ideas have been used in other different contexts (see [17], [47]).

For convex equations elliptic with respect to $\mathcal{L}_2$ (i.e., with kernels in $\mathcal{L}_0$ which are C^2 away from the origin), Caffarelli and Silvestre obtained $C^{2s+\alpha}$ interior regularity in [15]. This is the nonlocal extension of the Evans–Krylov theorem. The same result in the class $\mathcal{L}_0$ has been recently proved by the second author in [45]. (Other important references concerning interior regularity for nonlocal equations in nondivergence form are [2], [18], [22], [24], [25].)

Our main result is the following.

THEOREM 1.2
Let $s_0 \in (0,1)$ and let $s \in [s_0, 1)$. Let $\bar\alpha$ be the exponent given by Proposition 1.1. Assume that I *is a fully nonlinear and translation-invariant operator of the form (1.8)–(1.3)–(1.4). In addition, assume that the spectral measures satisfy*

$$\|\mu_{ab}\|_{C^\gamma(S^{n-1})} \le \Lambda.$$

Let $f \in C^\gamma(B_1^+)$, and let $u \in L^\infty(\mathbb{R}^n) \cap C(B_1)$ be any viscosity solution of

$$\begin{cases} \mathrm{I}u = f & \text{in } B_1^+, \\ u = 0 & \text{in } B_1^-. \end{cases} \tag{1.9}$$

Assume that $\gamma \in (0, 1-s+\bar\alpha)$ and that $s+\gamma$ is not an integer. Then, $u/(x_n)^s$ belongs to $C^{s+\gamma}(\overline{B_{1/2}^+})$ with the estimate

$$\big\|u/(x_n)^s\big\|_{C^{s+\gamma}(B_{1/2}^+)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{C^\gamma(B_1^+)}\big),$$

where the constant C depends only on n, s_0, γ, λ, and Λ.

Notice that taking $\gamma = 1-s+\alpha$ in the previous result (with $\alpha > 0$ small), we find that $u/(x_n)^s$ is $C^{1,\alpha}$ up to the boundary. Thus, it gives an estimate of order $1+s+\alpha$ on the boundary, and not only $2s+\alpha$ (which is the regularity in the interior of the domain for convex equations; see [15], [45]).

As stated above, before our results almost nothing was known about the boundary regularity of solutions to fully nonlinear integro-differential equations. It was only known that solutions u to these equations are C^α up to the boundary for some small $\alpha > 0$ (a result for u but not for the quotient u/d^s).

Theorem 1.2 gives a sharp boundary regularity estimate for fully nonlinear equations elliptic with respect to $\mathfrak{L}_*$, recovering in the limit $s \uparrow 1$ the celebrated boundary regularity estimate of order $2+\alpha$ of Krylov.

Note also that our result is not only an a priori estimate for classical solutions but also applies to viscosity solutions. For local equations of second order, the boundary regularity for viscosity solutions to fully nonlinear equations has been recently obtained by Silvestre and Sirakov [48]. The methods that we introduce here to prove Theorem 1.2 can be used to give a new proof of the results for second-order fully nonlinear equations.

Apart from their intrinsic interest, we believe that the results and proofs in this paper will be useful in order to make progress in the study of other important models with nonlocal diffusion where the boundary behavior of solutions plays a crucial role. For instance, the precise understanding of the boundary behavior of solutions given by Theorem 1.2 (and Theorems 1.3 and 1.4 stated later on in the Introduction) opens the door to a detailed study of regularity properties for natural free boundary models such as:

- The obstacle problem with diffusion operator in $L \in \mathfrak{L}_*$:
$$\min\{-Lu, u-\varphi\} = 0 \quad \text{in } \mathbb{R}^n.$$
- The (variational) *one-phase* problem, concerning local minimizers of
$$\frac{1}{2}\iint \big(u(x)-u(x+y)\big)^2 \frac{\mu(y/|y|)}{|y|^{n+2s}}\,dx\,dy + \int \chi_{\{u>0\}}(x)\,dx.$$

Currently, these models have only been studied for the fractional Laplacian $L = (-\Delta)^s$ (see [11], [12] and the references therein).

1.4. Estimates in $C^{2,\gamma}$ domains

We will also establish the following boundary regularity estimate in $C^{2,\gamma}$ domains. In this result, we consider operators
$$\mathrm{I}(u,x) = \inf_b \sup_a \big(L_{ab}u + c_{ab}(x)\big), \tag{1.10}$$
with L_{ab} of the form (1.3)–(1.4) and satisfying
$$\|\mu_{ab}\|_{C^{1,\gamma}(S^{n-1})} \le \Lambda. \tag{1.11}$$
Moreover, we assume also that
$$\|c_{ab}\|_{C^\gamma(\bar\Omega)} \le C_0. \tag{1.12}$$
Under these assumptions, we have the following.

THEOREM 1.3
Let $s_0 \in (0,1)$ and let $s \in [s_0, 1)$. Let $\bar\alpha$ be the exponent given by Proposition 1.1, and let $\gamma \in (0, 1-s+\bar\alpha)$. Assume in addition that $s+\gamma$ is not an integer and that $\gamma \le s$. Let Ω be any $C^{2,\gamma}$ domain, and let I *be a fully nonlinear operator of the form (1.10) with (1.3)–(1.4)–(1.11)–(1.12). Let $d(x)$ be a $C^{2,\gamma}(\bar\Omega)$ function that coincides with* $\operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)$ *in a neighborhood of $\partial\Omega$. Then let $u \in C(\bar\Omega)$ be any viscosity solution of*

$$\begin{cases} \mathrm{I}(u,x) = f & \textit{in } \Omega, \\ u = 0 & \textit{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

with $\|f\|_{C^\gamma(\bar\Omega)} \le C_0$. Then, u/d^s belongs to $C^{s+\gamma}(\bar\Omega)$ with the estimate

$$\|u/d^s\|_{C^{s+\gamma}(\bar\Omega)} \le C C_0,$$

where the constant C depends only on Ω, s_0, γ, λ, and Λ.

In case of local equations of second order, the Krylov estimate in $C^{2,\gamma}$ domains is usually proved by flattening the boundary; (see [26], [48]). However, we will not do this here, and thus we do not deduce Theorem 1.3 from Theorem 1.2.

Note that as a consequence of Theorem 1.3, one can immediately obtain estimates for solutions to

$$\begin{cases} \mathrm{I}u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

with $g \in C^{2s+\gamma}(\mathbb{R}^n)$ and $f \in C^\gamma(\bar\Omega)$. Indeed, one only needs to consider $\tilde u = u - g$ in $\mathbb{R}^n$ and then apply Theorem 1.3 to $\tilde u$ to find that $(u-g)/d^s \in C^{s+\gamma}(\bar\Omega)$.

1.5. Ingredients of the proof

Let us explain now the main ideas in the proofs of Theorems 1.2 and 1.3. Our proof of these results differs substantially from boundary regularity methods in second-order equations. Indeed, recall that for second-order equations, one first shows that D^2u is bounded on the boundary, and then the estimate for equations with bounded measurable coefficients implies immediately a $C^{2,\alpha}$ estimate on the boundary for solutions to fully nonlinear equations (see [10], [26]).

This is much more delicate for nonlocal equations, and it is not at all easy to prove Theorem 1.2 using Proposition 1.1. A main reason for this is not only the nonlocal character of the estimates, but also the fact that tangential and normal derivatives of the solution behave differently on the boundary; recall that the solution is C^s and not Lipschitz up to the boundary.

In our proof, the main step toward Theorem 1.2 is an iterative result of the form

$$u/(x_n)^s \in C^\beta(B^+_{3/4}) \quad \Longrightarrow \quad u/(x_n)^s \in C^{\alpha+\beta}(B^+_{1/2}), \tag{1.13}$$

where $\alpha \in (0,\bar{\alpha})$ is small, and $\beta \in [0,1]$ satisfies $\alpha + \beta \le \gamma + s$. Essentially, this is equivalent to an estimate *on* the boundary, which reads as follows. If u satisfies the hypotheses of Theorem 1.2 and if $u/(x_n)^s \in C^\beta(B^+_{3/4})$, then for all $z \in \{x_n = 0\} \cap \overline{B_{1/2}}$ there exist $b_z \in \mathbb{R}$ and $p_z \in \mathbb{R}^n$ for which

$$\left|u(x) - b_z(x_n)^s_+ - (p_z \cdot x)(x_n)^s_+\right| \le C|x-z|^{\alpha+\beta+s} \quad \text{for all } x \in B_1. \tag{1.14}$$

In the case that $\alpha + \beta < 1$, then the term $(p_z \cdot x)(x_n)^s_+$ does not appear.

The estimate on the boundary (1.14) relies heavily on two ingredients, as we explain next. The first ingredient is a Liouville-type theorem for solutions in a half-space. Essentially, we want a Liouville theorem that states that any solution to

$$\begin{cases} \mathrm{I}u = 0 & \text{in } \{x_n > 0\}, \\ u = 0 & \text{in } \{x_n < 0\}, \end{cases}$$

satisfying the growth control at infinity

$$\|u\|_{L^\infty(B_R)} \le CR^{s+\alpha+\beta} \quad \text{for all } R \ge 1$$

must be of the form

$$u(x) = (x_n)^s_+(p \cdot x + b).$$

However, for functions having such growth at infinity (recall that $s + \alpha + \beta$ could be $2s + \gamma$), the operator $\mathrm{I}u$ is not defined. The correct form of such a Liouville theorem is the following.

THEOREM 1.4
Let $\bar{\alpha} > 0$ be the exponent given by Proposition 1.1. Assume that $u \in C(\mathbb{R}^n)$ satisfies, in the viscosity sense,

$$\begin{cases} M^+_{\mathfrak{L}_*}\{u(\cdot + h) - u\} \ge 0 \quad \text{and} \quad M^-_{\mathfrak{L}_*}\{u(\cdot + h) - u\} \le 0 & \text{in } \{x_n > 0\}, \\ u = 0 & \text{in } \{x_n < 0\}, \end{cases}$$

for all $h \in \mathbb{R}^n$ such that $h_n \ge 0$. Assume that, for some $\beta \in (0,1)$ and $\alpha \in (0,\bar{\alpha})$, u satisfies

$$\left[u/(x_n)^s_+\right]_{C^\beta(B_R)} \le CR^\alpha \quad \text{for all } R \ge 1. \tag{1.15}$$

Then,

$$u(x) = (x_n)_+^s (p \cdot x + b)$$

for some $p \in \mathbb{R}^n$ *and* $b \in \mathbb{R}$.

To prove Theorem 1.4, we apply Proposition 1.1 to incremental quotients of u in the first $(n-1)$-variables. After this, rescaling the obtained estimates and using (1.15), we find that such incremental quotients are zero, and thus that u is a 1-dimensional solution. Then, we use the fact that, for 1-dimensional functions, all operators $L \in \mathcal{L}_*$ coincide up to a multiplicative constant with the fractional Laplacian $(-\Delta)^s$ (see Lemma 2.1). Therefore, we only need to prove a Liouville theorem for solutions to $(-\Delta)^s w = 0$ in $\mathbb{R}_+$, $w = 0$ in $\mathbb{R}_-$, satisfying a growth control at infinity. This is done in Lemmas 5.2 and 5.3.

The second ingredient toward (1.14) is a compactness argument. With u as in Theorem 1.2, and with $u/(x_n)^s \in C^\beta(B_{3/4}^+)$, we suppose by contradiction that (1.14) does not hold, and we blow up the fully nonlinear equation at a boundary point (after subtracting appropriate terms to the solution). We then show that the blow-up sequence converges to an entire solution in $\{x_n > 0\}$. Finally, the contradiction is reached by applying the Liouville-type theorem stated above to the entire solution in $\{x_n > 0\}$. For this, we need to develop a boundary version of a method introduced by the second author in [44], [45]. The method was conceived there to prove interior regularity for integro-differential equations with rough kernels.

These are the main ideas used to prove Theorem 1.2.

The proof of Theorem 1.3 follows the same ideas as the one of Theorem 1.2. However, since we will not flatten the boundary of Ω (since the equation would change too much), it requires one additional ingredient.

Indeed, we need to show that $L(d^s)$ is $C^\gamma(\bar{\Omega})$ for any linear operator L of the form (1.3)–(1.4)–(1.11). This is given by Proposition 9.1. Note that in the case of flat boundary, one has $L(x_n)_+^s = 0$ for any such operator, so that there is nothing to prove. To show this in general $C^{2,\gamma}$ domains, we need to flatten the boundary.

After flattening the boundary, any operator L of the form (1.3)–(1.4)–(1.11) becomes

$$\tilde{L}(u,x) = \mathrm{PV}\int_{\mathbb{R}^n} \big(u(x) - u(x+z)\big) K(x,z)\, dz,$$

where

$$K(x,z) = \frac{a_1(x, z/|z|)}{|z|^{n+2s}} + \frac{a_2(x, z/|z|)}{|z|^{n+2s-1}} \chi_{B_1}(z) + J(x,z),$$

and where a_1 is *even* in the second variable, a_2 is *odd* in the second variable, and J has a singularity of order $n+2s-1-\gamma$ near the origin. Moreover, a_1, a_2, and J are C^γ in the x-variable.

To prove that $\tilde{L}((x_n)_+^s, x)$ is a C^γ function, we have to take advantage of an important cancelation arising from the fact that a_2 is odd in the second variable.

This article has the following organization. In Section 2, we give some important results on $\mathcal{L}_*$ and $\mathcal{L}_0$. In Section 3, we construct some subsolutions and supersolutions that will be used later. In Section 4, we prove Proposition 1.1. In Sections 5 and 6, we show the Liouville Theorem 1.4. Then we prove Theorem 1.2 in Section 7, and in Section 8 we prove Theorem 1.3. Finally, in Section 9 we prove Proposition 9.1.

2. Properties of $\mathcal{L}_*$ and $\mathcal{L}_0$

This section has two main purposes: to show that the class $\mathcal{L}_* \subset \mathcal{L}_0$ is the appropriate one to obtain fine boundary regularity results, and to give some important results on $\mathcal{L}_*$ and $\mathcal{L}_0$.

2.1. *The class $\mathcal{L}_*$*

For $s \in (0,1)$, we define the ellipticity class $\mathcal{L}_* = \mathcal{L}_*(s)$ as the set of all linear operators L of the form (1.3)–(1.4). Throughout the current article, the extremal operators (as defined in [14]) for the class $\mathcal{L}_*$ are denoted by M^+ and M^-; that is,

$$M^+u(x) = M^+_{\mathcal{L}_*}u(x) = \sup_{L\in\mathcal{L}_*} Lu(x)$$

and

$$M^-u(x) = M^-_{\mathcal{L}_*}u(x) = \inf_{L\in\mathcal{L}_*} Lu(x). \tag{2.1}$$

The following useful formula writes an operator $L \in \mathcal{L}_*$ as a weighted integral of 1-dimensional fractional Laplacians in all directions:

$$\begin{aligned} Lu &= (1-s)\int_{S^{n-1}} d\theta \frac{1}{2}\int_{-\infty}^{\infty} dr\Big(\frac{u(x+r\theta)+u(x-r\theta)}{2} - u(x)\Big)\frac{\mu(\theta)}{|r|^{n+2s}} r^{n-1} \\ &= -\frac{1-s}{2c_{1,s}}\int_{S^{n-1}} d\theta\mu(\theta)(-\partial_{\theta\theta})^s u(x), \end{aligned} \tag{2.2}$$

where

$$-(-\partial_{\theta\theta})^s u(x) = c_{1,s}\int_{-\infty}^{\infty}\Big(\frac{u(x+\theta r)+u(x-\theta r)}{2} - u(x)\Big)\frac{dr}{|r|^{1+2s}}$$

is the 1-dimensional fractional Laplacian in the direction θ, whose Fourier symbol is $-|\theta\cdot\xi|^{2s}$. The following is an immediate consequence of the formula (2.2).

LEMMA 2.1

Let u be a function depending only on variable x_n (i.e., $u(x) = w(x_n)$, where $w : \mathbb{R} \to \mathbb{R}$). Then,

$$Lu(x) = -\frac{1-s}{2c_{1,s}} \Big(\int_{S^{n-1}} |\theta_n|^{2s} \mu(\theta)\, d\theta \Big) (-\Delta)^s_{\mathbb{R}} w(x_n),$$

where $(-\Delta)^s_{\mathbb{R}}$ denotes the fractional Laplacian in dimension 1.

Proof

Using (2.2), we have

$$\begin{aligned} Lu(x) &= \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} -(-\Delta)^s_{\mathbb{R}} \big(w(x_n + \theta_n \cdot) \big) \mu(\theta)\, d\theta \\ &= \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} -|\theta_n|^{2s} (-\Delta)^s_{\mathbb{R}} \big(w(x_n + \cdot) \big) \mu(\theta)\, d\theta, \end{aligned}$$

as desired. □

Another consequence of (2.2) is that M^+ and M^- admit the following closed formulae:

$$M^+ u(x) = \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} \big\{ \Lambda \big(-(-\partial_{\theta\theta})^s w(x) \big)^+ - \lambda \big(-(-\partial_{\theta\theta})^s w(x) \big)^- \big\}\, d\theta$$

and

$$M^- u(x) = \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} \big\{ \lambda \big(-(-\partial_{\theta\theta})^s w(x) \big)^+ - \Lambda \big(-(-\partial_{\theta\theta})^s w(x) \big)^- \big\}\, d\theta.$$

Throughout this paper, given that $\nu \in S^{n-1}$ and $\beta \in (0, 2s)$, we denote by $\varphi^\beta : \mathbb{R} \to \mathbb{R}$ and $\varphi^\beta_\nu : \mathbb{R}^n \to \mathbb{R}$ the functions

$$\varphi^\beta(x) := (x_+)^\beta \qquad \text{and} \qquad \varphi^\beta_\nu(x) := (x \cdot \nu)^\beta_+. \tag{2.3}$$

A very important property of $\mathcal{L}_*$ is the following.

LEMMA 2.2

For any unit vector $\nu \in S^{n-1}$, the function φ^s_ν satisfies $M^+ \varphi^s_\nu = M^- \varphi^s_\nu = 0$ in $\{x \cdot \nu > 0\}$ and $\varphi^s_\nu = 0$ in $\{x \cdot \nu < 0\}$.

Proof

We use Lemma 2.1 and the well-known fact that the function $\varphi^s(x) = (x_+)^s$ satisfies $(-\Delta)^s_{\mathbb{R}} \varphi^s = 0$ in $\{x > 0\}$ (see, e.g., [42, Proposition 3.1]). □

Next we give a useful property of M^+ and M^-.

LEMMA 2.3
Let $\beta \in (0,2s)$, and let M^+ and M^- be defined by (2.1). For any unit vector $\nu \in S^{n-1}$, the function φ_ν^β satisfies $M^+\varphi_\nu^\beta(x) = \overline{c}(s,\beta)(x\cdot\nu)^{\beta-2s}$ and $M^-\varphi_\nu^\beta(x) = \underline{c}(s,\beta)(x\cdot\nu)^{\beta-2s}$ in $\{x\cdot\nu > 0\}$, and $\varphi_\nu^\beta = 0$ in $\{x\cdot\nu<0\}$. Here, $\overline{c}$ and $\underline{c}$ are constants depending only on s, β, n, and the ellipticity constants. Moreover, $\overline{c}$ and $\underline{c}$ satisfy $\overline{c} \ge \underline{c}$, and they are continuous as functions of the variables (s,β) in $\{0 < s \le 1, 0 < \beta < 2s\}$. In addition, we have

$$\overline{c}(s,\beta) > \underline{c}(s,\beta) > 0 \quad \textit{for all } \beta \in (s,2s) \tag{2.4}$$

and

$$\lim_{\beta\nearrow 2s} \underline{c}(s,\beta) = \begin{cases} +\infty & \textit{for all } s \in (0,1), \\ C > 0 & \textit{for } s = 1. \end{cases} \tag{2.5}$$

Proof
Given $L \in \mathcal{L}_*$, by Lemma 2.1 we have

$$L\varphi_\nu^\beta(x) = -\frac{1-s}{2c_{1,s}}\Big(\int_{S^{n-1}} |\theta_n|^{2s}\mu(\theta)\,d\theta\Big)(-\Delta)^s_{\mathbb{R}}\varphi^\beta(x\cdot\nu).$$

Hence, using the scaling properties of the fractional Laplacian and of the function φ^β, we obtain that, for $x\cdot\nu>0$,

$$M^+\varphi_\nu^\beta(x) = C(x\cdot\nu)^{\beta-2s}\max\big\{-\Lambda(-\Delta)^s_{\mathbb{R}}\varphi^\beta(1), -\lambda(-\Delta)^s_{\mathbb{R}}\varphi^\beta(1)\big\}$$

and

$$M^-\varphi_\nu^\beta(x) = C(x\cdot\nu)^{\beta-2s}\min\big\{-\Lambda(-\Delta)^s_{\mathbb{R}}\varphi^\beta(1), -\lambda(-\Delta)^s_{\mathbb{R}}\varphi^\beta(1)\big\},$$

where $C = (1-s)/(2c_{1,s}) > 0$.

Therefore, to prove that the two functions $\overline{c}$ and $\underline{c}$ are continuous in the variables (s,β) in $\{0<s\le 1, 0<\beta<2s\}$ and that (2.4) and (2.5) hold, it is enough to prove the same for

$$(s,\beta) \longmapsto -(-\Delta)^s_{\mathbb{R}}\varphi^\beta(1).$$

We first prove continuity in β. If β and β' belong to $(0,2s)$, then as $\beta'\to\beta$, we have $\varphi^{\beta'} \to \varphi^\beta$ in $C^2([1/2,3/2])$ and

$$\int_{\mathbb{R}} |\varphi^{\beta'} - \varphi^\beta|(x)\big(1+|x|\big)^{-1-2s}\,dx \to 0.$$

As a consequence, $(-\Delta)^s_{\mathbb{R}}\varphi^{\beta'}(1) \to (-\Delta)^s_{\mathbb{R}}\varphi^{\beta}(1)$. It is easy to see that if s and s' belong to $(0,1]$, and $\beta < 2s$, then $(-\Delta)^{s'}_{\mathbb{R}}\varphi^{\beta}(1) \to (-\Delta)^s_{\mathbb{R}}\varphi^{\beta}(1)$ as $s' \to s$. Moreover, note that whenever $\beta > s$, the function φ^β is touched from below by the function $\varphi^s - C$ at some point $x_0 > 0$ for some constant $C > 0$. Hence, we have $(-\Delta)^s_{\mathbb{R}}\varphi^{\beta}(x_0) > (-\Delta)^s_{\mathbb{R}}\varphi^{s}(x_0) = 0$. This yields (2.4).

