1.

(see Krylov's book!)

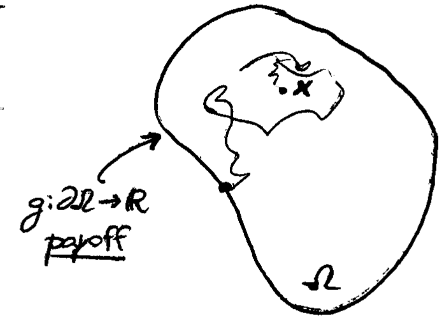
3. Fully nonlinear equations

3.1. Controlled diffusions

- We consider a random process X_t^γ, with a control parameter $\gamma : [0,\infty) \to \Gamma$. The set Γ is the set of all possible values of the parameter γ.
- Depending on the choice of $\gamma(t)$, we obtain different processes X_t^γ.
- Given $\gamma(t)$, we can consider the expected payoff

$$\mathbb{E}[g(X_\tau^\gamma)],$$

where τ is the first time that X_t^γ hits $\partial\Omega$.



- Now, we consider the following stochastic control problem:

- What is the best choice of the control γ, so that the expected payoff is maximum? And what is the maximum expected payoff?

- To answer those questions, we define

$$u(x) = \sup_{\substack{\text{all}\\ \text{possible}\\ \text{choices of } \gamma}} \mathbb{E}[g(X_\tau^\gamma)]$$

- The value of the control $\gamma(t)$ is to be chosen on the basis of observations of the controlled process X_t^γ before time t_0. (we don't know the future!). (Since the future is independent from past, we only care about the present!)
- Given a value $\gamma \in \Gamma$, the process X_t^γ (with γ constant in time) has infinitesimal generator L_γ. (L_γ is a second order operator $L_\gamma u(x) = \sum_{i,j} a_{ij}^{(\gamma)} \partial_{ij} u(x)$).

- It turns out that $u(x)$ solves

$$\sup_{\gamma\in\Gamma}\left(L_\gamma u(x)\right)=0 \quad \text{for all } x\in\Omega.$$

- That is, at each point $x\in\Omega$ the value γ is chosen so that $L_\gamma u(x)$ is maximum.

~~This can be seen easily: Let $\gamma=\gamma(x)$ be such that $L_\gamma u(x)$ is maximum~~

Let $\tilde\gamma=\tilde\gamma(x)$ be any choice of the control γ. Then, the expected payoff $\tilde v(x)$ with this choice of $\tilde\gamma=\tilde\gamma(x)$ solves

$$\left.\begin{aligned} L_{\tilde\gamma(x)}\tilde v(x)&=0 && \text{in } \Omega\\ \tilde v&=g && \text{on } \partial\Omega\end{aligned}\right\}.$$

Let now $\gamma(x)$ be such that $L_{\gamma(x)}u(x)=\sup_{\gamma\in\Gamma}L_\gamma u(x)$. Then, clearly

$$\left.\begin{aligned} \sup_{\gamma\in\Gamma}L_\gamma u(x)=L_{\gamma(x)}u(x)&=0 && \text{in } \Omega\\ u&=g && \text{on } \partial\Omega\end{aligned}\right\}$$

Moreover, we have

$$\left.\begin{aligned} L_{\tilde\gamma(x)}u(x)\le \sup_{\gamma\in\Gamma}L_\gamma u(x)&=0 && \text{in } \Omega\quad (=L_{\tilde\gamma(x)}\tilde v)\\ u&=g && \text{on } \partial\Omega \quad (=\tilde v)\end{aligned}\right\}\Rightarrow u\ge\tilde v \text{ in } \Omega.$$

This means that $u(x)$ is the maximum expected payoff.

