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Minimizers to reaction-diffusion PDEs, Sobolev inequalities, and monomial weights

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1 Introduction

The aim of this work is to present the results obtained by the author in the last months while studying reaction-diffusion equations and, as a consequence of it, some new Sobolev inequalities with monomial weights.

Our work concerns the regularity of minimizers to some nonlinear elliptic PDEs –a classical problem in the Calculus of Variations appearing, for instance, in Hilbert's 19th problem. An important example in Geometry is the regularity of minimal hypersurfaces of $\mathbb{R}^n$ which are minimizers of the area functional. A deep result from the seventies states that these hypersurfaces are smooth if $n \leq 7$, while in $\mathbb{R}^8$ the Simons cone

$$S = \{x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\} \tag{1.1}$$

is a minimizing minimal hypersurface with a singularity at 0. The same phenomenon –the fact that regularity holds in low dimensions– happens for other nonlinear equations in bounded domains. For instance, let u be a solution of

$$\begin{cases} -\Delta u &= f(u) \quad \text{in } \Omega \\ \quad u &= 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It is still an open question whether local minimizers of this equation have or not singularities if $n \leq 9$. In dimensions $n \geq 10$ there are examples of singular solutions to this problem which are local minimizers. Namely,

$$u(x) = \log \frac{1}{|x|^2} \quad \text{is a solution of (1.2) with} \quad f(u) = 2(n-2)e^u \quad \text{and} \quad \Omega = B_1,$$

which is stable if $n \geq 10$ and local minimizer if $n \geq 11$ (see [8]). Our goal is to make progress on the above open problem on the regularity of minimizers of (1.2) in dimensions $n \leq 9$.

Reaction-diffusion equations play a central role in PDE theory and its applications to other sciences. They model many problems, running from Physics (fluids, combustion, etc.), Biology and Ecology (population evolution, illness propagation, etc.), to Financial Mathematics and Economy (Black-Scholes equation, price formation, Lévy processes, etc.). They also play an important role in some geometric problems: the problem of prescribing a curvature on a manifold, conformal classification of varieties, and parabolic flows on manifolds. Similar questions for other nonlinear elliptic and parabolic PDEs on manifolds lead to the study of minimal varieties or to the Ricci flow –used in the recent proof of the Poincaré Conjecture.

The object of our study is a reaction-diffusion problem with interior reaction. We consider the semilinear elliptic equation

$$\left\{ \begin{array}{rcll} -\Delta u & = & \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u & > & 0 & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1.3}$$

with $\lambda > 0$, posed in a bounded domain Ω of $\mathbb{R}^n$ with smooth boundary and with zero Dirichlet data. We assume that the nonlinearity f is a continuous, positive, and increasing function, with $f(0) > 0$, and

$$\lim_{\tau \to +\infty} \frac{f(\tau)}{\tau} = +\infty.$$

Typical examples are $-\Delta u = \lambda e^u$ (known as Gelfand problem, used to model combustion processes) or $-\Delta u = \lambda(1+u)^p$, with $p > 1$.

Under these conditions, it is well known that there exists an extremal value $\lambda^* \in (0, +\infty)$ of the parameter λ such that for each $0 < \lambda < \lambda^*$ there exists a positive minimal solution u_λ of (1.3), while for $\lambda > \lambda^*$ the problem has no solution, even in the weak sense. Here, minimal means the smallest positive solution. For $\lambda = \lambda^*$, there exists a weak solution, called the extremal solution of (1.3), which is given by

$$u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_\lambda(x).$$

In 1997 H. Brezis and J.L. Vázquez [3] raised the question of studying the regularity of the extremal solution u^*, i.e., to decide whether u^* is or is not a classical solution depending on f and Ω. This is equivalent to determine whether u^* is bounded or unbounded. The importance of the problem stems in the fact that the existence of other non-minimal solutions for $\lambda < \lambda^*$ depends strongly on the regularity of the extremal solution.

The problem was studied in the nineties for different nonlinearities f, essentially exponential or power nonlinearities. In both cases a similar result holds: if $n \leq 9$ then the extremal solution u^* is bounded for every domain Ω, while for $n \geq 10$ there are examples of unbounded extremal solutions u^* even in the unit ball –as the one given before in this Introduction.

At present, it is known that this result holds true for all nonlinearities f when the domain Ω is a ball, and also in general domains for a class of nonlinearities that satisfy a quite restrictive condition at infinity which forces them to be very close to an exponential or a power.

The case of general f was studied first by G. Nedev in 2000 [27], who proved that the extremal solution of (1.3) is bounded for every convex nonlinearity f and domain Ω if $n \le 3$. He also gave L^p estimates for u^* for $n \ge 4$, and proved that $u^* \in H_0^1(\Omega)$ in every dimension when the domain is strictly convex. Finally, the best known result so far is the one proved by X. Cabré [6] in 2010. He proved that when $n \le 4$ and the domain is convex, the extremal solution of (1.3) is bounded. These results also apply to a more general class of solutions of (1.2): local minimizers or, more generally, stable solutions.

In the first part of our work we study the regularity of the extremal solution u^* of (1.3) in the class of domains that we call of double revolution. These are those domains which are invariant under rotations of the first m variables and of the last $n-m$ variables, that is,

$$\Omega = \{(x^1, x^2) \in \mathbb{R}^m \times \mathbb{R}^{n-m} : (s = |x^1|, t = |x^2|) \in \Omega_2\},$$

where $\Omega_2 \subset \mathbb{R}^2$ is a bounded domain even (or symmetric) with respect to each coordinate. Our main result is the following:

When Ω is a convex domain of double revolution and $n \le 7$, the extremal solution is bounded for each nonlinearity f.

The proof of this result uses the stability property of minimal solutions to obtain control on some integrals of the form

$$\int_{\Omega_2} \left(s^{-\alpha} u_s^2 + t^{-\beta} u_t^2 \right) dsdt.$$

From this, we want to deduce an L^∞ or L^p bound for u. Recall that s and t are the two radial coordinates describing Ω. After a change of variables, the problem transforms into the following: given $a > -1$ and $b > -1$, we want to find the greatest exponent $q > 2$ for which the inequality

$$\left(\int_{\widetilde{\Omega}_2} \sigma^a \tau^b |u|^q d\sigma d\tau \right)^{1/q} \le C \left(\int_{\widetilde{\Omega}_2} \sigma^a \tau^b |\nabla u|^2 d\sigma d\tau \right)^{1/2}$$

holds for all smooth functions u vanishing on the boundary of $\widetilde{\Omega}_2$. Note that when a and b are positive integers, this is exactly the classical Sobolev inequality in $\mathbb{R}^{a+1} \times \mathbb{R}^{b+1}$ for functions which are radially symmetric on the first $a+1$ variables and on the last $b+1$ variables. However, we need to establish it in the non-integer case.

Thus, we were led to study weighted Sobolev inequalities of the form

$$\left(\int_\Omega w(x)|u|^{p^*}dx\right)^{1/p^*} \le C\left(\int_\Omega w(x)|\nabla u|^p dx\right)^{1/p}, \qquad (1.4)$$

where

$$w(x) = |x_1|^{A_1}\cdots|x_n|^{A_n} =: x^A \qquad (1.5)$$

being Ω a domain in $\mathbb{R}^n$, u regular enough and vanishing on $\partial\Omega$, and $p^* > p$. The best exponent p^* will depend on $A_1, ..., A_n$. We denote the weight w as x^A, where $A = (A_1, ..., A_n)$. We realized that even in the case $n = 2$ there was no precise study of such Sobolev inequalities. To establish them for all domains and exponents $A_i \ge 0$ is the object of the second part of our work.

Sobolev-type inequalities play a key role in Analysis and in the study of solutions to partial differential equations. In fact, they are extremely flexible tools and are useful in many different settings. One of the most important results regarding weighted Sobolev inequalities is the one due to Fabes, Kenig, and Serapioni [18]. It states that, when Ω is bounded and $1 < p < \infty$, the Sobolev inequality (1.4) holds for any weight w satisfying the so-called Muckenhoupt condition A_p, that is, if there is a constant C such that, for all balls B in $\mathbb{R}^n$, we have

$$\left(\fint_B w(x)dx\right)\left(\fint_B w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \le C.$$

However, our weight (1.5) satisfies the Muckenhoupt condition A_p if and only if $-1 < A_i < 1$ for every i. Thus, for our weight with $A_i \ge 1$ for some i we cannot apply –at least in a direct way– this result to obtain (1.4).

Our main result is that inequality (1.4) with $\Omega = \mathbb{R}^n$ holds for any monomial weight (1.5) with every $A_i \ge 0$. That is,

$$\left(\int_{\mathbb{R}^n} x^A|u|^{p^*}dx\right)^{1/p^*} \le C\left(\int_{\mathbb{R}^n} x^A|\nabla u|^p dx\right)^{1/p}, \qquad (1.6)$$

where

$$p^* = \frac{pD}{D-p} \qquad \text{and} \qquad D = n + A_1 + \cdots + A_n,$$

for all $u \in C^1(\mathbb{R}^n)$ with compact support. Recall that the expression of p^* is exactly the one from the classical Sobolev inequality, but in this case the 'dimension' is given by D. Recall also that, as in the case of domains of double revolution explained above, the integer case is just the classical Sobolev inequality in $\mathbb{R}^D$ for functions depending only on some appropriate radial variables. Moreover, we obtain an explicit expression of the best constant C_p in inequality (1.6), as well as extremal functions for which the best constant is attained. For $p > D$, we prove a weighted version of the classical Morrey inequality on Hölder continuity.

The proof of inequality (1.6) is based on a new weighted isoperimetric inequality,

$$\left(\int_\Omega x^A dx \right)^{\frac{D-1}{D}} \leq C \int_{\partial\Omega} x^A d\sigma(x),$$

with the optimal constant C depending on $n, A_1, ..., A_n$. We establish it by adapting a recent proof of the classical Euclidean isoperimetric inequality due to X. Cabré [4, 13]. Our proof uses a linear Neumann problem for the operator $x^{-A}\text{div}(x^A \nabla \cdot)$ combined with the Alexandroff contact set method. The explicit expression of the best constant is given in terms of Gamma functions and it is attained, for example when $A_i > 0$ for each i, by domains of the form

$$\Omega = B_R(0) \cap (\mathbb{R}_+)^n.$$

The work is organized as follows. Section 2 is devoted to introduce in full detail the problem and known results on the regularity of the extremal solution, while in section 3 we prove our results on extremal solutions in domains of double revolution. In section 4 we summarize the known results on weighted Sobolev inequalities, introducing also the concept of maximal operator and of Muckenhoupt weights. Finally, in section 5 we prove our weighted isoperimetric inequality, as well as the Sobolev and Morrey inequalities with monomial weights.

In order to clarify which are our original results and which are known results, we have separated them in different sections:

- Section 2. Extremal solutions: Survey of known results.
- Section 3. Extremal solutions: Original results. The main result is Theorem 3.1.
- Section 4. Sobolev inequalities: Survey of known results and their applications to monomial weights.
- Section 5. Sobolev inequalities: Original results. The main results are Theorems 5.1, 5.2, and 5.3.

2 The extremal solution of $-\Delta u = \lambda f(u)$

Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain, and consider the problem

$$\begin{cases} -\Delta u &= \lambda f(u) & \text{in } \Omega \\ u &> 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where λ is a positive parameter and the nonlinearity $f : [0, \infty) \longrightarrow \mathbb{R}$ satisfies

$$f \text{ is } C^1, \text{ nondecreasing, } f(0) > 0, \text{ and } \lim_{\tau \to +\infty} \frac{f(\tau)}{\tau} = +\infty. \tag{2.2}$$

It is well known (see [15], [3] and references therein) that there exists a finite extremal parameter λ^* such that if $0 < \lambda < \lambda^*$ then problem (2.1) admits a minimal classical solution u_λ, while for $\lambda > \lambda^*$ it has no solution, even in the weak sense. Moreover, the set $\{u_\lambda : 0 < \lambda < \lambda^*\}$ is increasing in λ, and its pointwise limit $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ is a weak solution of problem (2.1) with $\lambda = \lambda^*$. It is called the extremal solution of (2.1).

When $f(u) = e^u$, it is well known that $u^* \in L^\infty(\Omega)$ if $n \leq 9$, while $u^*(x) = \log \frac{1}{|x|^2}$ and $\lambda^* = 2(n-2)$ if $n \geq 10$ and $\Omega = B_1$. An analogous result holds for $f(u) = (1+u)^p$, $p > 1$. In the nineties H. Brezis and J.L. Vázquez [3] raised the question of determining the regularity of u^*, depending on the dimension n, for general convex nonlinearities satisfying (2.2). The first general results were proved by G. Nedev [27, 28]– see [11] for the statement and proofs of the results of [28].

Theorem 2.1 ([27],[28])**.** *Let Ω be a smooth bounded domain, let f be a convex function satisfying (2.2), and let u^* be the extremal solution of (2.1).*

i) If $n \leq 3$, then $u^* \in L^\infty(\Omega)$.

ii) If $n \geq 4$, then $u^* \in L^p(\Omega)$ for every $p < \frac{n}{n-4}$.

iii) Assume either that $n \leq 5$ or that Ω is strictly convex. Then $u^* \in H_0^1(\Omega)$.

In 2006, X. Cabré and A. Capella studied the radial case [7]. Their result establishes optimal regularity results for general f.

Theorem 2.2 ([7])**.** *Let $\Omega = B_1$ be the unit ball in $\mathbb{R}^n$, f a function satisfying (2.2) and u^* the extremal solution of (2.1).*

i) If $n \leq 9$, then $u^* \in L^\infty(B_1)$.

ii) If $n \geq 10$, then $u^* \in L^p(B_1)$ for every $p < p_n$, where

$$p_n = 2 + \frac{4}{\frac{n}{2+\sqrt{n-1}} - 2}. \tag{2.3}$$

iii) For every dimension n, $u^* \in H^2(B_1)$, and if f is convex, then $u^* \in H^3(B_1)$.

The best known result was established in 2010 by X. Cabré [6] and establishes the boundedness of u^* in convex domains in dimension $n = 4$, while X. Cabré and M. Sanchón [11] prove L^p estimates for u^* when $n \geq 5$:

Theorem 2.3 ([6],[11]). *Let $\Omega \subset \mathbb{R}^n$ be a convex, smooth and bounded domain, f a function satisfying (2.2) and u^* the extremal solution of (2.1).*

i) If $n \leq 4$, then $u^* \in L^\infty(\Omega)$.

ii) If $n \geq 5$, then $u^* \in L^p(\Omega)$ for every $p < \frac{2n}{n-4} = 2 + \frac{4}{\frac{n}{2}-2}$.

The boundedness of extremal solutions remains an open question when $5 \leq n \leq 9$, even in the case of convex domains.

The proofs of the results in [27, 28, 7, 6, 11] use the semi-stability of the extremal solution u^*. In fact, one first proves estimates for any regular semi-stable solution u of

$$\begin{cases} -\Delta u &= f(u) \quad \text{in } \Omega \\ \quad u &= 0 \qquad \text{in } \partial\Omega, \end{cases} \tag{2.4}$$

then one applies these estimates to the minimal solutions u_λ, and finally by monotone convergence the estimates also hold for the extremal solution u^*.

