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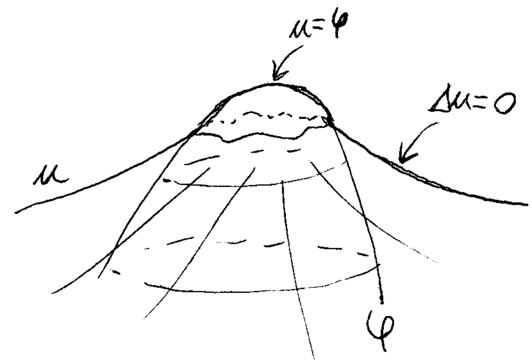
4. Obstacle problems

4.1 The classical obstacle problem



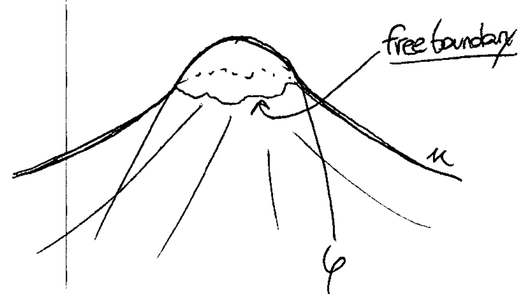
Given: $\begin{cases} \text{a domain } \Omega\subset\mathbb{R}^n \\ \text{a (smooth) obstacle } \varphi:\Omega\to\mathbb{R} \\ \text{boundary data } g:\partial\Omega\to\mathbb{R} \\ \text{a second-order elliptic operator } L \end{cases}$

the obstacle problem is:

$$\boxed{\left.\begin{aligned} u&\geq\varphi \text{ in } \Omega \\ \Delta u&=0 \text{ in } \{u>\varphi\} \\ -\Delta u&\geq 0 \text{ in } \Omega \\ u&=g \text{ on } \partial\Omega \end{aligned}\right\}}$$

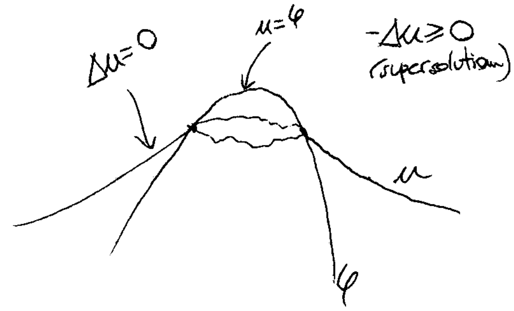
- In the simplest case, $\boxed{L=\Delta}$ (Laplacian)
- The problem has a unique solution, it can be constructed either by variational methods (minimizing an energy functional), or as a viscosity solution.

- There are two unknowns in this problem: solution u and contact set $\{u=\varphi\}$
  - The free boundary (FB) is $\partial\{u=\varphi\}$



- Questions to be understood:
  - Regularity of the solution u
  - Geometry and regularity of the FB

- Let us now briefly explain what are the main known results in case $L=\Delta$ (classical obstacle problem).



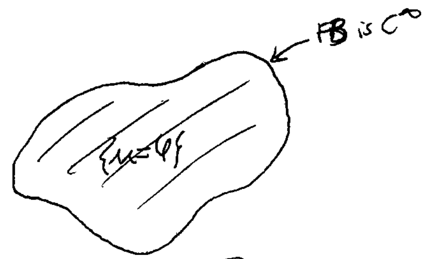
Main Known results:

- Solutions u are $C^{1,1}$ (optimal regularity)
- At every free boundary point we have $\Delta\varphi \leq 0$ (otherwise u does not detach from the obstacle)
- If $\underline{\Delta\varphi < 0}$ then at every free boundary point x_0 we have

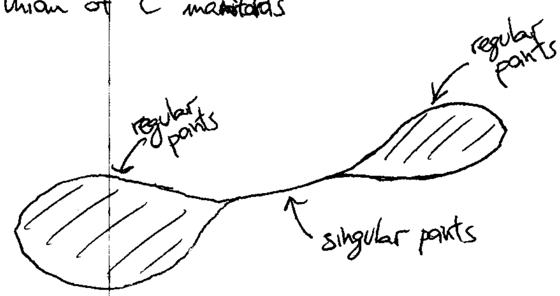
$$\left[0 < cr^2 \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^2 \right] \quad \forall r \in (0,1)$$

(u detaches from φ at a quadratic rate).

- If $\underline{\Delta\varphi < 0}$ then the free boundary splits into <u>regular</u> points and <u>singular</u> points.
- The set of <u>regular</u> points is an open subset of FB, and it is C^∞
- The set of <u>singular</u> points is contained in a union of C^1 manifolds

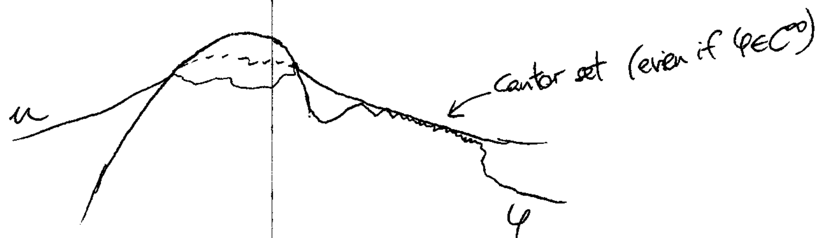


[FB in which all points are <u>regular</u>]



[At singular points the <u>set $\{u=\varphi\}$</u> has <u>density zero</u>]

- Finally, without the assumption $\Delta\varphi < 0$ then the free boundary could be very bad (even a fractal set like a Cantor set):



(• Essentially, the problem would be that without the assumption $\Delta\varphi < 0$ then one cannot prove $\underset{0}{cr^2} \leq \sup_{B_r(x_0)} (u - \varphi)$)

Blow-ups

- The proofs of all these results are by <u>blow-up</u> arguments: we consider

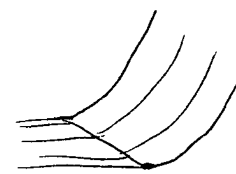
$$\left[\, u_r(x) := \frac{(u-\varphi)(x_0+rx)}{r^2} \,\right] \qquad \left(0 < c \le \|u_r\|_{L^\infty(B_1)} \le C\right)$$

and then show that $\left[\, u_r \longrightarrow u_0 \ \text{ in } C^1_{loc}(\mathbb{R}^n) \,\right]$

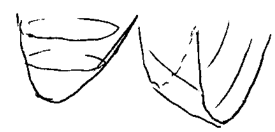
for some <u>global solution</u> u_0.