- Thus, u solves the nonlinear equation

$$\boxed{\left.\begin{aligned} Iu&=0 && \text{in } \Omega\\ u&=g && \text{on } \partial\Omega\end{aligned}\right\}}$$

where $\boxed{Iu(x)=\sup_{\gamma\in\Gamma}L_\gamma u(x)}$

- When the operators L_γ are nonlocal operators,

$$L_\gamma u(x)=\frac12\int_{\mathbb{R}^n}\{u(x+y)+u(x-y)-2u(x)\}K_\gamma(y)\,dy$$

then the problem is

$$\boxed{\left.\begin{aligned} Iu&=0 && \text{in } \Omega\\ u&=g && \text{in } \mathbb{R}^n\setminus\Omega\end{aligned}\right\}}.$$

(3.)

• Similar considerations with zero-sum two-player stochastic games lead to the nonlinear operators

$$\boxed{Iu(x) = \inf_{\beta\in B}\sup_{\alpha\in A} L_{\alpha\beta}u(x)}$$

• When running costs are considered, we get the problem

$$\boxed{\left\{\begin{array}{ll} Iu = 0 & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n\setminus\Omega \end{array}\right.}$$

$$\boxed{Iu = \sup_{\gamma\in\Gamma}(L_\gamma u + c_\gamma)} \quad \text{or} \quad \boxed{Iu = \inf_{\beta\in B}\sup_{\alpha\in A}(L_{\alpha\beta}u + c_{\alpha\beta})}$$

(stochastic control) (zero-sum games)

Second order equations

• When $L_\gamma u$ or $L_{\alpha\beta}u$ are second order uniformly elliptic operators of the form

$$\left[\, Lu(x) = \sum_{i,j=1}^{n} a_{ij}\,\partial_{ij}u(x), \quad \lambda\,\mathrm{Id} \leq (a_{ij}) \leq \Lambda\,\mathrm{Id} \,\right]$$

• Then Iu is a fully nonlinear uniformly elliptic operator $F(D^2u)=0$.

• When $Iu = \sup_{\gamma\in\Gamma}(L_\gamma u + c_\gamma)$ then F is convex.

• When $Iu = \inf_{\beta\in B}\sup_{\alpha\in A}(L_{\alpha\beta}u + c_{\alpha\beta})$ then F is in general non-convex.

Extremal operators

• In any case, we have:

Lemma. Assume that all L_γ and $L_{\alpha\beta}$ belong to a class of linear operators $\boxed{\mathcal{L}}$.

Let $Iu = \sup_{\gamma\in\Gamma}(L_\gamma u + c_\gamma)$ or $Iu = \inf_{\beta\in B}\sup_{\alpha\in A}(L_{\alpha\beta}u + c_{\alpha\beta})$.

Then,

$$\boxed{\inf_{L\in\mathcal{L}} Lv \leq I(u+v) - Iu \leq \sup_{L\in\mathcal{L}} Lv}$$

Proof. We check one case:

$$\sup_{\gamma\in\Gamma}(L_\gamma u + L_\gamma v + c_\gamma) - \sup_{\gamma\in\Gamma}(L_\gamma u + c_\gamma) \leq \sup_{\gamma\in\Gamma} L_\gamma v \leq \sup_{L\in\mathcal{L}} Lv$$

For the case $c_{\alpha\beta}$: $L_{\alpha\beta}(u+v) \leq L_{\alpha\beta}u + \sup_{L\in\mathcal{L}} Lv \rightarrow$ take inf sup $\rightarrow I(u+v) \leq Iu + \sup_{L\in\mathcal{L}} Lv$ //

- When $\mathcal{L}$ is the class of <u>second order</u> uniformly elliptic operators, these are called Pucci operators

$$M^{+}_{\Lambda,\lambda}(D^2u) = \sup_{\lambda \leq (a_{ij}) \leq \Lambda \mathrm{Id}} \left(\sum_{ij} a_{ij}\partial_{ij}u \right)$$

$$M^{-}_{\lambda,\Lambda}(D^2u) = \inf_{\lambda \leq (a_{ij}) \leq \Lambda \mathrm{Id}} \left(\sum a_{ij}\partial_{ij}u \right)$$