Recall that a solution u of (2.4) is said to be *semi-stable* if the second variation of energy at u is non-negative, i.e., if

$$Q_u(\xi) = \int_\Omega |\nabla \xi|^2 - f'(u)\xi^2 \geq 0 \tag{2.5}$$

for all $\xi \in C_0^1(\overline{\Omega})$, i.e., for all $C^1(\overline{\Omega})$ functions vanishing on $\partial\Omega$. Obviously, every local minimizer in the $C_0^1(\overline{\Omega})$ of the energy functional

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u),$$

where $F' = f$, is a semi-stable solution of (2.4). For this, recall that (2.4) and (2.5) are, respectively, the first and second variations of E.

The proof of the estimates in [7, 6, 11] by using the stability condition was inspired by the proof of Simons theorem on the nonexistence of singular minimal cones in $\mathbb{R}^n$ for $n \le 7$. The key idea is to take $\xi = |\nabla u|\eta$ (or $\xi = u_r\eta$ in the radial case) and compute $Q_u(|\nabla u|\eta)$ in the semi-stability property satisfied by u. Then, the expression of Q_u in terms of η does not depend on f, and a clever choice of the test function η leads to L^p and L^∞ bounds depending on the dimension n.

In the following subsections we give wider explanations of these results. In order to synthetize, we will center in the results which give boundedness of the extremal solution.

2.1 Exponential and power nonlinearities

In this subsection we explain the knwon results for exponential and power nonlinearities, that is, $f(u) = e^u$ and $f(u) = (1+u)^p$.

The main result is the boundedness in dimensions $n \le 9$ of the extremal solution.

Theorem 2.4 ([14],[25]). *Let Ω be a smooth and bounded domain in $\mathbb{R}^n$, and let u^* the extremal solution of (2.1). Assume that $f(u) = e^u$ or $f(u) = (1+u)^p$, with $p > 1$. If $n \le 9$, then u^* is bounded.*

Proof. We will prove the case $f(u) = e^u$.

Setting $\xi = e^{\alpha u_\lambda} - 1$ in the stability condition (2.5), we obtain that

$$\lambda \int_\Omega e^{u_\lambda}(e^{\alpha u_\lambda} - 1)^2 \le \alpha^2 \int_\Omega e^{2\alpha u_\lambda} |\nabla u_\lambda|^2.$$

Taking into account that $\lambda e^{u_\lambda} = -\Delta u_\lambda$ and integrating by parts,

$$\lambda \int_\Omega e^{u_\lambda}(e^{2\alpha u_\lambda} - 1) = -\int_\Omega \Delta u_\lambda (e^{2\alpha u_\lambda} - 1) = 2\alpha \int_\Omega e^{2\alpha u_\lambda} |\nabla u_\lambda|^2,$$

so

$$\int_\Omega e^{u_\lambda}(e^{\alpha u_\lambda} - 1)^2 \le \frac{\alpha}{2} \int_\Omega e^{u_\lambda}(e^{2\alpha u_\lambda} - 1)$$

and

$$\left(1 - \frac{\alpha}{2}\right) \int_\Omega e^{(2\alpha+1)u_\lambda} - 2\int_\Omega e^{(\alpha+1)u_\lambda} + \left(1 + \frac{\alpha}{2}\right) \int_\Omega e^{\alpha u_\lambda} \le 0.$$

But by Hölder's inequality we have that

$$\int_\Omega e^{(\alpha+1)u_\lambda} \le C \left(\int_\Omega e^{(2\alpha+1)u_\lambda} \right)^{\frac{\alpha+1}{2\alpha+1}},$$

and then

$$\left(1-\frac{\alpha}{2}\right)\int_\Omega e^{(2\alpha+1)u_\lambda}\le C\left(\int_\Omega e^{(2\alpha+1)u_\lambda}\right)^{\frac{\alpha+1}{2\alpha+1}}.$$

Hence, for each $\alpha<2$ one have $\|e^{u_\lambda}\|_{L^{2\alpha+1}}\le C$, and if $n\le 9$ we can take p such that $\frac{n}{2}<p<5$ so that

$$\|u_\lambda\|_{L^\infty}\le C_1\|u_\lambda\|_{W^{2,p}}\le C_2\|\Delta u_\lambda\|_{L^p}\le C_3\|e^{u_\lambda}\|_{L^p}\le C$$

for some constant which does not depend on λ. Finally, making $\lambda\to\lambda^*$ one gets that the extremal solution u^* is bounded and hence, classical. □

When the domain is a ball all solutions are known, as shown in the next two Theorems.

Theorem 2.5 ([23]). *Let Ω be the unit ball in $\mathbb{R}^n$ and $f(u)=e^u$. Then,*

1. *If $n=1,2$ for each $0<\lambda<\lambda^*$ there exist exactly 2 solutions, while for $\lambda=\lambda^*$ there exists exactly one solution, which is bounded.*
2. *If $3\le n\le 9$ we have that u^* is bounded and $\lambda^*>\lambda_0$, where $\lambda_0=2(n-2)$. For $\lambda=\lambda_0$ there exist infinitely many solutions which tend to $\log\frac{1}{|x|^2}$. For $|\lambda-\lambda_0|\ne 0$ but small there exist a big number of solutions.*
3. *If $n\ge 10$ then $\lambda_*=\lambda_0=2(n-2)$ and $u^*=\log\frac{1}{|x|^2}$. Moreover, for $0<\lambda<\lambda^*$ there exists only one solution.*

Theorem 2.6 ([23]). *Let Ω be the unit ball in $\mathbb{R}^n$ and $f(u)=(1+u)^p$, with $p>1$. Then,*

1. *If $p\le\frac{n+2}{n-2}$, for each $0<\lambda<\lambda^*$ there exist exactly 2 solutions, while for $\lambda=\lambda^*$ there exists exactly one solution, which is bounded.*
2. *If $p>\frac{n+2}{n-2}$ and $n<2+\frac{4p}{p-1}+4\sqrt{\frac{p}{p-1}}$, we have that u^* is bounded and that $\lambda^*>\lambda_p$, where $\lambda_p=\frac{2}{p-1}\left(n-\frac{2}{p-1}\right)$. For $\lambda=\lambda_p$ there exist infinitely many solutions which tend to $|x|^{\frac{2}{p-1}}-1$. For $|\lambda-\lambda_p|\ne 0$ but small, there exist a big number of solutions.*
3. *If $p>\frac{n+2}{n-2}$ and $n\ge 2+\frac{4p}{p-1}+4\sqrt{\frac{p}{p-1}}$, then $\lambda_*=\lambda_p$ and $u^*=|x|^{\frac{2}{p-1}}-1$. Moreover, for $0<\lambda<\lambda^*$ there exists only one solution.*

There are many results on the literature which generalize Theorem 2.4 to a larger class of nonlinearities f. The most general result which proves boundedness of the extremal solution in dimensions $n \leq 9$ is the following, proved in 2007 by M. Sanchón.

Theorem 2.7 ([29]). *Let Ω be a smooth domain in $\mathbb{R}^n$, and let f be a function satisfying (2.2) and such that the following limit exists and is finite*

$$\lim_{\tau \to \infty} \frac{[f(\tau) - f(0)]f''(\tau)}{f'(\tau)^2}.$$

If $n \leq 9$, then the extremal solution u^* of (2.1) is bounded.

In fact, a similar result holds not only for the Laplacian, but also for the p-Laplacian.

2.2 The radial case

In this subsection we explain the known results on regularity of the extremal solution when the domain is a ball, and we give the proof of its boundedness in dimensions $n \leq 9$.

First note that all semi-stable solutions of (2.4) are radially symmetric when $\Omega = B_1$. To prove it, let u be a semi-stable solution of (2.4), and define $v = x_i u_{x_j} - x_j u_{x_i}$, $i \neq j$. Note that u will be radial if and only if $v = 0$ for each $i \neq j$.

First, we see that v is a solution of the linearized equation of (2.4):

$$\begin{aligned}
\Delta v &= \Delta(x_i u_{x_j} - x_j u_{x_i}) \\
&= x_i \Delta u_{x_j} + 2\nabla x_i \cdot \nabla u_{x_j} - x_j \Delta u_{x_i} - 2\nabla x_j \cdot \nabla u_{x_i} \\
&= x_i (\Delta u)_{x_j} - x_j (\Delta u)_{x_i} \\
&= -f'(u)\{x_i u_{x_j} - x_j u_{x_i}\} \\
&= -f'(u)v.
\end{aligned}$$

Moreover, since $u = 0$ on ∂B_1 then $v = 0$ on ∂B_1. Now, multiplying the previous equation by v and integrating by parts, we obtain

$$\int_{B_1} \{|\nabla v|^2 - f'(u)v^2\} dx = 0.$$

But since u is semi-stable, $\lambda_1(\Delta + f'(u); B_1) \geq 0$.

If $\lambda_1(\Delta + f'(u); B_1) > 0$, then $v = 0$.

If $\lambda_1(\Delta + f'(u); B_1) = 0$, then it has to be $v = K\phi_1$, where ϕ_1 is the first eigenfunction, which is positive in B_1. But since v is the derivative of u along the vector field $\partial_t = x_i\partial_{x_j} - x_j\partial_{x_i}$, and its integral curves are closed, v can not have constant sign, and hence it has to be $K = 0$, that is, $v = 0$.

Thus, the extremal solution will be a radial solution, and in this case one can expects not only to obtain L^p estimates for the extremal solution, but pointwise estimates. The following Theorem gives optimal pointwise estimates which lead to the optimal L^p estimates of Theorem 2.2. As said before, in 2006 X. Cabré and A. Capella proved that the extremal solution is bounded in dimensions $n \leq 9$, and give pointwise estimates which lead to optimal L^p estimates for u^* in higher dimensions. However, the pointwise estimates were not optimal (by a logarithmic factor) for $n \geq 11$, and in 2010 S. Villegas established the optimal pointwise estimates in those dimensions.

Theorem 2.8 ([7, 35])**.** *Assume that $\Omega = B_1$, $n \geq 2$, and that f satisfies (2.2). Let u^* be the extremal solution of (2.1). Then, for some constant C, depending only on n, we have:*

i) If $n \leq 9$, then $u^*(r) \leq C$.

ii) If $n = 10$, then $u^*(r) \leq C|\log r|$.

iii) If $n \geq 11$, then $u^*(r) \leq C\left(r^{-n/2+\sqrt{n-1}+2} - 1\right)$.

iv) If $n \geq 10$, then $|\partial_r^{(k)} u^*(r)| \leq C r^{-n/2+\sqrt{n-1}+2-k}$ for $k = 1$ and $k = 2$. If f is convex, then it holds also for $k = 3$.

We now give the proof of the boundedness of the radial extremal solutions in dimensions $n \leq 9$, part i) of the previous Theorem. For it, we will need a Lemma.

Lemma 2.9 ([7])**.** *Let u be a semi-stable solution of (2.4). Then, for every $\eta \in H^1$ with compact support in B_1 we have that*

$$\int_{B_1} u_r^2 \left\{ |\nabla\eta|^2 - (n-1)\frac{\eta^2}{r^2} \right\} dx \geq 0.$$

Proof. It suffices to apply the stability condition to $\xi = u_r\eta$ and use that

$$\Delta u_r + f'(u)u_r = \frac{n-1}{r^2}u_r,$$

which can be seen by differentiating with respect to r the equation $\Delta u + f(u) = 0$ in radial coordinates. The fact that one can choose η to be H^1 instead of C^1 can be seen by approximation. □

Lemma 2.10 ([7]). *Let $n \geq 2$ and let u be a bounded semi-stable solution of (2.4). Then, for each $0 \leq \alpha < \sqrt{n-1}$,*

$$\int_{B_1} u_r^2 r^{-2\alpha-2} dx \leq C \|u\|_{H_0^1},$$

where the constant C depends only on n and α.

Proof. Take $\eta = r^{-\alpha} - 1$ in the last Lemma to obtain

$$(n-1) \int_{B_1} u_r^2 \frac{(r^{-\alpha}-1)^2}{r^2} dx \leq \int_{B_1} u_r^2 \left(\alpha r^{-\alpha-1}\right)^2 dx.$$

Then,

$$(n-1-\alpha^2) \int_{B_1} u_r^2 r^{-2\alpha-2} dx \leq (n-1) \int_{B_1} u_r^2 (2r^{-2\alpha}-1) dx,$$

and hence taking C such that

$$\frac{n-1}{n-1-\alpha^2}(2r^{-2\alpha}-1) \leq \frac{1}{2} r^{-2\alpha-2} + C,$$

we get

$$\frac{1}{2} \int_{B_1} u_r^2 r^{-2\alpha-2} dx \leq C \int_{B_1} u_r^2 dx.$$

□

Proof of Theorem 2.8. We will prove only part i).

Note that since the domain is strictly convex, then by Theorem 2.1 $u^* \in H_0^1$. Thus, applying Lemma 2.10 to u_λ and letting $\lambda \to \lambda^*$ we get that

$$\int_{B_1} (u_r^*)^2 r^{-2\alpha-2} dx \leq C.$$

Now, if $0 \leq \alpha < \sqrt{n-1}$, then

$$\begin{aligned} u^*(s) - u^*(1) &= \int_s^1 -u_r^* dr \\ &= \int_s^1 -u_r^* r^{-\alpha-1+\frac{n-1}{2}} r^{\alpha+1-\frac{n-1}{2}} dr \\ &\leq C \left(\int_s^1 (u_r^*)^2 r^{-2\alpha-2} r^{n-1} dr \right)^{1/2} \left(\int_s^1 r^{2\alpha+3-n} dr \right)^{1/2} \\ &= C \left(\int_{B_1} (u_r^*)^2 r^{-2\alpha-2} dx \right)^{1/2} \left(\int_s^1 r^{2\alpha+3-n} dr \right)^{1/2}. \end{aligned}$$

If $n \leq 9$, then one can choose $\alpha < \sqrt{n-1}$ such that $2\alpha + 4 - n > 0$, and hence $u^*(r) \leq C$. □

Remark 2.11. Although we have used Theorem 2.1, in the radial case it is not difficult to verify that $u^* \in H_0^1$ by using the equation written in the form $-(r^{n-1}u_r)_r = \lambda f(u) r^{n-1}$.

2.3 The general case

In this subsection we explain the results on regularity of the extremal solution in the general case, that is, for a general smooth bounded domain Ω and with a general nonlinearity f satisfying (2.2).

The first general result was the one given by G. Nedev in 2000 [27], who proved the regularity of the extremal solution in dimensions $n \leq 3$ with the only assumption of convexity of the nonlinearity f. With the same assumptions, he also proved L^p estimates in dimensions $n \geq 4$, and in another article [28], he proved that if the domain is stricly convex then $u^* \in H_0^1(\Omega)$.