Then, one shows:

<u>regular</u> points $\rightsquigarrow \left[\, u_0(x) = (x\cdot e)_+^2 \,\right]$



<u>singular</u> points $\longrightarrow u_0(x) = \sum_i \lambda_i x_i^2$ (paraboloid)



- Then, one has to transfer the information from u_0 to u: at <u>regular points</u> one has

$u_0(x) = (x\cdot e)_+^2 \rightsquigarrow \partial_{e'} u_0 \ge 0$ in $\mathbb{R}^n$ <u>and</u> $\partial_{e'} u_0 \ge c_1 > 0$ in $\{x\cdot e \ge c_2 > 0\}$ (if $e'\cdot e > 0$)

$\rightsquigarrow \partial_{e'} u_r \ge 0$ in B_1 (for some $r>0$ small) $\rightsquigarrow \partial_{e'}(u-\varphi) \ge 0$ in $B_r(x_0) \rightsquigarrow$ FB is Lipschitz $\rightsquigarrow$

$\rightsquigarrow$ (using boundary Harnack) $\rightsquigarrow$ FB is $C^{1,\alpha}$ in $B_r(x_0)$ $\rightsquigarrow$ FB is C^∞ in $B_r(x_0)$ //

- This is the (very short) sketch of the proof in the <u>classical</u> <u>obstacle problem</u> (for the Laplacian).

(4.)

4.2. Optimal stopping and obstacle problems

- Let X_t be a random process in $\mathbb{R}^n$, and φ be a payoff function.

- We can stop the process X_t any time we want $\tau \geq 0$, and we get a payoff $\varphi(X_\tau)$.
We want to find a strategy in order to maximize the expected payoff.

[Question: Should we stop if we are at $x \in \mathbb{R}^n$? At what points is it better to stop and at which ones better to continue?]

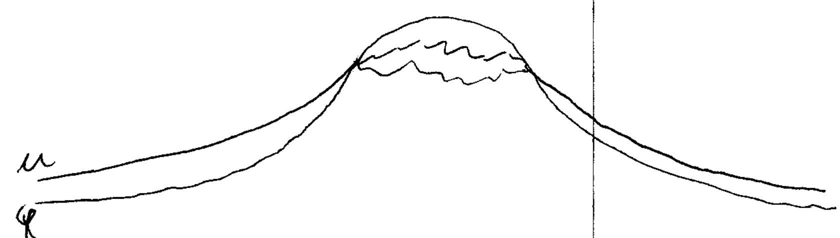
- We define the value function
$$\left[u(x) = \max_{\substack{\text{all choices} \\ \text{of } \tau}} \mathbb{E}\left[\varphi(X_\tau)\right] \right]$$

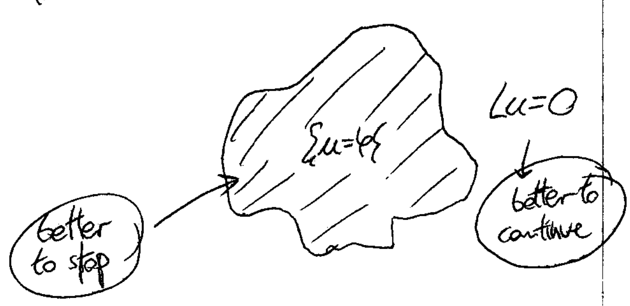
- It turns out that $u(x)$ solves an obstacle problem!

- L is the generator of the process; the "exercise region" is $\{u=\varphi\}$.
  - $\forall x \in \mathbb{R}^n$ we have $u \geq \varphi$ (one can always stop)
  - if $u(x) > \varphi(x) \rightsquigarrow$ better to continue $\rightsquigarrow$ $Lu = 0$ at x.
  - we can always continue $\rightsquigarrow$ $Lu \leq 0$ in $\mathbb{R}^n$.

$$\longrightarrow \boxed{\begin{array}{l} u \geq \varphi \text{ in } \mathbb{R}^n \\ -Lu \geq 0 \text{ in } \mathbb{R}^n \\ Lu = 0 \text{ in } \{u > \varphi\} \end{array}}$$

(and $u \to 0$ at ∞)





5.

- When X_t is a Brownian motion $\to$ L is the Laplacian $\to$ classical obstacle problem

- When X_t is a Lévy process $\to$ L is an integro-differential operator

- We will now study obstacle problems for

$$\left[Lu(x) = \frac{1}{2}\int_{\mathbb{R}^n} \{u(x+y)+u(x-y)-2u(x)\} K(y)\,dy \right]$$

$$\left[\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}} \right] \qquad (0<\lambda\leq\Lambda)$$

- The main result we want to prove is the following:

Theorem. Let L be an operator as above, with $K(y)$ homogeneous.

- Let φ be a smooth obstacle, and u be the solution of the obstacle problem

$$\begin{cases} u \geq \varphi & \text{in } \mathbb{R}^n \\ -Lu \geq 0 & \text{in } \mathbb{R}^n \\ Lu = 0 & \text{in } \{u>\varphi\} \end{cases}$$

- Then, for every free boundary point x_0, we have:

(i) either $\left[0 < cr^{1+s} \leq \sup_{B_r(x_0)} (u-\varphi) \leq Cr^{1+s} \right]$ (regular points)

(ii) or $\left[0 \leq \sup_{B_r(x_0)} (u-\varphi) \leq Cr^{2-\varepsilon} \right]$ $\forall \varepsilon>0$

- Moreover, the set of regular points (i) is an open subset of the FB, and it is $C^{1,\alpha}$ for some $\alpha>0$.