- For a general <u>class $\mathcal{L}$ of nonlocal operators</u> we define the <u>extremal operators</u>

$$\boxed{M^{+}_{\mathcal{L}}u = \sup_{L\in\mathcal{L}} Lu} \quad \text{and} \quad \boxed{M^{-}_{\mathcal{L}}u = \inf_{L\in\mathcal{L}} Lu}$$

- Notice that, thanks to the previous Lemma, if Iu is

$$Iu = \sup_{\gamma\in\Gamma}\left(L_\gamma u + c_\gamma\right) \quad \text{or} \quad Iu = \inf_{\beta\in\mathcal{B}} \sup_{\alpha\in\mathcal{A}} \left(L_{\alpha\beta}u + c_{\alpha\beta}\right),$$

with <u>$L_\gamma \in \mathcal{L}$</u> or <u>$L_{\alpha\beta}\in\mathcal{L}$</u> for all $\gamma\in\Gamma$ or all $\alpha\in\mathcal{A}$ and $\beta\in\mathcal{B}$, respectively, then

$$\boxed{M^{-}_{\mathcal{L}}(u-v) \leq Iu - Iv \leq M^{+}_{\mathcal{L}}(u-v)} \qquad (*)$$

- We say that <u>$\mathbf{I}$ is elliptic with respect to the class $\mathcal{L}$</u> when $(*)$ holds.

- Notice that $M^{+}_{\mathcal{L}}$ and $M^{-}_{\mathcal{L}}$ are themselves fully nonlinear operators, and they are elliptic with respect to $\mathcal{L}$.

1

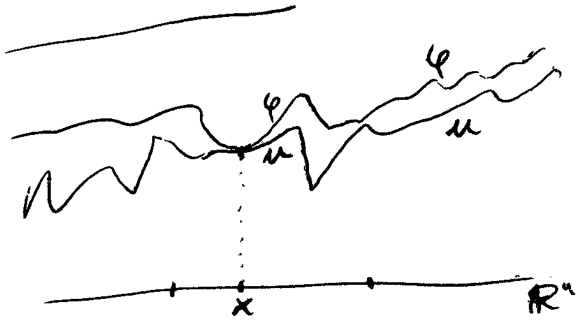
3.2. Viscosity solutions

Definition. A function $u : \mathbb{R}^n \to \mathbb{R}$, continuous in Ω, is said to be a <u>subsolution</u> <u>$Iu \geq f$ in Ω</u> if the following happens:

For any $x \in \Omega$ and test function $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfying

- φ is C^2 in a neighborhood of x
- $\varphi(x) = u(x)$
- $\varphi \geq u$ in all of $\mathbb{R}^n$

Then <u>$I\varphi(x) \geq f(x)$</u>.



[φ touches u from above]

The definition of <u>supersolution</u> <u>$Iu \leq f$ in Ω</u> is analogous.

u is a <u>solution</u> of <u>$Iu = f$ in Ω</u> if it is both a subsolution and supersolution.

- This definition only requires u to be <u>continuous</u> in Ω.

<u>Important properties of viscosity solutions</u>

- <u>Stability under uniform limits:</u> Assume: $\|u_k\|_{L^\infty(\mathbb{R}^n)} \leq C$, <u>$Iu_k \leq f_k$ in Ω</u>, (in the viscosity sense)

  $u_k \to u$ locally uniformly in Ω,

  $u_k \to u$ a.e. in $\mathbb{R}^n$,

  $f_k \to f$ locally uniformly in Ω

  Then, <u>$Iu \leq f$ in Ω</u> (in the viscosity sense).