Finally, (2.5) follows from an easy computation using the definition of $(-\Delta)^s_{\mathbb{R}}$, and thus the proof is finished. □

2.2. *The class $\mathcal{L}_0$*

As defined in [14], for $s \in (0,1)$ the ellipticity class $\mathcal{L}_0 = \mathcal{L}_0(s)$ consists of all operators L of the form

$$Lu(x) = (1-s)\int_{\mathbb{R}^n}\Big(\frac{u(x+y)+u(x-y)}{2} - u(x)\Big)\frac{b(y)}{|y|^{n+2s}}\,dy$$

where

$$b \in L^\infty(\mathbb{R}^n) \quad \text{satisfies } b(y) = b(-y) \qquad \text{and} \qquad \lambda \le b \le \Lambda.$$

It is clear that

$$\mathcal{L}_* \subsetneq \mathcal{L}_0.$$

The extremal operators for the class $\mathcal{L}_0$ are denoted here by $M^+_{\mathcal{L}_0}$ and $M^-_{\mathcal{L}_0}$. Since $\mathcal{L}_* \subset \mathcal{L}_0$, we have

$$M^-_{\mathcal{L}_0} \le M^- \le M^+ \le M^+_{\mathcal{L}_0}.$$

Hence, all elliptic equations with respect to $\mathcal{L}_*$ are elliptic with respect to $\mathcal{L}_0$ and all the definitions and results in [14] apply to the elliptic equations considered in these pages.

As in [14] and [16], we consider the weighted L^1 spaces $L^1(\mathbb{R}^n, \omega_s)$, where

$$\omega_s(x) = (1-s)\big(1+|x|\big)^{-n-2s}. \tag{2.6}$$

The utility of this weighted space is that, if $L \in \mathcal{L}_0(s)$, then $Lu(x)$ can be evaluated classically and is continuous in $B_{\epsilon/2}$, provided that $u \in C^2(B_\epsilon) \cap L^1(\mathbb{R}^n, \omega_s)$. One can then consider viscosity solutions to elliptic equations with respect to $\mathcal{L}_0(s)$ which are not bounded but belong to $L^1(\mathbb{R}^n, \omega_s)$. The weighted norm appears in stability results (see [16]). As noted in the Introduction, the definitions of viscosity solutions and viscosity inequalities that we follow are those from [14].

Next we state the interior Harnack inequality and the C^α estimate from [14].

THEOREM 2.4 ([14, Theorem 11.1])
Let $s_0 \in (0,1)$ and let $s \in [s_0, 1]$. Let $u \geq 0$ in $\mathbb{R}^n$ satisfy in the viscosity sense $M^-_{\mathcal{L}_0} u \leq C_0$ and $M^+_{\mathcal{L}_0} u \geq -C_0$ in B_R. Then,

$$u(x) \leq C\big(u(0) + C_0 R^{2s}\big) \quad \text{for every } x \in B_{R/2},$$

for some constant C depending only on n, s_0, and the ellipticity constants.

THEOREM 2.5 ([14, Theorem 12.1])
Let $s_0 \in (0,1)$ and $s \in [s_0, 1]$. Let $u \in C(\overline{B_1}) \cap L^1(\mathbb{R}^n, \omega_s)$ satisfy in the viscosity sense $M^-_{\mathcal{L}_0} u \leq C_0$ and $M^+_{\mathcal{L}_0} u \geq -C_0$ in B_1. Then, $u \in C^\alpha(\overline{B_{1/2}})$ with the estimate

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\big(C_0 + \|u\|_{L^\infty(B_1)} + \|u\|_{L^1(\mathbb{R}^n, \omega_s)}\big),$$

where α and C depend only on n, s, and the ellipticity constants.

2.3. No fine boundary regularity for $\mathcal{L}_0$

The aim of this section is to show that the class $\mathcal{L}_0$ is too large for all solutions to behave comparably near the boundary. Moreover, we give necessary conditions on a subclass $\mathcal{L} \subset \mathcal{L}_0$ in order to have comparability of all solutions near the boundary. These necessary conditions lead us to the class $\mathcal{L}_*$.

In the next result we show that, for any scale-invariant class $\mathcal{L} \subseteq \mathcal{L}_0$ that contains the fractional Laplacian $(-\Delta)^s$, and any unit vector ν, there exist powers $0 \leq \beta_1 \leq s \leq \beta_2$ such that $M^+_{\mathcal{L}} \varphi_\nu^{\beta_1} = 0$ and $M^-_{\mathcal{L}} \varphi_\nu^{\beta_2} = 0$ in $\{x \cdot \nu > 0\}$. Before stating this result, we give the following definition.

Definition 2.6
We say that a class of operators $\mathcal{L}$ is *scale-invariant* of order $2s$ if, for each operator L in $\mathcal{L}$ and for all $R > 0$, the rescaled operator L_R defined by

$$(L_R u)(R\cdot) = R^{-2s} L\big(u(R\cdot)\big)$$

also belongs to $\mathcal{L}$.

The proposition then is the following.

PROPOSITION 2.7
Assume that $\mathcal{L} \subset \mathcal{L}_0(s)$ is scale-invariant of order $2s$. Then the following hold.
(a) *For every $\nu \in S^{n-1}$ and $\beta \in (0, 2s)$, the function φ_ν^β defined in (2.3) satisfies*

$$\begin{aligned} M^+_{\mathcal{L}} \varphi_\nu^\beta(x) &= \overline{C}(\beta, \nu)(x \cdot \nu)^{\beta - 2s} \quad \text{in } \{x \cdot \nu > 0\}, \\ M^-_{\mathcal{L}} \varphi_\nu^\beta(x) &= \underline{C}(\beta, \nu)(x \cdot \nu)^{\beta - 2s} \quad \text{in } \{x \cdot \nu > 0\}. \end{aligned} \tag{2.7}$$

Here, $\overline{C}$ and $\underline{C}$ are constants depending only on s, β, ν, n, and the ellipticity constants.

(b) *The functions $\overline{C}$ and $\underline{C}$ are continuous in β and, for each unit vector ν, there are $\beta_1 \le \beta_2$ in $(0, 2s)$ such that*

$$\overline{C}(\beta_1, \nu) = 0 \qquad \text{and} \qquad \underline{C}(\beta_2, \nu) = 0. \tag{2.8}$$

Moreover, for all $\beta \in (0, 2s)$,

$$\overline{C}(\beta, \nu) - \overline{C}(\beta_1, \nu) \quad \text{has the same sign as } \beta - \beta_1 \tag{2.9}$$

and

$$\underline{C}(\beta, \nu) - \underline{C}(\beta_2, \nu) \quad \text{has the same sign as } \beta - \beta_2. \tag{2.10}$$

(c) *If in addition the fractional Laplacian $-(-\Delta)^s$ belongs to $\mathfrak{L}$, then we have $\beta_1 \le s \le \beta_2$.*

Proof

The scale invariance of $\mathfrak{L}$ is equivalent to a scaling property of the extremal operators $M^+_{\mathfrak{L}}$ and $M^-_{\mathfrak{L}}$. Namely, for all $R > 0$, we have

$$M^{\pm}_{\mathfrak{L}}\big(u(R\cdot)\big) = R^{2s}(M^{\pm}_{\mathfrak{L}} u)(R\cdot).$$

(a) By this scaling property, it is immediate to prove that given $\beta \in (0, 2s)$ and $\nu \in S^{n-1}$, the function φ^β_ν satisfies (2.7), where

$$\overline{C}(\beta, \nu) := M^+_{\mathfrak{L}} \varphi^\beta_\nu(\nu) \qquad \text{and} \qquad \underline{C}(\beta, \nu) := M^-_{\mathfrak{L}} \varphi^\beta_\nu(\nu).$$

Of course, $\overline{C}$ and $\underline{C}$ depend also on s and the ellipticity constants, but these are fixed constants in this proof.

(b) Note that, as $\beta' \to \beta \in [0, 2s)$, we have $\varphi^{\beta'}_\nu \to \varphi^\beta_\nu$ in $C^2(\overline{B_{1/2}(\nu)})$ and in $L^1(\mathbb{R}^n, \omega_s)$. As a consequence, $\underline{C}$ and $\overline{C}$ are continuous in β in the interval $[0, 2s)$. Since $\varphi^\beta_\nu \to \chi_{\{x\cdot\nu>0\}}$ as $\beta \to 0$, we have

$$\underline{C}(\nu, 0) \le \overline{C}(\nu, 0) < 0.$$

On the other hand, it is easy to see that

$$M^-_{\mathfrak{L}_0} \varphi^\beta_\nu(\nu) \longrightarrow +\infty \quad \text{as } \beta \nearrow 2s.$$

Hence, using that $M^-_{\mathfrak{L}_0} \le M^-_{\mathfrak{L}}$, we obtain

$$0 < \underline{C}(\nu, \beta) \le \overline{C}(\nu, \beta) \quad \text{for } \beta \text{ close to } 2s.$$

Therefore, by continuity, there are β_1 and β_2 in $(0,2s)$ such that

$$\overline{C}(\beta_1,\nu)=0 \qquad \text{and} \qquad \underline{C}(\beta_2,\nu)=0.$$

To prove (2.9), we observe that if $\beta>\beta_1$, then the function φ_ν^β is touched from below by $\varphi_\nu^{\beta_1}-C$ at some $x_0\in\{x\cdot\nu>0\}$ for some $C>0$. It follows that

$$M^+_{\mathcal{L}}\varphi_\nu^\beta(x_0)-M^+_{\mathcal{L}}\varphi_\nu^{\beta_1}(x_0)\ge M^-_{\mathcal{L}_0}(\varphi_\nu^\beta-\varphi_\nu^{\beta_1})(x_0)>0.$$

Since the sign of $M^+_{\mathcal{L}}\varphi_\nu^\beta$ is constant in $\{x\cdot\nu>0\}$, it follows that $\overline{C}(\nu,\beta)>0$ when $\beta>\beta_1$. Similarly, one proves that $\overline{C}(\nu,\beta)<0$ when $\beta<\beta_1$, and hence (2.10).

(c) This proof is an immediate consequence of the results of parts (a) and (b) and the fact that $-(-\Delta)^s\varphi_\nu^s=0$ in $\{x\cdot\nu>0\}$. □

Clearly, to hope for some good description of the boundary behavior of solutions to all elliptic equations with respect to a scale-invariant class $\mathcal{L}$, it must be $\beta_1=\beta_2$ for every direction ν. Typical classes $\mathcal{L}$ contain the fractional Laplacian $-(-\Delta)^s$. Thus, for them, we must have $\beta_1=\beta_2=s$ for all $\nu\in S^{n-1}$. If this happens, then

$$L\varphi_\nu^s=0 \quad \text{in } \{x\cdot\nu>0\} \text{ for all } L\in\mathcal{L}, \text{ and for all } \nu\in S^{n-1}, \tag{2.11}$$

since $M^-_{\mathcal{L}}\le L\le M^+_{\mathcal{L}}$ for all $L\in\mathcal{L}$.

As a consequence, we have the following.

COROLLARY 2.8

Let β_1, β_2 be given by (2.8) in Proposition 2.7. Then, for the classes $\mathcal{L}_0$, $\mathcal{L}_1$, and $\mathcal{L}_2$ we have $\beta_1<s<\beta_2$.

Proof

Let us show that for $\mathcal{L}=\mathcal{L}_0$ the condition (2.11) is not satisfied. Indeed, we may easily cook up $L\in\mathcal{L}_0$ so that $L\varphi_{e_n}^s(x',1)\ne 0$ for $x'\in\mathbb{R}^{n-1}$. Namely, if we take

$$b(y)=\big(\lambda+(\Lambda-\lambda)\chi_{B_{1/2}}(y)\big),$$

then at points $x=(x',1)$ we have

$$0>L\varphi_{e_n}^s(x)=(1-s)\int_{\mathbb{R}^n}\Big(\frac{u(x+y)+u(x-y)}{2}-u(x)\Big)\frac{b(y)}{|y|^{n+2s}}\,dy,$$

since $\varphi_{e_n}^s$ is concave in $B_{1/2}(x',1)$ and $(-\Delta)^s\varphi_{e_n}^s=0$ in $\{x_n>0\}$. By taking an smoothed version of $b(y)$, we see that both $\mathcal{L}_1$ and $\mathcal{L}_2$ fail to satisfy (2.11). □

By the results in Section 2.1, it holds that the class $\mathcal{L}_*$ satisfies the necessary condition (2.11). Although we do not have a rigorous mathematical proof, we believe that $\mathcal{L}_*$ is actually the largest scale-invariant subclass of $\mathcal{L}_0$ satisfying (2.11).

3. Barriers

In this section, we construct supersolutions and subsolutions that are needed in our analysis. From now on, all the results are for the class $\mathcal{L}_*$ (and not for $\mathcal{L}_0$). First we give two preliminary lemmas.

LEMMA 3.1
Let $s_0 \in (0,1)$ *and let* $s \in [s_0, 1)$. *Let*

$$\varphi^{(1)}(x) = \big(\mathrm{dist}(x, B_1)\big)^s \qquad \textit{and} \qquad \varphi^{(2)}(x) = \big(\mathrm{dist}(x, \mathbb{R}^n \setminus B_1)\big)^s.$$

Then,

$$\begin{aligned} 0 &\le M^-\varphi^{(1)}(x) \le M^+\varphi^{(1)}(x) \\ &\le C\big\{1 + (1-s)\big|\log\big(|x|-1\big)\big|\big\} \quad \textit{in } B_2 \setminus B_1 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} 0 &\ge M^+\varphi^{(2)}(x) \ge M^-\varphi^{(2)}(x) \\ &\ge -C\big\{1 + (1-s)\big|\log\big(1-|x|\big)\big|\big\} \quad \textit{in } B_1 \setminus B_{1/2}. \end{aligned} \tag{3.2}$$

The constant C *depends only on* s_0, n, *and the ellipticity constants.*

Note that the above bounds are much better than $||x|-1|^{-s}$, which would be the expected bound given by homogeneity. This is since $\varphi^{(1)}$ and $\varphi^{(2)}$ are in some sense close to the 1-dimensional solution $(x_+)^s$.

Proof of Lemma 3.1
Let $L \in \mathcal{L}_*$. For points $x \in \mathbb{R}^n$, we use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. To prove (3.1), let us estimate $L\varphi^{(1)}(x_\rho)$, where $x_\rho = (0, 1+\rho)$ for $\rho \in (0,1)$ and for a generic $L \in \mathcal{L}_*$. To do it, we subtract the function $\psi(x) = (x_n - 1)_+^s$, which satisfies $L\psi(x_\rho) = 0$. Note that

$$(\varphi^{(1)} - \psi)(x_\rho) = 0 \quad \text{for all } \rho > 0$$

and that, for $|y| < 1$,

$$\big|\mathrm{dist}(x_\rho + y, B_1) - (1 + \rho + y_n)_+\big| \le C|y'|^2.$$

This is because the level sets of the two previous functions are tangent on $\{y' = 0\}$.

Thus,

$$0 \le (\varphi_1^{(1)} - \psi)(x_\rho + y) \le \begin{cases} C\rho^{s-1}|y'|^2 & \text{for } y = (y', y_n) \in B_{\rho/2}, \\ C|y'|^{2s} & \text{for } y = (y', y_n) \in B_1 \setminus B_{\rho/2}, \\ C|y|^s & \text{for } y \in \mathbb{R}^n \setminus B_1. \end{cases}$$

The bound in $B_{\rho/2}$ follows from the inequality $a^s - b^s \le (a-b)b^{s-1}$ for $a > b > 0$.

Therefore, we have

$$\begin{aligned} 0 &\le L\varphi^{(1)}(x_\rho) \\ &= L(\varphi^{(1)} - \psi)(x_\rho) \\ &= (1-s)\int \frac{(\varphi_1^{(1)} - \psi)(x_\rho + y) + (\varphi_1^{(1)} - \psi)(x_\rho - y)}{2} \frac{\mu(y/|y|)}{|y|^{n+2s}}\, dy \\ &\le C(1-s)\Lambda\Big(\int_{B_{\rho/2}} \frac{\rho^{s-1}|y'|^2\, dy}{|y|^{n+2s}} + \int_{B_1 \setminus B_{\rho/2}} \frac{|y'|^{2s}\, dy}{|y|^{n+2s}} + \int_{\mathbb{R}^n \setminus B_1} \frac{|y|^s\, dy}{|y|^{n+2s}}\Big) \\ &\le C\big(1 + (1-s)|\log \rho|\big). \end{aligned}$$

This establishes (3.1). The proof of (3.2) is similar. □

In the next result, instead, the bounds are those given by the homogeneity. In addition, the constant in the bounds has the right sign to construct (together with Lemma 3.1) appropriate barriers.

LEMMA 3.2

Let $s_0 \in (0,1)$ and let $s \in [s_0, 1)$. Let

$$\varphi^{(3)}(x) = \big(\mathrm{dist}(x, B_1)\big)^{3s/2} \qquad \text{and} \qquad \varphi^{(4)}(x) = \big(\mathrm{dist}(x, \mathbb{R}^n \setminus B_1)\big)^{3s/2}.$$

Then,

$$M^-\varphi^{(3)}(x) \ge c\big(|x| - 1\big)^{-s/2} \quad \text{for all } x \in B_2 \setminus B_1 \tag{3.3}$$

and

$$M^-\varphi^{(4)}(x) \ge c\big(1 - |x|\big)^{-s/2} - C \quad \text{for all } x \in B_1 \setminus B_{1/2}. \tag{3.4}$$

The constants $c > 0$ and C depend only on n, s_0, and the ellipticity constants.

Proof

Let $L \in \mathcal{L}_*$. For points $x \in \mathbb{R}^n$, we use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. To prove (3.4) let us estimate $L\varphi^{(4)}(x_\rho)$, where $x_\rho = (0, 1+\rho)$ for $\rho \in (0,1)$ and for a generic $L \in \mathcal{L}_*$. To do it, we subtract the function $\psi(x) = (1 - x_n)_+^{3s/2}$, which by Lemma 2.3 satisfies $L\psi(x_\rho) = c\rho^{-s/2}$ for some $c > 0$. We note that

$$(\varphi^{(4)} - \psi)(x_\rho) = 0$$

and, similarly as in the proof of Lemma 3.1,

$$0 \geq (\varphi^{(4)} - \psi)(x_\rho + y) \geq \begin{cases} -C\rho^{3s/2-1}|y'|^2 & \text{for } y = (y', y_n) \in B_{\rho/2}, \\ -C|y'|^{3s} & \text{for } y = (y', y_n) \in B_1 \setminus B_{\rho/2}, \\ -C|y|^{3s/2} & \text{for } y \in \mathbb{R}^n \setminus B_1. \end{cases}$$

Hence,

$$\begin{aligned} &L\varphi^{(4)}(x_\rho) - c\rho^{-s/2} \\ &\quad = L(\varphi^{(4)} - \psi)(x_\rho) \\ &\quad \geq -C(1-s)\Lambda\Big(\int_{B_{\rho/2}} \frac{\rho^{3s/2-1}|y'|^2\,dy}{|y|^{n+2s}} + \int_{B_1\setminus B_{\rho/2}} \frac{|y'|^{3s}\,dy}{|y|^{n+2s}} + \int_{\mathbb{R}^n\setminus B_1} \frac{|y|^{s/2}\,dy}{|y|^{n+2s}}\Big) \\ &\quad \geq -C. \end{aligned}$$

This establishes (3.4). To prove (3.3), we now define $\psi(x) = (x_n - 1)_+^{3s/2}$, and we use Lemma 2.3 and the fact that $\varphi^{(3)} - \psi$ is nonnegative in all of $\mathbb{R}^n$ and vanishes on the positive x_n axis. □

We can now construct the subsolutions and supersolutions that will be used in the next section.

LEMMA 3.3
Let $s_0 \in (0,1)$ and let $s \in [s_0, 1)$. There are positive constants ϵ and C, and a radial, bounded, continuous function φ_1 which is $C^{1,1}$ in $B_{1+\epsilon} \setminus \overline{B}_1$ and which satisfies

$$\begin{cases} M^+\varphi_1(x) \leq -1 & \text{in } B_{1+\epsilon} \setminus \overline{B_1}, \\ \varphi_1(x) = 0 & \text{in } B_1, \\ \varphi_1(x) \leq C(|x|-1)^s & \text{in } \mathbb{R}^n \setminus B_1, \\ \varphi_1(x) \geq 1 & \text{in } \mathbb{R}^n \setminus B_{1+\epsilon}. \end{cases}$$

The constants ϵ, c, and C depend only on n, s_0, and the ellipticity constants.

Proof
Let

$$\psi = \begin{cases} 2\varphi^{(1)} - \varphi^{(3)} & \text{in } B_2, \\ 1 & \text{in } \mathbb{R}^n \setminus B_2. \end{cases}$$

By Lemmas 3.1 and 3.2, for $|x| > 1$ it is

$$M^+\psi \leq C\big\{1 + (1-s)\big|\log\big(|x|-1\big)\big|\big\} - c\big(|x|-1\big)^{-s/2} + C.$$

Hence, we may take $\epsilon > 0$ small enough so that $M^+\psi \leq -1$ in $B_{1+\epsilon} \setminus \overline{B_1}$. We then set $\varphi_1 = C\psi$ with $C \geq 1$ large enough so that $\varphi_1 \geq 1$ outside $B_{1+\epsilon}$. □

LEMMA 3.4
Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. There is $c > 0$, and a radial, bounded, continuous function φ_2 that satisfies

$$\begin{cases} M^-\varphi_2(x) \geq c & \text{in } B_1 \setminus B_{1/2}, \\ \varphi_2(x) = 0 & \text{in } \mathbb{R}^n \setminus B_1, \\ \varphi_2(x) \geq c(1-|x|)^s & \text{in } B_1, \\ \varphi_2(x) \leq 1 & \text{in } \overline{B_{1/2}}. \end{cases}$$

The constants ϵ, c, and C depend only on n, s_0, and the ellipticity constants.

Proof
We first construct a subsolution ψ in the annulus $B_1 \setminus \overline{B_{1-\epsilon}}$ for some small $\epsilon > 0$. Then, using it, we will construct the desired subsolution in $B_1 \setminus B_{1/2}$. Let

$$\psi = \varphi^{(2)} + \varphi^{(4)}.$$

By Lemmas 3.1 and 3.2, for $1/2 < |x| < 1$ we have

$$M^-\psi \geq -C\{1 + (1-s)\big|\log(1-|x|)\big|\} + c\big(1-|x|\big)^{-s/2} - C.$$

Hence, we can take $\epsilon > 0$ small enough so that $M^-\psi \geq 1$ in $B_1 \setminus \overline{B_{1-\epsilon}}$.

Let us now construct a subsolution in $B_1 \setminus \overline{B_{1/2}}$ from ψ, which is a subsolution only in $B_1 \setminus \overline{B_{1-\epsilon}}$. We consider

$$\Psi(x) = \max_{0 \leq k \leq N} C^k \psi(2^{k/N} x),$$

where N is a large integer and $C > 1$. Notice that, for C large enough, the set $\{x \in B_1 : \Psi(x) = \psi(x)\}$ is an annulus contained in $B_1 \setminus \overline{B_{1-\epsilon}}$.