- This means that:
  - $u \in C^{1+s}(\mathbb{R}^n)$ (optimal regularity)
  - the FB is ~~regular~~ $C^{1,\alpha}$ near regular points

⑥

• For the <u>fractional Laplacian</u>, the main known result is the following:

<u>Theorem</u>. Let φ be a smooth obstacle, and u be the solution to the obstacle problem for $L=(-\Delta)^s$. Then, for every free boundary point x_0 we have:

(i) either $\left[0 < cr^{1+s} \le \sup_{B_r(x_0)} (u-\varphi) \le Cr^{1+s}\right]$ (regular points)

(ii) or $\left[0 \le \sup_{B_r(x_0)} (u-\varphi) \le Cr^{2}\right]$

Moreover, the set of points satisfying (i) is open and $C^{1,\alpha}$. (in fact $\underline{C^\infty}$)

• When $\Delta\varphi < 0$ in $\{\varphi > 0\}$, then:

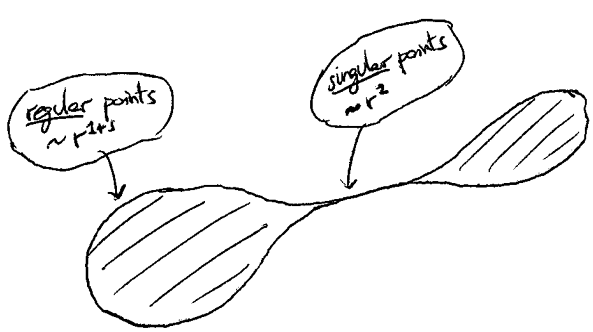
<u>Theorem</u>. Let φ be an obstacle satisfying

$$\left[\Delta\varphi < 0 \text{ in } \{\varphi > 0\}\right]$$

Then, for every point of <u>type (ii)</u> we have

$$\left[0 < cr^2 \le \sup_{B_r(x_0)} (u-\varphi) \le Cr^{2}\right] \quad \text{(singular points)}$$

and singular points are contained in a union of C^1 manifolds.



1.

4.3. Obstacle problems for integro-differential operators

- Let us consider from now on operators L with kernels

$$\left[\frac{\lambda}{|y|^{n+2s}} \le K(y) \le \frac{\Lambda}{|y|^{n+2s}}, \quad K(y) \text{ homogeneous} \right]$$

- We will study the obstacle problem in $\mathbb{R}^n$: given $\varphi \in C_c^\infty(\mathbb{R}^n)$, u will be the unique solution of

$$\boxed{\left.\begin{array}{ll} u \ge \varphi & \text{in } \mathbb{R}^n \\ -Lu \ge 0 & \text{in } \mathbb{R}^n \\ Lu = 0 & \text{in } \{u > \varphi\} \\ \lim_{|x|\to\infty} u(x) = 0 & \end{array}\right\}}$$

- The first three equations can be written as

$$\boxed{\min\{-Lu, u-\varphi\} = 0 \text{ in } \mathbb{R}^n} \quad \text{(obstacle problem)}$$

- We see in this way that the obstacle problem is a "fully nonlinear equation" (it is the minimum of two linear operators). (However, the theory of fully nonlinear operators cannot be applied here because one of the operators is just $u-\varphi$, of order 0.)

- We want to understand the regularity of solutions and the free boundaries.

- Let us start with some basic regularity properties of solutions.

2.

Semiconvexity and $C^{1,\alpha}$ estimates

- A first and very important property of the solution u is that:

  [u is the least supersolution which is above the obstacle φ]

~~This property can be proved for~~ · This property is essentially true by definition of the viscosity solution to the obstacle problem.

Lemma. Let $v \in C(\mathbb{R}^n)$ be a function satisfying $\left.\begin{array}{l} -Lv \geq 0 \text{ in } \mathbb{R}^n, \\ v \geq \varphi \text{ in } \mathbb{R}^n, \end{array}\right\}$.

Then, $v \geq u$ in $\mathbb{R}^n$

- This kind of comparison principle is useful to show the following.

Lemma. Let u be the solution to the obstacle problem. Then,

(a) u is Lipschitz, $\quad \|u\|_{Lip(\mathbb{R}^n)} \leq \|\varphi\|_{Lip(\mathbb{R}^n)}$

(b) u is semiconvex, $\quad \partial_{ee} u \geq -\|\varphi\|_{C^{1,1}(\mathbb{R}^n)}$ for all $e \in S^{n-1}$.

(c) Lu is bounded, $\quad \|Lu\|_{L^\infty(\mathbb{R}^n)} \leq C$.

Proof. (a) We use the previous Lemma.

Taking $v(x) = \max_{\mathbb{R}^n} \varphi$ we find $Lv = 0$ in $\mathbb{R}^n$, $v \geq \varphi$ in $\mathbb{R}^n$, so $v \geq u$ in $\mathbb{R}^n$. Thus, $\|u\|_{L^\infty(\mathbb{R}^n)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^n)}$.

Taking now $\left[v(x) = u(x+h) + C|h|\right]$ with $C = \|\varphi\|_{Lip(\mathbb{R}^n)}$.

Then $Lv(x) = Lu(x+h) \geq 0$ in $\mathbb{R}^n$, and $v(x) \geq u(x+h) + C|h| \geq \varphi(x+h) + C|h| \geq \varphi(x)$ in $\mathbb{R}^n$.

Thus, $v \geq u$ in $\mathbb{R}^n$, so $u(x+h) + C|h| \geq u(x)$ in $\mathbb{R}^n$. This means $\frac{-u(x+h)+u(x)}{|h|} \leq C$, so $\|u\|_{Lip} \leq C$.

(3)

(b) Take now $\left[v(x) = \dfrac{u(x+h)+u(x-h)}{2} + C|h|^2 \right]$

Then, $Lv(x) = \frac{1}{2} Lu(x+h) + \frac{1}{2} Lu(x-h) \geq 0$ in $\mathbb{R}^n$,

$$v(x) = \frac{u(x+h)+u(x-h)}{2} + C|h|^2 \geq \frac{\varphi(x+h)+\varphi(x-h)}{2} + C|h|^2 \geq \varphi(x) \text{ in } \mathbb{R}^n,$$

and thus $v \geq u$ in $\mathbb{R}^n$. This means that

$$\frac{u(x+h)+u(x-h)}{2} + C|h|^2 \geq u(x) \quad \forall x, h,$$

and this yields ($h = te$)

$$\frac{u(x+te)+u(x-te)-2u(x)}{2t^2} \geq -C,$$

Thus, $\partial_{ee} u \geq -C$ in $\mathbb{R}^n$.