②

- <u>Comparison principle</u>: If I is a fully nonlinear operator elliptic with respect to $\mathcal{L}$, (and $\mathcal{L}$ satisfies reasonable assumptions), then

$$\left.\begin{array}{l} Iu \geq f \text{ in } \Omega \\ Iv \leq f \text{ in } \Omega \\ u \leq v \text{ in } \mathbb{R}^n \setminus \Omega \end{array}\right\} \Rightarrow u \leq v \text{ in } \Omega.$$

(<u>Idea</u>: we have $0 \leq Iu - Iv \leq M^+_{\mathcal{L}}(u-v)$ in Ω, and $(u-v) \leq 0$ in $\mathbb{R}^n \setminus \Omega$.
If $(u-v)(x_0) > 0$ for some $x_0 \in \Omega$, then a contradiction because at a maximum we have $M^+_{\mathcal{L}}(u-v) < 0$.
The actual proof must be done for viscosity solutions.)

- <u>Existence of solutions</u>: The existence of viscosity solutions to $\left.\begin{array}{l} Iu = f \text{ in } \Omega \\ u = g \text{ in } \mathbb{R}^n \setminus \Omega \end{array}\right\}$ can be proved by using Perron's method (under reasonable assumptions on $\mathcal{L}$, g, and Ω).

- We will not prove those technical results, but we will use them.
- By definition, viscosity solutions u are continuous in $\overline{\Omega}$.

<u>Question</u>: If f is "nice enough", what is the regularity of u inside Ω?

- We will answer this question in the next sections.

3.3. Equations with bounded measurable coefficients: Harnack inequality and Hölder estimates

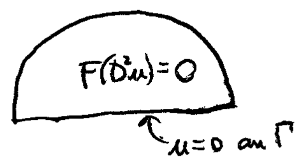
- For second-order fully nonlinear equations, the known main regularity results are the

If u is any viscosity sol. of $\boxed{F(D^2u)=0 \text{ in } B_1}$ then

(a) $\|u\|_{C^{1+\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}$ for some small $\alpha>0$.

(b). If in addition F is convex, then

$$\|u\|_{C^{2+\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)} \quad \text{for some small } \alpha>0.$$

(If u solves 

then $u\in C^{2+\alpha}$ up to the boundary.)

- We will prove that if Iu is an integro-differential fully nonlinear equation, then any viscosity solution u of $\boxed{Iu=0 \text{ in } B_1}$ satisfies

$$\|u\|_{C^{1+\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}.$$

- For this, let us recall the strategy for second order equations:

- Formally, if u solves $F(D^2u)=0$ in B_1, and we differentiate the equation in the direction e, we get

$$\frac{\partial F}{\partial M_{ij}}(D^2u(x))\cdot \partial_e(\partial_{ij}u(x))=0 \text{ in } B_1$$

where $\frac{\partial F}{\partial M_{ij}}$ is the derivative of $F(M)$ with respect to the variable M_{ij}.

Since F is uniformly elliptic, then $a_{ij}(x):=\frac{\partial F}{\partial M_{ij}}(D^2u(x))$

satisfy

$$\lambda\, \mathrm{Id} \leq (a_{ij}(x)) \leq \Lambda\, \mathrm{Id},$$

and hence $\partial_e u$ solves:

$$a_{ij}(x)\partial_{ij}(\partial_e u)=0 \text{ in } B_1.$$

- In other words, $\partial_e u$ solves an equation with bounded measurable coefficients.

(2.)

- Notice that the equation
$$a_{ij}(x)\partial_{ij} w = 0 \quad \text{in } B_1$$
is equivalent to
$$\left.\begin{array}{l} M^+_{\lambda,\Lambda}(D^2 w) \geq 0 \quad \text{in } B_1 \\ M^-_{\lambda,\Lambda}(D^2 w) \leq 0 \quad \text{in } B_1 \end{array}\right\} \circledast$$

- Indeed, heuristically those two inequalities imply that for some choice of $a_{ij}(x)$ we have $a_{ij}(x)\partial_{ij} w = 0$ in B_1.