Consider, for $k \geq 0$,

$$A_k = \big\{x \in B_1 : \Psi(x) = C^k \psi(2^{k/N} x)\big\}.$$

Since $A_0 \subset B_1 \setminus \overline{B_{1-\epsilon}}$, then Ψ satisfies $M^-\Psi \geq 1$ in A_0.

Observe that $A_k = 2^{-k/N} A_0$, since $C^{-1}\Psi(2^{1/n}x) = \Psi(x)$ in the annulus $\{1/2 < |x| < 2^{-1/n}\}$. Hence, for $x \in A_k$, we have $2^{k/N}x \in A_0 \subset B_1 \setminus \overline{B_{1-\epsilon}}$ and

$$M^-\Psi(x) > M^-\big(C^k\psi(2^{k/N}\cdot)\big)(x) = C^k 2^{2sk/N} M^-\psi(2^{k/N}x) > 1.$$

We then set $\varphi_2 = c\Psi$ with $c > 0$ small enough so that $\varphi_2(x) \leq 1$ in $\overline{B_{1/2}}$. □

Remark 3.5

Notice that the subsolution φ_2 constructed above is $C^{1,1}$ from below in $B_1 \setminus \overline{B}_{1/2}$, in the sense that it can be touched from below by paraboloids. This is important when considering nontranslation-invariant equations for which a comparison principle for viscosity solutions is not available.

4. The Caffarelli–Krylov method

The goal of this section is to prove Proposition 1.1. Its proof combines the interior Hölder regularity results of Caffarelli and Silvestre [14] and the next key lemma.

LEMMA 4.1

Let $s_0 \in (0,1)$, let $s \in [s_0,1)$, and let $u \in C(\overline{B_1^+})$ be a viscosity solution of (1.6). Then there exist $\alpha \in (0,1)$ and C depending only on n, s_0, and the ellipticity constants such that

$$\sup_{B_r^+} u/x_n^s - \inf_{B_r^+} u/x_n^s \le C r^\alpha \big(C_0 + \|u\|_{L^\infty(\mathbb{R}^n)}\big) \tag{4.1}$$

for all $r \le 3/4$.

To prove Lemma 4.1, we need two preliminary lemmas. We start with the first, which is a nonlocal version of Lemma 4.31 in [26]. Throughout this section, we denote

$$D_r^* := B_{9r/10} \cap \{x_n > 1/10\}.$$

LEMMA 4.2

Let $s_0 \in (0,1)$ and let $s \in [s_0,1)$. Assume that u satisfies $u \ge 0$ in all of $\mathbb{R}^n$ and that

$$M^- u \le C_0 \quad \text{in } B_r^+,$$

for some $C_0 > 0$. Then,

$$\inf_{D_r^*} u/x_n^s \le C\Big(\inf_{B_{r/2}^+} u/x_n^s + C_0 r^s\Big) \tag{4.2}$$

for all $r \le 1$, where C is a constant depending only on s_0, the ellipticity constants, and dimension.

Proof

The proof consists of two steps.

Step 1. Assume that $C_0 = 0$. Let us call

$$m = \inf_{D_r^*} u/x_n^s \geq 0.$$

We have

$$u \geq m x_n^s \geq m(r/10)^s \quad \text{in } D_r^*. \tag{4.3}$$

Let us scale and translate the subsolution φ_2 in Lemma 3.4 as follows to use it as a lower barrier:

$$\psi_r(x) := (r/10)^s \varphi_2\Big(\frac{10(x-x_0)}{2r}\Big). \tag{4.4}$$

We then have, for some $c > 0$,

$$\begin{cases} M^-\psi_r \geq 0 & \text{in } B_{2r/10}(x_0) \setminus B_{r/10}(x_0), \\ \psi_r = 0 & \text{in } \mathbb{R}^n \setminus B_{2r/10}(x_0), \\ \psi_r \geq c(\frac{2r}{10} - |x|)^s & \text{in } B_{2/10}(x_0), \\ \psi_r \leq (r/10)^s & \text{in } B_{r/10}(x_0). \end{cases}$$

It is immediate to verify that $B_{r/2}^+$ is covered by balls of radius $2r/10$ such that the concentric ball of radius $r/10$ is contained in D_r^*; that is,

$$B_{r/2}^+ \subset \bigcup \big\{ B_{2r/10}(x_0) : B_{r/10}(x_0) \subset D_r^* \big\}.$$

Now, if we choose some ball $B_{r/10}(x_0) \subset D_r^*$ and define ψ_r by (4.4), then by (4.3) we have $u \geq m\psi_r$ in $B_{r/10}(x_0)$. On the other hand, $u \geq m\psi_r$ outside $B_{2r/10}(x_0)$, since ψ_r vanishes there and $u \geq 0$ in all of $\mathbb{R}^n$ by assumption. Finally, $M^+\psi_r \leq 0$, and since $C_0 = 0$, then $M^- u \geq 0$ in the annulus $B_{2r/10}(x_0) \setminus B_{r/10}(x_0)$. It therefore follows from the comparison principle that $u \geq m\psi_r$ in $B_{2r/10}(x_0)$. Since these balls of radius $2r/10$ cover $B_{r/2}^+$ and $\psi_r \geq c(2r/10 - |x|)^s$ in $B_{2/10}(x_0)$, we obtain

$$u \geq c m x_n^s \quad \text{in } B_{r/2}^+,$$

which yields (4.2).

Step 2. If $C_0 > 0$, then we argue as follows. First, let

$$\phi(x) = \min\big\{1, 2(x_n)_+^s - (x_n)_+^{3s/2}\big\}.$$

By Lemma 2.3, we have $M^+\phi \leq -c$ in $\{0 < x_n < \epsilon\}$ for some $\epsilon > 0$ and some $c > 0$. By scaling ϕ and reducing c, we may assume that $\epsilon = 1$. We then consider

$$\tilde{u}(x) = u(x) + \frac{C_0}{c} r^{2s} \phi(x/r).$$

The function $\tilde{u}$ satisfies in $\{0 < x_n < r\}$

$$M^-\tilde{u} - M^-u \le M^+\Big(\frac{C_0}{c} r^{2s}\phi(x/r)\Big) \le -C_0,$$

and hence

$$M^-\tilde{u} \le 0.$$

Using the fact that $u(x) \le \tilde{u}(x) \le u(x) + CC_0 r^s (x_n)_+^s$ and applying Step 1 to $\tilde{u}$, we obtain (4.2). □

The second lemma in the direction of Proposition 4.1 is a nonlocal version of Lemma 4.35 in [26]. It is an immediate consequence of the Harnack inequality of Caffarelli and Silvestre in [14].

LEMMA 4.3
Let $s_0 \in (0,1)$, let $s \in [s_0, 1)$, let $r \le 1$, and let u satisfy $u \ge 0$ in all of $\mathbb{R}^n$ and

$$M^+u \ge -C_0 \qquad \text{and} \qquad M^-u \le C_0 \quad \text{in } B_r^+.$$

Then,

$$\sup_{D_r^*} u/x_n^s \le C\big(\inf_{D_r^*} u/x_n^s + C_0 r^s\big),$$

for some constant C depending only on n, s_0, and the ellipticity constants.

Proof
The lemma is a consequence of Theorem 2.4. Indeed, covering the set D_r^* with balls contained in B_r^+ and with radii comparable to r (using the same (scaled) covering for all r), Theorem 2.4 yields

$$\sup_{D_r^*} u \le C\big(\inf_{D_r^*} u + C_0 r^{2s}\big).$$

Then, the lemma follows by noting that x_n^s is comparable to r^s in D_r^*. □

Next we prove Lemma 4.1.

Proof of Lemma 4.1
First, dividing u by a constant, we may assume that $C_0 + \|u\|_{L^\infty(\mathbb{R}^n)} \le 1$.

We will prove that there exist constants $C_1 > 0$ and $\alpha \in (0,s)$, depending only on n, s_0, and the ellipticity constants, and monotone sequences $(m_k)_{k\ge1}$ and $(\overline{m}_k)_{k\ge1}$ satisfying the following. For all $k \ge 1$,

$$\overline{m}_k - m_k = 4^{-\alpha k}, \quad -1 \le m_k \le m_{k+1} < \overline{m}_{k+1} \le \overline{m}_k \le 1, \tag{4.5}$$

and

$$m_k \le C_1^{-1} u / x_n^s \le \overline{m}_k \quad \text{in } B_{r_k}^+, \text{ where } r_k = 4^{-k}. \tag{4.6}$$

Note that since $u = 0$ in B_1^-, we have that (4.6) is equivalent to the following inequality in B_{r_k} instead of $B_{r_k}^+$:

$$m_k (x_n)_+^s \le C_1^{-1} u \le \overline{m}_k (x_n)_+^s \quad \text{in } B_{r_k}, \tag{4.7}$$

where $r_k = 4^{-k}$. Clearly, if such sequences exist, then (4.1) holds for all $r \le 1/4$ with $C = 4^\alpha C_1$. Moreover, for $1/4 < r \le 3/4$, the result follows from (4.8) below. Hence, we only need to construct $\{m_k\}$ and $\{\overline{m}_k\}$.

Next we construct these sequences by induction. Using the supersolution φ_1 in Lemma 3.3, we find that

$$-\frac{C_1}{2}(x_n)_+^s \le u \le \frac{C_1}{2}(x_n)_+^s \quad \text{in } B_{3/4}^+ \tag{4.8}$$

whenever C_1 is large enough. Thus, we may take $m_1 = -1/2$ and $\overline{m}_1 = 1/2$. Assume now that we have sequences up to m_k and $\overline{m}_k$. We want to prove that there exist m_{k+1} and $\overline{m}_{k+1}$ which fulfill the requirements. Let

$$u_k = C_1^{-1} u - m_k (x_n)_+^s.$$

We will consider the positive part u_k^+ of u_k in order to have a nonnegative function in all of $\mathbb{R}^n$ to which we can apply Lemmas 4.2 and 4.3. Let $u_k = u_k^+ - u_k^-$. Observe that, by induction hypothesis,

$$u_k^+ = u_k \qquad \text{and} \qquad u_k^- = 0 \quad \text{in } B_{r_k}.$$

Moreover, $C_1^{-1} u \ge m_j (x_n)_+^s$ in B_{r_j} for each $j \le k$. Therefore, we have

$$\begin{aligned} u_k &\ge (m_j - m_k)(x_n)_+^s \ge (m_j - \overline{m}_j + \overline{m}_k - m_k)(x_n)_+^s \\ &= (-4^{-\alpha j} + 4^{-\alpha k})(x_n)_+^s \quad \text{in } B_{r_j}. \end{aligned}$$

But clearly $0 \le (x_n)_+^s \le r_j^s$ in B_{r_j}, and therefore by using $r_j = 4^{-j}$, we have

$$u_k \ge -r_j^s (r_j^\alpha - r_k^\alpha) \quad \text{in } B_{r_j} \text{ for each } j \le k.$$

Thus, since for every $x \in B_1 \setminus B_{r_k}$ there is $j < k$ such that

$$|x| < r_j = 4^{-j} \le 4|x|,$$

we find

$$u_k(x) \geq -r_k^{\alpha+s}\left|\frac{4x}{r_k}\right|^s\Big(\left|\frac{4x}{r_k}\right|^\alpha - 1\Big) \quad \text{outside } B_{r_k}. \tag{4.9}$$

Now let $L \in \mathcal{L}_*$. Using (4.9) and the fact that $u_k^- \equiv 0$ in B_{r_k}, for all $x \in B_{r_k/2}$ we have

$$\begin{aligned} 0 &\leq L u_k^-(x) \\ &= (1-s)\int_{x+y\notin B_{r_k}} u_k^-(x+y)\frac{\mu(y/|y|)}{|y|^{n+2s}}\,dy \\ &\leq (1-s)\int_{|y|\geq r_k/2} r_k^{\alpha+s}\left|\frac{8y}{r_k}\right|^s\Big(\left|\frac{8y}{r_k}\right|^\alpha - 1\Big)\frac{\Lambda}{|y|^{n+2s}}\,dy \\ &= (1-s)\Lambda r_k^{\alpha-s}\int_{|z|\geq 1/2}\frac{|8z|^s(|8z|^\alpha-1)}{|z|^{n+2s}}\,dz \\ &\leq \varepsilon_0 r_k^{\alpha-s}, \end{aligned}$$

where $\varepsilon_0 = \varepsilon_0(\alpha) \downarrow 0$ as $\alpha \downarrow 0$ since $|8z|^\alpha \to 1$. Since this can be done for all $L \in \mathcal{L}_*$, then u_k^- vanishes in B_{r_k} and satisfies pointwise $0 \leq M^- u_k^- \leq M^+ u_m^- \leq \varepsilon_0 r_k^{\alpha-s}$in $B^+_{r_k/2}$. Therefore, recalling that

$$u_k^+ = C_1^{-1}u - m_k(x_n)_+^s + u_k^-,$$

and using the fact that $M^+(x_n)_+^s = M^-(x_n)_+^s = 0$ in $\{x_n > 0\}$, we obtain

$$\begin{aligned} M^- u_k^+ &\leq C_1^{-1}M^- u + M^+(u_k^-) \\ &\leq C_1^{-1} + \varepsilon_0 r_k^{\alpha-s} \quad \text{in } B^+_{r_k/2}. \end{aligned}$$

Also clearly,

$$M^+ u_k^+ \geq M^+ u_k \geq -C_1^{-1} \quad \text{in } B^+_{r_k/2}.$$

Now we can apply Lemmas 4.2 and 4.3 with u in its statements replaced by u_k^+. Recalling that

$$u_k^+ = u_k = C_1^{-1}u - m_k x_n^s \quad \text{in } B^+_{r_k},$$

we obtain

$$\begin{aligned} \sup_{D^*_{r_k/2}} (C_1^{-1}u/x_n^s - m_k) &\leq C\Big(\inf_{D^*_{r_k/2}} (C_1^{-1}u/x_n^s - m_k) + C_1^{-1}r_k^s + \varepsilon_0 r_k^\alpha\Big) \\ &\leq C\Big(\inf_{B^+_{r_k/4}} (C_1^{-1}u/x_n^s - m_k) + C_1^{-1}r_k^s + \varepsilon_0 r_k^\alpha\Big). \end{aligned} \tag{4.10}$$

On the other hand, we can repeat the same reasoning "upside down"—that is, considering the functions $\overline{u}_k = \overline{m}_k (x_n)_+^s - u$ instead of u_k. In this way we obtain, instead of (4.10), the following:

$$\sup_{D^*_{r_k/2}} (\overline{m}_k - C_1^{-1} u/x_n^s) \le C\Big(\inf_{B^+_{r_k/4}} (\overline{m}^k - C_1^{-1} u/x_n^s) + C_1^{-1} r_k^s + \varepsilon_0 r_k^\alpha \Big). \quad (4.11)$$

Adding (4.10) and (4.11), we obtain

$$\begin{aligned}\overline{m}_k - m_k &\le C\Big(\inf_{B^+_{r_k/4}} (C_1^{-1} u/x_n^s - m_k) + \inf_{B^+_{r_k/4}} (\overline{m}_k - C_1^{-1} u/x_n^s) + C_1^{-1} r_k^s + \varepsilon_0 r_k^\alpha \Big) \\ &= C\Big(\inf_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s - \sup_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s + \overline{m}_k - m_k + C_1^{-1} r_k^s + \varepsilon_0 r_k^\alpha \Big).\end{aligned}$$

Thus, using $\overline{m}_k - m_k = 4^{-\alpha k}$, $\alpha < s$, and $r_k = 4^{-k} \le 1$, we obtain

$$\sup_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s - \inf_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s \le \Big(\frac{C-1}{C} + C_1^{-1} + \varepsilon_0 \Big) 4^{-\alpha k}.$$

Now we choose α small and C_1 large enough so that

$$\frac{C-1}{C} + C_1^{-1} + \varepsilon_0(\alpha) \le 4^{-\alpha}.$$

This is possible since $\varepsilon_0(\alpha) \downarrow 0$ as $\alpha \downarrow 0$ and the constant C depends only on n, s_0, and the ellipticity constants. Then we find

$$\sup_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s - \inf_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s \le 4^{-\alpha(k+1)},$$

and thus we are able to choose m_{k+1} and $\overline{m}_{k+1}$ satisfying (4.5) and (4.6). □

To end this section, we give the following proof.

Proof of Proposition 1.1
Let $x \in B^+_{1/2}$, and let x_0 be its nearest point on $\{x_n = 0\}$. Let

$$d = \mathrm{dist}(x, x_0) = x_n = \mathrm{dist}(x, B_1^-).$$

By Theorem 2.5 (rescaled), we have

$$\|u\|_{C^\alpha(B_{d/2}(x))} \le C d^{-\alpha} \big(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0 \big).$$

Hence, since $\|(x_n)^{-s}\|_{C^\alpha(B_{d/2}(x))} \le C d^{-s}$, then for $r \le d/2$

$$\operatorname{osc}_{B_r(x)} u/x_n^s \le C r^\alpha d^{-s-\alpha}\big(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\big). \tag{4.12}$$

On the other hand, by Lemma 4.1, for all $r \ge d/2$, we have

$$\operatorname{osc}_{B_r(x)\cap B^+_{3/4}} u/x_n^s \le C r^\alpha\big(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\big). \tag{4.13}$$

In both previous estimates, $\alpha \in (0,1)$ depends only on n, s_0, and the ellipticity constants. Let us call

$$M = \big(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\big).$$

Then, given $\theta > 1$, we have the following alternatives.

(i) If $r \le d^\theta/2$, then by (4.12),

$$\operatorname{osc}_{B_r(x)} u/x_n^s \le C r^\alpha d^{-s-\alpha} M \le C r^{\alpha-(s+\alpha)/\theta} M.$$

(ii) If $d^\theta/2 < r \le d/2$, then by (4.13),

$$\operatorname{osc}_{B_r(x)} u/x_n^s \le \operatorname{osc}_{B_{d/2}(x)} u/x_n^s \le C d^\alpha M \le C r^{\alpha/\theta} M.$$

(iii) If $d/2 < r$, then by (4.13),

$$\operatorname{osc}_{B_r(x)\cap B^+_{3/4}} u/x_n^s \le C r^\alpha M.$$

Choosing $\theta > (s+\alpha)/\alpha$ (so that the exponent in (i) is positive), we obtain

$$\operatorname{osc}_{B_r(x)\cap B^+_{3/4}} u/x_n^s \le C r^{\alpha'} M \quad \text{whenever } x \in B^+_{1/2} \text{ and } r > 0, \tag{4.14}$$

for some $\alpha' \in (0,\alpha)$. This means that $\|u/x_n^s\|_{C^{\alpha'}(B^+_{1/2})} \le CM$, as desired. □

5. Liouville theorems in $\mathbb{R}^n_+$

The goal of this section is to prove Theorem 1.4.

First, as a consequence of Proposition 1.1, we show the following Liouville-type result involving the extremal operators. Note that the growth condition CR^β in this lemma holds for $\beta < s + \bar\alpha$ (with $\bar\alpha$ small), in contrast with the Liouville Theorem 1.4.

PROPOSITION 5.1

Let $s_0 \in (0,1)$ and let $s \in [s_0, 1)$. Let $\bar\alpha > 0$ be the exponent given by Proposition 1.1. Assume that $u \in C(\mathbb{R}^n)$ is a viscosity solution of

$$\begin{aligned} M^+ u \ge 0 \quad &\text{and} \quad M^- u \le 0 \quad \text{in } \{x_n > 0\},\\ u = 0 \quad &\text{in } \{x_n < 0\}. \end{aligned}$$

Assume that, for some positive $\beta < s + \bar{\alpha}$, u satisfies the growth control at infinity

$$\|u\|_{L^\infty(B_R)} \le CR^\beta \quad \text{for all } R \ge 1. \tag{5.1}$$

Then,

$$u(x) = K(x_n)_+^s$$

for some constant $K \in \mathbb{R}$.

Proof
Given $\rho \ge 1$, let $v_\rho(x) = \rho^{-\beta} u(\rho x)$. Note that for all $\rho \ge 1$, the function v_ρ satisfies the same growth control (5.1) as u. Indeed,

$$\|v_\rho\|_{L^\infty(B_R)} = \rho^{-\beta}\|u\|_{L^\infty(B_{\rho R})} \le \rho^{-\beta} C(\rho R)^\beta = CR^\beta.$$

In particular $\|v_\rho\|_{L^\infty(B_1)} \le C$ and $\|v_\rho\|_{L^1(\mathbb{R}^n,\omega_s)} \le C$, with C independent of ρ. Hence, the function $\tilde{v}_\rho = v_\rho \chi_{B_1}$ satisfies $M^+\tilde{v}_\rho \ge -C$ and $M^-\tilde{v}_\rho \le C$ in $B_{1/2} \cap \{x_n > 0\}$, and $\tilde{v}_\rho = 0$ in $\{x_n < 0\}$. Also, $\|\tilde{v}_\rho\|_{L^\infty(B_{1/2})} \le C$. Therefore, by Proposition 1.1, we obtain

$$\|v_\rho / x_n^s\|_{C^\alpha(B_{1/4}^+)} = \|\tilde{v}_\rho / x_n^s\|_{C^\alpha(B_{1/4}^+)} \le C.$$

Scaling this estimate back to u, we obtain

$$\begin{aligned}[u/x_n^s]_{C^\alpha(B_{\rho/4}^+)} &= \rho^{-\alpha}\big[u(\rho x)/(\rho x_n)^s\big]_{C^\alpha(B_{1/4}^+)} \\ &= \rho^{\beta-s-\alpha}\big[v_\rho/(x_n)^s\big]_{C^\alpha(B_{1/4}^+)} \le C\rho^{\beta-s-\alpha}.\end{aligned}$$

Using the fact that $\beta < s + \alpha$ and letting $\rho \to \infty$, we obtain

$$[u/x_n^s]_{C^\alpha(\mathbb{R}^n \cap \{x_n > 0\})} = 0,$$

which means that $u = K(x_n)_+^s$. □

Proposition 5.1 will be applied to tangential derivatives of a solution to u in the situation of Theorem 1.4. It will give that u is in fact a function of x_n alone. To proceed, we will need the following crucial lemmas. These are Liouville-type results for the fractional Laplacian in dimension 1, and they will be proved in Section 6.

In the first one, the growth of the solution u still allows us to compute $(-\Delta)^s u$.

LEMMA 5.2
Let $u \in C(\mathbb{R})$ be a function satisfying $(-\Delta)^s u = 0$ in $\mathbb{R}_+$, $u \equiv 0$ in $\mathbb{R}_-$, and $|u(x)| \le C(1 + |x|^\beta)$ for some $\beta < 2s$. Then, $u(x) = K(x_+)^s$.

The second one is for functions that grow too much at infinity, so that one cannot compute $(-\Delta)^s u$—as in Theorem 1.4.