(c) We have $-Lu \geq 0$ in $\mathbb{R}^n$, so we only need to show $Lu \geq -C$ in $\mathbb{R}^n$.
But since u is bounded and semiconvex then,

$$\left.\begin{array}{ll} u(x+y)+u(x-y)-2u(x) \geq -C|y|^2 & \text{in } B_1 \\ u(x+y)+u(x-y)-2u(x) \geq -C & \text{in } \mathbb{R}^n \setminus B_1 \end{array}\right\} \to \int_{\mathbb{R}^n} \{u(x+y)+u(x-y)-2u(x)\} K(y)\,dy \geq$$

$$\geq C \int_{\mathbb{R}^n} \min\{1, |y|^2\} K(y)\,dy \geq -C,$$

Thus, $\|Lu\|_{L^\infty(\mathbb{R}^n)} \leq C$.

- Thus, we just showed some initial regularity on u: u is Lipschitz and semiconvex.

Moreover, when $s > \frac{1}{2}$ the fact that $Lu \in L^\infty(\mathbb{R}^n)$ yields $\boxed{u \in C^{2s}(\mathbb{R}^n)}$

(Recall: $Lu = f$, $f \in L^\infty \Rightarrow u \in C^{2s}$ if $2s$ is not integer).

- Thus, when $s > \frac{1}{2}$ this gives that solutions are $C^{1,\alpha}$, with $\alpha = 2s-1$.
- When $s=1$ this is optimal: solutions are $C^{1,1}$. However, when $s<1$ solutions are C^{1+s}, so we still need more work in order to show that result!

4.

· A first step towards the regularity is the following:

Lemma. Let u be the solution to the obstacle problem (for any $s\in(0,1)$).
Then, $u\in C^{1+\alpha}$ for some $\alpha>0$ small.

· The idea is that u is Lipschitz, then the equation in $\{u>\varphi\}$ and the semiconvexity lead to some more regularity of u.

(· Such $C^{1+\alpha}$ estimate is important in order to have convergence of the blow-up sequences in the $\underline{C^1_{loc}(\mathbb{R}^n)}$ norm.)

The boundary Harnack inequality

· A very important tool in the study of obstacle problems is the boundary Harnack inequality:

Theorem. Let $\Omega\subset\mathbb{R}^n$ be any open set, and u_1, u_2 be two solutions of

$$\left.\begin{array}{l} Lu_1=0 \text{ in } \Omega\cap B_1 \\ u_1=0 \text{ in } B_1\setminus\Omega \end{array}\right\} \qquad \left.\begin{array}{l} Lu_2=0 \text{ in } \Omega\cap B_1 \\ u_2=0 \text{ in } B_1\setminus\Omega \end{array}\right\}$$

satisfying $\boxed{u_i\geq 0 \text{ in } \mathbb{R}^n}$ and $\boxed{\int_{\mathbb{R}^n}\frac{u_i(x)}{1+|x|^{n+2s}}=1}$

Then,

$$\boxed{0<c\leq\frac{u_1}{u_2}\leq C \quad \text{in } \Omega\cap B_{1/2}}$$

· It says: if two solutions are non-negative, then they are comparable in a smaller ball.

• As a consequence, one has the following.

Theorem. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, and u_1, u_2, be solutions of

$$\left.\begin{array}{ll} Lu_i = f_i & \text{in } \Omega \cap B_1 \\ u_i = 0 & \text{in } B_1 \setminus \Omega \end{array}\right\}, \quad \text{with } \|f_i\|_{L^\infty} \leq \delta, \text{ and } \delta \text{ small enough.}$$

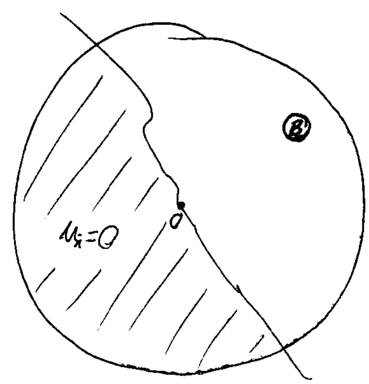
Assume $u_i \geq 0$ in $\mathbb{R}^n$ and $\displaystyle\int_{\mathbb{R}^n} \frac{u_i(x)}{1+|x|^{n+2s}}\,dx = 1$.

Then,

$$\|u_1/u_2\|_{C^\alpha(\overline{\Omega} \cap B_{1/2})} \leq C$$

for some small $\alpha > 0$.

• The proof of this result is by iterating the boundary Harnack inequality.



Proof of the boundary Harnack:

• Let $B = B_r(x_0)$ be such that $B_{2r}(x_0) \subset \Omega \cap B_{1/2}$

• By the half-Harnack (interior), we have that

$$\left[\inf_B u_i \geq c > 0\right]$$

(we are using here $\int_{\mathbb{R}^n} \frac{u(x)}{1+|x|^{n+2s}}\,dx = 1$).

• On the other hand, since $u \geq 0$ and $Lu = 0$ in $\{u > 0\}$, it turns out that $Lu \geq 0$ in B_1 (u is a subsolution in B_1). Thus, using the other half-Harnack, we get

$$\left[\sup_{B_{3/4}} u_i \leq C\right]$$

• Let $b \in C_c^\infty(B_{3/4})$, $0 \leq b \leq 1$ in $B_{3/4}$ and $b \equiv 1$ in $B_{1/2}$. Let $\eta \in C_c^\infty(B)$, $\eta \geq 0$.