- Now, for solutions w of $\circledast$ we have a $C^{0,\alpha}$ estimate,
$$\|w\|_{C^{0,\alpha}(B_{1/2})} \leq C\|w\|_{L^\infty(B_1)},$$
and since w was any derivative of u, we find a $C^{1,\alpha}$ estimate for u.

- To make this argument rigorous, (u is only continuous! we cannot differentiate the equation), we observe that $u(x+h)$ solve the equation, too, and thus
$$M^-_{\lambda,\Lambda}(u(x+h)-u(x)) \leq \underbrace{F(D^2u(x+h)) - F(D^2u(x))}_{\overset{\shortparallel}{0}} \leq M^+_{\Lambda,\lambda}\left(u(x+h)-u(x)\right) \quad \text{in } B_{1-|h|}$$

This means that
$$w(x) = \frac{u(x+h)-u(x)}{|h|^\alpha} \quad \text{solves} \quad \left.\begin{array}{l} M^+_{\Lambda,\lambda}(D^2 w) \geq 0 \quad \text{in } B_{1-|h|} \\ M^-_{\Lambda,\lambda}(D^2 w) \leq 0 \quad \text{in } B_{1-|h|} \end{array}\right\}$$
in the <u>viscosity sense</u>.

- Using the estimate for equations with bounded meas. coeff., we first find that $u \in C^{0,\alpha}$, then $C^{2\alpha}$, ..., until we reach $u \in Lip$ and $u \in C^{1,\alpha}$ for some small $\alpha > 0$.

③

- For nonlocal equations the idea will be the same:

if $\quad Iu=0$ in B_1, then

$$M^-_{\mathcal{L}}(u(x+h)-u(x)) \leq \underbrace{I(u(x+h)) - I(u(x))}_{=0} \leq M^+_{\mathcal{L}}(u(x+h)-u(x)) \quad \text{in } B_{1-|h|}.$$

- Thus, the function $\quad w(x) = \dfrac{u(x+h)-u(x)}{|h|^\alpha}$ solves $\left.\begin{array}{l} M^+_{\mathcal{L}} w \geq 0 \quad \text{in } B_{1-|h|} \\ M^-_{\mathcal{L}} w \leq 0 \quad \text{in } B_{1-|h|} \end{array}\right\}$

- This is (heuristically) equivalent to

$$\frac{1}{2}\int_{\mathbb{R}^n} \{w(x+y)+w(x-y)-2w(x)\} K(x,y)\,dy$$

for some kernel $K(x,y)$ (with no regularity in x!).

- The class $\mathcal{L}$ that we will take is

$$\boxed{\mathcal{L}_0 := \left\{ \text{operators } Lu(x) = \frac{1}{2}\int_{\mathbb{R}^n} \{u(x+y)+u(x-y)-2u(x)\}K(y)\,dy \text{ with kernels } K \text{ satisfying } \frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}} \right\}}$$

- To prove $C^{1+\alpha}$ regularity of solutions of $\underline{Iu=0 \text{ in } B_1}$, we need a C^α estimate for solutions of

$$\boxed{\left.\begin{array}{l} M^+_{\mathcal{L}_0} u \geq 0 \quad \text{in } B_1 \\ M^-_{\mathcal{L}_0} u \leq 0 \quad \text{in } B_1 \end{array}\right\}.}$$

- To prove such C^α estimate we will first establish a <u>Harnack inequality</u> for such equation.

④

<u>Remark.</u> For the class of Kernels $\mathcal{L}_0$, we have the following explicit expressions for $M^{\pm}_{\mathcal{L}_0}$:

$$M^{+}_{\mathcal{L}_0} u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \Lambda\big(u(x+y)+u(x-y)-2u(x)\big)_{+} \frac{dy}{|y|^{n+2s}} + $$

$$- \frac{1}{2}\int_{\mathbb{R}^n} \lambda\big(u(x+y)+u(x-y)-2u(x)\big)_{-} \frac{dy}{|y|^{n+2s}}.$$