LEMMA 5.3
Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\phi_{a,b}(x) = a(x_+)^s + b(x_+)^{1+s}.$$

Then we have the following.

(a) *For any a and b, the function $\phi_{a,b}$ satisfies*

$$(-\Delta)^s\big\{\phi_{a,b}(\cdot + h) - \phi_{a,b}\big\} = 0 \quad \text{in } (0,\infty).$$

(b) *Let $u \in C(\mathbb{R})$ be any function such that $u \equiv 0$ in $\mathbb{R}_-$ and satisfying*

$$(-\Delta)^s\big\{u(\cdot + h) - u\big\} = 0 \quad \text{in } \mathbb{R}_+$$

for any $h > 0$. Assume in addition that, for some $\gamma \in (0,1)$ and $\beta \in (0,2s)$,

$$\big[u/(x_+)^s\big]_{C^\gamma([0,R])} \le CR^\beta \quad \text{for all } R \ge 1.$$

Then,

$$u(x) = \phi_{a,b}(x)$$

for some a and b.

We will also need the following observation.

LEMMA 5.4
Assume that u is a function in $\mathbb{R}^n$ depending only of one variable (i.e., $u(x) = \zeta(x_n)$). Then, we have

$$M^+u(x) = -c_1(-\Delta)^s_{\mathbb{R}}\zeta(x_n)$$

and

$$M^-u(x) = -c_2(-\Delta)^s_{\mathbb{R}}\zeta(x_n)$$

in the viscosity sense, where c_1 and c_2 are positive constants.

Proof
The proof follows immediately from Lemma 2.1. □

Furthermore, we will also use the following lemma.

LEMMA 5.5
Let $a \in \mathbb{R}^n$ and let $b \in \mathbb{R}$, and define

$$\phi(x) = (x_n)_+^s (a \cdot x + b).$$

Then, for all $h \in \mathbb{R}^n$ with $h_n \geq 0$, we have

$$M^+\{\phi(\cdot + h) - \phi\} = M^-\{\phi(\cdot + h) - \phi\} = 0 \quad \text{in } \{x_n > 0\}.$$

Proof
The proof follows immediately from Lemmas 5.4 and 5.3(a). □

We can now give the following proof.

Proof of Theorem 1.4
First take $h \in S^{n-1}$ such that $h_n = 0$, and define

$$v(x) = u(x + h) - u(x).$$

Then, v satisfies

$$\begin{cases} M^+ v \geq 0 \quad \text{and} \quad M^- v \leq 0 & \text{in } \{x_n > 0\}, \\ v = 0 & \text{in } \{x_n < 0\}. \end{cases}$$

Moreover, by (1.15) it also satisfies the growth control

$$\left\| v/(x_n)_+^s \right\|_{L^\infty(B_R)} \leq C R^\alpha \quad \text{for all } R \geq 1,$$

and hence

$$\|v\|_{L^\infty(B_R)} \leq C R^{\alpha+s} \quad \text{for all } R \geq 1.$$

Thus, it follows from Proposition 5.1 that

$$v(x) = K(x_n)_+^s.$$

Therefore, we have

$$u(x + h) - u(x) = K(x_n)_+^s$$

whenever $h_n = 0$, and this implies that

$$u(x) = (x_n)_+^s (a \cdot x + b) + \psi(x_n)$$

for some 1-dimensional function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$.

Now, by Lemmas 5.5 and 5.4, we have, for all $h \in \mathbb{R}^n_+$ and all $x \in \mathbb{R}^n_+$,

$$M^+\big\{u(\cdot + h) - u\big\}(x) = -c_1(-\Delta)^s\big\{\psi(\cdot + h_n) - \psi\big\}(x_n),$$

and the same holds with M^-. Thus, for any $h > 0$, this 1-dimensional function ψ satisfies

$$\begin{cases} (-\Delta)^s\{\psi(\cdot + h) - \psi\} = 0 & \text{in } (0, +\infty), \\ \psi = 0 & \text{in } (-\infty, 0). \end{cases}$$

Moreover, note that by the assumptions of the theorem, the function ψ satisfies

$$\big[\psi/(x_+)^s\big]_{C^\beta([0,R])} \le CR^\alpha \quad \text{for all } R \ge 1.$$

Hence, by Lemma 5.3, we find that $\psi(x_n) = K_1(x_n)_+^{1+s} + K_2(x_n)_+^s$, and

$$u(x) = (x_n)_+^s(\tilde{a} \cdot x + \tilde{b}),$$

as desired. □

6. Liouville theorems in dimension 1

The aim of this section is to prove Lemmas 5.2 and 5.3.

To prove them, we need the following result. It classifies all homogeneous solutions (with no growth condition) that vanish in a half line of the extension problem of Caffarelli and Silvestre [13] in dimension $1 + 1$.

LEMMA 6.1
Let $s \in (0, 1)$. Let (x, y) denote a point in $\mathbb{R}^2$, and let $r > 0$, $\theta \in (-\pi, \pi)$ be polar coordinates defined by the relations $x = r\cos\theta$, $y = r\sin\theta$. Assume that $\nu > -s$ and that $q_\nu = r^{s+\nu}\Theta_\nu(\theta)$ is even with respect y (or equivalently with respect to θ) and solves

$$\begin{cases} \operatorname{div}(|y|^{1-2s}\nabla q_\nu) = 0 & \text{in } \{y \neq 0\}, \\ \lim_{y\to 0} |y|^{1-2s}\partial_y q_\nu = 0 & \text{on } \{y = 0\} \cap \{x > 0\}, \\ q_\nu = 0 & \text{on } \{y = 0\} \cap \{x < 0\}. \end{cases} \tag{6.1}$$

Then we have the following.

(a) *ν belongs to $\mathbb{N} \cup \{0, -1\}$ and*

$$\Theta_\nu(\theta) = K|\sin\theta|^s P_\nu^s(\cos\theta),$$

where P_ν^μ is the associated Legendre function of the first kind. Equivalently,

$$\Theta_\nu(\theta) = C\left|\cos\left(\frac{\theta}{2}\right)\right|^{2s} {}_2F_1\left(-\nu, \nu+1; 1-s; \frac{1-\cos\theta}{2}\right),$$

where ${}_2F_1$ is the hypergeometric function.

(b) *The functions $\{\Theta_\nu\}_{\nu\in\mathbb{N}\cup\{0\}}$ are a complete orthogonal system in the subspace of even functions of the weighted space $L^2((-\pi,\pi), |\sin\theta|^{1-2s})$.*

Proof

We defer the proof to the Appendix. □

Using the previous computation, we can now prove Lemma 5.2.

Proof of Lemma 5.2

Let

$$P_s(x,y) = \frac{p_{1,s}}{y}\frac{1}{(1+(x/y)^2)^{(1+2s)/2}}$$

be the Poisson kernel for the extension problem of Caffarelli and Silvestre (see [8], [13]). Given the growth control $|u(x)| \le C|x|^\beta$ at infinity with $\beta < 2s$ and given $|u(x)| \le C|x|^{\delta-1}$ near the origin with $\delta > 0$, the convolution

$$v(\cdot, y) = u * P_s(\cdot, y)$$

is well defined and is a solution of the extension problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \{y>0\}, \\ v(x,0) = u(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Since $(-\Delta)^s u = 0$ in $\{x>0\}$ and $u = 0$ in $\{x<0\}$, the function v satisfies

$$\lim_{y\searrow 0} y^{1-2s}\partial_y v(x,y) = 0 \quad \text{for } x>0 \qquad \text{and} \qquad v(x,0) = 0 \quad \text{for } x<0.$$

Hence, v solves (6.1). Let Θ_ν, $\nu \in \mathbb{N}\cup\{0\}$, be as in Lemma 6.1. Recall that $r^{s+\nu}\Theta_\nu(\theta)$ also solves (6.1). By standard separation of variables, in every ball $B_R^+(0)$ of $\mathbb{R}^2$ the function v can be written as a series

$$v(x,y) = v(r\cos\theta, r\sin\theta) = \sum_{\nu=0}^{\infty} a_\nu r^{s+\nu}\Theta_\nu(\theta). \tag{6.2}$$

To obtain this expansion, we use the fact that, by Lemma 6.1(b), the functions $\{\Theta_\nu\}_{\nu\in\mathbb{N}\cup\{0,-1\}}$ are a complete orthogonal system in the subspace of even functions in the weighted space $L^2((-\pi,\pi), |\sin\theta|^{1-2s})$, and hence they are complete in $L^2((0,\pi),(\sin\theta)^{1-2s})$. Moreover, by uniqueness, the coefficients a_ν are independent of R and hence the series (6.2) provides a representation formula for $v(x,y)$ in the whole $\{y>0\}$.

Now, we claim that the growth control $\|u\|_{L^\infty(-R,R)} \le CR^\beta$ with $\beta \in (0,2s)$ is transferred to v (perhaps with a bigger constant C); that is,

$$\|v\|_{L^\infty(B_R^+)} \le CR^\beta.$$

To see this, consider the rescaled function $u_R(x) = R^{-\beta}u(Rx)$, which satisfies the same growth control of u. Then,

$$v_R = R^{-\beta}v(R\cdot) = u_R * P_s.$$

Since the growth control for u_R is independent of R, we find a bound for $\|v_R\|_{L^\infty(B_1^+)}$ that is independent of R, and this means that v is controlled by CR^β in B_R^+, as claimed. Next, since we may assume that $\int_0^\pi |\Theta_\nu(\theta)|^2|\sin\theta|^a\,d\theta = 1$ for all $\nu \ge 0$, Parseval's identity yields

$$\int_{\partial^+ B_R} \big|v(x,y)\big|^2 y^a\,d\sigma = \sum_{\nu=0}^{\infty} |a_\nu|^2 R^{2s+2\nu+1+a}, \tag{6.3}$$

where $\partial^+ B_R = \partial B_R \cap \{y>0\}$. But by the growth control, we have

$$\int_{\partial^+ B_R} \big|v(x,y)\big|^2 y^a\,d\sigma \le CR^{2\beta}\int_{\partial^+ B_R} y^a\,d\sigma = CR^{2\beta+1+a}. \tag{6.4}$$

Finally, since $2\beta < 4s < 2s+2$, this implies that $a_\nu = 0$ for all $\nu \ge 1$, and hence $u(x) = K(x_+)^s$, as desired. □

To establish Lemma 5.3, we will need the following extension of Lemma 5.2.

LEMMA 6.2
Let u satisfy $(-\Delta)^s u = 0$ in $\mathbb{R}_+$ and let $u = 0$ in $\mathbb{R}_-$. Assume that, for some $\delta > 0$ and $\beta \in (0,2s)$, u satisfies the growth conditions

- $|u(x)| \le C|x|^{\delta-1}$ *for all* $x \in (0,1)$,
- $|u(x)| \le C|x|^\beta$ *for all* $x \ge 1$.

Then $u(x) = a(x_+)^s + b(x_+)^{s-1}$.

Proof
The proof is a slight modification of the proof of Lemma 5.2. Indeed, we may consider the extension $v(x,y)$ of $u(x)$, which solves (6.1). Then we consider $\tilde v(x,y) = \int_{-\infty}^x v(x,y)\,dx$, which also satisfies (6.1) as well as satisfying the growth condition

$$\|v\|_{L^\infty(B_R^+)} \le CR^{\beta+1},$$

with $\beta+1<1+2s$. Finally, writing $\tilde{v}$ as in (6.2) and using (6.3) and (6.4) and the fact that $2(\beta+1)<2+4s$, we find that $a_\nu = 0$ for all $\nu \ge 2$. This yields $\tilde{v}(x,0) = (x_+)^s(ax+b)$, and hence $u(x) = a(x_+)^s + b(x_+)^{s-1}$. □

We finally give the following proof.

Proof of Lemma 5.3
(a) The proof follows easily by using the extension of Caffarelli–Silvestre from [13].

(b) First, notice that $u \in C^\delta([0,1])$, with $\delta = \min\{\gamma, s\}$. Hence, for each $h>0$, the function $v(x) = u(x+h) - u(x)$ satisfies $v \equiv 0$ in $(-\infty,-h)$, $|v| \le C|h|^\delta$ in $[-h,0]$, and $(-\Delta)^s v = 0$ in $(0,\infty)$, and $|v(x)| \le C|h|^\gamma(1+|x|^{\beta+s})$ in $(0,\infty)$. Thus, by standard interior regularity (see for example [42]), we get $[v]_{C^{0,1}([h,2h])} \le C|h|^{\delta-1}$. In particular,

$$\left|u'(x+h) - u'(x)\right| = \left|v'(x)\right| \le C|h|^{\delta-1} \quad \text{for all } x \in [h,2h], h \in (0,1).$$

And this implies (summing a geometric series) that

$$\left|u'(x)\right| \le C|x|^{\delta-1} \quad \text{for } x \in (0,1).$$

On the other hand, since $|v(x)| \le C|h|^\gamma |x|^{\beta+s}$ for $x>1$, we have

$$\left|v'(R)\right| \le [v]_{C^{0,1}([R,2R])} \le \frac{C}{R}\|v\|_{L^\infty([R/2,3R])} \le C|h|^\gamma R^{\beta+s-1} \quad \text{for } R \ge 1.$$

Therefore, it follows that, for all $x>1$,

$$\begin{aligned} \left|u'(x)\right| &\le \left|u'(1) - u'(2)\right| + \cdots + \left|u'(x-1) - u'(x)\right| \\ &\le C(1^{\beta+s-1} + 2^{\beta+s-1} + \cdots + x^{\beta+s-1}) \\ &\le C|x|^{\beta+s}. \end{aligned}$$

Thus, the function u' satisfies

- $(-\Delta)^s(u') = 0$ in $(0,\infty)$,
- $|u'(x)| \le C|x|^{\delta-1}$ for $x \in (0,1)$,
- $|u'(x)| \le C|x|^{s+\beta+\gamma-1}$ for $x>1$,

and then it follows from Lemma 6.1 that

$$u'(x) = a(x_+)^s + b(x_+)^{s-1}.$$

Hence, since $u(0)=0$, we have

$$u(x) = a(x_+)^{s+1} + b(x_+)^s,$$

as desired. □

7. Boundary regularity: Flat boundary

In this section we prove Theorem 1.2. The main step in the direction of this result will be Proposition 7.1. Note that throughout this section the operator I will not be translation-invariant but of the form (7.1), with L_{ab} translation-invariant. Hence, $\mathrm{I}(u,x)$ belongs to a restricted class of nontranslation-invariant operators. Within this class we can truncate solutions. Thanks to this, we may assume, for example, that in $u/(x_n)^s$ we find C^β in all of $\mathbb{R}^n_+$ and not only in $B^+_{3/4}$ (recall that we want to show (1.13)).

In the following, given $\beta \in [0,1]$ and $A \subset \mathbb{R}^n$, we denote

$$[u]_{\beta,A} := \sup_{x,y\in A, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\beta}.$$

That is, for $\beta = 0$, this gives the oscillation, and for $\beta \in (0,1)$, this gives the C^β seminorm. We also denote

$$\|u\|_{\beta,A} := \|u\|_{L^\infty(A)} + [u]_{\beta;A}.$$

This notation is appropriate in order to treat at the same time the cases $\beta = 0$ and $\beta \in (0,1)$.

PROPOSITION 7.1

Let $s_0 \in (0,1)$, and let $\bar{\alpha} > 0$ be the constant given by Proposition 1.1. Let $s \in (s_0,1)$, let $\alpha \in (0,\bar{\alpha})$, let $\gamma \in [0,1)$, and let $\beta \in [0,1]$ such that $\alpha+\beta \le \gamma + s$. Assume in addition that $\alpha+\beta \neq 1$. Let u be a solution of $\mathrm{I}(u,x) = 0$ in B_1^+ and $u = 0$ in $\mathbb{R}^n_-$, where I is any fully nonlinear operator elliptic with respect to $\mathfrak{L}_*(s)$ of the form

$$\mathrm{I}(u,x) = \inf_{b\in\mathcal{B}} \sup_{a\in\mathcal{A}} \big(L_{ab}u(x) + c_{ab}(x)\big). \tag{7.1}$$

Assume that

$$\big\|u/(x_n)^s\big\|_{\beta;\{x_n\ge 0\}} \le 1 \qquad \text{and} \qquad \|c_{ab}\|_{\gamma;B_1^+} \le 1$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then, for all $r > 0$,

$$r^{-\alpha}\big[u/(x_n)^s - P_r(\cdot)\big]_{\beta;B_r^+} \le C,$$

for some constant C that depends only on n, s_0, the ellipticity constants, α, and β, where $P_r(x)$ is defined as the polynomial of degree at most $\lfloor \alpha+\beta \rfloor$ (0 or 1) which best fits the function $u/(x_n)^s$ in B_r^+. That is,

$$P_r := \operatorname*{arg\,min}_{P\in\mathcal{P}} \int_{B_r^+} \big(u(x)/(x_n)^s - P(x)\big)^2\,dx,$$

where $\mathcal{P}$ is the space of polynomials with real coefficients and degree at most $\lfloor \alpha+\beta \rfloor$.

We will need the following preliminary results.

LEMMA 7.2
Let $s_0 \in (0,1)$*, let* $s_m \in [s_0, 1]$ *be a converging sequence with* $s_m \to s$*, and let* $M^+_{s_m}$ *and* $M^-_{s_m}$ *denote the extremal Pucci-type operators for the class* $\mathcal{L}_*(s_m)$ *of order* $2s_m$*. Then,* $M^+_{s_m} \to M^+_s$ *and* $M^-_{s_m} \to M^-_s$ *weakly with respect to the weight* $\omega_{s_0}(y) = (1+|y|)^{-n-2s_0}$.

Proof
The proof follows in a straightforward manner from the definition of weak convergence of nonlocal elliptic operators in [16]. □

The second needed lemma is the following.

LEMMA 7.3
Let $s_0 \in (0,1)$ *and let* $s \in (s_0, 1)$*. There exists* $\delta > 0$ *such that the following statement holds. If* $M^+ w \ge -C_0$ *and* $M^- w \le C_0$ *in* B_1^+*,* $w = 0$ *in* B_1^-*, and* $\|w\|_{L^1(\mathbb{R}^n,\omega_s)} \le C_0$*, then*

$$\|w\|_{C^\delta(B^+_{3/4})} \le CC_0,$$

where C *and* δ *depend only on* n*,* s_0*, and the ellipticity constants.*

Proof
First, using the barrier given by Lemma 3.3, we find that $|w(x)| \le CC_0(x_n)^s_+$ in $B^+_{7/8}$. Then, the result follows by using the interior estimates in [14] (see also [16, Section 3]). □

We next give the following proof.

Proof of Proposition 7.1
The proof is by contradiction. If the conclusion of the proposition is false, then there are sequences u_k, I_k, s_k, and γ_k satisfying
- $\mathrm{I}_k(u_k, x) = 0$ in B_1^+ and $u_k = 0$ in B_1^-,
- $\mathrm{I}_k(u_k, x) = \inf_{b\in\mathcal{B}_k} \sup_{a\in\mathcal{A}_k} (L_{ab}u_k(x) + c_{ab}(x))$,
- $\{L_{ab} : a \in \mathcal{A}_k, b \in \mathcal{B}_k\} \subset \mathcal{L}(s_k)$ with $s_k \in [s_0, 1]$,
- $\|u/(x_n)^{s_k}\|_{\beta;\{x_n>0\}} \le 1$ and $\|c_{ab}\|_{\gamma_k;B_1^+} \le 1$ for all $a \in \mathcal{A}_k$ and $b \in \mathcal{B}_k$,
- $\gamma_k \ge \min\{0, \alpha + \beta - s_k\}$

for which

$$\sup_k \sup_{r>0} r^{-\alpha}\big[u_k/(x_n)^{s_k} - P_{k,r}\big]_{\beta;B_r^+} = +\infty, \tag{7.2}$$

where

$$P_{k,r} := \operatorname*{arg\,min}_{P\in\mathcal{P}} \int_{B_r^+} \big(u_k(x)/(x_n)^{s_k} - P\big)\,dx$$

(recall that $\mathcal{P}$ denotes the real polynomials of degree at most $\lfloor \alpha+\beta\rfloor$).

To prove that this is impossible, let us start by defining

$$\theta(r) := \sup_k \sup_{r'>r} (r')^{-\alpha}\big[u_k/(x_n)^{s_k} - P_{k,r}\big]_{\beta;B_{r'}^+}.$$

The function θ is monotone nonincreasing and we have $\theta(r) < +\infty$ for $r > 0$ since we are assuming that

$$\big\|u_k/(x_n)^{s_k}\big\|_{\beta;\{x_n>0\}} \le 1. \tag{7.3}$$

In addition, by (7.2) we have $\theta(r) \nearrow +\infty$ as $r \searrow 0$. For every positive integer m, by definition of $\theta(1/m)$ there are $r'_m \ge 1/m$ and k_m for which

$$(r'_m)^{-\alpha}\big[u_k/(x_n)^{s} - P_{k_m,r'_m}\big]_{\beta;B_{r'_m}} \ge \frac{1}{2}\theta(1/m) \ge \frac{1}{2}\theta(r'_m). \tag{7.4}$$

Here we have used that θ is nonincreasing. Note that we will have $r'_m \searrow 0$ since $\theta(1/m) \nearrow +\infty$ and (7.3) holds.

From now on in this proof we will use the notation

$$u_m = u_{k_m}, \qquad P_m = P_{k_m,r'_m}, \qquad s_m = s_{k_m}, \qquad \text{and} \qquad \gamma_m = \gamma_{k_m}.$$

Let us consider the blow-up sequence

$$v_m(x) = \frac{u_m(r'_m x)/(r'_m x_n)^s - P_m(r'_m x)}{(r'_m)^{\alpha+\beta}\theta(r'_m)}. \tag{7.5}$$

For all $m \ge 1$, we have

$$\int_{B_1^+} v_m(x)P(x)\,dx = 0 \quad \text{for all } P \in \mathcal{P}, \tag{7.6}$$

since this is the optimality condition in the least squares minimization. Note also that (7.4) implies the following inequality for all $m \ge 1$:

$$[v_m]_{\beta;B_1^+} \ge 1/2. \tag{7.7}$$

Let us now show that

$$[v_m]_{\beta;B_R^+} \le CR^{\alpha} \tag{7.8}$$

for all $R \ge 1$.