Theorem 2.12 ([27]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and let f be a convex function satisfying (2.2). If $n = 2$ or $n = 3$, then u^* is a classical solution. If $n \geq 4$ we have $u^* \in W^{2,p}(\Omega)$ for $p < \frac{n}{n-2}$ and $u^* \in L^p(\Omega)$ for $p < \frac{n}{n-4}$.*

Proof. Let $g(\tau) = \int_0^\tau f'(t)^2 dt$, and multiply the equation (2.1) by $g(u_\lambda)$ to obtain

$$\int_\Omega |\nabla u_\lambda|^2 f'(u_\lambda)^2 = \lambda \int_\Omega f(u_\lambda) g(u_\lambda).$$

Set for convenience $\widetilde{f}(\tau) = f(\tau) - f(0)$. The inequality (2.5) applied with $\xi = \widetilde{f}(u_\lambda)$ then yields

$$\int_\Omega f'(u_\lambda)\widetilde{f}(u_\lambda)^2 \leq \int_\Omega \widetilde{f}(u_\lambda) g(u_\lambda) + f(0) \int_\Omega g(u_\lambda). \tag{2.6}$$

Now, defining

$$h(\tau) = \int_0^\tau f'(t)[f'(\tau) - f'(t)]dt,$$

we have that

$$\begin{aligned} \widetilde{f}(\tau)^2 f'(\tau) - \widetilde{f}(\tau) g(\tau) &= \widetilde{f}(\tau) f'(\tau) \int_0^\tau f'(t)dt - \widetilde{f}(\tau) \int_0^\tau f'(t)^2 dt \\ &= \widetilde{f}(\tau) h(\tau), \end{aligned}$$

which combined with (2.6) yields

$$\int_\Omega \widetilde{f}(u_\lambda)h(u_\lambda) \le f(0)\int_\Omega g(u_\lambda). \tag{2.7}$$

Moreover, it is not difficult to see that

$$\lim_{\tau\to+\infty} \frac{h(\tau)}{f'(\tau)} = +\infty \tag{2.8}$$

and

$$\lim_{\tau\to+\infty} \frac{\widetilde{f}(\tau)h(\tau)}{g(\tau)} = +\infty.$$

This last inequality combined with (2.7) leads to

$$\int_\Omega \widetilde{f}(u_\lambda)h(u_\lambda) \le C,$$

with C independent of λ. Hence, using (2.8) and that $\lim_{\tau\to+\infty} f(\tau)/\widetilde{f}(\tau) = 1$, we obtain

$$\int_\Omega f(u_\lambda)f'(u_\lambda) \le C. \tag{2.9}$$

Now, on the one hand we have that, since f is convex,

$$-\Delta\widetilde{f}(u_\lambda) = -f''(u_\lambda)|\nabla u_\lambda|^2 - f'(u_\lambda)\Delta u_\lambda \le h(x),$$

where $h(x) = f'(u_\lambda)f(u_\lambda)$.

On the other hand, let v the solution of the problem

$$\begin{cases} -\Delta v &= h(x) & \text{in } \Omega \\ \quad v &= 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

By (2.9) and standard regularity theory,

$$\|v\|_{L^p} \le \|h\|_{L^1} \le C \text{ for each } p < \frac{n}{n-2},$$

and since $\widetilde{f}(u_\lambda)$ is a subsolution of this problem, then $0 \le \widetilde{f}(u_\lambda) \le v$. Therefore, $\|f(u_\lambda)\|_{L^p} \le C$ for each $p < \frac{n}{n-2}$, and using that $\|u\|_{W^{2,p}} \le \|\Delta u\|_p$ and making $\lambda \to \lambda^*$, we obtain that

$$u^* \in W^{2,p} \text{ for each } p < \frac{n}{n-2}.$$

Finally, using Sobolev embeddings we obtain that if $n \le 3$ then $u^* \in L^\infty$, while if $n \ge 4$ then $u^* \in L^p$ for $p < \frac{n}{n-4}$. □

Between 2000 and 2010 many results were published regarding regularity of the extremal solution, but none of them covered the case of general f. In 2010, X. Cabré proved the best known result at present, that is, the boundedness of the extremal solution up to dimension four. The proof does not need to assume the convexity of the nonlinearity, but it needs the domain to be convex when $n = 3$ and $n = 4$.

Theorem 2.13 ([6]). *Let f satisfy (2.2) and $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Assume that $n \leq 4$, and that Ω is convex in the case $n \in \{3, 4\}$. Then, $u^* \in L^\infty(\Omega)$.*

The proof of the Theorem is based in the following estimate.

Theorem 2.14 ([6]). *Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain, f a smooth nonlinearity, and u a semi-stable solution of (2.4), with $u > 0$ in Ω. Then, for every $t > 0$,*

$$\|u\|_{L^\infty(\Omega)} \leq t + \frac{C}{t}|\Omega|^{\frac{4-n}{2n}} \left(\int_{\{u<t\}} |\nabla u|^4 dx \right)^2,$$

where C is a universal constant.

The first key idea in the proof of this result is the one explained in the introduction, that is, to take $\xi = |\nabla u|\eta$ in the stability condition. Then, the obtained inequality does not depend on the nonlinearity f.

Proposition 2.15 ([30],[31]). *Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain, and let u be a semi-stable solution of (2.4). Then, the stability condition (2.5) applied to $\xi = |\nabla u|\eta$ gives*

$$\int_{\Omega \cap \{|\nabla u|=0\}} \left(|\nabla_T |\nabla u||^2 + |A|^2 |\nabla u|^2 \right) \eta^2 dx \leq \int_{\Omega \cap \{|\nabla u|=0\}} |\nabla u|^2 |\nabla \eta|^2 dx,$$

where ∇_T denotes the tangential or Riemannian gradient along the level set of u and where $|A|^2 = |A(x)|^2$ is the sum of the squares of the principal curvatures of the level set of u passing through $x \in \Omega \cap \{|\nabla u| = 0\}$.

Then, one takes $\eta = \phi(u)$, with ϕ to be determined later, uses the coarea formula, and takes into account that $|H| \leq |A|$, where H is the mean curvature of the level sets of u. Finally, the following Sobolev-type inequality and the clever choice of ϕ finishes the proof of Theorem 2.14.

Theorem 2.16 ([1],[24])**.** *Let $M \subset \mathbb{R}^n$ be a smooth $(n-1)$-dimensional compact hypersurface without boundary. Then, for every $p \in (1, n-1)$ there exists a constant C, depending only on n and p such that, for every smooth function $v : M \to \mathbb{R}$,*

$$\left(\int_M |v|^{p^*} dV \right)^{1/p^*} \leq C \left(\int_M |\nabla v|^p + |Hv|^p \, dV \right)^{1/p},$$

where H is the mean curvature of M and $p^* = (n-1)p/(n-1-p)$.

By taking t small enough in Theorem 2.14, the question of regularity of the extremal solution reduces to a question of regularity near the boundary of Ω. Here, and only here, is where it is needed the convexity of the domain in dimensions $n = 3$ and $n = 4$. The classical way to prove boundary estimates is the moving planes method, which is the one used to prove the following:

Theorem 2.17 ([19])**.** *Let Ω be a smooth, bounded, and convex domain, let f be any Lipschitz function and let u be a bounded solution of (2.4). Then, there exists $\delta > 0$ and C, depending only on Ω, such that*

$$\|u\|_{L^\infty(\Omega_\delta)} \leq C \|u\|_{L^1(\Omega)},$$

where $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$.

In dimension $n = 2$ it is possible to prove the same result for non-convex domains (with the only assumption $f \geq 0$) by using the Kelvin transform.

3 Regularity of minimizers in domains of double revolution

The aim of this section is to study the regularity of the extremal solution u^* of (2.1) in a class of domains that we call of double revolution. The class contains much more general domains than balls but is much simpler than general convex domains.

Let $n \geq 4$ and

$$\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k \quad \text{with} \quad n = m + k, \; m \geq 2 \quad \text{and} \quad k \geq 2.$$

For each $x \in \mathbb{R}^n$ we define the variables

$$\begin{cases} s &= \sqrt{x_1^2 + \cdots + x_m^2} \\ t &= \sqrt{x_{m+1}^2 + \cdots + x_n^2}. \end{cases}$$

We say that a domain $\Omega \subset \mathbb{R}^n$ is a *domain of double revolution* if it is of the form $\Omega = \{x \in \mathbb{R}^n : (s,t) \in U\}$, where U is a domain in $(\mathbb{R}_+)^2 = \{s > 0, \; t > 0\}$. We will make an abuse of notation and will make no difference between U and Ω. Equivalently, Ω is a domain of double revolution if it is invariant under rotations of the first m variables and also rotations of the last k variables.

We will see that any semi-stable and regular solution u of (2.4) depends only on s and t, and hence we can identify it with a function $u(s,t)$ defined in $(\mathbb{R}_+)^2$. Moreover, in the convex case we will also have $u_s \leq 0$ and $u_t \leq 0$ and hence, $u(0) = \|u\|_\infty$ (see Remark 3.7).

The following is our main result. We prove that in convex domains of double revolution the extremal solution u^* is bounded when $n \leq 7$, and that it belongs to L^p and H_0^1 spaces for certain exponents p when $n \geq 8$. We also prove that in dimension $n = 4$ the convexity of the domain is not required for the boundedness of u^*.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain of double revolution, f a function satisfying (2.2) and u^* the extremal solution of (2.1).*

a) Assume either that $n = 4$ or that $n \leq 7$ and Ω is convex. Then, $u^* \in L^\infty(\Omega)$.

b) If $n \geq 8$ and Ω is convex, then $u^* \in L^p(\Omega)$ for all $p < p_{m,k}$, where

$$p_{m,k} = 2 + \frac{4}{\frac{m}{2+\sqrt{m-1}} + \frac{k}{2+\sqrt{k-1}} - 2}. \tag{3.1}$$

c) Assume either that $n \le 6$ or that Ω is convex. Then, $u^* \in H_0^1(\Omega)$.

Remark 3.2. By convexity on m it can be easily seen that, asymptotically as $n \to \infty$,

$$2 + \frac{2\sqrt{2}}{\sqrt{n}} \lesssim p_{m,k} \lesssim 2 + \frac{4}{\sqrt{n}}.$$

Instead, in a general convex domain, L^p estimates are known for $p \simeq 2 + \frac{8}{n}$, while in the radial case one has estimates for $p \simeq 2 + \frac{4}{\sqrt{n}}$. In fact, the optimal exponent (2.3) in the radial case can be obtained by setting $m = n$ and $k = 0$ in (3.1), but recall that in our results we always assume $m \ge 2$ and $k \ge 2$.

We will proceed in a way very similar to [7, 6, 11], proving first results for general semi-stable solutions of (2.4) and then applying them to u_λ to deduce estimates for u^*. The only difference is that we will take $\xi = u_s\eta$ and $\xi = u_t\eta$ separately instead of $\xi = |\nabla u|\eta$ to obtain our estimates.

Our result for general semi-stable solutions of (2.4) reads as follows.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain of double revolution, f a function satisfying (2.2), and u a bounded semi-stable solution of (2.4).*

Let δ be a positive real number, and define $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$. Then, for some constant C depending only on Ω, δ, n, and p, one has:

a) If $n \le 7$ and Ω is convex, then $\|u\|_{L^\infty(\Omega)} \le C\|f(u)\|_{L^\infty(\Omega_\delta)}$.

b) If $n \ge 8$ and Ω is convex, then $\|u\|_{L^p(\Omega)} \le C\|f(u)\|_{L^\infty(\Omega_\delta)}$ for each $p < p_{m,k}$, where $p_{m,k}$ is given by (3.1).

c) For all $n \ge 4$, $\|u\|_{H_0^1(\Omega)} \le C\|u\|_{H_0^1(\Omega_\delta)}$.

Part b) of Proposition 3.3 will be a consequence of a new weighted Sobolev inequality in $(\mathbb{R}_+)^2 = \{(s,t) \in \mathbb{R}^2 : s > 0, t > 0\}$. We prove it in this section and it states the following.

Proposition 3.4. *Let $a > -1$ and $b > -1$ be real numbers, being positive at least one of them, and let $D = 2 + a + b$. Let $u \in C_c^1(\mathbb{R}^2)$ be a positive function such that*

$$u_s \le 0 \quad \text{and} \quad u_t \le 0 \quad \text{in} \quad (\mathbb{R}_+)^2.$$

Then, for each $1 \leq p < D$ there exist a constant C, depending only on a, b, and p, such that

$$\left(\int_{(\mathbb{R}_+)^2} s^a t^b |u|^{p^*} dsdt\right)^{1/p^*} \leq C \left(\int_{(\mathbb{R}_+)^2} s^a t^b |\nabla u|^p dsdt\right)^{1/p}, \tag{3.2}$$

where $p^* = \frac{Dp}{D-p}$.

In subsection 3.3 we establish this weighted Sobolev inequality as a consequence of a weighted isoperimetric inequality.

Remark 3.5. When a and b are nonnegative integers, this inequality is a direct consequence of the classical Sobolev inequality in $\mathbb{R}^D$. Namely, define in $\mathbb{R}^D = \mathbb{R}^{a+1} \times \mathbb{R}^{b+1}$ the variables s and t as before, with $m = a + 1$ and $k = b + 1$. Then, for functions u defined in $\mathbb{R}^D$ depending only on the variables s and t, write the integrals appearing in the classical Sobolev inequality in $\mathbb{R}^D$ in terms of s and t. Since $dx = c(a, b)s^a t^b dsdt$, the obtained inequality is precisely the one given in Proposition 3.4.

Thus, the previous proposition extends the classical Sobolev inequality to the case of non-integer exponents a and b. In section 5 we prove inequality (3.2) in $\mathbb{R}^n$, where the weight is $x^A = |x_1|^{A_1} \cdots |x_n|^{A_n}$. We also determine the best constant and extremal functions of this inequality, prove a related isoperimetric inequality, and give the weighted version of the Sobolev-Morrey embeddings.

3.1 Proofs of the estimates

In this subsection we prove the estimates of Proposition 3.3. For this, we will need two preliminary results.

Lemma 3.6 ([6])**.** *Let u be a semi-stable solution of (2.4), and let c be a $L^\infty(\Omega)$ function. Then,*

$$\int_\Omega c\{\Delta c + f'(u)c\}\eta^2 dx \leq \int_\Omega c^2 |\nabla \eta|^2 dx$$

for all $\eta \in Lip(\overline{\Omega})$, $\eta|_{\partial\Omega} = 0$.

Proof. It suffices to set $\xi = c\eta$ in the semi-stability condition and then integrate by parts. The fact that we can take $\eta \in \mathrm{Lip}(\overline{\Omega})$ can be deduced by density arguments. □

Remark 3.7. The symmetry of the domain implies the symmetry of all semi-stable solutions of (2.4). It can be proved with a similar argument as in subsection 2.2, where it is proved that when $\Omega = B_1$ all semi-stable solutions of (2.4) are radially symmetric.

When Ω is convex with respect to each coordinate, one has that u satisfies $u_{x_i} \leq 0$ when $x_i \geq 0$, for $i = 1, ..., n$ by the classical result in [19]. In particular, if Ω is convex, we have that $u_s \leq 0$, $u_t \leq 0$ and $\|u\|_{L^\infty(\Omega)} = u(0)$.

We use now Lemma 3.6 to establish the following result. More precisely, we apply Lemma 3.6 separately with $c = u_s$ and with $c = u_t$, and then we choose appropriately the test function η to get the desired result. This estimate is the key ingredient in the proof of Proposition 3.3.

Lemma 3.8. *Let u be a bounded semi-stable solution of (2.1), and let α, β be such that $\alpha^2 < m-1$ and $\beta^2 < k-1$. Then, for each $\delta > 0$ there exists a constant $C > 0$, which depends only on α, β, δ, and Ω, such that*

$$\left(\int_\Omega \left\{ u_s^2 s^{-2\alpha-2} + u_t^2 t^{-2\beta-2} \right\} dx \right)^{1/2} \leq C \|f(u)\|_{L^\infty(\Omega_\delta)}. \tag{3.3}$$

Proof. We will prove only the estimate for $u_s^2 s^{-2\alpha-2}$, the other term can be estimated analogously.