$$M^{-}_{\mathcal{L}_0} u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \lambda\big(u(x+y)+u(x-y)-2u(x)\big)_{+} \frac{dy}{|y|^{n+2s}}$$

$$- \frac{1}{2}\int_{\mathbb{R}^n} \Lambda\big(u(x+y)+u(x-y)-2u(x)\big)_{-} \frac{dy}{|y|^{n+2s}}$$

such expression follow from

$$M^{+}_{\mathcal{L}_0} u(x) = \sup_{L\in\mathcal{L}_0} Lu(x) = \sup_{\frac{\lambda}{|y|^{n+2s}} \le K(y) \le \frac{\Lambda}{|y|^{n+2s}}} \left\{ \frac{1}{2}\int_{\mathbb{R}^n} \big(u(x+y)+u(x-y)-2u(x)\big) K(y)\, dy \right\}.$$

(• It is not very important to have those expressions, but it is sometimes useful.)

(5.) Half Harnack inequality ~~and Hölder regularity~~ for supersolutions

Theorem. Assume that $\boxed{u \geq 0 \text{ in } \mathbb{R}^n}$ and it solves

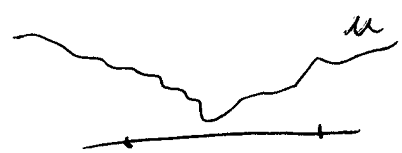
$$\boxed{M^-_{\mathcal{L}_0} u \leq C_0 \text{ in } B_1}$$

in the viscosity sense. Then,

$$\boxed{\int_{\mathbb{R}^n} \frac{u(x)}{1+|x|^{n+2s}}\,dx \leq C\left(\inf_{B_{1/2}} u + C_0\right)}$$

- The theorem says: if $\underline{u \geq 0}$ is a supersolution, then the infimum is controlled (by below) by the L^1 norm of u.

- Thus, if $u \geq 0$ is a supersolution in B_1, it cannot be close to 0 in $B_{1/2}$ unless it is close to 0 everywhere (in an L^1 sense).



Proof. Let $b \in C^\infty_c(B_{3/4})$ be such that $0 \leq b \leq 1$ and $b \equiv 1$ in $B_{1/2}$.

Let $t>0$ be the maximum value for which $\boxed{u \geq tb}$

Notice that $\boxed{t \leq \inf_{B_{1/2}} u}$

Moreover, since u and b are continuous in B_1, then there is $x_0 \in B_{3/4}$ such that $u(x_0) = t\, b(x_0)$.

Now, on the one hand we have

$$M^-_{\mathcal{L}_0}(u-tb) \leq M^-_{\mathcal{L}_0} u - t M^-_{\mathcal{L}_0} b \leq C_0 + C_1 t \quad \text{in } B_1.$$

where $C_1 = \|M^-_{\mathcal{L}_0} b\|_{L^\infty(B_1)}$.

On the other hand, since $(u-tb) \geq 0$ in $\mathbb{R}^n$ and $(u-tb)(x_0) = 0$, then

$$M^-_{\mathcal{L}_0}(u-tb)(x_0) = \frac{1}{2}\int_{\mathbb{R}^n} \lambda\big(u(x+y)+u(x-y) - tb(x+y) - tb(x-y)\big)\frac{dy}{|y|^{n+2s}} \geq c\int_{\mathbb{R}^n} \big(u(x+y)+u(x-y)\big)\frac{dy}{1+|y|^{n+2s}} - Ct\int_{\mathbb{R}^n} \cdots$$