We will do the proof of (7.8) in the most difficult case $\alpha+\beta>1$ (the case $\alpha+\beta\in(0,1)$ is very similar). To prove (7.8), we need to estimate the difference in the coefficients of $P_{k_m,Rr'_m} - P_{k_m,r'_m}$. Let us denote

$$P_{k,r}(x) = p_{k,r}\cdot x + b_{k,r} \quad \text{where } p_{k,r}\in\mathbb{R}^n \text{ and } b_{k,r}\in\mathbb{R}.$$

Note that since we are doing the case $\alpha+\beta>1$, we will have $\lfloor\alpha+\beta\rfloor = 1$ and hence $\mathcal{P}$ contains all affine functions. Let $R=2^k$, and let us show that

$$|p_{k_m,Rr'_m} - p_m| = |p_{k_m,2^k r'_m} - p_{k_m,r'_m}| \le C\theta(r'_m)(Rr'_m)^{\alpha+\beta-1}. \tag{7.9}$$

Indeed, by definition of $\theta(2r)$ and $\theta(r)$, we have

$$\begin{aligned}
\frac{|p_{k,2r}-p_{k,r}|r^{1-\beta}}{r^{\alpha}\theta(r)} &\le \frac{[(p_{k,2r}-p_{k,r})\cdot x]_{\beta;B_r^+}}{r^{\alpha}\theta(r)}\\
&= \frac{[P_{k,2r}-P_{k,r}]_{\beta;B_r^+}}{r^{\alpha}\theta(r)}\\
&\le \frac{2^{\alpha}\theta(2r)}{\theta(r)}\frac{[u_k/(x_n)^{s_k}-P_{k,2r}]_{\beta;B_{2r}^+}}{(2r)^{\alpha}\theta(2r)}\\
&\quad + \frac{[u_k/(x_n)^{s_k}-P_{k,r}]_{\beta;B_r^+}}{r^{\alpha}\theta(r)}\\
&\le 2^{\alpha}+1\le 3,
\end{aligned}$$

where we have used the definition of θ and its monotonicity. Thus,

$$|p_{k_m,2^k r'_m} - p_{k_m,r'_m}| \le 3\sum_{j=0}^{k-1}\theta(2^j r'_m)(2^j r'_m)^{\alpha+\beta-1} \le C\theta(r'_m)(2^k r'_m)^{\alpha+\beta-1}.$$

Note that it is here where we use the fact that $\alpha+\beta-1>0$.

Therefore, we can estimate as follows:

$$\begin{aligned}
[v_m]_{\beta;B_R^+} &= \frac{1}{\theta(r'_m)(r'_m)^{\alpha}}\big[u_m/(x_n)^{s_m} - P_m\big]_{\beta;B^+_{Rr'_m}}\\
&= \frac{R^{\alpha}}{\theta(r'_m)(Rr'_m)^{\alpha}}\Big(\big[u_{k_m}/(x_n)^{s_m} - P_{k_m,Rr'_m}\big]_{\beta;B^+_{Rr'_m}}\\
&\quad + [P_{k_m,Rr'_m} - P_m]_{\beta;B^+_{Rr'_m}}\Big)
\end{aligned}$$

$$\leq R^\alpha \frac{\theta(Rr'_m)}{\theta(r'_m)} + 2|p_{k_m,Rr'_m} - p_m|(Rr'_m)^{1-\beta}$$
$$\leq R^\alpha + CR^\alpha,$$

where we have used (7.9). This proves (7.8) in the case $\alpha+\beta>1$. As noted above, the proof of (7.8) in the case $\alpha+\beta\in(0,1)$ is easier since in this case $[P_{k_m,Rr'_m} - P_m]_{\beta;B^+_{Rr'_m}} = 0$ and we do not need to estimate the difference of the coefficients.

When $R=1$, (7.8) implies that $\|v_m - b\|_{L^\infty(B_1)} \leq C$ for some $b\in\mathbb{R}$. Thus, (7.6) implies that

$$\|v_m\|_{L^\infty(B_1^+)} \leq C. \tag{7.10}$$

Then, using (7.8) and (7.10), we easily obtain that

$$\|v_m\|_{L^\infty(B_R^+)} \leq CR^{\alpha+\beta}. \tag{7.11}$$

Note that in the case $\beta=0$, the difference between (7.8) and (7.11) is that we pass from a oscillation bound to an L^∞ bound. Next we prove the following.

CLAIM

Let $w_m(x) = v_m(x)(x_n)_+^s$. Then, up to subsequences we have $s_m \to s \in [s_0,1]$ and $w_m \to w$ locally uniformly in $\{x_n \geq 0\}$, where $w \in C(\mathbb{R}^n)$. Moreover, the limiting function w satisfies the assumptions of the Liouville-type Theorem 1.4.

In the case $\beta>0$, it follows from (7.8), (7.11), and the Arzelà–Ascoli theorem that a subsequence of v_m converges locally uniformly in $\mathbb{R}^n_+$ to some $v\in C(\{x_n\geq 0\})$. The convergence of w_m to $w=v(x_n)_+^s$ is then immediate. In the case $\beta=0$, the functions w_m satisfy $M^+w_m \leq C(K)$ and $M^-w_m \geq -C(K)$ in every half-ball B_K^+ and also satisfy $\|w_m\|_{L^\infty(B_R)} \leq CR^{s_m+\alpha}$ for every $R\geq 1$. Thus $\|w_m\|_{C^\delta(B_K)} \leq C(K)$ by Lemma 7.3. Therefore, the functions w_m converge to some w uniformly in compact sets. Moreover, by passing to the limit (7.8), we find that the assumption (1.15) of Theorem 1.4 is satisfied by w.

Now, each u_k satisfies

$$\mathrm{I}_k(u_k,x) := \inf_{b\in\mathcal{B}_k}\sup_{a\in\mathcal{A}_k}\big(L_{ab}u_k(x) + c_{ab}(x)\big) = 0 \quad \text{in } B_1^+.$$

Thus, for every $\bar h \in B^+_{1/2}$, we have

$$\inf_{b\in\mathcal{B}_{k_m}}\sup_{a\in\mathcal{A}_{k_m}}\big(L_{ab}u_m(\bar x+\bar h) + c_{ab}(\bar x+\bar h)\big) = 0 \quad \text{for } \bar x \in B^+_{1/2}$$

in the viscosity sense. Using the fact that $[c_{ab}]_{C^{\gamma_m}} \le 1$ (for all $a \in \mathcal{A}_{k_m}$ and $b \in \mathcal{B}_{k_m}$), it follows that

$$\inf_{b\in\mathcal{B}_{k_m}} \sup_{a\in\mathcal{A}_{k_m}} \big(L_{ab}u_m(\bar{x}+\bar{h}) + c_{ab}(0)\big) \ge -|\bar{x}+\bar{h}|^{\gamma_m} \quad \text{for } \bar{x} \in B_{1/2}^+$$

and

$$\inf_{b\in\mathcal{B}_{k_m}} \sup_{a\in\mathcal{A}_{k_m}} \big(L_{ab}u_m(\bar{x}) + c_{ab}(0)\big) \le |\bar{x}|^{\gamma_m} \quad \text{for } \bar{x} \in B_{1/2}^+$$

in the viscosity sense.

Therefore, using Lemma 5.8 in [14], we obtain

$$M_{s_m}^+\big(u_m(\cdot+\bar{h}) - u_m\big) \ge -|\bar{x}|^{\gamma_m} - |\bar{x}+\bar{h}|^{\gamma_m} \quad \text{in } B_{1/2}^+ \tag{7.12}$$

in the viscosity sense. Next, by Lemma 5.5, for every affine or constant function $P \in \mathcal{P}$, the function $\varphi(x) = P(x_n)_+^s$ satisfies

$$M^+\big(\phi(\cdot+\bar{h}) - \phi\big) = M^-\big(\phi(\cdot+\bar{h}) - \phi\big) = 0 \quad \text{in } \mathbb{R}_+^n$$

pointwise in the classical sense for every $\bar{h}$ with $\bar{h}_n \ge 0$. Using this property, the value of the operator does not change when adding to test functions multiples of $\phi(\cdot+\bar{h}) - \phi$. Hence, recalling that

$$w_m(x) = v_m(x)(x_n)_+^{s_m} = \frac{v_m(x)(r_m' x_n)_+^{s_m}}{(r_m')^{s_m}} = \frac{u_m(r_m' x) - P_m(r_m' x_n)_+^{s_m}}{(r_m')^{\alpha+\beta+s_m}\theta(r_m')}$$

and the definition of v_m from (7.5), we can translate (7.12) from u_m to w_m. Indeed, setting $\bar{h} = r_m' h$ and $\bar{x} = r_m' x$, we obtain

$$-(r_m')^{\gamma_m} 3K^{\gamma_m} \le \frac{\theta(r_m')(r_m')^{\alpha+\beta+s_m}}{(r_m')^{2s_m}} M^+\big(w_m(\cdot+h) - w_m\big) \quad \text{in } B_K^+$$

whenever $h_n \ge 0$, $|h| < K$, and $r_m' < 1/2K$. Therefore

$$-3K^{\gamma_m}\frac{(r_m')^{s_m+\gamma_m}}{\theta(r_m')(r_m')^{\alpha+\beta}} \le M^+\big(w_m(\cdot+h) - w_m\big) \quad \text{in } B_K^+ \tag{7.13}$$

in the viscosity sense for all $h \in B_K$ whenever $r_m' < 1/2K$. Since $w_m \to w$ locally uniformly in $\{x_n \ge 0\}$ (up to subsequences), we then have

$$\big(w_m(\cdot+h) - w_m\big) \to \big(w(\cdot+h) - w\big) \quad \text{locally uniformly in } \mathbb{R}^n. \tag{7.14}$$

Let us check that, for some $\epsilon > 0$ small enough, we have, for every $h \in \mathbb{R}^n$ with $h_n \ge 0$,

$$\big(w_m(\cdot+h)-w_m\big)\to\big(w(\cdot+h)-w\big)\quad \text{in } L^1(\mathbb{R}^n,\omega_{s-\epsilon}). \tag{7.15}$$

Recall that throughout the present article we denote $\omega_s(y)=(1-s)(1+|y|)^{-n-2s}$.

To show (7.15), observe that

$$\begin{aligned} &w_m(x+h)-w_m(x)\\ &\quad=\big(v_m(x+h)-v_m(x)\big)(x_n+h_n)_+^{s_m}+v_m(x)\big((x_n+h_n)_+^{s_m}-(x_n)_+^{s_m}\big)\end{aligned}$$

and hence, using (7.8) and (7.11),

$$\begin{aligned}&\big|w_m(x+h)-w_m(x)\big|\\ &\quad\le\begin{cases} C|h|^\beta(1+|x|)^\alpha(x_n+h_n)^{s_m} & \\ \qquad +C(1+|x|)^{\alpha+\beta}h_n(x_n)^{s_m-1} & \text{if } x_n>0,\\ C(1+|x|)^{\alpha+\beta}(h_n)^{s_m} & \text{if } -h_n<x_n<0,\\ 0 & \text{if } x_n<-h_n.\end{cases}\end{aligned}$$

Therefore, we have

$$\begin{aligned}\big|w_m(x+h)-w_m(x)\big|\ &\le\ g(x)\\ &:=C\big(1+\big(|x|^\alpha(x_n)^{s_m}+|x|^\alpha(x_n)^{s_m-1}\big)\chi_{(0,+\infty)}(x_n)\\ &\qquad+|x|^{\alpha+\beta}\chi_{(-C,0)}(x_n)\big),\end{aligned}$$

where C (and g) depend on h. Since $s_m\to s$, we will have $s_m\ge s-\epsilon$ for m large enough, and using the facts that $\beta\le 1$ and $\bar\alpha<s_0$, we readily show that $g\in L^1(\mathbb{R}^n,\omega_{s-\epsilon})$. Therefore, (7.15) follows from (7.14) using the dominated convergence theorem.

Finally, using (7.15) and (7.14), it follows from Lemma 7.2 and Lemma 5 in [16] we can pass to the limit in (7.13) in every ball B_K^+ to obtain

$$0\le M^+\big\{w(\cdot+h)-w\big\}\quad \text{in } B_K^+.$$

Thus, since this can be done for any $K>0$, we have

$$0\le M^+\big\{w(\cdot+h)-w\big\}\quad \text{in } \mathbb{R}^n_+.$$

Analogously, we will have

$$0\ge M^-\big\{w(\cdot+h)-w\big\}\quad \text{in } \mathbb{R}^n_+,$$

and this finishes the proof the claim.

We have thus proved that w satisfies all the assumptions of Theorem 1.4 and hence we conclude that $v=w/(x_n)^s$ is an affine function. On the other hand, passing

(7.6) to the limit, we obtain that v is orthogonal to every affine function and hence it must be $v \equiv 0$. But then passing (7.7) to the limit, we obtain that v cannot be constantly zero in B_1, which is a contradiction. □

To prove Theorem 1.2, we will need the following lemma, which matches Proposition 7.1.

LEMMA 7.4
Let $\alpha \in (0,1]$ and let $\beta \in [0,1]$ with $\alpha + \beta \neq 1$, and let v satisfy, for all $r > 0$,

$$\sup_{r>0} r^{-\alpha}[v - P_r]_{\beta;B_r^+} \le C_0,$$

where P_r is some polynomial of degree at most $\lfloor \alpha + \beta \rfloor$ (0 or 1) depending on r. In the case $\alpha + \beta > 1$, assume in addition that $P_1(x) = p_1 \cdot x + b_1$, with $|p_1| \le C_0$. Then, the limit $P = \lim_{r \searrow 0} P_r$ exists, and for all $r > 0$ we have

$$\|v - P\|_{L^\infty(B_r^+)} \le CC_0 r^{\beta+\alpha} \qquad \textit{and} \qquad |p| \le CC_0,$$

where $P(x) = p \cdot x + b$ and $p = 0$ if $\alpha + \beta < 1$. The constant C depends only on α and β.

Proof
We will show the most difficult case, $\alpha + \beta > 1$.

Let $P_r(x) = p_r \cdot x + b_r$. We have, for all $r > 0$,

$$[v - p_r \cdot x]_{\beta;B_r^+} \le C_0 r^\alpha.$$

Thus,

$$\begin{aligned} |p_r - p_{r/2}|(r/2)^{1-\beta} &\le \big[(p_r - p_{r/2}) \cdot x\big]_{\beta;B_{r/2}^+} \\ &\le [v - p_r \cdot x]_{\beta;B_{r/2}^+} + [v - p_r \cdot x]_{C^\beta;B_{r/2}^+} \\ &\le C_0 r^\alpha + C_0 (r/2)^\alpha \end{aligned}$$

and hence

$$|p_r - p_{r/2}| \le CC_0 r^{\alpha+\beta-1}.$$

It follows (developing the expressions as telescopic sums and summing the geometric series) that $p = \lim_{r \searrow 0} p_r$ exists and that

$$|p_r - p| \le CC_0 r^{\alpha+\beta-1}.$$

In particular,

$$|p| \le |p_1| + |p_1 - p| \le C_0 + CC_0.$$

Then we obtain

$$\begin{aligned}\operatorname{osc}_{B_r^+}[v - p\cdot x] &\le \operatorname{osc}_{B_r^+}[v - p_r\cdot x] + \operatorname{osc}_{B_r^+}\big[(p_r - p)\cdot x\big]\\ &\le [v - p_r\cdot x]_{\beta;B_r^+} r^\beta + |p_r - p| r\\ &\le CC_0 r^{\beta+\alpha}\end{aligned}$$

and the lemma now follows. □

We give now a second step toward Theorem 1.2. This result follows from the interior estimates.

LEMMA 7.5
Let $s_0 \in (0,1)$, and let $\bar\alpha > 0$ be the constant given by Proposition 1.1. Let $s \in (s_0, 1)$, let $\alpha \in (0,\bar\alpha)$, let $\gamma \in (0,1)$, and let $\beta \in [0,1)$ such that $\alpha + \beta \le \gamma + s$. Assume in addition that $\alpha + \beta \ne 1$. Let $e_n = (0,\dots,0,1)$ and let w be a solution of $\mathrm{I}(w,x) = 0$ in $B_1(e_n)$, where I is any fully nonlinear operator elliptic with respect to $\mathcal{L}_*(s)$ of the form

$$\mathrm{I}(w,x) = \inf_{b\in\mathcal{B}}\sup_{a\in\mathcal{A}}\big(L_{ab}w(x) + c_{ab}(x)\big),$$

with L_{ab} given by (1.3) and (1.4),

$$\|c_{ab}\|_{\gamma;B_1(e_n)} \le 1 \qquad \text{and} \qquad \|\mu_{ab}\|_{C^\gamma(S^{n-1})} \le \Lambda.$$

Assume that

$$\|w\|_{L^\infty(\mathbb{R}^n)} < \infty, \qquad \|w\|_{L^\infty(B_r(0))} \le C_0 r^{\alpha+\beta+s}, \qquad \|w\|_{\beta;B_r(2re_n)} \le C_0 r^{\alpha+s}$$

for all $r > 0$. Then,

$$[w]_{C^{\alpha+\beta}(B_{r/2}(2re_n))} \le C_0 r^s,$$

for all $r \in (0,1)$, for some constant C that depends only on n, s_0, the ellipticity constants, α, and β.

Proof
We defer the proof to the Appendix. □

We next show the following result. It follows from Proposition 7.1 and Lemma 7.4 by truncating the solution u. We will also use the interior estimates from Lemma 7.5.

PROPOSITION 7.6

Let $s_0 \in (0,1)$, and let $\bar\alpha > 0$ be the constant given by Proposition 1.1. Let $s \in (s_0, 1)$, let $\alpha \in (0, \bar\alpha)$, let $\gamma \in (0,1)$, and let $\beta \in [0,1]$ such that $\alpha + \beta \le \gamma + s$. Assume in addition that $\alpha + \beta \neq 1$. Let $u \in L^\infty(\mathbb{R}^n)$ be a solution of $\mathrm{I}(u,x) = 0$ in B_1^+ and $u = 0$ in B_1^-, where I is any fully nonlinear operator elliptic with respect to $\mathfrak{L}_*(s)$ of the form

$$\mathrm{I}(u,x) = \inf_{b\in\mathcal{B}} \sup_{a\in\mathcal{A}} \big(L_{ab}u(x) + c_{ab}(x)\big),$$

with $\|\mu_{ab}\|_{C^\gamma(S^{n-1})} \le 1$. Assume that

$$\big\|u/(x_n)^s\big\|_{\beta;B_{3/4}^+} \le 1, \qquad \|c_{ab}\|_{\gamma;B_1^+} \le 1 \qquad \text{and} \qquad \|u\|_{L^\infty(\mathbb{R}^n)} \le 1.$$

Then,

$$\big\|u/(x_n)^s\big\|_{C^{\beta+\alpha}(B_{1/4}^+)} \le C,$$

for some constant C that depends only on n, s_0, the ellipticity constants, α, and β.

Proof

Let us consider $\bar u = u\eta$, where $\eta \in C_c^\infty(B_{3/4})$ satisfies $\eta \equiv 1$ in $B_{5/8}$. Then, using that the kernels of the operators are C^γ outside the origin, we find that $L_{ab}\bar u = L_{ab}u + \bar c_{ab}(x)$ in $B_{1/2}^+$, where $\bar c_{ab}$ satisfies

$$\|\bar c_{ab}\|_{\gamma;B_{1/2}^+} \le C_0.$$

Hence, we have

$$\bar{\mathrm{I}}(\bar u, x) = \inf_{b\in\mathcal{B}} \sup_{a\in\mathcal{A}} \big(L_{ab}\bar u(x) - \bar c_{ab}(x) + c_{ab}(x)\big) = 0 \quad \text{in } B_{1/2}^+.$$

Moreover, we have

$$\big\|\bar u/(x_n)^s\big\|_{\beta;\{x_n\ge 0\}} \le C_0.$$

Hence, by Proposition 7.1 and Lemma 7.4, we find the following. For each $z \in B_{1/4} \cap \{x_n = 0\}$ there exist $p(z) \in \mathbb{R}^n$ and $b(z) \in \mathbb{R}$ such that

$$\big\|u/(x_n)^s - p(z)\cdot x - b(z)\big\|_{L^\infty(B_r^+)} \le CC_0 r^{\beta+\alpha} \quad \text{for all } r < 1/4, \tag{7.16}$$

and

$$\big\|u/(x_n)^s - p(z)\cdot x - b(z)\big\|_{\beta;B_r^+} \le CC_0 r^\alpha \quad \text{for all } r < 1/4, \tag{7.17}$$

with

$$\big|p(z)\big| \le CC_0, \qquad \big|b(z)\big| \le CC_0.$$

We have used the fact that $\bar{u} = u$ in $B_{1/2}$.

Let us see next that (7.16) and (7.17) imply

$$\big\|u/(x_n)^s\big\|_{C^{\alpha+\beta}(B_{1/4}^+)} \le CC_0.$$

For this, we define

$$Q_z(x) := \big(p(z)\cdot x + b(z)\big)(x_n)_+^s \chi_{B_2}(x),$$

and we observe that (7.16) and (7.17) give

$$\|u - Q_z\|_{L^\infty(B_R(0))} \le CR^{s+\alpha+\beta}$$

and

$$\|u - Q_z\|_{\beta;B_R(x_0)} \le CR^{s+\alpha}$$

for every ball $B_R(x_0)$ such that $2R = \mathrm{dist}(x_0, \{x_n = 0\})$ and $B_{2R}(x_0) \subset B_{1/4}^+$, where z is the projection of x_0 on $\{x_n = 0\}$. Using the previous two inequalities, Lemma 7.5 yields

$$\|u - Q_z\|_{C^{\beta+\alpha}(B_R(x_0))} \le CR^s \tag{7.18}$$

in every such ball $B_R(x_0)$. Finally, using $\|(x_n)^{-s}\|_{L^\infty(B_R(x_0))} \le CR^{-s}$ and $\|(x_n)^{-s}\|_{C^{\alpha+\beta}(B_R(x_0))} \le CR^{-s-\alpha-\beta}$, we find

$$\big\|u/(x_n)^s\big\|_{C^{\alpha+\beta}(B_R(x_0))} \le C.$$

Since this can be done for all balls $B_R(x_0)$ with $2R = \mathrm{dist}(x_0, \{x_n = 0\})$ and $B_{2R}(x_0) \subset B_{1/4}^+$, we have $\|u/(x_n)^s\|_{C^{\alpha+\beta}(B_{1/4}^+)} \le C$, and thus the proposition is proved. □

Finally, we give the following proof.

Proof of Theorem 1.2

We will show the result by applying inductively Proposition 7.6. First, using the supersolution in Lemma 3.3, we find that

$$\|u/(x_n)^s\|_{L^\infty(B^+_{3/4})} \le CC_0.$$

Hence, using Proposition 7.6 with $\beta = 0$, we find that

$$\|u/(x_n)^s\|_{C^\alpha(B^+_{1/4})} \le CC_0.$$

Since this can be done for any solution u, then by a standard covering argument, we will have the estimate

$$\|u/(x_n)^s\|_{C^\alpha(B^+_{3/4})} \le CC_0.$$

Using this and Proposition 7.6, we find that

$$\|u/(x_n)^s\|_{C^{2\cdot\alpha}(B^+_{1/4})} \le CC_0.$$

Iterating this procedure, we find that $u/(x_n)^s \in C^{k\cdot\alpha}(B_{1/4})$ whenever $k \cdot \alpha \le s + \gamma$. More precisely, after a finite number of steps, we find that, for any solution u, we have the estimate

$$\|u/(x_n)^s\|_{C^{s+\gamma}(B^+_{1/4})} \le CC_0.$$

Thus, the theorem is proved. □

8. Boundary regularity: Curved boundary

In this section, we prove Theorem 1.3 by following the same steps as in the preceding section.