Differentiating the equation $-\Delta u = f(u)$ with respect to s, we obtain

$$-\Delta u_s + (m-1)\frac{u_s}{s^2} = f'(u) u_s,$$

and hence, setting $c = u_s$ in Lemma 3.6, we have that

$$(m-1)\int_\Omega u_s^2 \frac{\eta^2}{s^2} dx \leq \int_\Omega u_s^2 |\nabla \eta|^2 dx$$

for all $\eta \in \text{Lip}(\overline{\Omega})$, $\eta|_{\partial\Omega} = 0$. Let us set in the last equation $\eta = \eta_\epsilon$, where

$$\eta_\epsilon = \begin{cases} s^{-\alpha}\rho & \text{if } s > \epsilon \\ \epsilon^{-\alpha}\rho & \text{if } s \leq \epsilon, \end{cases} \qquad \text{and} \qquad \rho = \begin{cases} 0 & \text{in } \Omega_{\delta/3} \\ 1 & \text{in } \Omega \backslash \Omega_{\delta/2}. \end{cases}$$

Then,

$$|\nabla \eta_\epsilon|^2 \leq \begin{cases} \alpha^2 s^{-2\alpha-2}\rho^2 & \text{in } (\Omega\backslash\Omega_{\delta/2}) \cap \{s > \epsilon\} \\ \frac{1}{2}(\alpha^2 + m - 1)s^{-2\alpha-2}\rho^2 + Cs^{-2\alpha} & \text{in } \Omega_{\delta/2} \cap \{s > \epsilon\} \\ C\epsilon^{-2\alpha} & \text{in } \Omega \cap \{s \leq \epsilon\}, \end{cases}$$

and therefore

$$\frac{m-1-\alpha^2}{2} \int_{\Omega\cap\{s>\epsilon\}} u_s^2 s^{-2\alpha-2}\rho^2 dx \leq C \int_{\Omega_{\delta/2}\cap\{s>\epsilon\}} u_s^2 s^{-2\alpha} dx + C\epsilon^{m-2\alpha},$$

where C denote different constants. Now, making $\epsilon \to 0$ and using that $2\alpha < 2\sqrt{m-1} \leq m$,

$$\int_\Omega u_s^2 s^{-2\alpha-2}\rho^2 dx \leq C \int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha} dx,$$

and then,

$$\int_{\Omega\setminus\Omega_{\delta/2}} u_s^2 s^{-2\alpha-2} dx \leq C \int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha} dx. \tag{3.4}$$

Finally, using that $u_s(0,t) = 0$ we obtain that, if δ is small enough, $\|s^{-\gamma}u_s\|_{L^\infty(\Omega_{\delta/2})} \leq C\|f(u)\|_{L^\infty(\Omega_\delta)}$ for each $\gamma < 1$. Hence, taking γ such that $m - 2\alpha - 2 + 2\gamma > 0$, we have

$$\int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha-2} dx \leq \|f(u)\|^2_{L^\infty(\Omega_\delta)} \int_{\Omega_{\delta/2}} s^{-2\alpha-2+2\gamma} dx \leq C\|f(u)\|^2_{L^\infty(\Omega_\delta)},$$

and therefore

$$\int_\Omega u_s^2 s^{-2\alpha-2} dx \leq C\|f(u)\|^2_{L^\infty(\Omega_\delta)}.$$

□

Using Lemma 3.8 we can now establish Proposition 3.3.

Proof of Proposition 3.3. a) On the one hand, making the change of variables $\sigma = s^{2+\alpha}$, $\tau = t^{2+\beta}$ in the integral appearing in (3.3), one has

$$\begin{cases} s^{m-1}ds &= c(\alpha)\sigma^{\frac{m}{2+\alpha}-1}d\sigma \\ t^{k-1}dt &= c(\beta)\tau^{\frac{k}{2+\beta}-1}d\tau, \end{cases}$$

and thus,

$$\int_\Omega \sigma^{\frac{m}{2+\alpha}-1}\tau^{\frac{k}{2+\beta}-1}(u_\sigma^2 + u_\tau^2)d\sigma d\tau \leq C\|f(u)\|^2_{L^\infty(\Omega_\delta)}. \tag{3.5}$$

Therefore, setting $\rho = \sqrt{\sigma^2+\tau^2}$ and taking into account that in $\{\tau < \sigma < 2\tau\}$ we have $\sigma > \frac{\rho}{2}$ and $\tau \geq \frac{\rho}{3}$, we obtain

$$\int_{\Omega\cap\{\tau<\sigma<2\tau\}} \rho^{\frac{m}{2+\alpha}+\frac{k}{2+\beta}-2}(u_\sigma^2 + u_\tau^2)d\sigma d\tau \leq C\|f(u)\|^2_{L^\infty(\Omega_\delta)}. \tag{3.6}$$

On the other hand, for each angle θ we have

$$u(0) \leq \int_{l_\theta} |\nabla u| d\rho,$$

where l_θ is the segment of angle θ in the (σ, τ)-plane from the origin to $\partial\Omega$, and integrating for $\frac{\pi}{4} < \theta < \frac{\pi}{3}$,

$$u(0) \le C \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_{l_\theta} |\nabla u| d\rho d\theta = C \int_{\Omega\cap\{\tau<\sigma<2\tau\}} \frac{|\nabla u|}{\rho} d\sigma d\tau. \tag{3.7}$$

Now, applying Schwarz's inequality and taking into account (3.6) and (3.7),

$$u(0) \le C\|f(u)\|_{L^\infty(\Omega_\delta)} \left(\int_{\Omega\cap\{\tau<\sigma<2\tau\}} \rho^{-\left(\frac{m}{2+\alpha}+\frac{k}{2+\beta}\right)} d\sigma d\tau \right)^{1/2}.$$

This integral is finite when

$$\frac{m}{2+\alpha} + \frac{k}{2+\beta} < 2,$$

and therefore if

$$\frac{m}{2+\sqrt{m-1}} + \frac{k}{2+\sqrt{k-1}} < 2 \tag{3.8}$$

we will be able to choose $\alpha < \sqrt{m-1}$ and $\beta < \sqrt{k-1}$ such that the integral is finite. Hence, since $\|u\|_{L^\infty(\Omega)} = u(0)$, if condition (3.8) is satisfied then

$$\|u\|_{L^\infty(\Omega)} \le C\|f(u)\|_{L^\infty(\Omega_\delta)}.$$

It is immediate to verify that (3.8) is satisfied for all $m \ge 2$ and $k \ge 2$ if $n \le 7$.

b) Since $n \ge 8$ then $\frac{m}{2+\sqrt{m-1}} + \frac{k}{2+\sqrt{k-1}} > 2$, and hence, given $p < p_{m,k}$ we can choose α, β such that $\alpha^2 < m-1$, $\beta^2 < k-1$,

$$p = 2 + \frac{4}{\frac{m}{2+\alpha} + \frac{k}{2+\beta} - 2},$$

and such that one of the numbers $\frac{m}{2+\alpha} - 1$ or $\frac{k}{2+\beta} - 1$ is positive.

Making the change of variables $\sigma = s^{2+\alpha}$, $\tau = t^{2+\beta}$ we obtain inequality (3.5), and hence, using Proposition 3.4 with $a = \frac{m}{2+\alpha} - 1$, $b = \frac{k}{2+\beta} - 1$ and $p = 2$, we deduce that

$$\left(\int_\Omega \sigma^{\frac{m}{2+\alpha}-1} \tau^{\frac{k}{2+\beta}-1} |u|^p d\sigma d\tau \right)^{1/p} \le C\|f(u)\|_{L^\infty(\Omega_\delta)}.$$

Finally, since

$$\int_\Omega \sigma^{\frac{m}{2+\alpha}-1}\tau^{\frac{k}{2+\beta}-1}|u|^p d\sigma d\tau = c(\alpha,\beta)\|u\|^p_{L^p(\Omega)},$$

then

$$\|u\|_{L^p(\Omega)} \le C\|f(u)\|_{L^\infty(\Omega_\delta)}.$$

c) Setting $\alpha = 0$ in (3.4), we obtain

$$\int_{\Omega\setminus\Omega_{\delta/2}} u_s^2 s^{-2} dx \le C\int_{\Omega_{\delta/2}} u_s^2 dx,$$

and therefore

$$\int_\Omega u_s^2 dx \le C\int_{\Omega_{\delta/2}} u_s^2 dx.$$

□

3.2 Regularity of the extremal solution

This subsection is devoted to give the proof of Theorem 3.1. The estimates for convex domains will follow easily from Proposition 3.3 and the boundary estimates from Theorem 2.17.

The main part of the proof are the estimates for non-convex domains. They will be proved by interpolating the Nedev's $W^{1,p}$ and $W^{2,p}$ estimates from Theorem 2.12 and our estimate proved in Lemma 3.8, and by applying then Sobolev inequality.

Proof of Theorem 3.1. As we have pointed out, the estimates for convex domains are a consequence of Proposition 3.3 and Theorem 2.17. Namely, we can apply estimates of Proposition 3.3 to the bounded solutions u_λ of (2.1) for $\lambda < \lambda^*$, and then by monotone convergence the estimates hold for the extremal solution u^*. Next we prove the estimates for non-convex domains.

a) Let $n = 4$, i.e. $m = k = 2$.

From Theorem 2.12 we deduce that the extremal solution satisfies $u^* \in W^{1,p}(\Omega)$ for all $p < \frac{n}{n-3}$, so in this case for each $p < 4$ we have

$$\int_\Omega |u_s^*|^p dx \le C, \qquad \int_\Omega |u_t^*|^p dx \le C.$$

Assume $\|u^*\|_{L^\infty(\Omega_\delta)} \leq C$. Then, by Lemma 3.8

$$\int_\Omega s^{-\gamma}|u_s^*|^2 dx \leq C \qquad \int_\Omega t^{-\gamma}|u_t^*|^2 dx \leq C$$

for all $\gamma < 2+2\sqrt{3}$. Hence, for each $\lambda \in [0,1]$,

$$\int_\Omega (s^{-\lambda\gamma}|u_s^*|^{p-\lambda(p-2)} + t^{-\lambda\gamma}|u_t^*|^{p-\lambda(p-2)})dx \leq C.$$

Setting now $\sigma = s^\alpha$, $\tau = t^\alpha$ and

$$\alpha = 1 + \frac{\lambda\gamma}{p-\lambda(p-2)},$$

we obtain

$$\int_\Omega \sigma^{\frac{2}{\alpha}-1}\tau^{\frac{2}{\alpha}-1}|\nabla_{(\sigma,\tau)}u^*|^{p-\lambda(p-2)} d\sigma d\tau \leq C,$$

and taking $p=3$, $\gamma=5$ and $\lambda=1/2$, we obtain

$$\int_\Omega |\nabla_{(\sigma,\tau)}u^*|^{5/2} d\sigma d\tau \leq C.$$

Finally, applying Sobolev's inequality in dimension 2, $u^* \in L^\infty(\Omega)$.

It remains to prove that $\|u^*\|_{L^\infty(\Omega_\delta)} \leq C$. We have that $u^* \in W^{1,p}$ for $p<4$, and in particular

$$\int_{\Omega_\delta} st|\nabla u|^p dsdt \leq C.$$

Since the domain is smooth, then it has to be $0 \notin \partial\Omega$ (otherwise the boundary will have an isolated point) and hence, there exist $r_0 > 0$ and $\delta > 0$ such that $\Omega_\delta \cap B(0,r_0) = \emptyset$. Thus, $s \geq c$ in $\Omega_\delta \cap \{s>t\}$ and $t \geq c$ in $\Omega_\delta \cap \{s<t\}$, and

$$\int_{\Omega_\delta\cap\{s>t\}} t|\nabla u|^p dsdt \leq C, \qquad \int_{\Omega_\delta\cap\{s<t\}} s|\nabla u|^p dsdt \leq C.$$

If $p>3$ then we can apply Sobolev's inequality in dimension 3 (as explained in Remark 3.5), to obtain $u^* \in L^\infty(\Omega_\delta \cap \{s>t\})$ and $u^* \in L^\infty(\Omega_\delta \cap \{s<t\})$. Thus, $u^* \in L^\infty(\Omega_\delta)$, as claimed.

c) Let $n \leq 6$. By Proposition 3.3, it suffices to prove that $u^* \in H_0^1(\Omega_\delta)$ for some $\delta > 0$. Take r_0 and δ such that $\Omega_\delta \cap B(0,r_0) = \emptyset$, as in part a).

From Theorem 2.12 we have that $u^* \in W^{2,p}(\Omega)$ for $p < \frac{n}{n-2}$, and hence

$$\int_{\Omega_\delta \cap \{s>t\}} t^{k-1}|D^2u^*|^p dsdt \le C, \qquad \int_{\Omega_\delta \cap \{s<t\}} s^{m-1}|D^2u^*|^p dsdt \le C.$$

Taking $p = \frac{2k+2}{k+3}$ and $p = \frac{2m+2}{m+3}$ respectively, and applying Sobolev's inequality in dimension $k+1$ and $m+1$ respectively, we obtain $u^* \in H_0^1(\Omega_\delta \cap \{s > t\})$ and $u^* \in H_0^1(\Omega_\delta \cap \{s < t\})$. Therefore, $u^* \in H_0^1(\Omega_\delta)$. $\square$

3.3 Sobolev inequality with the weight $s^a t^b$

As we will see in section 4, the classical Sobolev inequality can be deduced from an isoperimetric inequality. This can done in the following way: first, by applying the isoperimetric inequality to the level sets of a function and using the coarea formula, one deduces the Sobolev inequality with $p = 1$. Then, by applying Hölder's inequality one deduces the general Sobolev inequality.

Since in our case we have $u_s < 0$ and $u_t < 0$ for $s > 0$ and $t > 0$, it suffices to prove a weighted isoperimetric inequality for domains $\Omega \subset (\mathbb{R}_+)^2$ satisfying the following properties:

a) If $(s,t) \in \Omega$ then $(s',t') \in \Omega$ for all s' and t' such that $0 < s' < s$ and $0 < t' < t$.

b) $\Omega(\cdot,t) = \{s : (s,t) \in \Omega\}$ and $\Omega(s,\cdot) = \{t : (s,t) \in \Omega\}$ are strictly decreasing in t and s, respectively.

We will say that a domain satisfies the P-property when it satisfies both properties. We will denote

$$m(\Omega) = \int_\Omega s^a t^b dsdt \qquad \text{and} \qquad m(\partial\Omega) = \int_{\partial\Omega} s^a t^b d\sigma.$$

Proposition 3.9. *Let $\Omega \subset (\mathbb{R}_+)^2$ be a smooth domain satisfying the P-property, and let $a > -1$ and $b > -1$ be real numbers, being positive at least one of them. Then, there exists a constant C, depending only on a and b, such that*

$$m(\Omega)^{\frac{D-1}{D}} \le Cm(\partial\Omega),$$

where $D = a + b + 2$.

Proof. First, by symmetry we can suppose $a > 0$.