$$\geq c\int_{\mathbb{R}^n} \frac{u(z)}{1+|z|^{n+2s}}\,dz - C_2 t$$

8

Combining the previous identities, we get

$$c\int_{\mathbb{R}^n} \frac{u(z)}{1+|z|^{n+2s}}\,dz - C_2 t \le C_0 + C_1 t$$

$$\downarrow$$

$$\int_{\mathbb{R}^n} \frac{u(z)}{1+|z|^{n+2s}}\,dz \le C(t+C_0)$$

Using that $t \le \inf_{B_{1/2}} u$, we are done. //

Hölder regularity estimate

· Using the weak Harnack inequality, we now want to prove that solutions of

$$\left.\begin{array}{l} M^+u \ge C_0 \text{ in } B_1 \\ M^-u \le C_0 \text{ in } B_1 \end{array}\right\} \text{ satisfy } \left[\|u\|_{C^\alpha(B_{1/2})} \le C\left(C_0 + \|u\|_{L^\infty(\mathbb{R}^n)}\right)\right]$$

for some small $\alpha > 0$.

· Let us see first a sketch of the ~~proof~~ in the local case:

- Assume we have a weak Harnack $\int_{B_{1/2}} u \le C \inf_{B_{1/2}} u$ for solutions $u \ge 0$ in B_1 to some local equation.

- To prove a Hölder estimate, it suffices to show that $\left[\operatorname{osc}_{B_{1/2}} u \le (1-\theta) \operatorname{osc}_{B_1} u\right]$, and iterate. $\left(\operatorname{osc}_{B_1} = \sup_{B_1} - \inf_{B_1}\right)$

- To show this, assume $\sup_{B_1} u = 1$, and $\inf_{B_1} u = -1$.

  Assume also $\int_{B_{1/2}} u \ge 0$ (if $\int_{B_{1/2}} u \le 0$ just take $-u$ instead of u).

- Then the function $w = u + 1 = u - \inf_{B_1} u$ satisfies $w \ge 0$ in B_1, and thus

$$0 < c \le \int_{B_{1/2}} (u+1) = \int_{B_{1/2}} w \le C \inf_{B_{1/2}} w = C\left(\inf_{B_{1/2}} u + 1\right)$$

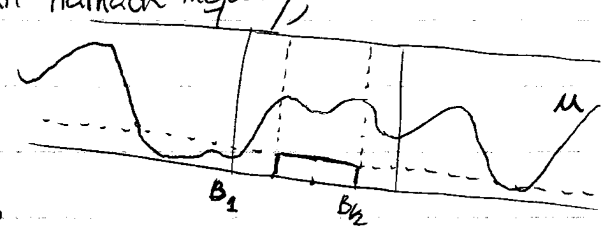
Thus, $\inf_{B_{1/2}} u \ge c - 1$, for some $c > 0$.

This yields

$$\operatorname{osc}_{B_{1/2}} u = \sup_{B_{1/2}} u - \inf_{B_{1/2}} u \le 1 - (c-1) = 2 - c \overset{(\theta = \frac{c}{2})}{=} 2(1-\theta) = (1-\theta) \operatorname{osc}_{B_1} u \;//$$

(7)

- We now want to do the same but with our half Harnack inequality, which requires $u \geq 0$ in $\mathbb{R}^n$ (instead of B_1).



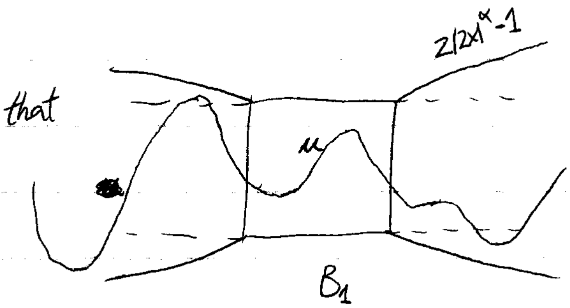
- Thus, the non-locality affects this iterative procedure, and the proof needs to be done carefully in the non-local case.

Lemma. Assume $\|u\|_{L^\infty(B_1)} \leq 1$, and $\boxed{|u(x)| \leq 2|2x|^\alpha - 1 \text{ for } |x| \geq 1}$, with $\alpha > 0$ small enough.