The main ingredient in the direction of Theorem 1.3 will be Proposition 8.2. Before stating it, we need the following.

Definition 8.1

We say that Γ is a global $C^{2,\gamma}$ surface given by a graph of $C^{2,\gamma}$ norm smaller than 1, splitting $\mathbb{R}^n$ into Ω^+ and Ω^-, if the following steps happen.

- The surface $\Gamma \subset \mathbb{R}^n$ is the graph of a global $C^{2,\gamma}$ and bounded function, whose $C^{2,\gamma}$ norm is smaller than 1.
- The two disjoint domains Ω^+ and Ω^- partition $\mathbb{R}^n$ (i.e., $\mathbb{R}^n = \overline{\Omega^+} \cup \overline{\Omega^-}$).
- We have $\Gamma = \partial\Omega^+ = \partial\Omega^-$, and $0 \in \Gamma$.
- The origin 0 belongs to Γ and the normal vector to Γ at 0 is $\nu(0) = e_n$.

Moreover, we let $d(x)$ be any $C^{2,\gamma}(\overline{\Omega^+})$ function that coincides with $\mathrm{dist}(x, \Omega^-)$ in a neighborhood of $\Gamma \cap B_4$ and $d \equiv 0$ outside B_5.

We then have the following proposition.

PROPOSITION 8.2

Let $s_0 \in (0,1)$, and let $\bar{\alpha} > 0$ be the constant given by Proposition 1.1. Let $s \in (s_0, 1)$, let $\alpha \in (0, \bar{\alpha})$, let $\gamma \in (0, s]$, and let $\beta \in [0,1)$ such that $\alpha + \beta \le \gamma + s$. Assume in addition that $\alpha + \beta \neq 1$. Assume that Γ is a global $C^{2,\gamma}$ surface given by a graph of $C^{2,\gamma}$ norm smaller than 1, splitting $\mathbb{R}^n$ into Ω^+ and Ω^- (see Definition 8.1). Let $u \in C_c(B_2)$ be a solution of $\mathrm{I}(u,x) = 0$ in $B_1 \cap \Omega^+$ and $u = 0$ in all of Ω^-, where I is any fully nonlinear operator elliptic with respect to $\mathcal{L}_*(s)$ of the form

$$\mathrm{I}(u,x) = \inf_{b \in \mathcal{B}} \sup_{a \in \mathcal{A}} \big(L_{ab} u(x) + c_{ab}(x) \big).$$

Assume that

$$\|u/d^s\|_{\beta;\Omega^+} \le 1 \qquad \text{and} \qquad \|c_{ab}\|_{\gamma; B_1 \cap \Omega^+} \le 1 \quad \text{for all } a \in \mathcal{A} \text{ and } b \in \mathcal{B},$$

and that $\|\mu_{ab}\|_{C^\gamma(S^{n-1})} \le \Lambda$. Then,

$$r^{-\alpha} \big[u/d^s - P_r(\cdot) \big]_{\beta; B_r \cap \Omega^+} \le C$$

for some constant C that depends only on n, s_0, the ellipticity constants, α, and β, where $P_r(x)$ is defined as the polynomial of degree at most $\lfloor \alpha + \beta \rfloor$ which best fits the function $u/(x_n)^s$ in $B_r \cap \Omega^+$. That is,

$$P_r := \operatorname*{arg\,min}_{P \in \mathcal{P}} \int_{B_r \cap \Omega^+} \big(u(x)/d^s - P(x) \big)^2 \, dx,$$

where $\mathcal{P}$ is the space of polynomials with real coefficients and degree at most $\lfloor \alpha + \beta \rfloor$.

An important ingredient of this proof is the following.

LEMMA 8.3

Let $s_0 \in (0,1)$, let $s \in (s_0, 1)$, and let $\gamma \in (0, s]$. Let Γ and d be as in Definition 8.1. Let L be any operator of the form (1.3) with $\|\mu_{ab}\|_{C^{1,\gamma}(S^{n-1})} \le 1$. Then, for any function $\eta \in C^{2,\gamma}(\mathbb{R}^n)$, we have

$$\big\| L(d^s \eta) \big\|_{C^\gamma(B_1 \cap \Omega^+)} \le C \|\eta\|_{C^{2,\gamma}},$$

where C is a constant that depends only on n and s_0.

Proof

The proof follows easily from Proposition 9.2. □

Remark 8.4

The hypothesis $\gamma \le s$ of Proposition 8.2 and of Theorem 1.3 is only needed to show

Lemma 8.3. In particular, if one can show this result for all $\gamma \in (0,1)$, then the hypothesis $\gamma \le s$ in Theorem 1.3 can be removed.

Let us now proceed with the proof of Proposition 8.2. We will skip some details of this proof, since it is very similar to the one of Proposition 7.1.

Proof of Proposition 8.2
We argue by contradiction. If the conclusion of the proposition is false, then there are sequences u_k, I_k, Γ_k, γ_k, and s_k satisfying

- Γ_k is a global C^{2,γ_k} surface given by a graph of C^{2,γ_k} norm smaller than 1, splitting $\mathbb{R}^n$ into Ω_k^+ and Ω_k^- (see Definition 8.1),
- $\mathrm{I}_k(u_k, x) = 0$ in $B_1 \cap \Omega_k^+$ and $u_k = 0$ in Ω_k^-,
- $\mathrm{I}_k(u_k, x) = \inf_{b\in\mathcal{B}_k} \sup_{a\in\mathcal{A}_k} (L_{ab}u_k(x) + c_{ab}(x))$,
- $\{L_{ab} : a \in \mathcal{A}_k, b \in \mathcal{B}_k\} \subset \mathcal{L}(s_k)$ with $s_k \in [s_0, 1]$,
- $\|u/d_k^{s_k}\|_{\beta;\Omega_k^+} \le 1$ and $\|c_{ab}\|_{\gamma_k;B_1\cap\Omega_k^+} \le 1$ for all $a \in \mathcal{A}_k$ and $b \in \mathcal{B}_k$,
- $\gamma_k + s_k \ge \alpha + \beta$

for which

$$\sup_k \sup_{r>0} r^{-\alpha}[u_k/d_k^{s_k} - P_{k,r}]_{\beta;B_r\cap\Omega_k^+} = +\infty, \tag{8.1}$$

where

$$P_{k,r} := \arg\min_{P\in\mathcal{P}} \int_{B_r\cap\Omega_k^+} \left(u_k(x)/d_k^{s_k} - P\right) dx.$$

To prove that this is impossible, we proceed as in the Proof of Proposition 7.1. We define

$$\theta(r) := \sup_k \sup_{r'>r} (r')^{-\alpha}[u_k/d_k^{s_k} - P_{k,r}]_{\beta;B_{r'}\cap\Omega_k^+}.$$

The function θ is monotone nonincreasing, and $\theta(r) < +\infty$ for $r > 0$ since

$$\|u_k/d_k^{s_k}\|_{\beta;\Omega_k^+} \le 1. \tag{8.2}$$

In addition, by (8.1) we have $\theta(r) \nearrow +\infty$ as $r \searrow 0$, and there are sequences $r'_m \searrow 0$ and k_m for which

$$(r'_m)^{-\alpha}[u_k/d_k^{s_k} - P_{k_m,r'_m}]_{\beta;B_{r'_m}\cap\Omega_{k_m}^+} \ge \frac{1}{2}\theta(r'_m). \tag{8.3}$$

For the rest of this article, we will use the notation

$$u_m = u_{k_m}, \qquad P_m = P_{k_m,r'_m}, \qquad s_m = s_{k_m}, \qquad \text{and} \qquad \gamma_m = \gamma_{k_m},$$

and

$$\bar{\Gamma}_m = \frac{1}{r'_m}\Gamma_{k_m}, \qquad \bar{\Omega}^+_m = \frac{1}{r'_m}\Omega^+_{k_m}, \qquad \bar{d}_m(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus \bar{\Omega}^+_m) = \frac{d_{k_m}(r'_m \cdot)}{r'_m}.$$

(Note that $\bar{\Gamma}_m$ is a *rescaled* version of Γ_{k_m}, so that, d_{k_m} does *not* coincide with $\bar{d}_m$.) Since $r_m \to 0$, then $\bar{\Gamma}_m$ will converge to $\{x_n = 0\}$ as $m \to \infty$. Also, $\bar{\Omega}^+_m$ will converge to $\mathbb{R}^n_+$ as $m \to \infty$. We consider the blow-up sequence

$$\begin{aligned} v_m(x) &= \frac{u_m(r'_m x)/[d_{k_m}^{s_m}(r'_m x)] - P_m(r'_m x)}{(r'_m)^{\alpha+\beta}\theta(r'_m)} \\ &= \frac{u_m(r'_m x)/[(r'_m)^{s_m}\bar{d}_m^{s_m}(x)] - P_m(r'_m x)}{(r'_m)^{\alpha+\beta}\theta(r'_m)}. \end{aligned} \tag{8.4}$$

As in the proof of Proposition 7.1, for all $m \geq 1$ we have

$$\int_{B_1 \cap \bar{\Omega}^+_m} v_m(x) P(x)\, dx = 0 \quad \text{for all } P \in \mathcal{P}, \tag{8.5}$$

$$[v_m]_{\beta; B_1 \cap \bar{\Omega}^+_m} \geq 1/2 \tag{8.6}$$

and

$$[v_m]_{\beta; B_R \cap \bar{\Omega}^+_m} \leq C R^{\alpha} \tag{8.7}$$

for all $R \geq 1$. Furthermore, we also have

$$\|v_m\|_{L^\infty(B_R \cap \bar{\Omega}^+_m)} \leq C R^{\alpha+\beta}. \tag{8.8}$$

Next we prove the following.

CLAIM

Let $w_m(x) = v_m(x)\bar{d}_m^s$. Then, up to subsequences we have $s_m \to s \in [s_0, 1]$ and $w_m \to w$ locally uniformly in $\mathbb{R}^n$, where $w \in C(\mathbb{R}^n)$. Moreover, the limiting function w satisfies the assumptions of the Liouville-type Theorem 1.4.

First recall that, by definition of $\bar{\Gamma}_m$ and $\bar{d}_m$, we have that $\bar{d}_m^s$ converges locally uniformly to $(x_n)_+^s$.

In the case $\beta > 0$, it follows from (8.7), (8.8), and the Arzelà–Ascoli theorem that a subsequence of v_m converges locally uniformly in $\mathbb{R}^n_+$ to some $v \in C(\{x_n > 0\})$. The convergence of w_m to $w = v(x_n)_+^s$ is then immediate.

In the case $\beta = 0$, the functions w_m satisfy $M^+ w_m \leq C(K)$ and $M^- w_m \geq -C(K)$ in $B_K \cap \bar{\Omega}^+_m$ for any $K > 0$, and they satisfy $w_m = 0$ in $\bar{\Omega}^-_m$ and

$\|w_m\|_{L^\infty(B_R)} \le CR^{s_m+\alpha}$ for every $R \ge 1$. Hence, we will have $\|w_m\|_{C^\delta(B_K)} \le C(K)$ for some $\delta > 0$. Thus, the functions w_m converge to some w uniformly in compact sets.

Passing to the limit (8.7), we find that the assumption (i) of Theorem 1.4 is satisfied by w. Let $\mathcal{L}_*^{\gamma_m}(s_m)$ be the class consisting of all the operators in $\mathcal{L}_*$ whose spectral measures have $C^{1,\gamma_m}(S^{n-1})$ norm smaller or equal than Λ, and let $M^+_{\mathcal{L}_*^{\gamma_m}(s_m)}$ and $M^-_{\mathcal{L}_*^{\gamma_m}(s_m)}$ denote the extremal Pucci operators for this class. Note that $M_{\mathcal{L}_*^{\gamma_m}(s_m)} \le M^+$. Let us prove that, similarly as in Proposition 7.1, there is a function $\delta(r)$ with $\lim_{r\searrow 0}\delta(r) = 0$ such that, for all $h \in B_K$ and $r'_m < 1/2K$, we have

$$-C(K)\delta(r'_m) \le M^+_{\mathcal{L}_*^{\gamma_m}(s_m)}\big(w_m(\cdot + h) - w_m\big) \quad \text{in } \bar\Omega^+_m \cap (\bar\Omega^+_m - h) \cap B_K. \tag{8.9}$$

To prove (8.9) we use the fact that, by definition of θ,

$$[u_k/d_k^{s_k} - P_{k,r}]_{\beta,B_r\cap\Omega_k} \le \theta(r)r^\alpha \quad \text{for all } k \text{ and } r > 0.$$

Thus, recognizing that $P_{k,r}$ is the best fitting polynomial in $\mathcal{P}$ for u_k in B_r, we obtain

$$\|u_k/d_k^{s_k} - P_{k,r}\|_{L^\infty(B_r\cap\Omega_k)} \le Cr^{\alpha+\beta}\theta(r),$$

where C depends only on the dimension. Hence, for all $r > 0$ and k, we have

$$\begin{aligned}\|P_{k,2r} - P_{k,r}\|_{L^\infty(B_r\cap\Omega_k)} &\le \|P_{k,2r} - u_k\|_{L^\infty(B_{2r}\cap\Omega_k)} + \|u_k - P_{k,r}\|_{L^\infty(B_r\cap\Omega_k)}\\ &\le C\theta(r)r^{\alpha+\beta}.\end{aligned} \tag{8.10}$$

Let us now denote

$$P_{k,r}(x) = p_{k,r}\cdot x + b_{k,r},$$

where $p_{k,r} \in \mathbb{R}^n$ is nonzero only if $\alpha + \beta > 1$ and where $b_{k,r} \in \mathbb{R}$. Using (8.2) and observing that $b_{k,r} = \fint_{B_r\cap\Omega_k^+} u_k\,dr$, we obtain

$$|b_{k,r}| \le 1 \quad \text{for all } k \text{ and } r > 0. \tag{8.11}$$

On the other hand, when $\alpha + \beta > 1$ using (8.10) we obtain

$$|p_{k,2r} - p_{k_r}| \le C\theta(r)r^{\alpha+\beta-1}.$$

Therefore, for $r = 2^{-i}$, we have

$$\frac{|p_{k,r} - p_{k,1}|}{\theta(r)} \le C\sum_{j=0}^{i}\frac{\theta(2^{-j})}{\theta(r)}(1/2)^{j(\alpha+\beta-1)}. \tag{8.12}$$

But again using (8.2), we obtain $|p_{k,1}| \le C$, and thus from (8.12) and (8.11) we have, for $r \in [2^{-i}, 2^{-i+1}]$,

$$\frac{|p_{k,r}| + |b_{k,r}|}{\theta(r)} \le C \sum_{j=0}^{i} \frac{\theta(2^{-j})}{\theta(r)} (1/2)^{j(\alpha+\beta-1)} =: \psi(r). \tag{8.13}$$

Note that $\psi(r) \le C$ for all $r \le 1$ and that, moreover, $\psi(r) \to 0$ as $r \searrow 0$ since $\frac{\theta(2^{-j})}{\theta(r)} \to 0$ for every fixed j. Hence, using Lemma 8.3 and the assumption that Γ_k is a global C^{2,γ_k} surface given by a graph of C^{2,γ_k} norm smaller than 1, we obtain

$$\Big[L\Big(\frac{d_{k_m}^{s_m} P_m}{\theta(r'_m)}\Big)\Big]_{C^{\gamma_m}(B_1 \cap \Omega_{k_m}^+)} \le C\psi(r'_m) \quad \text{for all } L \in \mathfrak{L}_*^{\gamma_m}(s_m), \tag{8.14}$$

which via rescaling is

$$(r'_m)^{-\gamma_m} \Big[(r'_m)^{-2s_m} L\Big(\frac{d_{k_m}^{s_m}(r'_m \cdot) P_m(r'_m \cdot)}{\theta(r'_m)}\Big)\Big]_{C^{\gamma_m}((r'_m)^{-1}(B_1 \cap \Omega_{k_m}^+))} \le C\psi(r'_m).$$

Equivalently, since $\bar\Omega_m = \frac{1}{r'_m}\Omega_{k_m}^+$ and $\bar d_m^{s_m} = (r'_m)^{-s_m} d_{k_m}^{s_m}(r'_m \cdot)$, we obtain

$$\Big[L\Big(\frac{\bar d_m^{s_m} P_m(r'_m \cdot)}{\theta(r'_m)(r'_m)^{\alpha+\beta}}\Big)\Big]_{C^{\gamma_m}(B_{1/r'_m} \cap \bar\Omega_m^+)} \le \frac{C\psi(r'_m)(r'_m)^{s_m+\gamma_m}}{(r'_m)^{\alpha+\beta}} \quad \text{for all } L \in \mathfrak{L}_*^{\gamma_m}(s_m). \tag{8.15}$$

Now, recall that $\gamma_m + s_m \ge \alpha + \beta$ (for all m) and that

$$w_m(x) = v_m(x)\bar d_m^{s_m}(x) = \frac{u_m(r'_m x)}{(r'_m)^{\alpha+\beta+s_m}\theta(r'_m)} - \frac{P_m(r'_m x) \cdot \bar d_m^{s_m}(x)}{(r'_m)^{\alpha+\beta}\theta(r'_m)}.$$

Therefore, using (8.15) and the same argument as in the proof of Proposition 7.1, we find

$$M^+_{\mathfrak{L}_*^{\gamma_m}(s_m)}\big(w_m(\cdot + h) - w_m\big) \ge -\frac{3K_m^\gamma}{\theta(r'_m)} - CK^{\gamma_m}\psi(r'_m) \quad \text{in } \bar\Omega_m^+ \cap (\bar\Omega_m^+ - h) \cap B_K$$

for all $h \in B_K$. Since $\theta(r'_m) \to \infty$ and $\psi(r'_m) \to 0$ as $r'_m \to 0$, (8.9) follows.

On the other hand, since $w_m \to w$ locally uniformly in $\mathbb{R}^n$ (up to subsequences), then we have

$$\big(w_m(\cdot + h) - w_m\big) \to \big(w(\cdot + h) - w\big) \quad \text{locally uniformly in } \mathbb{R}^n. \tag{8.16}$$

Also, similarly as in Proposition 7.1, we have, for every $h \in \mathbb{R}^n$ with $h_n \ge 0$,

$$\big(w_m(\cdot+h)-w_m\big)\to\big(w(\cdot+h)-w\big)\quad\text{in } L^1(\mathbb{R}^n,\omega_{s-\epsilon})\tag{8.17}$$

for some $\epsilon>0$. Thus, using (8.17) and (8.16) we can pass to the limit in (8.9) to obtain that, for every $K\ge 1$ and for every $h\in B_K^+$,

$$0\le M^+_{\mathfrak{L}^\gamma_*(s)}\big\{w(\cdot+h)-w\big\}\quad\text{in } B_K^+.$$

This yields

$$0\le M^+_{\mathfrak{L}^\gamma_*(s)}\big\{w(\cdot+h)-w\big\}\quad\text{in } \mathbb{R}^n_+$$

whenever $h_n\ge 0$. Analogously, we have

$$0\ge M^-_{\mathfrak{L}^\gamma_*(s)}\big\{w(\cdot+h)-w\big\}\quad\text{in } \mathbb{R}^n_+.$$

Since $M^+_{\mathfrak{L}^\gamma_*(s)}\le M^+$ and $M^-_{\mathfrak{L}^\gamma_*(s)}\ge M^-$, this finishes the proof the claim.

Hence, w satisfies all the assumptions of Theorem 1.4, and thus $v=w/(x_n)^s$ is an affine function. On the other hand, passing (8.5) to the limit we obtain that v is orthogonal to every affine function and hence it must be $v\equiv 0$. But then passing (8.6) to the limit, we obtain that v cannot be constantly zero in B_1, which is a contradiction. □

Proof of Theorem 1.3
Using Proposition 8.2 instead of Proposition 7.1, the proof follows exactly the same steps as those of Theorem 1.2. □

9. Flattening the boundary: Proof of Proposition 9.1

The aim of this section is to prove Propositions 9.1 and 9.2. Throughout this section, $d(x)$ is any $C^{2,\gamma}(\bar\Omega)$ function that coincides with $\mathrm{dist}(x,\mathbb{R}^n\setminus\Omega)$ in a neighborhood of $\partial\Omega$.

PROPOSITION 9.1
Let $s_0\in(0,1)$, let $s\in(s_0,1)$, and let $\gamma\in(0,s]$. Let Ω be any $C^{2,\gamma}$ bounded domain, and let L be any operator of the form (1.3), with $\mu\in C^{1,\gamma}(S^{n-1})$. Then, the function d^s satisfies

$$\big\|L(d^s)\big\|_{C^\gamma(\bar\Omega)}\le C,$$

where C is a constant that depends only on n, s_0, Ω, and $\|\mu\|_{C^{1,\gamma}(S^{n-1})}$.

In fact, we will also need the following.

PROPOSITION 9.2
Let $s_0 \in (0,1)$ and $s \in (s_0,1)$, and let $\gamma \in (0,s]$. Let Ω be any $C^{2,\gamma}$ bounded domain, and let L be any operator of the form (1.3), with $\mu \in C^{1,\gamma}(S^{n-1})$. Then, for any function $\eta \in C^{2,\gamma}(\mathbb{R}^n)$, we have

$$\left\| L(d^s\eta) \right\|_{C^\gamma(\bar{\Omega})} \le C \|\eta\|_{C^{2,\gamma}},$$

where C is a constant that depends only on n, s_0, Ω, and $\|\mu\|_{C^{1,\gamma}(S^{n-1})}$.

Remark 9.3
The dependence on Ω of the constant C in Propositions 9.1 and 9.2 is through the $C^{2,\gamma}$ norm of the diffeomorphism that flattens the boundary $\partial\Omega$. In particular, this constant C is uniform among all domains with a uniform bound on this $C^{2,\gamma}$ norm.

To prove these two propositions, we will need to flatten the boundary of Ω. In the following result we show how these operators change when we flatten the boundary.