Let

$$\left[w := u_1 \cdot \chi_{B_{3/4}} + C_1(b-1) + C_2\eta\right]$$

• Notice $\left[w \leq 0 \text{ in } \mathbb{R}^n \setminus B_{3/4}\right]$ $\left[Lw \geq -C - CC_1 + cC_2 \geq 0 \text{ in } B_{3/4}.\right]$ $\rightsquigarrow$ $w \leq Cu_2$ in $B_{3/4}$ $\to$ $u_1 \leq Cu_2$ in $B_{1/2}$

(1)

Regular points and blow-ups

· We want to show the following: $\left[\frac{\lambda}{|y|^{n+2s}} \le K(y) \le \frac{\Lambda}{|y|^{n+2s}},\ K \text{ homogeneous}\right]$

Theorem. Let u be the solution to the obstacle problem.

Let $x_0 \in \partial\{u=\varphi\}$ be any free boundary point. Then, there is $\alpha>0$ such that:

(i) either $\left[0 < cr^{1+s} \le \sup_{B_r(x_0)} (u-\varphi) \le Cr^{1+s}\right]$ (regular points)

(ii) or $\left[0 \le \sup_{B_r(x_0)} (u-\varphi) \le Cr^{1+s+\alpha}\right]$

Moreover, the set of regular points (i) is open and $C^{1,\alpha}$.

Global strategy of the proof:

- Assume that (ii) does not happen at x_0. Then, show that we can do a blow-up

$$\left[u_r(x) := \frac{(u-\varphi)(x_0+rx)}{r\|\nabla(u-\varphi)\|_{L^\infty(B_r(x_0))}}\right] \quad \left(\text{notice } \|\nabla u_r\|_{L^\infty(B_1)}=1\right)$$

along an appropriate subsequence $r_k \to 0$ so that the functions u_{r_k} have a "good" growth at infinity (uniform in k). $\left[|\nabla u_{r_k}(x)| \le C(1+|x|^{s+\alpha})\right]$

- In the limit $r_k \to 0$ we get a global, convex solution u_0 with the same growth.

$$\boxed{\begin{array}{l} L(\nabla u_0)=0 \text{ in } \{u_0>0\} \\ u_0 \ge 0 \\ u_0 \in C^{1+\alpha} \\ u_0 \text{ convex} \end{array}}$$

(equation for ∇u_0 because u_0 grows too much!!)

- Show that it must be $\boxed{u_0(x) = (x\cdot e)_+^{1+s}}$ for some $e \in S^{n-1}$.

- Transfer the information to u: prove that the free boundary is $C^{1,\alpha}$ near x_0, and then by estimates in $C^{1,\alpha}$ domains we get $0 < cr^{1+s} \le \sup_{B_r(x_0)} (u-\varphi) \le Cr^{1+s}$.

In fact, $\left[u(x)-\varphi(x) = c_0 d^{1+s}(x) + o(|x-x_0|^{1+s+\alpha})\right]$

(this implies also that the set of regular points is open).

(2.)

Blow-ups at regular points

- Assume that $0 \in \partial\{u > \varphi\}$ is a free boundary point.
- We take the function $\boxed{\tilde{u} = u - \varphi}$

which satisfies

$$\left.\begin{aligned} \tilde{u} &\geq 0 \quad \text{in } \mathbb{R}^n \\ -L\tilde{u} &\geq f \quad \text{in } \mathbb{R}^n \\ -L\tilde{u} &= f \quad \text{in } \{\tilde{u} > 0\} \\ D^2\tilde{u} &\geq -C \quad \text{in } \mathbb{R}^n \end{aligned}\right\}$$

(where $\underline{f = L\varphi}$)
($f \in C^{\alpha}(\mathbb{R}^n)$)

Lemma. Assume

$$\left[\limsup_{r \downarrow 0} \frac{\|\tilde{u}\|_{L^\infty(B_r)}}{r^{1+s+\alpha}} = \infty\right]$$

(that is, (ii) does not happen)

Then, there exists a sequence $r_k \to 0$ such that $\|\nabla \tilde{u}\|_{B_{r_k}} \geq r_k^{s+\alpha}$ and for which the rescaled functions

$$\left[u_{r_k}(x) = \frac{u(r_k x)}{r_k \|\nabla \tilde{u}\|_{B_{r_k}}}\right]$$

satisfy

$$\left[|\nabla u_{r_k}(x)| \leq C\left(1 + |x|^{s+\alpha}\right) \quad \text{in } \mathbb{R}^n\right]$$

Proof. Define

$$\left[\theta(\rho) := \max_{r \geq \rho} \frac{\|\nabla \tilde{u}\|_{B_r}}{r^{s+\alpha}}\right].$$

Notice that, since $\tilde{u}(0) = 0$, then $\dfrac{\|\tilde{u}\|_{B_r}}{r^{1+s+\alpha}} \leq \dfrac{\|\nabla \tilde{u}\|_{B_r}}{r^{s+\alpha}}$.

Therefore, $\underline{\theta(\rho) \to \infty}$ as $\rho \downarrow 0$. Notice also $\theta(\rho)$ is non-increasing.

Now, there is $r_k \to 0$ such that $r_k^{-s-\alpha}\|\nabla \tilde{u}\|_{B_{r_k}} = \theta(r_k)$, so in particular $\|\nabla \tilde{u}\|_{B_{r_k}} \geq r_k^{s+\alpha}$ (for k large).

Finally, $\|\nabla u_{r_k}\|_{B_R} = \dfrac{\|\nabla \tilde{u}\|_{B_{Rr_k}}}{\|\nabla \tilde{u}\|_{B_{r_k}}} \leq \dfrac{\theta(r_k R) \cdot (r_k R)^{s+\alpha}}{\theta(r_k) \cdot r_k^{s+\alpha}} \leq R^{s+\alpha}$ //

(i) $0 < c r^{2+s} \le \sup_{B_r(x_0)} (u-\varphi) \le C r^{2+s}$

(ii) $\sup_{B_r(x_0)} (u-\varphi) \le C r^{2+s+\alpha}$.

Proposition. Assume x_0 is a free boundary point at which (ii) does not hold.