The P-property implies that there exists a non-increasing function ψ such that $\Omega = \{(s,t) \in \mathbb{R}^2_+ : s < \psi(s)\}$. Then,

$$m(\Omega) = \int_0^{+\infty} s^a \psi^{b+1} ds, \qquad m(\partial\Omega) = \int_0^{+\infty} s^a \psi^b \sqrt{1+\dot{\psi}^2} ds.$$

Let $\lambda > 0$ be such that $m(\Omega) = \frac{\lambda^D}{a+1}$. Then, we claim that $\psi(s) < \lambda$ for $s > \lambda$. Otherwise, we would have $v(s') \geq \lambda$ for some $s' > \lambda$, and

$$m(\Omega) \geq \int_0^{s'} s^a \psi^{b+1} ds > \int_0^{\lambda} s^a \lambda^{b+1} ds = \frac{\lambda^D}{a+1},$$

a contradiction. On the other hand, since $b+1 > 0$ and $\dot{\psi} \leq 0$,

$$\begin{aligned} m(\partial\Omega) &= \int_0^{+\infty} s^a \psi^b \sqrt{1+\dot{\psi}^2} ds \\ &\geq c \int_0^{+\infty} s^a \psi^b \left[1 - (b+1)\dot{\psi}\right] ds \\ &= c \int_0^{+\infty} s^a \left[\psi^b - \frac{d}{ds}(\psi^{b+1})\right] ds \\ &= c \int_0^{+\infty} s^a \psi^{b+1} \left(\frac{1}{\psi} + \frac{1}{s}\right) ds. \end{aligned}$$

Finally, taking into accout that $\psi(s) < \lambda$ for $s > \lambda$, we obtain that $\frac{1}{\psi} + \frac{1}{s} \geq \lambda^{-1}$ for each $s > 0$, and

$$m(\partial\Omega) \geq c \int_0^{+\infty} s^a \psi^{b+1} \left(\frac{1}{\psi} + \frac{1}{s}\right) ds \geq c\lambda^{-1} m(\Omega) = c m(\Omega)^{\frac{D-1}{D}},$$

as claimed. □

Since a weighted isoperimetric inequality implies a weighted Sobolev inequality (as we prove in Theorem 4.2 in the next section), Proposition 3.4 follows.

4 Weighted Sobolev inequalities

Let us recall the definition of the Sobolev spaces. Let $u \in L^p(\mathbb{R}^n)$, where $1 \leq p \leq \infty$. We say that u belongs to the Sobolev space $W^{1,p}(\mathbb{R}^n)$ if its distributional derivatives of first order belong to $L^p(\mathbb{R}^n)$. Note that this definition easily extends to the setting of Riemannian manifolds, as the gradient is well defined there.

The fundamental result in the theory of Sobolev spaces is the so-called Sobolev embedding theorem. It states that, for $1 \leq p < n$,

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n),$$

where $p^* = np/(n-p)$. Moreover, for $p > n$ one has the Morrey embedding

$$W^{1,p}(\mathbb{R}^n) \subset C^{0,\alpha}(\mathbb{R}^n),$$

where $\alpha = 1 - n/p$. These embeddings are equivalent to the following inequalities. The first one is the classical Sobolev inequality, and the second one the Morrey inequality.

Theorem 4.1 (see, for instance, [16])**.** *Let $p \geq 1$ be a real number, and let $u \in W^{1,p}(\mathbb{R}^n)$. There exist a constant C, depending only on n and p, such that*

i) if $p < n$, then

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{1/p^*} \leq C \left(\int_{\mathbb{R}^n} |\nabla u|^p dx \right)^p,$$

where $p^* = \frac{pn}{n-p}$.

ii) if $p > n$, then

$$\sup_{y \neq z} \frac{|u(y) - u(z)|}{|y - z|^\alpha} \leq C \left(\int_{\mathbb{R}^n} |\nabla u|^p dx \right)^p,$$

where $\alpha = 1 - \frac{n}{p}$.

Sobolev inequalities are a central tool in the study of various aspects of partial differential equations and calculus of variations. There are several generalizations of the classical Sobolev inequalities as they are very basic tools in the study of the existence, regularity, and uniqueness of the solutions of all sorts of PDEs, linear and nonlinear, elliptic, parabolic, and hyperbolic.

Moreover, the scope of its applications is much wider, including questions on differential geometry, algebraic topology, complex analysis, and probability theory.

The aim of this section is to explain some known results which lead to some weighted Sobolev inequalities. We will explain these results and how can they be applied in order to prove Proposition 3.4, needed in the previous section to study extremal solutions in domains of double revolution. In fact, we will be able to prove the Sobolev inequality with monomial weight (1.4) for $p = 2$ by deducing it from a result of P. Hajlasz. However, this proof does not work with all exponents p nor gives the best constant, and it is in section 5 where we give the proof of inequality (1.4) with best constant and for each exponent $p \geq 1$.

4.1 Relation between isoperimetric and Sobolev inequalities

One of the several ways to prove the classical Sobolev inequality in $\mathbb{R}^n$ is to deduce it from the isoperimetric inequality via the coarea formula.

Next we prove that this can be done for an arbitrary measure m. That is, by applying an isoperimetric inequality with some measure m to the level sets of a function, one can deduce a Sobolev inequality with this measure. In particular, when the measure is given by $dm = w(x)dx$ it gives a powerful tool to prove weighted Sobolev inequalities.

Proposition 4.2 ([15]). *Let m be a measure in $X \subset \mathbb{R}^n$ such that, for some $D > 1$,*

$$m(E)^{\frac{D-1}{D}} \leq Cm(\partial E) \tag{4.1}$$

for each smooth set $E \subset X$. Then, for each $1 \leq p < D$ there exist a constant C such that

$$\left(\int_X |u|^{p^*} d\mu \right)^{1/p^*} \leq C \left(\int_X |\nabla u|^p d\mu \right)^{1/p},$$

for all differentiable functions $u \in C_c^1(X)$, where $p^* = \frac{pD}{D-p}$.

Proof. We will prove first the case $p = 1$.

Letting χ_A the characteristic function of the set A, we have

$$u(x) = \int_0^{+\infty} \chi_{[u(x)>\tau]} d\tau.$$

Thus, by Minkowski's integral inequality

$$\begin{aligned}\left(\int_X |u|^{\frac{D}{D-1}} d\mu\right)^{\frac{D-1}{D}} &\leq \int_0^{+\infty} \left(\int_X \chi_{[u(x)>\tau]} d\mu\right)^{\frac{D-1}{D}} d\tau \\ &= \int_0^{+\infty} m(\{u(x)>\tau\})^{\frac{D}{D-1}} d\tau.\end{aligned}$$

Inequality (4.1) together with Sard's Theorem imply

$$m(\{u(x)>\tau\})^{\frac{D}{D-1}} \leq C m(\{u(x)=\tau\}),$$

whence

$$\left(\int_X |u|^{\frac{D}{D-1}} d\mu\right)^{\frac{D-1}{D}} \leq C \int_0^{+\infty} m(\{u(x)=\tau\}) d\tau = C \int_X |\nabla u| d\mu,$$

where we have used the coarea formula.

It remains to prove the case $1 < p < D$. Take $u \in C_c^1(X)$, and define $v = u^\gamma$, where $\gamma = \frac{p^*}{1^*}$. In particular, $\gamma > 1$, so that $v \in C_c^1(X)$, and we can apply the weighted Sobolev inequality with $p = 1$ to get

$$\left(\int_X |u|^{p^*} d\mu\right)^{1/1^*} = \left(\int_X |v|^{\frac{D}{D-1}} d\mu\right)^{\frac{D-1}{D}} \leq C \int_X |\nabla v| d\mu.$$

Now, $|\nabla v| = \gamma u^{\gamma-1} |\nabla u|$, and by Hölder's inequality it follows that

$$\int_X |\nabla v| d\mu \leq C \left(\int_X |\nabla u|^p d\mu\right)^{1/p} \left(\int_X |u|^{(\gamma-1)p'} d\mu\right)^{1/p'}.$$

But from the definition of γ and p^* it follows that

$$\frac{1}{1^*} - \frac{1}{p^*} = \frac{1}{p'}, \qquad (\gamma-1)p' = p^*,$$

and hence,

$$\left(\int_X |u|^{p^*} d\mu\right)^{1/p^*} \leq C \left(\int_X |\nabla u|^p d\mu\right)^{1/p}.$$

□

Remark 4.3. When the constant appearing in (4.1) is optimal, the proof gives the optimal constant for the weighted Sobolev inequality for $p = 1$. It can be seen by proving that when one has an increasing sequence of smooth functions $u_\varepsilon \to \chi_E$, then one has $\|\nabla u_\varepsilon\|_{L^1} \to m(\partial E)$.

4.2 The maximal operator and Muckenhoupt weights

Next we present the most classical and known weighted Sobolev inequalities. We introduce a class of weights A_p, known as Muckenhoupt weights, such that a weighted Sobolev inequality with weight w and exponent p holds for all $w \in A_p$. For it, we introduce first the notion of maximal operator.

The Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{r>0} ⨍_{B(x,r)} |f(y)| dy.$$

There are variations of this definition such as replacing balls by cubes or consider balls containing x, not only centered at x, but all of them are equivalent with dimensional constants.

The maximal operator arises naturally when proving theorems of existence a.e. of limits or when controlling pointwise important objects. For example, the Lebesgue differentiation Theorem,

$$f(x) = \lim_{r\to 0} ⨍_{B(x,r)} f(y) dy \qquad \text{a.e.},$$

follows from the following weak type estimate:

Theorem 4.4 (see, for instance, [34]). *There exists a constant C such that for every $\lambda > 0$ the following inequality holds,*

$$|\{x \in \mathbb{R}^n : Mf(x) < \lambda\}| \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| dx.$$

As a consequence of Theorem 4.4 one can derive:

Corollary 4.5 (see, for instance, [34]). *Let $1 < p < \infty$ then there exists a constant C such that*

$$\int_{\mathbb{R}^n} (Mf)^p dx \leq C \int_{\mathbb{R}^n} |f|^p dx.$$

Sometimes it is not enough to have the maximal operator bounded in $L^p(\mathbb{R}^n)$ and one needs it to be bounded in $L^p(\mathbb{R}^n, w(x)dx)$, where w is a weight. So the problem that arises is to find the necessary and sufficient condition for the maximal operator to be bounded in $L^p(\mathbb{R}^n, w(x)dx)$. This problem was solved by Muchenhoupt in 1972 and is known as Muckenhoupt Theorem.

Theorem 4.6 ([26])**.** *Let $1 < p < \infty$ and let w be a positive L^1_{loc} function. Then, the following conditions are equivalent:*

(a) There exists a constant C such that

$$\int_{\mathbb{R}^n} (Mf)^p w(x)dx \leq C \int_{\mathbb{R}^n} |f|^p w(x)dx$$

for all $f \in L^p(\mathbb{R}^n, w(x)dx)$.

(b) There exists a constant C such that for all balls B,

$$\left(\fint_B w(x)dx \right) \left(\fint_B w(x)^{-\frac{1}{p-1}} \right)^{p-1} \leq C.$$

When these equivalent conditions are satisfied, we say that w belongs to the Muckenhoupt class A_p.

The Muckenhoupt Theorem is the basic tool to prove the following weighted Sobolev inequality of Fabes, Kenig, and Serapioni.

Theorem 4.7 ([18])**.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain, let $1 < p < \infty$, and let $w \in A_p(\Omega)$ be a Muckenhoupt weight. Then, there exist $p^* > np/(n-1)$ and a positive constant C such that*

$$\left(\int_\Omega |u|^{p^*} w(x)dx \right)^{1/p^*} \leq C \left(\int_\Omega |\nabla u|^p w(x)dx \right)^{1/p}$$

for all $u \in C^1_c(\Omega)$.

As we have said, this inequality is the most general weighted Sobolev inequality known. However, the exponent p^* depends on the weight w, and the theorem gives no information on which is the optimal exponent p^* in the inequality.

Moreover, the constant C depends on Ω, and hence the Theorem does not give a weighted Sobolev inequality in all $\mathbb{R}^n$.

Note also that there are many weights which satisfy a weighted Sobolev inequality but which do not belong to the Muckenhoupt class. For example, our weight $s^a t^b$ belongs to $A_p(\mathbb{R}^2)$ if and only if $-1 < a < p-1$ and $-1 < b < p-1$.

In the following subsection we will se a more precise version of Theorem 4.7 and how it can be used to prove the weighted Sobolev inequality with the weight x^A.

4.3 Sobolev inequality in general metric spaces

Next we introduce the notion of Sobolev space in a general metric space, we see that it coincides with the classical definition in the Euclidean space, and we prove a Sobolev inequality in a general metric space. All these results of P. Hajlasz can be found in [22].

Definition 4.8. Let X be a metric space, μ a measure in X, and $1 < p < \infty$ a real number. We say that a function f belongs to the Sobolev space $W^{1,p}(X)$ if $f \in L^p(X)$ and there exist a function $g \in L^p(X)$ such that

$$|f(x) - f(y)| \leq |x-y|\{g(x) + g(y)\} \qquad a.e.$$

It can be proved that the space $W^{1,p}(X)$ is a Banach space with the norm

$$\|u\|_{W^{1,p}(X)} = \|f\|_{L^p(X)} + \|g\|_{L^p(X)}.$$

The next Lemma proves that when $\Omega \subset \mathbb{R}^n$ and $d\mu = w(x)dx$, with $w \in A_p$, this space coincides with the classical one.

Lemma 4.9 ([22])**.** *Let $1 < p < \infty$ and $w \in A_p$ a Muckenhoupt weight. Assume that $u \in W^{1,p}(\Omega, \omega(x)dx)$ in the classical sense, that is, $\nabla u \in L^p(\Omega, \omega(x)dx)$, where Ω is a bounded Lipschitz domain. Then, there exists a nonnegative function $g \in L^p(\Omega, \omega(x)dx)$ such that*

$$|u(x) - u(y)| \leq |x-y|\{g(x) + g(y)\} \ a.e. \tag{4.2}$$

and $\|g\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$.

Proof. It is well known [20] that if B is a ball in $\mathbb{R}^n$ then the following inequality holds a.e.

$$|u(x) - u_B| \leq C \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}} dx,$$

where $u_B = ⨍_B u dx$. Moreover, we have that

$$\begin{aligned} \int_{B(x,R)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dx &= \sum_{k\geq 0} \int_{2^{-k-1}R \leq |x-y| \leq 2^{-k}R} \frac{|\nabla u|}{|x-y|^{n-1}} dx \\ &\leq C \sum_{k\geq 0} 2^{-k}R(2^{-k}R)^{-n} \int_{|x-y|\leq 2^{-k}R} |\nabla u| dx \\ &\leq CR \sup_{r>0} ⨍_{B(x,r)} |\nabla u| dx \end{aligned}$$

and then

$$|u(x) - u(y)| \leq |u(x) - u_B| + |u(y) - u_B| \leq C|x - y|\{g(x) + g(y)\},$$

where g is the maximal operator of $|\nabla u|$, i.e.

$$g(x) = M|\nabla u|(x) = \sup_{r>0} \fint_{B(x,r)} |\nabla u| dx.$$

Finally, since $\omega \in A_p$ and $1 < p < \infty$, by Theorem 4.6 the maximal operator is bounded on $L^p(\mathbb{R}^n, \omega(x)dx)$. Since the domain is Lipschitz, it will satisfy the extension property, so the maximal operator will be bounded in $L^p(\Omega, \omega(x)dx)$ and $\|g\|_{L^p} \leq C\|\nabla u\|_{L^p}$. □

We now give a proof of the Sobolev inequality in general metric space.