PROPOSITION 9.4
Let $\bar{L}$ be any operator of the form (1.3), with $\mu \in C^{1,\gamma}(S^{n-1})$. Let Ω be any bounded $C^{2,\gamma}$ domain, and let $\bar{u}$ be any function satisfying

$$\bar{L}\bar{u} = \bar{f} \quad \text{in } \Omega, \qquad \bar{u} = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.$$

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^{2,\gamma}$ diffeomorphism that flattens the boundary $\partial\Omega$ and such that $(\phi_n)_+^s = d^s$. In particular, $\phi(B_1^+) = \Omega \cap \{d < 1\}$. Then, the function $u = \bar{u} \circ \phi$ satisfies the equation

$$L(u,x) = f(x) \quad \text{in } B_1^+, \qquad u = 0 \quad \text{in } B_1^-,$$

where $f = \bar{f} \circ \phi$ and

$$L(u,x) := \bar{L}(u \circ \phi^{-1})\big(\phi(x)\big).$$

Moreover, $L(u,x)$ can be written as

$$L(u,x) = \int_{\mathbb{R}^n} \big(u(x) - u(x+z)\big) K(x,z) \frac{dz}{|z|^{n+2s}},$$

and

$$K(x,z) = a_1\Big(x, \frac{z}{|z|}\Big) + |z| a_2\Big(x, \frac{z}{|z|}\Big) + |z|^{1+\gamma} J(x,z) \quad \text{for } |z| \le 2.$$

The functions a_1 and a_2 belong to $C^{1,\gamma}(S^{n-1})$ and $C^\gamma(S^{n-1})$ respectively, and J is C^γ with respect to x. Furthermore,

$$a_1(x,-\theta) = a_1(x,\theta) \quad \textit{for all } \theta \in S^{n-1},$$

and

$$a_2(x,-\theta) = -a_2(x,\theta) \quad \textit{for all } \theta \in S^{n-1}.$$

The C^γ norms of a_1, a_2, and J, depend only on n, s, the $\|\phi\|_{C^{2,\gamma}}$, and $\|\mu\|_{C^{1,\gamma}(S^{n-1})}$.

Proof
By definition,

$$L(u,x) = \int_{B_2} \{u(x) - u(\phi^{-1}(\phi(x)+y))\} \frac{\mu(y/|y|)}{|y|^{n+2s}}\, dy.$$

Thus, making the change of variables

$$y = \phi(x+z) - \phi(x),$$

(i.e., $z = \phi^{-1}(\phi(x)+y) - x$), we have

$$y = A(x)z + z^t B(x) z + |z|^{2+\gamma}\psi(x,z),$$

where

$$A(x) = D\phi(x), \qquad B(x) = D^2\phi(x),$$

and where $\psi(x,y)$ is bounded and C^γ in the x-variable. We have used the fact that ϕ is $C^{2,\gamma}$. Moreover, notice also that $A(x)$ is $C^{1+\gamma}$ and that $B(x)$ is C^γ. Writing now $z = r\theta$, with $r = |z|$ and $\theta \in S^{n-1}$, one finds

$$y = rA(x)\theta + r^2\theta^t B(x)\theta + r^{2+\gamma}\psi(x,r,\theta).$$

Therefore, this yields

$$|y| = r\big|A(x)\theta\big| + r^2\Big[\frac{A(x)\theta}{|A(x)\theta|}\cdot\big(\theta^t B(x)\theta\big)\Big] + r^{2+\gamma}\psi_1(x,r,\theta)$$

and also

$$\frac{1}{|y|} = \frac{1}{r|A(x)\theta|}\Big\{1 - \frac{r}{|A(x)\theta|^2}\big[\big(A(x)\theta\big)\cdot\big(\theta^t B(x)\theta\big)\big] + r^{1+\gamma}\psi_2(x,r,\theta)\Big\}.$$

Thus,

$$\frac{y}{|y|} = \frac{A(x)\theta}{|A(x)\theta|} + r\Big\{\frac{\theta^t B(x)\theta}{|A(x)\theta|} - \frac{A(x)\theta}{|A(x)\theta|^3}\big[\big(A(x)\theta\big)\cdot\big(\theta^t B(x)\theta\big)\big]\Big\} + r^{1+\gamma}\psi_3(x,r,\theta). \tag{9.1}$$

Moreover, the functions ψ_1, ψ_2, and ψ_3 are bounded and C^γ in the x-variable.

Now, recognizing that $\mu \in C^{1,\gamma}(S^{n-1})$ and (9.1), we have

$$\mu\big(y/|y|\big) = a_1(x,\theta) + r a_2(x,\theta) + r^{1+\gamma}\psi_4(x,r,\theta),$$

where

$$a_1(x,\theta) = a\Big(\frac{A(x)\theta}{|A(x)\theta|}\Big),$$

and

$$a_2(x,\theta) = \nabla_{S^{n-1}} a\Big(\frac{A(x)\theta}{|A(x)\theta|}\Big)\cdot\Big\{\frac{\theta^t B(x)\theta}{|A(x)\theta|} - \frac{A(x)\theta}{|A(x)\theta|^3}\big[\big(A(x)\theta\big)\cdot\big(\theta^t B(x)\theta\big)\big]\Big\}.$$

Moreover, the function ψ_4 is bounded and it is C^γ in the x-variable.

Finally notice that, since $\mu(y/|y|) = \mu(-y/|y|)$, it immediately follows from the expressions of a_1 and a_2 that $a_1(x,\theta) = a_1(x,-\theta)$ and that $a_2(x,-\theta) = -a_2(x,\theta)$. □

We will also need the following lemmas.

LEMMA 9.5

Let $s_0 \in (0,1)$, let $s \in (s_0,1)$, and let $\gamma \in (0,s]$. Let $a_1(x,\theta)$ be a function in $L^\infty(\mathbb{R}^n \times S^{n-1})$ which is C^γ in x and which satisfies

$$a_1(x,-\theta) = a_1(x,\theta) \quad \textit{for all } \theta \in S^{n-1}.$$

Define

$$I_1(x) := \int_{B_2} \big((x_n)_+^s \eta(x) - (x_n + y_n)_+^s \eta(x+y)\big)\frac{a_1(x,y/|y|)}{|y|^{n+2s}}\,dy.$$

Then, we have $I_1 \in C^\gamma(\overline{B_1^+})$, and

$$\|I_1\|_{C^\gamma(B_1^+)} \le \frac{C}{1-s},$$

where C depends only on n, s_0, the $C^{2,\gamma}$ norm of η, and the C^γ norm of a_1.

Proof

Case 1: Assume that $\eta \equiv 1$. Then, since a_1 is even, we have

$$\int_{\mathbb{R}^n} \big((x_n)_+^s - (x_n + y_n)_+^s\big)\frac{a_1(x,y/|y|)}{|y|^{n+2s}}\,dy = c(x)(-\Delta)^s(x_+)^s = 0. \qquad (9.2)$$

Therefore,

$$I_1(x) = \int_{\mathbb{R}^n \setminus B_2} \big((x_n)_+^s - (x_n + y_n)_+^s\big) \frac{a_1(x, y/|y|)}{|y|^{n+2s}}\, dy.$$

Now, using that $(x_n)_+^s$ is C^γ, we have

$$\big|(x_n^{(1)})_+^s - (x_n^{(1)} + y_n)_+^s - (x_n^{(2)})_+^s + (x_n^{(2)} + y_n)_+^s\big| \le C|x_1 - x_2|^\gamma |y|^{s-\gamma}$$

for any x_1 and x_2 in B_1^+. Thus, recognizing also that a_1 is C^γ with respect to x, we find

$$\begin{aligned}\big|I_1(x_1) - I_1(x_2)\big| &\le \int_{\mathbb{R}^n \setminus B_2} C|x_1 - x_2|^\gamma |y|^{s-\gamma} \frac{C}{|y|^{n+2s}}\, dy \\ &\quad + \int_{\mathbb{R}^n \setminus B_2} C|y|^s \frac{C|x_1 - x_2|^\gamma}{|y|^{n+2s}}\, dy \\ &\le C|x_1 - x_2|^\gamma.\end{aligned}$$

Case 2: Assume that η is a linear function, $\eta(x) = b \cdot x + c$. Then,

$$\begin{aligned}I_1(x) = \mathrm{PV}&\int_{B_2} (x_n + y_n)_+^s (b \cdot y) \frac{a_1(x, y/|y|)}{|y|^{n+2s}}\, dy \\ &+ (b \cdot x + c) \int_{\mathbb{R}^n \setminus B_2} \big((x_n)_+^s - (x_n + y_n)_+^s\big) \frac{a_1(x, y/|y|)}{|y|^{n+2s}}\, dy,\end{aligned}$$

where we have used (9.2). The second term is C^γ, as we already proved that in Case 1. Hence, it remains to see that the first term is C^γ also. Since a_1 is C^γ in x, it suffices to prove that

$$\left|\mathrm{PV}\int_{B_2} \big[(x_n + y_n)_+^s - (x_n + h + y_n)_+^s\big](b \cdot y) \frac{a_1(x, y/|y|)}{|y|^{n+2s}}\, dy\right| \le \frac{C}{1-s}|h|.$$

To prove this, we show next that the function

$$I_{1,2}(x) := \mathrm{PV}\int_{B_2} (x_n + y_n)_+^{s-1} (b \cdot y) \frac{a_1(x, y/|y|)}{|y|^{n+2s}}\, dy$$

satisfies

$$\big|I_{1,2}(x)\big| \le \frac{C}{1-s}.$$

Here, we denoted $(t)_+^{s-1} = (|t|^{s-2} t)_+$. To bound $I_{1,2}(x)$, we first notice that

$$\mathrm{PV}\int_{\mathbb{R}^n} (x_n + y_n)_+^{s-1} (b \cdot y) \frac{a_1(x, y/|y|)}{|y|^{n+2s}}\, dy = 0.$$

Indeed, this follows from

$$\begin{aligned}
&\mathrm{PV}\int_{\mathbb{R}^n}(x_n+y_n)_+^{s-1}(b\cdot y)\frac{a_1(x,y/|y|)}{|y|^{n+2s}}\,dy\\
&=\mathrm{PV}\int_{S^{n-1}}|\theta_n|^{2s}|b\cdot\theta|a_1(x,\theta)\,d\theta\int_{-\infty}^{+\infty}(x_n+r)_+^{s-1}\frac{r\,dr}{|r|^{1+2s}}\\
&=c(x)(x_n)^{-s}\,\mathrm{PV}\int_{-\infty}^{+\infty}(1+r)_+^{s-1}\frac{r\,dr}{|r|^{1+2s}}
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{PV}\int_{-\infty}^{+\infty}(1+r)_+^{s-1}\frac{r\,dr}{|r|^{1+2s}}\\
&=\mathrm{PV}\int_0^{+\infty}t^{s-1}\frac{t-1}{|t-1|^{1+2s}}\,dr\\
&=\lim_{\epsilon\to 0}\left\{-\int_0^{1-\epsilon}t^{s-1}\frac{dt}{(1-t)^{2s}}+\int_{1+\epsilon}^{\infty}t^{s-1}\frac{dt}{(t-1)^{2s}}\right\}\\
&=\lim_{\epsilon\to 0}\left\{-\int_0^{1-\epsilon}t^{s-1}\frac{dt}{(1-t)^{2s}}+\int_0^{\frac{1}{1+\epsilon}}z^{1-s}\frac{z^{2s-2}\,dz}{(1-z)^{2s}}\right\}\\
&=\lim_{\epsilon\to 0}\int_{1-\epsilon}^{\frac{1}{1+\epsilon}}t^{s-1}\frac{dt}{(1-t)^{2s}}\\
&=0. \qquad (9.3)
\end{aligned}$$

Thus,

$$I_{1,2}(x)=-\int_{\mathbb{R}^n\setminus B_2}(x_n+y_n)_+^{s-1}(b\cdot y)\frac{a_1(x,y/|y|)}{|y|^{n+2s}}\,dy,$$

and using the fact that $x\in B_1$,

$$\left|I_{1,2}(x)\right|\le\int_{\mathbb{R}^n\setminus B_2}|y_n|_+^{s-1}|y|\frac{C}{|y|^{n+2s}}\,dy=\frac{C}{1-s},$$

as desired.

Case 3: Now let us proceed to the general case $\eta\in C^{2,\gamma}$. We have

$$\begin{aligned}
I_1(x)=\ &\mathrm{PV}\int_{B_2}(x_n+y_n)_+^s\big[\eta(x)-\eta(x+y)\big]\frac{a_1(x,y/|y|)}{|y|^{n+2s}}\,dy\\
&+\eta(x)\int_{\mathbb{R}^n\setminus B_2}\big((x_n)_+^s-(x_n+y_n)_+^s\big)\frac{a_1(x,y/|y|)}{|y|^{n+2s}}\,dy,
\end{aligned}$$

where we have used (9.2). The second term is C^γ, as we already proved that in Case 1. Hence, it remains to see that the first term is C^γ also. Let us denote $I_{1,3}(x)$ this first term; that is,

$$I_{1,3}(x) = \mathrm{PV}\int_{B_2} \xi(x,y)\frac{a_1(x,y/|y|)}{|y|^{n+2s}}\,dy,$$

with

$$\xi(x,y) = (x_n+y_n)_+^s\big[\eta(x)-\eta(x+y)\big].$$

Using that a_1 is C^γ with respect to x, and that

$$\begin{aligned}\big|\xi(x,y)+\xi(x,-y)\big| &\le (x_n+y_n)_+^s\big|2\eta(x)-\eta(x+y)-\eta(x-y)\big| \\ &\qquad + \big|(x_n+y_n)_+^s-(x_n-y_n)_+^s\big|\cdot\big|\eta(x)-\eta(x-y)\big| \\ &\le C|y|^{1+s}, \end{aligned} \tag{9.4}$$

we have

$$\begin{aligned}\big|I_{1,3}(x)-I_{1,3}(x+he_i)\big| &\le \Big|\mathrm{PV}\int_{B_2}\big[\xi(x,y)-\xi(x+he_i,y)\big]\frac{a_1(x,y/|y|)}{|y|^{n+2s}}\,dy\Big| \\ &\qquad + \int_{B_2} C|y|^{1+s}\frac{C|h|^\gamma}{|y|^{n+2s}}\,dy.\end{aligned}$$

We now claim, for any $i=1,\dots,n$, that

$$\Big|\mathrm{PV}\int_{B_2}\big[\xi(x,y)-\xi(x+he_i,y)\big]\frac{a_1(x,y/|y|)}{|y|^{n+2s}}\,dy\Big| \le \frac{C}{1-s}|h|^\gamma. \tag{9.5}$$

Indeed, since η is $C^{2,\gamma}$, we have

$$\big\|2\eta(\cdot)-\eta(\cdot+y)-\eta(\cdot-y)\big\|_{C^\gamma} \le C|y|^2,$$

and hence a similar computation as in (9.4) yields

$$\big|\xi(x,y)+\xi(x,-y)-\xi(x+he_i,y)-\xi(x+he_i,-y)\big| \le C|y|^{1+s}|h|^\gamma,$$

and therefore (9.5) follows. Hence, we have showed that

$$\big|I_{1,3}(x)-I_{1,3}(x+he_i)\big| \le \frac{C}{1-s}|h|^\gamma,$$

and thus the lemma is proved. □

LEMMA 9.6

Let $s_0 \in (0,1)$, let $s \in (s_0, 1)$, and let $\gamma \in (0, s]$. Let $a_2(x,\theta)$ be a function in $L^\infty(\mathbb{R}^n \times S^{n-1})$ which is C^γ in x and which satisfies

$$a_2(x,-\theta) = -a_2(x,\theta) \quad \text{for all } \theta \in S^{n-1}.$$

Let $\eta \in C_c^{2,\gamma}(\mathbb{R}^n)$, and define

$$I_2(x) := \int_{B_2} \big((x_n)_+^s \eta(x) - (x_n + y_n)_+^s \eta(x+y)\big) \frac{a_2(x, y/|y|)}{|y|^{n+2s-1}}\, dy.$$

Then, we have $I_2 \in C^\gamma(\overline{B_1^+})$, and

$$\|I_2\|_{C^\gamma(B_1^+)} \le \frac{C}{1-s},$$

where C depends only on n, s_0, the $C^{2,\gamma}$ norm of η, and the C^γ norm of a_2.

Proof

Case 1: Assume that $\eta \equiv 1$. Since the function $(x_n)_+^s$ does not depend on the first $n-1$ variables and a_2 is C^γ with respect to x, then it is clear that

$$\big|I_2(x) - I_2(x + he_i)\big| \le \int_{B_2} |y|^s \frac{C|h|^\gamma}{|y|^{n+2s-1}}\, dy \le \frac{C}{1-s}|h|^\gamma$$

for $i = 1, 2, \dots, n-1$. Moreover, we also have

$$\begin{aligned} &I_2(x) - I_2(x+he_n) \\ &\quad = \int_{B_2} \big((x_n+h)_+^s - (x_n+h+y_n)_+^s\big) \frac{a_2(x,y/|y|) - a_2(x+he_n, y/|y|)}{|y|^{n+2s-1}}\, dy \\ &\qquad + \int_{B_2} \big\{(x_n)_+^s - (x_n+y_n)_+^s - (x_n+h)_+^s + (x_n+h+y_n)_+^s\big\} \frac{a_2(x,y/|y|)}{|y|^{n+2s-1}}\, dy. \end{aligned}$$

As before, the first term is bounded by

$$\begin{aligned} &\Big|\int_{B_2} \big((x_n+h)_+^s - (x_n+h+y_n)_+^s\big) \frac{a_2(x,y/|y|) - a_2(x+he_n, y/|y|)}{|y|^{n+2s-1}}\, dy\Big| \\ &\quad \le \frac{C}{1-s}|h|^\gamma. \end{aligned}$$

Thus, it only remains to see that the second term is also bounded by $C|h|^\gamma$.

We will show that, in fact, we have

$$\left|\int_{B_2}\left\{(x_n)_+^s-(x_n+y_n)_+^s-(x_n+h)_+^s+(x_n+h+y_n)_+^s\right\}\frac{a_2(x,y/|y|)}{|y|^{n+2s-1}}\,dy\right| \\ \leq \frac{C}{1-s}|h|. \tag{9.6}$$

Indeed, it is clear that (9.6) is equivalent to

$$\left|\int_{B_2}\left\{(x_n)_+^{s-1}-(x_n+y_n)_+^{s-1}\right\}\frac{a_2(x,y/|y|)}{|y|^{n+2s-1}}\,dy\right|\leq\frac{C}{1-s}, \tag{9.7}$$

where we denoted (abusing a little bit the notation) $(x_n)_+^{s-1}=(|x_n|^{s-2}x_n)_+$. Let us define

$$\tilde{I}_2(x):=\int_{B_2}\left\{(x_n)_+^{s-1}-(x_n+y_n)_+^{s-1}\right\}\frac{a_2(x,y/|y|)}{|y|^{n+2s-1}}\,dy.$$

Notice that, since a_2 is odd, then

$$\tilde{I}_2=-\mathrm{PV}\int_{B_2}(x_n+y_n)_+^{s-1}\frac{a_2(x,y/|y|)}{|y|^{n+2s-1}}\,dy.$$

We now claim that

$$\hat{I}_2(x):=\mathrm{PV}\int_{\mathbb{R}^n}(x_n+y_n)_+^{s-1}\frac{a_2(x,y/|y|)}{|y|^{n+2s-1}}\,dy=0.$$

Indeed, we have

$$\begin{aligned}\hat{I}_2(x)&=\mathrm{PV}\int_{-\infty}^{+\infty}\int_{S^{n-1}}(x_n+r\theta_n)_+^{s-1}a_2(x,\theta)\frac{r}{|r|^{1+2s}}\,d\theta\,dr\\ &=\mathrm{PV}\int_{-\infty}^{+\infty}\int_{S^{n-1}}(x_n+r)_+^{s-1}|\theta_n|^{2s-1}\theta_n a_2(x,\theta)\frac{r}{|r|^{1+2s}}\,d\theta\,dr\\ &=c(x)\,\mathrm{PV}\int_{-\infty}^{+\infty}(x_n+r)_+^{s-1}\frac{r}{|r|^{1+2s}}\,dr\\ &=c(x)(x_n)^{-s}\,\mathrm{PV}\int_{-\infty}^{+\infty}(1+r)_+^{s-1}\frac{r}{|r|^{1+2s}}\,dr,\end{aligned}$$

and hence the claim follows from (9.3). Therefore, we have

$$\tilde{I}_2(x)=\int_{\mathbb{R}^n\setminus B_2}(x_n+y_n)_+^{s-1}\frac{a_2(x,y/|y|)}{|y|^{n+2s-1}}\,dy,$$

and then, using the fact that $x\in B_1$,

$$\begin{aligned}\left|\tilde{I}_2(x)\right| &= C\int_{\mathbb{R}^n\setminus B_2}(x_n+y_n)_+^{s-1}\frac{dy}{|y|^{n+2s-1}}\\ &\le C\int_{\mathbb{R}^n_+\setminus B_2^+}(y_n)_+^{s-1}\frac{dy}{|y|^{n+2s-1}}\le\frac{C}{1-s}.\end{aligned}$$

Hence, (9.7) is proved, and the lemma follows.

Case 2: Assume now that η is any $C^{2,\gamma}$ function. Then, using the result in Case 1, it suffices to show that the function

$$I_{2,2}(x):=\int_{B_2}(x_n+y_n)_+^s\big[\eta(x)-\eta(x+y)\big]\frac{a_2(x,y/|y|)}{|y|^{n+2s-1}}\,dy$$

is C^γ.

For $i=1,\dots,n-1$, we have

$$\begin{aligned}&\left|I_{2,2}(x)-I_{2,2}(x+he_i)\right|\\ &\quad\le\int_{B_2}(x_n+y_n)_+^s\left|\eta(x+h)-\eta(x+h+y)\right|\\ &\qquad\times\frac{|a_2(x,y/|y|)-a_2(x+h,y/|y|)|}{|y|^{n+2s-1}}\,dy\\ &\qquad+\left|\int_{B_2}(x_n+y_n)_+^s\big[\eta(x)-\eta(x+y)-\eta(x+h)+\eta(x+h+y)\big]\right.\\ &\qquad\left.\times\frac{a_2(x,y/|y|)}{|y|^{n+2s-1}}\,dy\right|.\end{aligned}$$

Since a_2 is C^γ, then

$$\begin{aligned}&\int_{B_2}(x_n+y_n)_+^s\left|\eta(x+h)-\eta(x+h+y)\right|\frac{|a_2(x,y/|y|)-a_2(x+h,y/|y|)|}{|y|^{n+2s-1}}\,dy\\ &\quad\le\int_{B_2}C|y|\frac{C|h|^\gamma}{|y|^{n+2s-1}}\,dy\le\frac{C}{1-s}|h|^\gamma.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\left|\int_{B_2}(x_n+y_n)_+^s\big[\eta(x)-\eta(x+y)-\eta(x+h)+\eta(x+h+y)\big]\frac{a_2(x,y/|y|)}{|y|^{n+2s-1}}\,dy\right|\\ &\quad\le\frac{C}{1-s}|h|.\end{aligned}$$

Indeed, this is equivalent to

$$\left|\int_{B_2}(x_n+y_n)_+^s\big[\partial_{x_i}\eta(x)-\partial_{x_i}\eta(x+y)\big]\frac{a_2(x,y/|y|)}{|y|^{n+2s-1}}\,dy\right|\le\frac{C}{1-s},$$

and this follows immediately from the fact that η is $C^{2,\gamma}$.