Then, the functions (u_k) given by the Lemma satisfy:

$$\boxed{\|\nabla u_k\|_{L^\infty(B_1)} = 1} \qquad \text{(nondegeneracy)}$$

$$\boxed{|\nabla u_k(x)| \le C(1+|x|^{s+\alpha}) \text{ in } \mathbb{R}^n} \qquad \text{(growth control)}$$

$$\boxed{|L(\nabla u_k)| \le C r_k^{s-\alpha} \quad \text{in } \{u_k > 0\}} \qquad \text{(equation).}$$

Moreover, in the limit $k \to \infty$ we have

$$\boxed{u_k \longrightarrow u_0 \quad \text{in } C^1_{loc}(\mathbb{R}^n)}$$

for some function u_0 satisfying

$$\boxed{\begin{gathered} \|\nabla u_0\|_{L^\infty(B_1)} = 1 \\ |\nabla u_0(x)| \le C(1+|x|^{s+\alpha}) \text{ in } \mathbb{R}^n \\ L(\nabla u_0) = 0 \text{ in } \{u_0 > 0\} \end{gathered}}$$

Proof. Nondegeneracy and growth control follow from the Lemma.

The equation for u_k is

$$L u_k(x) = \frac{r_k^{2s} f(r_k x)}{r_k \|\nabla \tilde{u}\|_{L^\infty(B_{r_k})}} \quad \text{in } \{u_k > 0\},$$

so

$$|L(\nabla u_k)(x)| \le \frac{r_k^{2s} |\nabla f|(r_k x)}{\|\nabla \tilde{u}\|_{L^\infty(B_{r_k})}} \le \frac{C r_k^{2s}}{r_k^{s+\alpha}} \le C r_k^{s-\alpha} \left(\longrightarrow 0 \text{ as } r_k \to 0\right).$$

By $C^{1,\alpha}$ estimates, we get that $\left[u_k \to u_0 \text{ in } C^1_{loc}(\mathbb{R}^n)\right]$ for some u_0 satisfying nondegeneracy and growth control.

Finally, passing the equation to the limit we get $L(\nabla u_0) = 0$ in $\{u_0 > 0\}$. //

(1)

Classification of blow-ups

Theorem. Assume that u_0 solves

$$u_0 \geq 0 \text{ in } \mathbb{R}^n, \quad u_0 \text{ convex},$$
$$\|\nabla u_0\|_{L^\infty(B_1)} = 1, \qquad |\nabla u_0(x)| \leq 1 + |x|^{s+\alpha},$$
$$L(\nabla u_0) = 0 \text{ in } \{u_0 > 0\}.$$

Then, $\boxed{u_0(x) = (x \cdot e)_+^{1+s}}$

To prove this, we need the following:

Prop. Let Σ be a closed convex cone in $\mathbb{R}^n$ (with nonempty interior)

Let w_1, w_2 be two non-negative solutions of

$$\begin{cases} L w_i = 0 & \text{in } \mathbb{R}^n \setminus \Sigma \\ w_i = 0 & \text{in } \Sigma \\ w_i > 0 & \text{in } \mathbb{R}^n \setminus \Sigma, \end{cases} \qquad \int_{\mathbb{R}^n} \frac{w_i(x)}{1+|x|^{n+2s}}\,dx < \infty.$$

Then, $\boxed{w_1 \equiv K w_2 \text{ in } \mathbb{R}^n.}$

Proof. We use the boundary Harnack inequality.

Take $p \in \mathbb{R}^n \setminus \Sigma$ with $|p| = 1$.

$Lw_i = 0$

~~Rescale w_1 and w_2 so that $\int w_1 = \int w_2 = 1$.~~ Rescale $\int \underset{\cdot\cdot}{w_1} = \int \underset{\cdot\cdot}{w_2} = 1$.

~~We want to prove $w_1 \equiv w_2$ in $\mathbb{R}^n$~~. Then, $[w_1(p) \sim w_2(p) \sim 1]$

$\cancel{w_i = 0}$

Given $R \geq 1$ define $\quad \overline{w}_i(x) := \dfrac{w_i(Rx)}{C_i}$

with C_i such that $\displaystyle\int_{\mathbb{R}^n} \frac{\overline{w}_i(x)}{1+|x|^{n+2s}}\,dx = 1.$

By the boundary Harnack inequality, we have $[C^{-1}\overline{w}_2 \leq \overline{w}_1 \leq C\overline{w}_2 \text{ in } B_{1/2}]$

(2)

Since $w_1(p)$ and $w_2(p)$ are comparable, then $\bar{w}_1(P/R)$ and $\bar{w}_2(P/R)$ are comparable, and hence C_1 and C_2 are comparable.
Thus, it follows that

$$C^{-1} w_2 \le w_1 \le C w_2 \quad \text{in } B_{R/2},$$

with C independent of R. Taking $R \to \infty$, we get

$$\left[C^{-1} w_2 \le w_1 \le C w_2 \quad \text{in } \mathbb{R}^n \right]$$

Now define
$$K = \sup\{c > 0 : w_1 \ge c w_2 \text{ in } \mathbb{R}^n\}.$$

and
$$\left(w_3 := w_1 - K w_2 \right] \ge 0 \quad \text{in } \mathbb{R}^n.$$

Since w_3 satisfies the equation, either $w_3 \equiv 0$ in $\mathbb{R}^n$, or $w_3 > 0$ in $\mathbb{R}^n \setminus \Sigma$.
If $w_3 \equiv 0$ we are done. If not, we would get

$$w_2 \le C w_3 \quad \text{in } \mathbb{R}^n,$$

but then $w_3 - \frac{1}{C} w_2 = w_1 - K w_2 - \frac{1}{C} w_2 \ge 0$,

contradiction. Thus, $w_3 \equiv 0$, and $w_1 \equiv w_2$ in $\mathbb{R}^n$ //

(3.)

<u>Proof of the Theorem.</u> Let us denote $\boxed{\Omega = \{u_0 = 0\}}$

Since u_0 is <u>convex</u>, then <u>Ω is convex</u>.