Definition 4.10. Let X be a metric space with finite diameter. We say that a measure μ in X satisfies the D-regularity property if there exist a constant $b > 0$ such that $\mu(B(x,r)) \geq br^D$ for all $x \in X$ and $r < \operatorname{diam}(X)$, where $\mu(B) = \int_B d\mu$.

Theorem 4.11 ([22])**.** *Let X be a metric space with finite diameter and let μ be a measure in X satisfying the D-regularity property. Then, there exist a constant C depending only on Ω, ω and q such that*

(a) if $1 < p < D$,

$$\|u\|_{L^{p^*}(X)} \leq C\|u\|_{W^{1,p}(X)}$$

for all $u \in W^{1,p}(X)$, where $p^* = \frac{pD}{D-p}$.

(b) if $p > D$,

$$\|u\|_{L^\infty(X)} \leq C\|u\|_{W^{1,p}(X)}$$

for all $u \in W^{1,p}(X)$.

Proof. Let $g \in L^p(X)$ be such that

$$|f(x) - f(y)| \leq |x - y|\{g(x) + g(y)\} \qquad a.e.,$$

and define $E_k = \{x \in X : g(x) \leq 2^k\}$ and $a_k = \sup_{E_k} |f|$ for each $k \in \mathbb{Z}$.

We will estimate a_k in terms of a_{k-1}.

Let $x \in E_k$ and let $B_r(x)$ be the ball with radius

$$r = b^{-1/D}\mu(X \backslash E_{k-1})^{1/D}$$

and centered at x. Then the D-regularity property implies $\mu(B_r(x)) \geq \mu(X\backslash E_{k-1})$, and there exists $\overline{x} \in B_r(x) \cap E_{k-1}$. Hence,

$$\begin{aligned} |u(x)| &\leq |u(x) - u(\overline{x})| + |u(\overline{x})| \leq 2^{k+1}|x - \overline{x}| + a_{k-1} \\ &\leq C2^k \mu(X\backslash E_{k-1})^{1/D} + a_{k-1}, \end{aligned}$$

and since

$$\mu(X\backslash E_{k-1}) \leq 2^{-kp}\|g\|_{L^p}^p$$

then

$$a_k \leq C2^{k\left(1-\frac{p}{D}\right)}\|g\|_{L^p}^{p/D} + a_{k-1}. \tag{4.3}$$

Now we will estimate a_{k_0} for some $k_0 \in \mathbb{Z}$. Let k_0 be such that $\mu(E_{k_0-1}) \leq \mu(X)/2 \leq \mu(E_{k_0})$. Then,

$$2^{(k_0-1)p}\mu(X)/2 \leq 2^{(k_0-1)p}\mu(X\backslash E_{k_0-1}) \leq \|g\|_{L^p}^p,$$

and hence $2^{k_0} \leq C\|g\|_{L^p}$. Since $f|_{E_{k_0}}$ is Lipschitz with constant 2^{k_0+1} then

$$a_{k_0} \leq \left(\fint_{E_{k_0}} |u|^p \mu(x)dx \right)^{1/p} + C2^{k_0},$$

and

$$a_{k_0} \leq \mu(E_{k_0})^{-1/p}\|u\|_{L^p} + C\|g\|_{L^p} \leq C\|u\|_{W^{1,p}}.$$

If $p < D$, adding up (4.3) from k_0 to k we get

$$a_k \leq C2^{k\left(1-\frac{p}{D}\right)}\|u\|_{W^{1,p}}^{p/D} + a_{k_0},$$

and therefore

$$a_k^{p^*} \leq C2^{kp}\|u\|_{W^{1,p}}^{pp^*/D} + C\|u\|_{W^{1,p}}^{p^*}.$$

Finally, since

$$\sum_{k\in\mathbb{Z}} 2^{kp}\mu(E_k\backslash E_{k-1}) \leq C\int_X g^p \mu(x)dx,$$

we will have

$$\begin{aligned} \int_X |u|^{p^*}d\mu &\leq \sum_{k\in\mathbb{Z}} a_k^{p^*}\mu(E_k\backslash E_{k-1}) \\ &\leq C\|u\|_{W^{1,p}}^{pp^*/D}\sum_{k\in\mathbb{Z}} 2^{kp}\mu(E_k\backslash E_{k-1}) + C\|u\|_{W^{1,p}}^{p^*} \\ &\leq C\|u\|_{W^{1,p}}^{pp^*/D}\|g\|_{L^p}^p + C\|u\|_{W^{1,p}}^{p^*} \\ &\leq C\|u\|_{W^{1,p}}^{p^*}. \end{aligned}$$

If $p > D$, adding up (4.3) from k_0 to k we get

$$a_k \leq C 2^{k_0(1-p/D)} \|u\|_{W^{1,p}}^{p/D} + a_{k_0} \leq C\|u\|_{W^{1,p}} + a_{k_0},$$

and then

$$\|u\|_{L^\infty} = \sup_{k\in\mathbb{Z}} a_k \leq C\|u\|_{W^{1,p}}.$$

□

Next we will see how these results of P. Hajlasz can be applied to prove the weighted Sobolev inequality (1.4) for $p = 2$. This was the first result we obtained towards proving Proposition 3.4, but it does not give the optimal exponent p^* in the case $a < 0$ or $b < 0$ in the Proposition. Some time after finding this result, we obtained the proof of Proposition 3.4 given in subsection 3.3, and after that, the proof of inequality (1.4) given in section 5, which works with each exponent $p \geq 1$ and gives the best constant C_p. As said before, the following proof does not work with each exponent p, it only works in bounded domains, and it does not give the optimal constant in the inequality.

Note that $A = (A_1, ..., A_n) \in \mathbb{R}^n$, and that we are using the notation $|A| = |A_1| + \cdots + |A_n|$ and $A^+ = (A_1^+, ..., A_n^+)$, where $x^+ = \max\{x, 0\}$.

As said before, the weight x^A does not belong to the Muckenhoupt class A_2 when $A_i \geq 1$ for some i, and thus we cannot apply directly Theorem 4.11. The key idea in the following proof is to apply it not in $\mathbb{R}^n$ but in a higher dimensional space $\mathbb{R}^N$ in which the transformed weight will be Muckenhoupt. Note first that when we write an integral from $(\mathbb{R}^N, dz)$ in radial coordinates $x_1, ..., x_n$, it transforms into $(\mathbb{R}^n, x^B dx)$. Hence, when we have an integral with $x^A dx$ in $\mathbb{R}^n$ one can transform it into an integral in $\mathbb{R}^N$ with $x^{A-B}dz$, where x_i are radial variables in $\mathbb{R}^N$. If one can take B such that the weight x^{A-B} is Muckenhoupt in $\mathbb{R}^N$, then one can apply Theorem 4.11. Finally, once we have the weighted Sobolev inequality in $\mathbb{R}^N$, by writing it in the original variables we will obtain the desired result.

Proposition 4.12. *Let $\Omega \subseteq (\mathbb{R}_+)^n$ be a bounded and smooth domain, with $n \geq 2$, and let $A \in \mathbb{R}^n$ be such that $A > -1$. Let $D = n + |A^+|$ and $u \in C_0^1(\Omega)$. Then, there exist a constant C, depending only on A, Ω, and n such that*

$$\left(\int_\Omega x^A |u|^q dx\right)^{1/q} \leq C \left(\int_\Omega x^A |\nabla u|^2 dx\right)^{1/2},$$

where $q = \frac{2D}{D-2}$ if $D > 2$ and $q < \infty$ if $D = 2$.

Proof. Let us introduce the new variables $z_{i,j}$, $i = 1, ..., n$ and $j = 1, ..., B_i$ where $B_i = [A_i^+]$. Then, in the space $\mathbb{R}^N$, $N = B_1 + \cdots + B_n + n$, let

$$s_i = \sqrt{s_{i,1}^2 + ... + s_{i,B_i+1}^2}, \qquad i = 1, ..., n,$$

and define the function $\widetilde{u}(z) = u(s_1, ..., s_n)$. Will have that $|\nabla\widetilde{u}| = |\nabla u|$ and therefore

$$\int_\Omega s_1^{A_1-B_1} \cdots s_n^{A_n-B_n} |\widetilde{u}|^q dz = C \int_\Omega x_1^{A_1} \cdots x_n^{A_n} |u|^q dx,$$

$$\int_\Omega s_1^{A_1-B_1} \cdots s_n^{A_n-B_n} |\nabla\widetilde{u}|^2 dz = C \int_\Omega x_1^{A_1} \cdots x_n^{A_n} |\nabla u|^2 dx.$$

Hence, if we want to apply Theorem 5.2, we have to prove that the weight $s_1^{A_1-B_1} \cdots s_n^{A_n-B_n}$ satisfies the Muckenhoupt condition $A_2(\mathbb{R}^N)$, i.e. that for all ball B,

$$\left(\fint_B s_1^{A_1-B_1} \cdots s_n^{A_n-B_n} dz \right) \left(\fint_B s_1^{-A_1+B_1} \cdots s_n^{-A_n+B_n} dz \right) \le C.$$

Making the change of variables $s_i = x_i$, we see that this condition is equivalent to

$$\left(\fint_B x_1^{A_1} \cdots x_n^{A_n} dx \right) \left(\fint_B x_1^{-A_1+2B_1} \cdots x_n^{-A_n+2B_n} dx \right) \le C$$

for all ball $B \subseteq \Omega$. Moreover, we can reduce to prove it to cubes, and then we only have to see that

$$\prod_{i=1}^n \left(\int_{a_i}^{b_i} x_i^{A_i} dx_i \right) \left(\int_{a_i}^{b_i} x_i^{2B_i-A_i} dx_i \right) \le C. \tag{4.4}$$

Since $x^t \in A_2(\mathbb{R})$ if $-1 < t < 1$, and $-1 < A_i - B_i < 1$, then we have that for each i

$$\left(\int_{a_i}^{b_i} x_i^{A_i-B_i} dx_i \right) \left(\int_{a_i}^{b_i} x_i^{B_i-A_i} dx_i \right) \le C,$$

and since $B_i \ge 0$, then we will have

$$\left(\int_{a_i}^{b_i} x_i^{A_i} dx_i \right) \left(\int_{a_i}^{b_i} x_i^{2B_i-A_i} dx_i \right) \le C,$$

and hence (4.4).

Define now $\widetilde{\Omega} = \{s_i < R,\ i = 1, ..., d\}$, where R is such that $\Omega \subseteq \widetilde{\Omega}$, and extend u by zero in $\widetilde{\Omega} \backslash \Omega$. Theorem 5.2 applied to $L^2(\widetilde{\Omega}, w(y)dy)$ with $D = |A^+| + n$ and $w(y) = s_1^{A_1 - B_1} \cdots s_n^{A_n - B_n}$ will gives us the desired result.

Hence, it only remains to prove that $w(B(x, r)) \geq br^{|A^+|+n}$, i.e. we want to see that

$$\int_{B(x,r)\cap\widetilde{\Omega}} s_1^{A_1-B_1} \cdots s_n^{A_n-B_n} dy \geq br^{|A^+|+n} \tag{4.5}$$

for some $b > 0$.

Making the change of variables $s_i = x_i$, we have to prove that

$$\int_{B(x,r)\cap\widetilde{\Omega}} x_1^{A_1} \cdots x_n^{A_n} dx \geq br^{|A^+|+n}.$$

Since $\widetilde{\Omega}$ is a cube in $\mathbb{R}^n$, it reduces to prove that

$$\int_{Q_r} x_1^{A_1} \cdots x_n^{A_n} dx \geq br^{|A^+|+n}$$

for each cube Q_r of radius r and each $r < \operatorname{diam}(\widetilde{\Omega})$. As before, we can write Q_r as $(a_1, b_1) \times \cdots \times (a_n, b_n)$, with $b_i - a_i = r$ for each i, and then

$$\int_{Q_r} x_1^{A_1} \cdots x_n^{A_n} dx = \prod_{i=1}^{n} \int_{a_i}^{b_i} x_i^{A_i} dx_i.$$

For $A_i \geq 0$, we have that $b_i^{A_i+1} - a_i^{A_i+1} \geq r^{A_i+1}$, while for $A_i < 0$ we have $\int_{a_i}^{b_i} x_i^{A_i} dx_i \geq cr$, and (4.5) follows. □

Remark 4.13. Note that this proposition gives the optimal exponent q in Proposition 3.4 when $A \geq 0$, but the exponent obtained in subsection 3.3 for $a < 0$ or $b < 0$ is better than this one.

Moreover, this proof can be adapted to show the Sobolev inequality in $W^{1,p}(\Omega)$ for $p \geq 2$, but not for $1 \leq p < 2$ since in this case the weight x^A does not belong to A_p for each $A < 1$.

Finally note also that this Proposition can be used only to prove Sobolev inequalities in bounded domains, not in the whole $\mathbb{R}^n$, and that it does not give the best constant nor extremal functions in any case.

In section 5 we give the proof of the Sobolev inequality with monomial weight x^A, with $A \geq 0$, for each exponent $p \geq 1$, and we obtain best constant and extremal functions of this inequality.

5 Sobolev inequalities with monomial weights

In this section we study the Sobolev embeddings and the isoperimetric inequality in $(\mathbb{R}^n, x^A dx)$, where A is a nonnegative vector in $\mathbb{R}^n$. As before, we will denote

$$x^A = |x_1|^{A_1} \cdots |x_n|^{A_n},$$

$$m(\Omega) = \int_\Omega x^A dx, \qquad \text{and} \qquad m(\partial\Omega) = \int_{\partial\Omega} x^A d\sigma.$$

Recall that when the numbers A_i are positive integers, the weighted Sobolev inequality

$$\left(\int_{(\mathbb{R}_+)^n} x^A |u|^{p^*} dx \right)^{1/p^*} \leq C \left(\int_{(\mathbb{R}_+)^n} x^A |\nabla u|^p dx \right)^p, \tag{5.1}$$

where $p^* = \frac{pD}{D-p}$ and $D = n + A_1 + \cdots + A_n$, is precisely the classical one in suitable radial coordinates. Namely, assume A_i are positive integers, and consider $\mathbb{R}^D = \mathbb{R}^{A_1+1} \times \cdots \times \mathbb{R}^{A_n+1}$, with $D = n + A_1 + \cdots + A_n$. For each $z \in \mathbb{R}^D$, write $z = (z^1, ..., z^n)$, with $z^i \in \mathbb{R}^{A_i+1}$, and define $x = (x_1, ..., x_n) = (|z^1|, ..., |z^n|) \in \mathbb{R}^n$. Now, on the one hand, each function u defined in $\mathbb{R}^n$ can be extended to $\mathbb{R}^D$ by defining $u(z) = u(|z_1|, ..., |z_n|)$, and then $|\nabla_z u| = |\nabla_x u|$. On the other hand, an integral in $\mathbb{R}^D$ of a function depending only on the variables x_i can be written as an integral in $(\mathbb{R}_+)^n$, where $dz = c(A) x^A dx$. Therefore, writing in the coordinates $x_1, ..., x_n$ the classical Sobolev inequality in $\mathbb{R}^D$ for the extended function u, one obtains (5.1). Analogously, the isoperimetric or the Morrey inequality with the monomial weight x^A is exactly the classical one in suitable radial coordinates.