For $i=n$, the reasoning is very similar, and we only have to bound the additional term

$$\begin{aligned}\int_{B_2}&\big|(x_n+y_n+h)_+^s-(x_n+y_n)_+^s\big|\cdot\big|\eta(x)-\eta(x+y)\big|\frac{|a_2(x,y/|y|)|}{|y|^{n+2s-1}}\,dy\\&\le\int_{B_2}|h|^s|y|\frac{dy}{|y|^{n+2s-1}}\le\frac{C}{1-s}|h|^s.\end{aligned}$$

Thus, the lemma is proved. □

LEMMA 9.7
Let $s_0\in(0,1)$, let $s\in(s_0,1)$, and let $\gamma\in(0,s]$. Let $J(x,y)$ be a function in $L^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ which is C^γ in x. Let $\eta\in C_c^{2,\gamma}(\mathbb{R}^n)$, and define

$$I_3(x):=\int_{B_2}\big((x_n)_+^s\eta(x)-(x_n+y_n)_+^s\eta(x+y)\big)\frac{J(x,y)}{|y|^{n+2s-1-\gamma}}\,dy.$$

Then, we have $I_3\in C^\gamma(\overline{B_1^+})$, and

$$\|I_3\|_{C^\gamma(B_1^+)}\le\frac{C}{1-s},$$

where C depends only on n, s_0, the $C^{2,\gamma}$ norm of η, and the C^γ norm of J.

Proof
Case 1: Assume first that $\eta\equiv 1$. Using (9.2) and recognizing that J is C^γ with respect to x, we have

$$\begin{aligned}\big|I_3(x_1)-I_3(x_2)\big|&\le\int_{B_2}C|x_1-x_2|^\gamma|y|^{s-\gamma}\frac{C}{|y|^{n+2s-1-\gamma}}\,dy\\&\quad+\int_{B_2}C|y|^s\frac{C|x_1-x_2|^\gamma}{|y|^{n+2s-1-\gamma}}\,dy\\&\le\frac{C}{1-s}|x_1-x_2|^\gamma+C|x_1-x_2|^\gamma,\end{aligned}$$

and the result follows.

Case 2: Assume now that η is any $C^{2,\gamma}$ function. Then, one only needs to use that $g(x):=(x_n)_+^s\eta(x)$ is a C^s function. Indeed, we then have

$$\big|g(x_1)-g(x_1+y)-g(x_2)+g(x_2+y)\big|\le C|x_1-x_2|^s$$

and

$$\big|g(x_1)-g(x_1+y)-g(x_2)+g(x_2+y)\big|\le C|y|^s,$$

and interpolating these two inequalities,

$$\big|g(x_1) - g(x_1 + y) - g(x_2) + g(x_2 + y)\big| \le C|x_1 - x_2|^\gamma |y|^{s-\gamma}.$$

Using this, the proof is the same as in Case 1. $\square$

Using the previous lemmas, we can now give the following.

Proofs of Propositions 9.1 and 9.2
The proofs follow immediately from Proposition 9.4 and Lemmas 9.5, 9.6, and 9.7. $\square$

Remark 9.8
In case both the domain Ω and the spectral measure μ are C^∞, the result in Proposition 9.1 is well known and can be proved by Fourier transform methods (see [21]). In this case, one has that $L(d^s)$ is $C^\infty(\bar{\Omega})$.

Appendices

A. Proof of Lemma 6.1

In this appendix we give the following.

Proof of Lemma 6.1
Let us show first the statement (a). Denote

$$a = 1 - 2s.$$

We first note that the Caffarelli–Silvestre extension equation $\Delta u + \frac{a}{y}\partial_y u = 0$ is written in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$, $\theta \in (0,\pi)$ as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{a}{r\sin\theta}\Big(\sin\theta\, u_r + \cos\theta\frac{u_\theta}{r}\Big) = 0.$$

Note the homogeneity of the equation in the variable r. If we seek for (bounded at 0) solutions of the form $u = r^{s+\nu}\Theta_\nu(\theta)$, then it must be $\nu > -s$ and

$$\Theta_\nu'' + a\operatorname{cotg}\theta\,\Theta_\nu' + (s+\nu)(s+\nu+a)\Theta_\nu = 0.$$

If we want u to satisfy the boundary conditions

$$u(x,0) = 0 \quad \text{for } x < 0 \qquad \text{and} \qquad |y|^a\partial_y u(x,y) \to 0 \quad \text{as } y \to 0,$$

then Θ_ν must satisfy

$$\begin{cases} \Theta_\nu(\theta) = \Theta_\nu(0) + o((\sin\theta)^{2s}) \to 0 & \text{as } \theta \searrow 0, \\ \Theta_\nu(\pi) = 0. \end{cases} \tag{A.1}$$

We have used the fact that, for $x > 0$,

$$\lim_{y \searrow 0} y^a \partial_y u(x,y) = 0 \quad \Rightarrow \quad u(x,y) = u(x,0) + o(y^{2s}),$$

since $a = 1 - 2s$.

To solve this ordinary differential equation (ODE), consider

$$\Theta_\nu(\theta) = (\sin\theta)^s h(\cos\theta).$$

After some computations and the change of variable $z = \cos\theta$, we obtain the following ODE for $h(z)$:

$$(1-z^2)h''(z) - 2zh'(z) + \Big(\nu + \nu^2 - \frac{s^2}{1-z^2}\Big)h(z) = 0.$$

This is the so-called *associated Legendre differential equation*. All solutions to this second-order ODE solutions are given by

$$h(z) = C_1 P_\nu^s(z) + C_2 Q_\nu^s(z),$$

where P_ν^s and Q_ν^s are the *associated Legendre functions* of first and second kind, respectively. Translating (A.1) to the function h and using the fact that $\sin\theta \sim (1 - \cos\theta)^{1/2}$ as $\theta \searrow 0$ and $\sin\theta \sim (1+\cos\theta)^{1/2}$ as $\theta \nearrow \pi$, we obtain

$$\begin{cases} (1-z)^{s/2}h(z) = c + o((1-z)^s) & \text{as } z \nearrow 1, \\ \lim_{z\searrow -1}(1+z)^{s/2}h(z) = 0. \end{cases} \tag{A.2}$$

Let us prove that P_ν^s fulfills all these requirements only for $\nu = 0, 1, 2, 3, \ldots$, while Q_ν^s has to be discarded. To have a good description of the singularities of $P_\nu^s(z)$ at $z = \pm 1$, we use its expression as an hypergeometric function

$$P_\nu^s(z) = \frac{1}{\Gamma(1-s)} \frac{(1+z)^{s/2}}{(1-z)^{s/2}} {}_2F_1\Big(-\nu, \nu+1; 1-s; \frac{1-z}{2}\Big).$$

Using this and the definition of ${}_2F_1$ as a power series, we obtain

$$P_\nu^s(z) = \frac{1}{\Gamma(1-s)} \frac{2^{s/2}}{(1-z)^{s/2}} \Big\{1 - \frac{\nu(\nu+1)}{1-s}\frac{1-z}{2} + o\big((1-z)^2\big)\Big\} \quad \text{as } z \nearrow 1.$$

Hence, $(1-z)^{s/2}P_\nu^s(z) = c + O(1-z) = c + o((1-z)^s)$, as desired.

For the analysis as $z \searrow -1$, we need to use Euler's transformation

$${}_2F_1(a,b;c;x) = (1-x)^{c-b-a}{}_2F_1(c-a, c-b; c; x),$$

obtaining

$$P_\nu^s(z) = \frac{1}{\Gamma(1-s)} \frac{(1+z)^{s/2}}{2^{s/2}} \Big(\frac{1+z}{2}\Big)^{-s} \big\{ {}_2F_1(1-s-\nu, -s-\nu; 1-s; 1) + o(1) \big\}$$

as $z \searrow -1$. It follows that the zero-boundary condition is satisfied if and only if

$${}_2F_1(1-s-\nu, -s-\nu; 1-s; 1) = \frac{\Gamma(1-s)\Gamma(s)}{\Gamma(-\nu)\Gamma(1+\nu)} = 0.$$

This implies that $\nu = 0, 1, 2, 3, \ldots$, so that $\Gamma(-\nu) = \infty$. With a similar analysis one easily finds that the functions $Q_\nu^s(x)$ do not satisfy (A.2) for any $\nu \geq -s$.

The statement (b) of the lemma could be proved for example by using singular Sturm–Liouville theory after observing that the ODE

$$\Theta_\nu'' + a \operatorname{cotg} \theta \Theta_\nu' - \lambda \Theta_\nu = 0$$

can be written as

$$\big(|\sin\theta|^a \Theta_\nu'\big)' = \lambda |\sin\theta|^a \Theta_\nu.$$

However, it is not necessary to do it because we have already computed the eigenfunctions to this ODE, and they are given by

$$\Theta_k(\theta) = (\sin\theta)^s P_k^s(\cos\theta),$$

where P_ν^s are the associated Legendre functions of first kind. The functions $\{P_k^s(x)\}_{k\geq 0}$ have been well studied, and they are known to be a complete orthogonal system in $L^2((0,1), dx)$ (see [30], [51]). Therefore, it immediately follows (after a change of variables) that $\{\Theta_k(\theta)\}_{k\geq 0}$ are a complete orthogonal system in $L^2((0,\pi), (\sin\theta)^a\, d\theta)$. Thus, the lemma is proved. □

B. Interior regularity

We give here the proof of the interior estimate in Lemma 7.5. For it, we will need the following.

LEMMA B.1

Let $\bar\alpha > 0$ be the exponent given by Proposition 1.1. Assume that $u \in C(\mathbb{R}^n)$ satisfies in the viscosity sense

$$M^+\big\{u(\cdot + h) - u\big\} \geq 0 \qquad \text{and} \qquad M^-\big\{u(\cdot + h) - u\big\} \leq 0 \quad \text{in } \mathbb{R}^n$$

for all $h \in \mathbb{R}^n$. Assume that for some $\beta \in (0,1)$ and $\alpha \in (0, \bar\alpha)$, u satisfies

$$[u]_{C^\beta(B_R)} \leq CR^\alpha \quad \text{for all } R \geq 1. \tag{B.1}$$

Then,

$$u(x) = p \cdot x + b$$

for some $p \in \mathbb{R}^n$ *and* $b \in \mathbb{R}$.

Proof
Given that $\rho \geq 1$, let $v(x) = \frac{u(\rho x+\rho h)-u(\rho x)}{\rho^\alpha |\rho h|^\beta}$. By assumption, we have

$$M^+ v \geq 0 \qquad \text{and} \qquad M^- v \leq 0 \quad \text{in } B_1$$

and

$$\|v\|_{L^\infty(B_R)} \leq C R^\alpha$$

for all $R \geq 1$. Hence it follows from the interior estimate in [14] (recall that $M^-_{\mathcal{L}_0} \leq M^- \leq M^+ \leq M^+_{\mathcal{L}_0}$) that

$$\|v\|_{C^{\bar\alpha}(B_1)} \leq C.$$

We use now the following well-known result: if all incremental quotients of order β of u are uniformly C^α, then u is $C^{\beta+\alpha}$ (unless $\beta + \alpha$ is an integer) (see, e.g., [4, Proposition 2.1] or [10, Lemma 5.6]). This yields

$$[u]_{C^{\bar\alpha+\beta}(B_\rho)} \leq C\rho^{\alpha-\bar\alpha}.$$

Letting $\rho \to \infty$, we conclude that $[u]_{C^{\bar\alpha+\beta}(\mathbb{R}^n)} = 0$, and the lemma follows. □

We next show the following.

PROPOSITION B.2
Let $s_0 \in (0,1)$*, and let* $\bar\alpha \in (0, s_0)$ *be the constant given by Proposition 1.1. Let* $s \in (s_0, 1)$*, let* $\alpha \in (0, \bar\alpha)$*, let* $\gamma \in (0,1)$*, and let* $\beta \in (0,1)$ *such that* $\alpha + \beta \leq \gamma + 2s$*. Assume in addition that* $\alpha + \beta \neq 1$*. Let* $w \in C^\beta(\mathbb{R}^n)$ *be a solution of* $\mathrm{I}(w,x) = 0$ *in* B_1*, where* I *is any fully nonlinear operator elliptic with respect to* $\mathcal{L}_*(s)$ *of the form*

$$\mathrm{I}(w,x) = \inf_{b\in\mathcal{B}} \sup_{a\in\mathcal{A}} \big(L_{ab} w(x) + c_{ab}(x)\big).$$

Suppose that L_{ab} *are given by (1.3) and (1.4) and that*

$$\big|\inf c_{ab}(x)\big| < \infty, \quad [c_{ab}]_{\gamma;B_1} \leq 1. \tag{B.2}$$

Then,

$$\|w\|_{C^{\beta+\alpha}(B_{1/2})} \le C\|w\|_{C^{\beta}(\mathbb{R}^n)}, \tag{B.3}$$

for some constant C that depends only on n, s_0, the ellipticity constants, α, and β.

Proof

It suffices to prove the estimate

$$\sup_{r>0} r^{-\alpha}[w-P_r]_{C^{\beta};B_r} \le C\|w\|_{\beta;\mathbb{R}^n}, \tag{B.4}$$

where

$$P_r := \operatorname*{arg\,min}_{P\in\mathcal{P}} \int_{B_r} (w_k - P)^2\,dx,$$

$\mathcal{P}$ begin the linear space of polynomials of degree at most $\lfloor \alpha+\beta\rfloor$ with real coefficients. Using (B.4), (B.3) follows easily.

The proof of (B.4) is by contradiction. If it did not hold, then there would be sequences w_k, I_k, s_k, and γ_k satisfying

- $\|w_k\|_{\beta;\mathbb{R}^n} \le 1$,
- $\mathrm{I}_k(w_k, x) = 0$ in B_1,
- $\mathrm{I}_k(w_k, x) = \inf_{b\in\mathcal{B}_k}\sup_{a\in\mathcal{A}_k}(L_{ab}u_k(x) + c_{ab}(x))$,
- $\{L_{ab} : a\in\mathcal{A}_k, b\in\mathcal{B}_k\} \subset \mathcal{L}(s_k)$ with $s_k\in[s_0,1]$,
- $|\inf c_{ab}(x)| < \infty$, $[c_{ab}]_{\gamma;B_1} \le 1$ for all $a\in\mathcal{A}_k$ and $b\in\mathcal{B}_k$,
- $\gamma_k + 2s_k \ge \alpha+\beta$

for which

$$\sup_k\sup_{r>0} r^{-\alpha}[w_k - P_{k,r}]_{\beta;B_r} = +\infty, \tag{B.5}$$

where

$$P_{k,r} := \operatorname*{arg\,min}_{P\in\mathcal{P}} \int_{B_r} \big(w_k(x) - P\big)\,dx.$$

To prove that this is impossible, we proceed similarly as in the Proof of Proposition 7.1. We define

$$\theta(r) := \sup_k\sup_{r'>r}(r')^{-\alpha}[w_k - P_{k,r}]_{\beta;B_{r'}}.$$

The function θ is monotone-nonincreasing, and $\theta(r) < +\infty$ for $r > 0$ since $\|w_k\|_{\beta;\mathbb{R}^n} \le 1$. In addition, by (B.5) we have $\theta(r) \nearrow +\infty$ as $r \searrow 0$, and there are sequences $r'_m \searrow 0$ and k_m for which

$$(r'_m)^{-\alpha}[w_k - P_{k_m,r'_m}]_{\beta;B_{r'_m}} \geq \frac{1}{2}\theta(r'_m). \tag{B.6}$$

For the rest of this proof, we will use the notation

$$u_m = u_{k_m}, \qquad P_m = P_{k_m,r'_m}, \qquad s_m = s_{k_m}, \qquad \gamma_m = \gamma_{k_m}.$$

We consider the blow-up sequence

$$v_m(x) = \frac{w_m(r'_m x) - P_m(r'_m x)}{(r'_m)^{\alpha+\beta}\theta(r'_m)}. \tag{B.7}$$

As in the proof of Proposition 7.1, for all $m \geq 1$ we have

$$\int_{B_1} v_m(x)P(x)\,dx = 0 \quad \text{for all } P \in \mathcal{P}, \tag{B.8}$$

$$[v_m]_{\beta;B_1} \geq 1/2 \tag{B.9}$$

and

$$[v_m]_{\beta;B_R} \leq CR^{\alpha} \tag{B.10}$$

for all $R \geq 1$. Furthermore, we also have

$$\|v_m\|_{L^\infty(B_R)} \leq CR^{\alpha+\beta}. \tag{B.11}$$

Next we prove the following.

CLAIM
Up to subsequences, we have $s_m \to s \in [s_0, 1]$ and $v_m \to v$ locally uniformly in $\mathbb{R}^n$, where $w \in C(\mathbb{R}^n)$. Moreover, the limiting function v satisfies the assumptions of the Liouville-type Lemma B.1.

Since $\beta > 0$, it follows from (B.10), (B.11), and the Arzelà–Ascoli theorem that a subsequence of v_m converges locally uniformly in $\mathbb{R}^n$ to some $v \in C(\{\mathbb{R}^n\})$. Passing to the limit (B.10) we find that the assumption (B.1) of Theorem B.1 is satisfied by v. Similarly as in Proposition 7.1, using $[c_{ab}]_{\gamma_k;B_1} \leq 1$, we show that

$$\begin{aligned} 0 &= \inf_{b\in\mathcal{B}_k} \sup_{a\in\mathcal{A}_k} \big(L_{ab}u_k(\bar{x}) + c_{ab}(\bar{x})\big) \\ &\geq \inf_{b\in\mathcal{B}_k} \sup_{a\in\mathcal{A}_k} \big(L_{ab}u_k(\bar{x}) + c_{ab}(0)\big) + |\bar{x}|^{\gamma_k} \end{aligned}$$

and that

$$
\begin{aligned}
0 &= \inf_{b\in\mathcal{B}_k}\sup_{a\in\mathcal{A}_k}\big(L_{ab}u_k(\bar{x}+\bar{h})+c_{ab}(\bar{x}+\bar{h})\big)\\
&\le \inf_{b\in\mathcal{B}_k}\sup_{a\in\mathcal{A}_k}\big(L_{ab}u_k(\bar{x}+\bar{h})+c_{ab}(0)\big)+|\bar{x}+\bar{h}|^{\gamma_k},
\end{aligned}
$$

and thus that

$$
-|\bar{x}|^{\gamma_m}-|\bar{x}+\bar{h}|^{\gamma_m}\le M^+_{\mathcal{L}_*(s_m)}\big(v_m(\cdot+h)-v_m\big)\quad\text{in } B_{1/2}.
$$

Therefore, by rescaling we obtain

$$
-\frac{3K^{\gamma_m}(r'_m)^{2s_m+\gamma_m}}{\theta(r'_m)(r'_m)^{\alpha+\beta}}\le M^+_{\mathcal{L}_*(s_m)}\big(v_m(\cdot+h)-v_m\big)\quad\text{in } B_K \tag{B.12}
$$

whenever $|h|<K$ and $r'_m<1/2K$.

On the other hand, since $v_m\to v$ locally uniformly in $\mathbb{R}^n$ (up to subsequences), we then have

$$
\big(v_m(\cdot+h)-v_m\big)\to\big(v(\cdot+h)-v\big)\quad\text{locally uniformly in } \mathbb{R}^n. \tag{B.13}
$$

Also, similarly as in Proposition 7.1, (B.10) and the dominated convergence theorem imply that

$$
\big(v_m(\cdot+h)-v_m\big)\to\big(v(\cdot+h)-v\big)\quad\text{in } L^1(\mathbb{R}^n,\omega_{s_0}), \tag{B.14}
$$

since $|v_m(\cdot+h)-v_m|\le C(1+|x|)^\alpha\le C(1+|x|)^{s_0}\in L^1(\mathbb{R}^n,\omega_{s_0})$. Thus, using (B.14) and (B.13), we can pass to the limit in (B.12) (for each $K\ge 1$) to obtain

$$
0\le M^+_{\mathcal{L}_*(s)}\big\{v(\cdot+h)-v\big\}\quad\text{in } \mathbb{R}^n.
$$

Analogously, we have

$$
0\ge M^-_{\mathcal{L}_*(s)}\big\{v(\cdot+h)-v\big\}\quad\text{in } \mathbb{R}^n_+.
$$

Hence, w satisfies all the assumptions of Lemma B.1, and thus v is an affine function. On the other hand, passing (B.8) to the limit we obtain that v is orthogonal to every affine function and hence it must be $v\equiv 0$. But then passing (B.9) to the limit we obtain that v cannot be constantly zero in B_1, which is a contradiction. ☐

Finally, we give the following.

Proof of Lemma 7.5

The result follows by rescaling from Proposition B.2. Indeed, let

$$
\bar{w}(x)=r^{-\alpha-\beta-s}w(rx).
$$

We have

$$\|\bar{w}\|_{L^\infty(\mathbb{R}^n)} < \infty \qquad \text{and} \qquad \|\bar{w}\|_{L^\infty(B_R)} \le C_0 R^{\alpha+\beta+s}$$

for all $R \ge 1$. Since the spectral measures μ_{ab} satisfy $\|\mu_{ab}\|_{C^\gamma(S^{n-1})} \le \Lambda$, we have

$$\Big[\frac{\mu_{ab}(y/|y|)}{|y|^{n+2s}}\Big]_{C^\gamma(B_{2R}\setminus B_R)} \le \frac{C\Lambda}{R^{n+2s+\gamma}}.$$

Hence, if $\eta \in C_c^\infty(B_1(e_n))$ is such that $\eta \equiv 1$ on $\overline{B_{5/6}}$, it follows that

$$\tilde{\mathrm{I}}(\bar{w}\eta, \bar{x}) = \inf_{b\in\mathcal{B}} \sup_{a\in\mathcal{A}} \big(L_{ab}w(\bar{x}) + \tilde{c}_{ab}(\bar{x})\big) = 0 \quad \text{in } B_{4/6},$$

where

$$\tilde{c}_{ab}(\bar{x}) = r^{2s} r^{-\alpha-\beta-s} c_{ab}(r\bar{x}) + L_{ab}(1-\eta)\bar{w}.$$

It then easily follows that

$$[\tilde{c}_{ab}]_{C^\gamma(B_{4/6}(e_n))} \le C r^{\gamma+s-\alpha-\beta} + CC_0 \le C(1+C_0).$$

Therefore, Proposition B.2 yields

$$\|\bar{w}\eta\|_{C^{\alpha+\beta}(B_{1/2}(e_n))} \le C\|\bar{w}\eta\|_{C^\beta(B_1(e_n))}.$$

Rescaling back to w, we have

$$\|w\|_{C^{\alpha+\beta}(B_{r/2}(re_n))} \le Cr^{-\alpha}\|w\|_{\beta;B_r(re_n)} \le Cr^{-\alpha}r^{\alpha+s} = Cr^s,$$

and thus the lemma is proved. □

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