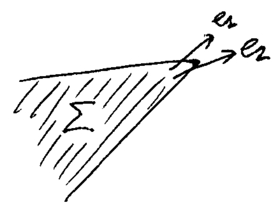
$L(D u_0) = 0$

We divide the proof:

<u>Case 1</u>: Assume that $\Omega = \Sigma$, a convex <u>cone</u> (with nonempty interior)

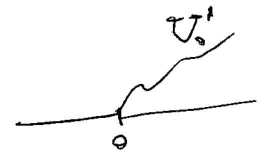
Then, there are n independent vectors e_i such that $\underline{-e_i \in \Sigma}$

Define $\boxed{w_i = \partial_{e_i} u_0}$



By convexity, $w_i \geq 0$, and thus
$$\begin{cases} L w_i = 0 & \text{in } \mathbb{R}^n \setminus \Sigma \\ w_i = 0 & \text{in } \Sigma \\ w_i > 0 & \text{in } \mathbb{R}^n \setminus \Sigma \end{cases}$$
$\int_{\mathbb{R}^n} \frac{w_i}{1+|x|^{n+2s}} dx < \infty$ (by <u>growth</u>)

By the previous Proposition, all w_i are equal up to multiplicative constant.
This means that all derivatives $\partial_{e_i} u_0$ are equal (up to multiplicative constants), and thus <u>u_0 is a 1D function</u>: $u_0(x) = U_0(x \cdot e)$.

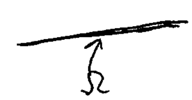


But then the 1D function U_0 solves
$$\left.\begin{array}{l} (-\Delta)^s (U_0') = 0 \text{ in } (0,\infty) \\ U_0' = 0 \text{ in } (-\infty, 0) \end{array}\right\}$$
with $|U_0'(x)| \leq 1 + |x|^{s+\alpha}$.

This yields $U_0'(x) = (x_+)^s \rightsquigarrow U_0(x) = (x_+)^{1+s} \rightsquigarrow u_0(x) = (x \cdot e)_+^{1+s}$ //

<u>Case 2</u>: Assume that Ω is contained in a hyperplane.

$L(D u_0) = 0$



Then, it is not difficult to show that $\left[L(D u_0) = 0 \text{ in all of } \mathbb{R}^n.\right]$
(it is solution across Ω).
But then, by a Liouville theorem we get $D u_0(x)$ is linear, so $u_0(x)$ is a paraboloid. By the growth condition $|D u_0(x)| \leq 1 + |x|^{s+\alpha}$ this is not possible.

(4.)

Case 3: Let now Ω be any convex set. We will see that by a blow-down argument we end up in case 1 or case 2.

Namely, we define

$$u_m(x) = \frac{u_0(R_m x)}{R_m \cdot \|\nabla u_0\|_{L^\infty(B_{R_m})}} \qquad \boxed{R_m \to \infty}$$

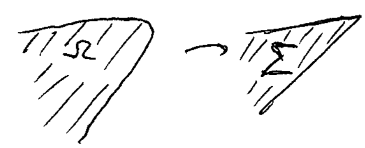
along an appropriate subsequence so that we keep the growth

$$|\nabla u_m(x)| \leq C(1+|x|^{s+\alpha}) \text{ in } \mathbb{R}^n, \quad \|\nabla u_m\|_{L^\infty(B_1)} = 1.$$

In the limit $m \to \infty$, we get a function $\boxed{u_\infty}$ satisfying the same assumptions as u_0, but with one extra property:

$$\Sigma = \{u_\infty = 0\} \text{ is a } \underline{\text{convex cone}}$$

This is because the blow-down of a convex set is a cone.

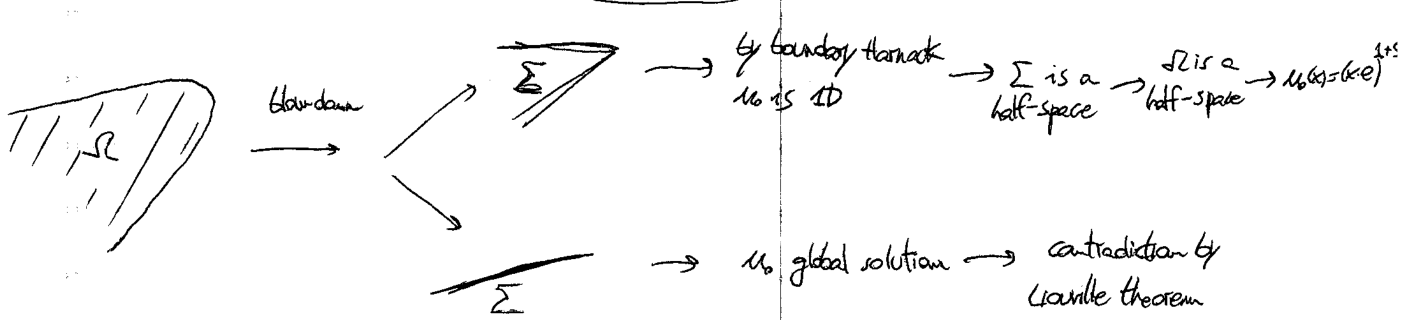


Moreover, the cone is a half-space if and only if Ω itself was a half-space, since $\boxed{\Sigma \subset \Omega.}$

Thus, by the Cases 1 and 2 we deduce that Ω is a half-space, and then

$$u_0(x) = (x \cdot e)_+^{1+s}$$

as desired.



- We have now proved the following:

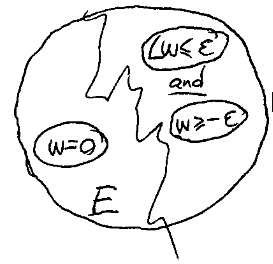
If (ii) does <u>not</u> hold, then there is a subsequence $r_k \to 0$

$$\left[\, u_{r_k}(x) := \frac{(u-\varphi)(x_0+r_k x)}{r_k \|\nabla(u-\varphi)\|_{L^\infty(B_{r_k}(x_0))}} \longrightarrow (x\cdot e)_+^{1+s} \quad \text{in } C^1_{loc}(\mathbb{R}^n). \right]$$

- We now want to deduce the $C^{1,\alpha}$ regularity of the free boundary. For this, we need:

<u>Lemma</u>. There is $\varepsilon>0$ small for which the following happens. Assume $E\subset \overline{B_1}$ is a closed set, and w satisfies

$$\left.\begin{array}{l} Lw \le \varepsilon \ \text{ in } B_1\setminus E \\ w = 0 \ \text{ in } E\cup(B_2^c) \\ w \ge -\varepsilon \ \text{ in } E^c \end{array}\right\}$$



and $\boxed{\int_{B_1} w_+ \ge 1}$. Then, $\boxed{w \ge 0 \text{ in } B_{1/2}}$

Proof. Let $b\in C_c^\infty(B_{3/4})$ be a radial bump function, $0\le b\le 1$ and $b\equiv 1$ in $B_{1/2}$.