In this section we show that Sobolev, Morrey, and isoperimetric inequalities with monomial weights hold also for noninteger exponents $A \geq 0$.

Let us define

$$\mathbb{R}^n_* = \{(x_1, ..., x_n) \in \mathbb{R}^n : A_i x_i \geq 0 \;\text{ for all }\; i\}, \tag{5.2}$$

that is, the set of points x in $\mathbb{R}^n$ such that $x_i \geq 0$ for those indices i for which $A_i > 0$.

Our first result is the following weighted isoperimetric inequality.

Theorem 5.1. *Let $A \geq 0$ be a vector in $\mathbb{R}^n$, and let $D = n + |A|$. Then, for each bounded smooth domain $\Omega \subset \mathbb{R}^n$,*

$$\frac{m(\partial\Omega)}{m(\Omega)^{\frac{D-1}{D}}} \geq \frac{m(\partial B_*)}{m(B_*)^{\frac{D-1}{D}}}\ , \tag{5.3}$$

where $B_* = B_1(0) \cap \mathbb{R}^n_*$ *and* $B_1(0)$ *is the unit ball centered at 0 of* $\mathbb{R}^n$*.*

Note that equality holds when $\Omega = rB_*$, where r is any positive number, but we do not know yet if these are the only domains for which equality holds.

We have seen in section 4 that an isoperimetric inequality implies a Sobolev inequality, so as an immediate consequence of the isoperimetric inequality we obtain the Sobolev inequality with monomial weights. Moreover, since our isoperimetric inequality is optimal (it has the best constant), by applying two results of G. Talenti we will be able to find the best constant and extremal functions in the Sobolev inequality.

Theorem 5.2. *Let* A *be a nonnegative vector in* $\mathbb{R}^n$*, let* $D = |A| + n$ *and let* $p \geq 1$ *be a real number.*

a) If $p < D$*, there exists a constant* C_p *such that for all* $u \in C^1_c(\mathbb{R}^n)$*,*

$$\left(\int_{\mathbb{R}^n_*} x^A |u|^{p^*} dx \right)^{1/p^*} \leq C_p \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^p, \tag{5.4}$$

where $p^* = \frac{pD}{D-p}$*.*

b) The best constant C_p *is given by the explicit expression (5.12-5.13) below. Such constant is not attained when* $p = 1$*, and is attained by the functions*

$$u(x) = \left(a + b|x|^{\frac{p}{p-1}} \right)^{1-\frac{D}{p}}$$

when $1 < p < D$*, where* a *and* b *are positive parameters.*

In subsection 5.2, as a Corollary of the last Theorem (see Corollary 5.10) we will obtain precisely the inequality which motivated our study of this type of weights.

The following is the weighted version of the Morrey inequality, which we prove at the end of this section.

Theorem 5.3. *Let* A *be a nonnegative vector in* $\mathbb{R}^n$*, let* $D = |A| + n$ *and let* p *be a real number. If* $p > D$*, there exists a constant* C *such that for all* $u \in C^1_c(\mathbb{R}^n)$*,*

$$\sup_{y,z \in \mathbb{R}^n_*} \frac{|u(y) - u(z)|}{|y - z|^\alpha} \leq C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^{1/p},$$

where $\alpha = 1 - \frac{D}{p}$*.*

Adding up the results of the last two propositions, we get the following continuous embbedings:

Corollary 5.4. *Let A be a nonnegative vector in $\mathbb{R}^n$, let $D = |A| + n$, and let p be a real number. Then, we have the following continuous embeddings:*

(i) If $1 \leq p < D$

$$W^{1,p}(\mathbb{R}^n_*, x^A dx) \subset L^{p^*}(\mathbb{R}^n_*, x^A dx),$$

where p^* is given by $p^* = \frac{pD}{D-p}$.

(ii) If $D < p \leq +\infty$ then

$$W^{1,p}(\mathbb{R}^n_*, x^A dx) \subset C^{0,\alpha}(\mathbb{R}^n_*),$$

where $\alpha = 1 - \frac{D}{p}$.

This section is organized as follows. In subsection 5.1 we give the proof of the weighted isoperimetric inequality and some of their consequences. In subsection 5.2 we prove the weighted Sobolev inequality, while in subsection 5.3 we obtain best constants and extremal functions of this inequality. Finally, in subsection 5.4 we prove the weighted Morrey inequality.

5.1 Proof of the isoperimetric inequality with monomial weight

In this subsection we give the proof of the isoperimetric inequality with monomial weight. This proof is a generalization of the the proof due to X. Cabré of the classical isoperimetric inequality. In fact, by setting $A = 0$ in the following proof we obtain exactly the original proof. It is quite surprising (and fortunate) that the proof (which gives the best constant) can be adapted it to make it work with the monomial weight.

The main changes in the proof are that the Laplacian operator is replaced by $x^{-A}\mathrm{div}(x^A \nabla u)$, and that we apply a weighted version of the inequality between the arithmetic and the geometric means instead of the classical one.

Proof of Theorem 5.1. First of all, note that we can suppose that Ω is contained in $\mathbb{R}^n_*$. Otherwise, we can split the domain in (at most) 2^n domains Ω_i, each one contained in $\{A_i x_i \epsilon_i \geq 0\}$ for different $\epsilon_i \in \{-1, 1\}$. Then, since the weight is zero on the x_i-axis when $A_i > 0$, one has that $m(\partial\Omega) = \sum_i m(\partial\Omega_i)$ and $m(\Omega) = \sum_i m(\Omega_i)$, and then

$$\frac{m(\partial\Omega)}{m(\Omega)^{\frac{D-1}{D}}} \geq \min_i \left\{ \frac{m(\partial\Omega_i)}{m(\Omega_i)^{\frac{D-1}{D}}} \right\}.$$

Moreover, since every smooth domain Ω of $\mathbb{R}^n_*$ is the increasing limit of smooth domains Ω_k such that $\overline{\Omega}_k \subset \mathbb{R}^n_*$, then it suffices to prove it for domains $\Omega \subset\subset \mathbb{R}^n_*$.

Let u be a solution of the Neumann problem

$$\begin{cases} \operatorname{div}(x^A \nabla u) = c x^A & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

where the constant c is chosen so that the problem has a unique solution up to an additive constant, i.e.

$$c = \frac{m(\partial\Omega)}{m(\Omega)}.$$

Since $\overline{\Omega} \subset \mathbb{R}^n_*$ the operator is uniformly elliptic, and thus u is smooth in $\overline{\Omega}$.

We consider the lower contact set of u, defined by

$$\Gamma_u = \{x \in \Omega \;:\; u(y) \geq u(x) + \nabla u(x) \cdot (y - x) \;\text{ for all } y \in \overline{\Omega}\}. \tag{5.6}$$

It is the set of points where the tangent hyperplane to the graph of u lies below u in all $\overline{\Omega}$. We claim that

$$B_1(0) \subset \nabla u(\Gamma_u), \tag{5.7}$$

where $B_1(0)$ denotes the unit ball of $\mathbb{R}^n$ with center 0.

To show (5.7), take any $p \in \mathbb{R}^n$ satisfying $|p| < 1$. Let $x \in \overline{\Omega}$ be a point such that

$$\min_{y \in \overline{\Omega}} \{u(y) - p \cdot y\} = u(x) - p \cdot x$$

(this is, up to a sign, the Legendre transform of u). If $x \in \partial\Omega$ then the exterior normal derivative of $u(y) - p \cdot y$ at x would be nonpositive and hence $(\partial u/\partial \nu)(x) \leq |p| < 1$, a contradiction with (5.5). It follows that $x \in \Omega$ and, therefore, that x is an interior minimum of the function $u(y) - p \cdot y$. In particular, $p = \nabla u(x)$ and $x \in \Gamma_u$. Claim (5.7) is now proved. It is interesting to visualize geometrically the proof of the claim, by considering the graphs of the functions $p \cdot y + c$ for $c \in \mathbb{R}$. These are parallel hyperplanes which lie, for c close to $-\infty$, below the graph of u. We let c increase and consider the first c for which there is contact or "touching" at a point x. It is clear geometrically that $x \notin \partial\Omega$, since $|p| < 1$ and $\partial u/\partial \nu = 1$ on $\partial\Omega$.

Moreover, denoting $\Gamma_u^* = \Gamma_u \cap (\nabla u)^{-1}(B^*)$, we immediately deduce from (5.7) that

$$B_* = \nabla u(\Gamma_u^*). \tag{5.8}$$

From this we deduce

$$m(B_*) = \int_{\nabla u(\Gamma_u^*)} p^A dp \leq \int_{\Gamma_u^*} (\nabla u)^A \det D^2 u(x)\ dx. \tag{5.9}$$

We have applied the area formula to the map $\nabla u : \Gamma_u^* \to \mathbb{R}^n$, and we have used that its Jacobian, $\det D^2 u$, is nonnegative in Γ_u by definition of this set.

We now use the weighted version of the geometric and arithmetic means inequality applied to numbers $\frac{u_i}{x_i}$ and the eigenvalues of $D^2u(x)$ (which are nonnegative numbers for $x \in \Gamma_u^*$). We obtain

$$\left(\frac{u_1}{x_1}\right)^{A_1} \cdots \left(\frac{u_n}{x_n}\right)^{A_n} \det D^2 u \leq \left(\frac{A_1 \frac{u_1}{x_1} + \cdots + A_n \frac{u_n}{x_n} + \Delta u}{A_1 + \cdots + A_n + n}\right)^{A_1 + \cdots + A_n + n}.$$

This, combined with

$$A_1 \frac{u_1}{x_1} + \cdots + A_n \frac{u_n}{x_n} + \Delta u = \frac{\operatorname{div}(x^A \nabla u)}{x^A} \equiv c,$$

gives

$$\int_{\Gamma_u^*} (\nabla u)^A \det D^2 u(x)\ dx \leq \int_{\Gamma_u^*} x^A \left(\frac{c}{D}\right)^D\ dx,$$

and

$$m(B_*) \leq \left(\frac{m(\partial\Omega)}{Dm(\Omega)}\right)^D m(\Gamma_u^*) \leq \left(\frac{m(\partial\Omega)}{Dm(\Omega)}\right)^D m(\Omega). \tag{5.10}$$

Finally, it is immediate to see that $m(\partial B_*) = Dm(B_*)$, and thus we conclude the isoperimetric inequality

$$\frac{m(\partial B_*)}{m(B_*)^{\frac{D-1}{D}}} = Dm(B_*)^{\frac{1}{D}} \leq \frac{m(\partial\Omega)}{m(\Omega)^{\frac{D-1}{D}}}. \tag{5.11}$$

□

Remark 5.5. Note that if $A \neq 0$ the isoperimetric quotient of B^* is strictly less than the isoperimetric quotient of the entire ball:

$$\frac{m(\partial B^*)}{m(B^*)^{\frac{D-1}{D}}} = k^{-1/D} \frac{m(\partial B)}{m(B)^{\frac{D-1}{D}}} < \frac{m(\partial B)}{m(B)^{\frac{D-1}{D}}},$$

where k is the number of positive entries in the vector A.

Remark 5.6. The fact that $m(\Omega)^{\frac{D-1}{D}} \leq Cm(\partial\Omega)$ for some nonoptimal constant C is an interesting consequence of a result of [21]. It is the following.

We say that a manifold M satisfies the m-isoperimetric inequality if there exists a positive constant c such that $\mu(\partial\Omega) \geq c\mu(\Omega)^{\frac{m-1}{m}}$ for each $\Omega \subset M$. In [21], the author proves that if M_1 and M_2 are manifolds that satisfy m_1-isoperimetric and m_2-isoperimetric inequalities, then the product manifold $M_1 \times M_2$ satisfies the $(m_1 + m_2)$-isoperimetric inequality. By applying this result to $M_i = (\mathbb{R}, x_i^{A_i} dx_i)$, this allows us to reduce the problem to $n = 1$, and in this case the inequality is easy to verify.

An important consequence of the weighted isoperimetric inequality is that it allows to make a weighted rearrangement, very useful to prove some inequalities concerning $\int x^A f(u)dx$ and $\int x^A g(|\nabla u|)dx$:

Proposition 5.7. *Let u be a Lipschitz continuous function in $\mathbb{R}^n_*$ with compact support in $\overline{\mathbb{R}^n_*}$. Then, denoting $m(E) = \int_E x^A dx$, there exists a radial rearrangement u^* of u such that $m(\{|u| > t\}) = m(\{u^* > t\})$for all t, u^* is radially decreasing, and*

$$\int_{\mathbb{R}^n_*} x^A \Phi(|\nabla u^*|)dx \leq \int_{\mathbb{R}^n_*} x^A \Phi(|\nabla u|)dx$$

for every Young function Φ (i.e. convex and increasing function that vanishes at 0).

Proof. Is a direct consequence of Theorem 1 in [34]. □

5.2 Sobolev inequality with monomial weight

As said before, the Sobolev inequality is an immediate consequence of Theorem 5.1. However, here we give an alternative proof of the weighted Sobolev inequality for the case $A_i > 0$ and $u_i \leq 0$. This alternative proof is more elementary than the one given by the isoperimetric inequality, since it does not use an elliptic problem, but it does not give the best constant in the inequality. Note that this proof is very similar to the one used in the proof of Proposition 5.3 in section 3.

Proposition 5.8. *Let $A_1, ..., A_n$ be positive numbers, and $D = n + A_1 + \cdots + A_n$. Then, for each $p < D$ there exist a constant C such that for each $u \in C^1_c(\mathbb{R}^n)$ satisfying $u_i \leq 0$ for $x_i \geq 0$,*

$$\left(\int_{(\mathbb{R}_+)^n} x^A |u|^{p^*} dx \right)^{1/p^*} \leq C \left(\int_{(\mathbb{R}_+)^n} x^A |\nabla u|^p dx \right)^{1/p},$$

where $p^* = \frac{pD}{D-p}$.

Proof. We will prove the case $p = 1$, the case $1 < p < D$ follows from the case $p = 1$ by Hölder's inequality.

Integrating by parts, we have

$$\int_{(\mathbb{R}_+)^n} x^A |u| \left(\frac{A_1}{x_1} + \cdots + \frac{A_n}{x_n} \right) dx \leq \int_{(\mathbb{R}_+)^n} x^A (|u_1| + \cdots + |u_n|) dx,$$

and then

$$\int_{(\mathbb{R}_+)^n} x^A |u| \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) \leq C \int_{(\mathbb{R}_+)^n} x^A |\nabla u| dx.$$

Let $\lambda > 0$ be such that

$$\int_{(\mathbb{R}_+)^n} x^A |u|^{\frac{D}{D-1}} dx = b\lambda^D,$$

where $b = \int_{0 \leq z_i \leq 1} z^A dz$.

Then, we claim that for each $x \in (\mathbb{R}_+)^n$ there exists i such that $|u|^{\frac{1}{D-1}} \leq \frac{\lambda}{x_i}$. Otherwise, it would exist $y \in (\mathbb{R}_+)^n$ such that $|u(y)|^{\frac{1}{D-1}} > \frac{\lambda}{y_i}$ for each i. Hence,

$$|u(y)|^{\frac{D}{D-1}} > \frac{\lambda^D}{y^{A+1}},$$

and since $u(x) \geq u(y)$ for $x \leq y$, then

$$\int_{\{0 \leq x \leq y\}} x^A |u|^{\frac{D}{D-1}} dx > \lambda^D \int_{\{x_i \leq y_i\}} x^A y^{-A-1} dx = b\lambda^D,$$

a contradiction.

Hence, it has to be

$$|u|^{\frac{1}{D-1}} \leq \lambda \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right),$$

and therefore

$$\int_{(\mathbb{R}_+)^n} x^A |u|^{\frac{D}{D-1}} dx \leq \lambda \int_{(\mathbb{R}_+)^n} x^A |u| \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) dx.$$

Finally, since

$$\lambda = C \left(\int_{(\mathbb{R}_+)^n} x^A u|^{\frac{D}{D-1}} dx \right)^{1/D},$$

we have that

$$\left(\int_{(\mathbb{R}_+)^n} x^A |u|^{\frac{D}{D-1}} dx\right)^{\frac{D-1}{D}} \le C \left(\int_{(\mathbb{R}_+)^n} x^A |\nabla u| dx\right),$$

and we are done. $\square$

Remark 5.9. One can think on adapting the classical proof of the Sobolev inequality due to L. Nirenberg (see [16] for instance). But using that

$$x_i^{A_i} |u(x)| \le \int_{\mathbb{R}} y_i^{A_i} |\nabla u(x_1, ..., y_i, ..., x_n)| dy_i$$

and following the proof, one obtains

$$\left(\int_{\mathbb{R}^n} x^{\frac{nA}{n-1}} |u|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \le C \int_{\mathbb{R}^n} x^A |\nabla u| dx,$$

which is an interesting –but not the desired– inequality.

To end this subsection, we state the following inequality, which is precisely the one in section 3 which led us to study Sobolev inequalities with monomial weights.

Corollary 5.10. *Let $B_1, ..., B_n$ be real numbers such that $0 \le B_i < 2$. There exists a constant C such that for all $u \in C_c^1(\mathbb{R}^n)$,*

$$\left(\int_{\mathbb{R}^n} |u|^q dx\right)^{1/q} \le C \left(\int_{\mathbb{R}^n} \left\{|x_1|^{B_1} |u_1|^2 + \cdots + |x_n|^{B_n} |u_n|^2\right\} dx\right)^{1/2},$$

where $q = \frac{2D}{D-2}$ and $D = n + \frac{B_1}{2-B_1} + \cdots + \frac{B_n}{2-B_n}$.

Proof. It suffices to make the change of variables $y_i = x_i^{\frac{2-B_i}{2}}$ and apply Theorem 5.2 with $A_i = \frac{B_i}{2-B_i}$. $\square$

5.3 Best constant and extremal functions in the Sobolev inequality with monomial weight

In this subsection we obtain best constants and extremal functions in the weighted Sobolev inequality.

We start by calculating $m(B_*)$, which will lead us to the optimal constant in the isoperimetric inequality, and therefore, to the optimal constant in Sobolev inequality for $p = 1$ (see Remark 4.3).

Lemma 5.11. *Let $A_1, ..., A_n$ be nonnegative real numbers. Then,*

$$m(B_*) = \frac{\Gamma\left(\frac{A_1+1}{2}\right)\Gamma\left(\frac{A_2+1}{2}\right)\cdots\Gamma\left(\frac{A_n+1}{2}\right)}{2^k\Gamma\left(1+\frac{D}{2}\right)},$$

where $D = A_1 + \cdots + A_n$ and k is the number of positive entries in A.

Proof. We will prove by induction on n that

$$\int_{B_1} x^A dx = \frac{\Gamma\left(\frac{A_1+1}{2}\right)\Gamma\left(\frac{A_2+1}{2}\right)\cdots\Gamma\left(\frac{A_n+1}{2}\right)}{\Gamma\left(1+\frac{D}{2}\right)},$$

where B_1 is the unit ball in $\mathbb{R}^n$.

For $n = 1$ it is immediate. Assume it is true for $n-1$ and let us prove it for n. Let us denote $x = (x', x_n)$, $A = (A', A_n)$, with $x', A' \in \mathbb{R}^{n-1}$, and $D' = |A'| + n - 1$.

$$\begin{aligned}
\int_{B_1} x^A dx &= \int_{-1}^{1} x_n^{A_n} \left(\int_{|x'| \le \sqrt{1-x_n^2}} x'^{A'} dx' \right) dx_n \\
&= \int_{-1}^{1} x_n^{A_n} \left(\left(1-x_n^2\right)^{\frac{D'}{2}} \int_{|x'|\le 1} x'^{A'} dx' \right) dx_n \\
&= \int_{|x'|\le 1} x'^{A'} dx' \int_{-1}^{1} x_n^{A_n} \left(1-x_n^2\right)^{\frac{D'}{2}} dx_n,
\end{aligned}$$

and hence it remains to calculate

$$\int_{-1}^{1} x_n^{A_n} \left(1-x_n^2\right)^{\frac{D'}{2}} dx_n.$$

Making the change of variables $x_n^2 = t$ one obtains

$$\begin{aligned}
\int_{-1}^{1} x_n^{A_n} \left(1-x_n^2\right)^{\frac{D'}{2}} dx_n &= 2\int_{0}^{1} x_n^{A_n} \left(1-x_n^2\right)^{\frac{D'}{2}} dx_n \\
&= \int_0^1 t^{\frac{A_n-1}{2}} (1-t)^{\frac{D'}{2}} dt \\
&= B\left(\frac{A_n+1}{2}, 1+\frac{D'}{2}\right),
\end{aligned}$$

where B is the Beta function. Since

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

then

$$
\begin{aligned}
\int_{B_1} x^A dx &= \int_{|x'|\leq 1} x'^{A'} dx' \int_{-1}^{1} x_n^{A_n} \left(1 - x_n^2\right)^{\frac{D'}{2}} dx_n \\
&= \frac{\Gamma\left(\frac{A_1+1}{2}\right)\cdots\Gamma\left(\frac{A_{n-1}+1}{2}\right)}{\Gamma\left(1+\frac{D'}{2}\right)} \cdot \frac{\Gamma\left(\frac{A_n+1}{2}\right)\Gamma\left(1+\frac{D'}{2}\right)}{\Gamma\left(1+\frac{D}{2}\right)} \\
&= \frac{\Gamma\left(\frac{A_1+1}{2}\right)\Gamma\left(\frac{A_2+1}{2}\right)\cdots\Gamma\left(\frac{A_n+1}{2}\right)}{\Gamma\left(1+\frac{D}{2}\right)},
\end{aligned}
$$

and the proof finishes by taking into account that

$$
m(B_*) = \frac{m(B_1)}{2^k}.
$$

□

We can now find the best constant in the weighted Sobolev inequality for $p \geq 1$. The proof is based on Proposition 5.7, which allows us to reduce the problem to radial functions. Then, we obtain that the functional which we have to minimize is exactly the same as in the classical Sobolev inequality, and hence by applying the results of G. Talenti [33] we will be done.

Proposition 5.12. *The best constant in Sobolev inequality (5.4) is given by*

$$
C_p = C_1 \left(\frac{D(p-1)}{D-p}\right)^{\frac{p}{p-1}} \left\{ \frac{\Gamma(D)}{\Gamma\left(\frac{D}{p}\right)\Gamma\left(1+D-\frac{D}{p}\right)} \right\}^{1/D} \tag{5.12}
$$

$$
C_1 = D \left\{ \frac{\Gamma\left(\frac{A_1+1}{2}\right)\Gamma\left(\frac{A_2+1}{2}\right)\cdots\Gamma\left(\frac{A_n+1}{2}\right)}{2^k\Gamma\left(1+\frac{D}{2}\right)} \right\}^{1/D}, \tag{5.13}
$$

where k is the number of positive entries in the vector A. Moreover, for $p = 1$ this constant is not attained by any smooth function, while for $1 < p < D$ the constant is attained by

$$
u(x) = \left(a + b|x|^{\frac{p}{p-1}}\right)^{1-\frac{D}{p}},
$$

where a and b are arbitrary positive constants.

Proof. Let $p = 1$. Then, as we have seen in Remark 4.3, the best constant in Sobolev inequality is the same than in the isoperimetric inequality, which is given by $Dm(B^*)^{1/D}$. The value of C_1 follows from the previous Lemma.

Let $1 < p < D$, and let u a smooth function in $\mathbb{R}^n_*$, and let u_* be its radial rearrangement given by Proposition 5.7. Then,

$$\frac{\|\nabla u\|_{L^p(\mathbb{R}^n_*, x^A dx)}}{\|u\|_{L^{p^*}(\mathbb{R}^n_*, x^A dx)}} \geq \frac{\|\nabla u_*\|_{L^p(\mathbb{R}^n_*, x^A dx)}}{\|u_*\|_{L^{p^*}(\mathbb{R}^n_*, x^A dx)}}.$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}^n_*} x^A |u_*|^{p^*} dx &= \int_0^\infty \left(\int_{r\partial B_*} x^A |u_*|^{p^*} d\sigma \right) dr \\
&= \int_0^\infty r^{D-1} |u_*|^{p^*} \left(\int_{\partial B_*} x^A d\sigma \right) dr \\
&= m(\partial B_*) \int_0^\infty r^{D-1} |u_*|^{p^*} dr
\end{aligned}$$

and

$$\int_{\mathbb{R}^n_*} x^A |\nabla u_*|^p dx = m(\partial B_*) \int_0^\infty r^{D-1} |u_*'|^p dr,$$

and therefore the best constant in Sobolev inequality can be computed as

$$\inf_{u \in C^1_c(\mathbb{R}^n)} \frac{\|\nabla u\|_{L^p(\mathbb{R}^n_*, x^A dx)}}{\|u\|_{L^{p^*}(\mathbb{R}^n_*, x^A dx)}} = D^{\frac{D-1}{D}} C_1 \inf_{u \in C^1_c(\mathbb{R})} \frac{\left(\int_0^\infty r^{D-1} |u'|^p dr\right)^{1/p}}{\left(\int_0^\infty r^{D-1} |u|^{p^*} dr\right)^{1/p^*}}, \tag{5.14}$$

where we have used that $\frac{1}{p} - \frac{1}{p^*} = \frac{1}{D}$ and that

$$m(\partial B_*)^{1/D} = D^{\frac{D-1}{D}} C_1.$$

In [33] the same quotient for radial functions is minimized, and the result obtained is that

$$\inf_{u \in C^1_c(\mathbb{R})} \frac{\left(\int_0^\infty r^{D-1} |u'|^p dr\right)^{1/p}}{\left(\int_0^\infty r^{D-1} |u|^{p^*} dr\right)^{1/p^*}} = \left(\frac{D(p-1)}{D-p} \right)^{\frac{p}{p-1}} \left\{ \frac{\Gamma(D)}{\Gamma\left(\frac{D}{p}\right) \Gamma\left(1 + D - \frac{D}{p}\right)} \right\}^{1/D},$$

while equality holds if and only if

$$u(r) = \left(a + b r^{\frac{p}{p-1}} \right)^{1 - \frac{D}{p}},$$

for some positive constants a and b. □

5.4 Morrey inequality with monomial weight

We finish this work proving the weighted Morrey inequality. The proof is based in the following lemma, which is the analog of the one used in the proof of the classical Morrey inequality.

Lemma 5.13. *If $y \in \mathbb{R}^n_*$ and $r = |y|$, then*

$$|u(y) - u(0)| \leq C \int_{B^*_{2r}} \frac{|\nabla u(x)|}{|x|^{D-1}} x^A dx.$$

Proof. By introducing new variables (as explained in the introduction of this section), we have that the inequality is true when A_i are nonnegative integers. Hence, if we denote $B = (B_1, ..., B_n)$, with $B_i = \lceil A_i \rceil$ (the upper integer part of A_i), then we have that

$$|u(y) - u(0)| \leq C \int_{B^*_{2r}} \frac{|\nabla u(x)|}{|x|^{|B|+n-1}} x^B dx.$$

But since $x^C \leq |x|^{|C|}$ for each positive vector $C \in \mathbb{R}^n$ and $x \in \mathbb{R}^n_*$, then

$$\frac{x^B}{|x|^{|B|}} \leq \frac{x^A}{|x|^{|A|}},$$

and therefore

$$|u(y) - u(0)| \leq C \int_{B^*_{2r}} \frac{|\nabla u(x)|}{|x|^{|B|+n-1}} x^B dx \leq C \int_{B^*_{2r}} \frac{|\nabla u(x)|}{|x|^{D-1}} x^A dx.$$

□

Finally, we can give the:

Proof of Proposition 5.3. By Lemma 5.13 and by Hölder's inequality, we have that if $y \in \mathbb{R}^n_*$ and $r = |y|$, then

$$\begin{aligned} |u(y) - u(0)| &\leq C \int_{B^*_{2r}} \frac{|\nabla u|}{|x|^{D-1}} x^A dx \\ &\leq C \left(\int_{B^*_{2r}} x^A |\nabla u|^p dx \right)^{1/p} \left(\int_{B^*_{2r}} \frac{x^A}{|x|^{p^*(D-1)}} dx \right)^{1/p^*} \\ &= C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^{1/p} |y|^{1-\frac{p}{D}}, \end{aligned}$$

so the inequality is proved for $z = 0$ and $y \in \mathbb{R}^n_*$.

Now, let $y, z \in \mathbb{R}^n_*$ such that $y - z \in \mathbb{R}^n_*$. Then, applying last inequality to $v(y) = u(y - z)$, we get that

$$|u(y) - u(z)| \leq C \left(\int_{\mathbb{R}^n_* + z} (x - z)^A |\nabla u|^p dx \right)^{1/p} |y - z|^{1 - \frac{p}{D}},$$

where $\mathbb{R}^n_* + z = \{x : x - z \in \mathbb{R}^n_*\}$. Since $(x - z)^A \leq x^A$ if $x, z, x - z \in \mathbb{R}^n_*$, the inequality is proved for $y, z \in \mathbb{R}^n_*$ such that $y - z \in \mathbb{R}^n_*$.

Let now $y, z \in \mathbb{R}^n_*$, and define $w = \min\{y, z\}$. Then, $w \in \mathbb{R}^n_*$, $y - w \in \mathbb{R}^n_*$ and $z - w \in \mathbb{R}^n_*$, and hence

$$|u(y) - u(w)| \leq C \left(\int_{\mathbb{R}^n_* + w} x^A |\nabla u|^p dx \right)^{1/p} |y - w|^{1 - \frac{p}{D}},$$

$$|u(z) - u(w)| \leq C \left(\int_{\mathbb{R}^n_* + w} x^A |\nabla u|^p dx \right)^{1/p} |z - w|^{1 - \frac{p}{D}},$$

from which we deduce

$$|u(y) - u(z)| \leq C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^{1/p} |y - z|^{1 - \frac{p}{D}},$$

and the inequality is proved for all $y, z \in \mathbb{R}^n_*$. □

Corollary 5.14. *Let Ω be a bounded domain in $\mathbb{R}^n$, and $u \in C^1_0(\Omega)$. Then, for each $p > D$ there exists a constant C such that*

$$\sup_{\Omega} |u| \leq C \, diam(\Omega)^{1 - \frac{D}{p}} \int_{\Omega} x^A |\nabla u|^p dx.$$

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