Let

$$\left(\psi_t(x) := -\varepsilon - t + \varepsilon b(x)\right) \qquad (\text{notice } \psi_t \le -t \text{ in } \mathbb{R}^n,\ \psi_t \equiv -t \text{ in } B_{1/2})$$

Assume the Lemma is false. Then, ψ_t touches from below w at some point $z\in B_{3/4}$, for some $t>0$.

That is, $w(z)=\psi_t(z)$, and $w\ge\psi$ in $\mathbb{R}^n$.

On the one hand, we will have

$$L(w-\psi_t)(z) \ge \lambda \int_{\mathbb{R}^n} (w-\psi_t)(z+y)\,\frac{dy}{|y|^{n+2s}} \ge c_0 \int_{B_1} w_+\,dx \ge c_0 > 0$$

On the other hand,

$$L(w-\psi_t)(z) \le Lw(z) + |L\psi_t(z)| \le \varepsilon + C\varepsilon.$$

If $\varepsilon>0$ is small enough we get a contradiction. //

2.

Proposition. If (ii) does not hold at x_0, then the free boundary is Lipschitz in a neighborhood of x_0. More precisely, there is $e \in S^{n-1}$ such that

$$\partial_{e'}(u-\varphi) \geq 0 \quad \text{in } B_r(x_0), \qquad \text{for all } e' \cdot e \geq \tfrac{1}{2}.$$

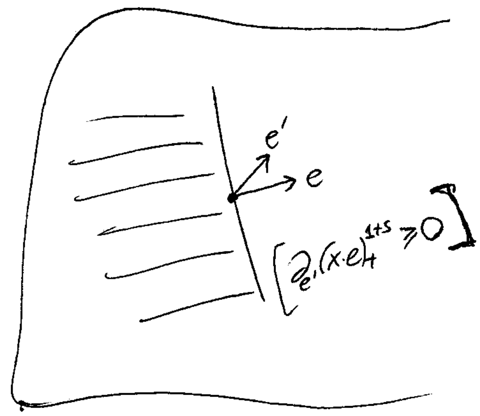
Proof. Let

$$u_{r_k}(x) := \frac{(u-\varphi)(x_0 + r_k x)}{r_k \|\nabla(u-\varphi)\|_{L^\infty(B_{r_k}(x_0))}} \longrightarrow (x\cdot e)_+^{1+s} \quad \text{in } C^1_{loc}(\mathbb{R}^n).$$

In particular, fixed $\delta > 0$ small and $R_0 >> 1$, we have

$$\|\partial_{e'} u_{r_k} - \partial_{e'} (x\cdot e)_+^{1+s}\| \leq \delta \quad \text{in } B_{R_0}$$

if r_k is small enough.



Let

$$\boxed{w = [C_1 \partial_{e'} u_{r_k}]\chi_{B_2}}$$

Then,

$$w \geq -\delta \quad \text{in } B_1$$

$$Lw \leq \delta \quad \text{in } \{w > 0\}$$

$$\int_{B_1} w_+ \geq 1$$

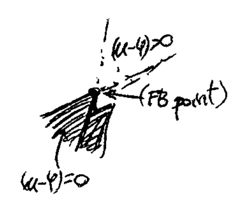
(Exercise)

Thus, by the previous Lemma, we get $w \geq 0$ in $B_{1/2}$, and thus

$$\partial_{e'}(u-\varphi) \geq 0 \quad \text{in } B_{r_k/2}(x_0), \quad (\forall e'\cdot e \geq \tfrac{1}{2})$$

provided that r_k is small (but fixed).

This means that the free boundary is Lipschitz (Exercise).



3.

[**Theorem.** If (iii) does not happen (at x_0), then the free boundary is $C^{1,\alpha}$ in a neighborhood of x_0.]

~~Proof. Using the boundary Harnack, we get that~~

Using the boundary Harnack, we get

$$\left[\frac{\partial_{e'}(u-\varphi)}{\partial_e(u-\varphi)} \in C^{0,\alpha}(B_{1/2}(x_0)) \right]$$

for all $e' \cdot e \geq \frac{1}{2}$. In particular, this yields

$$\left(\frac{\partial_{x_i}(u-\varphi)}{\partial_e(u-\varphi)} \in C^{0,\alpha}(B_{1/2}(x_0)) \quad \text{for } i=1,\dots,n \right)$$

Now, the normal vector to the free boundary (level set of $(u-\varphi)$) is

$$\nu^i(x) = \frac{\partial_{x_i}(u-\varphi)}{|\nabla(u-\varphi)|}(x) = \frac{\partial_{x_i}(u-\varphi)/\partial_e(u-\varphi)}{\left(\sum_{j=1}^{n}\left(\partial_{x_j}(u-\varphi)/\partial_e(u-\varphi)\right)^2\right)^{1/2}}$$

Since each term is $C^{0,\alpha}$, and the denominator can not vanish, then $\nu^i(x)$ is $C^{0,\alpha}$, and hence the free boundary is $C^{1,\alpha}$.

- Once we know that the free boundary is $C^{1,\alpha}$, the fact

$$\left[0 < c r^{1+s} \leq \sup_{B_r(x_0)} (u-\varphi) \leq C r^{1+s} \right]$$

follows from a barrier argument (a subsolution and a supersolution).

- Thus, this finishes the proof of the theorem.

OPEN PBS:

- Parabolic: $s > 1/2$, $s = 1/2$, $s < 1/2$

- Higher regularity: C^∞

- Complete structure for analytic obstacles: $1+s, 3+s, 5+s, \dots$; $2, 4, 6, \dots$; $3, 5, 7, \dots$