Assume $\varepsilon > 0$ is small enough, and u solves $\left.\begin{array}{l} M^+ u \geq -\varepsilon \text{ in } B_1 \\ M^- u \leq \varepsilon \text{ in } B_1 \end{array}\right\}$.

- If $\int_{B_{1/2}} u \geq 0$ then $u \geq \theta - 1$ in $B_{1/2}$. (If $\int_{B_{1/2}} u \leq 0$ then $u \leq 1 - \theta$ in $B_{1/2}$.)

Proof. Assume $\int_{B_{1/2}} u \geq 0$, and take $\boxed{w = (u+1)_+}$. Notice that



$w \geq 0$ in $\mathbb{R}^n$, and $w = (u+1) + (u+1)_-$.

Let L be any operator with Kernel $\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}$.

Then, in $B_{3/4}$ we have

$$Lw(x) = Lu(x) + L(u+1)_-(x) =$$
$$= Lu(x) + \int_{\mathbb{R}^n} \{(u+1)_-(x+y) + (u+1)_-(x-y)\} K(y)\,dy,$$

and since

$$0 \leq (u+1)_-(z) \leq 2|2z|^\alpha - 2, \quad \text{and } (u+1)_- \equiv 0 \text{ in } B_1, \text{ then}$$

$$0 \leq L(u+1)_-(x) \leq \int_{\mathbb{R}^n \setminus B_{1/4}} \{2|x+y|^\alpha + 2|x-y|^\alpha - 4\} \frac{\Lambda}{|y|^{n+2s}}$$

$$\leq \int_{\mathbb{R}^n \setminus B_{1/4}} 4\{|8y|^\alpha - 1\} \frac{\Lambda}{|y|^{n+2s}} \leq \varepsilon$$ if $\alpha > 0$ is small enough.

Since $\left.\begin{array}{l} M^+ u \geq -\varepsilon \\ M^- u \leq \varepsilon \end{array}\right\}$ then we get $\left.\begin{array}{l} M^+ w \geq -2\varepsilon \text{ in } B_{3/4} \\ M^- w \leq 2\varepsilon \text{ in } B_{3/4} \end{array}\right\}$.

9.

We will show by induction that, for some small $\alpha>0$, we have

$$\operatorname*{osc}_{B_{2^{-k}}} u \leq 2^{-k\alpha}$$

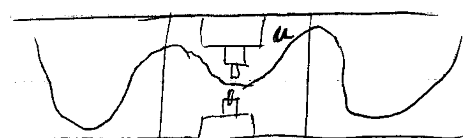
- More precisely, we will construct sequences a_k and b_k such that

$$\left[b_k \leq u(x) \leq a_k \text{ in } B_{2^{-k}}\right] \quad \text{and} \quad \left[a_k - b_k = 2^{-k\alpha}\right] \cdot \left[\leq b_0 \leq \dots \leq b_k \leq b_{k+1} \leq \dots \leq a_{k+1} \leq a_k \leq \dots\right]$$

We construct those sequences by induction. For $k \leq 0$ we may take $a_0 = \frac{1}{2}$ and $b_0 = -\frac{1}{2}$.
For $k<0$, just take $b_k = -\frac{1}{2}$, $a_k = b_k + 2^{-k\alpha}$.

- Assume now we have those sequences a_j, b_j for $j \leq k$, and let us construct a_{k+1}, b_{k+1}.

- Let $\left(m = \frac{a_k+b_k}{2}\right]$ and notice $\left[|u-m| \leq \frac{1}{2} 2^{-k\alpha} \text{ in } B_{2^{-k}}\right]$



Let

$$\left[v(x) := 2 \cdot 2^{k\alpha}\left(u(2^{-k}x) - m\right).\right]$$

Then, $\|v\|_{L^\infty(B_1)} \leq 1$ and v solves
$$\left.\begin{array}{ll} M^+ v \geq -\varepsilon (2^{-k})^{2s} \cdot 2^{k\alpha} & \text{in } B_1 \\ M^- v \leq \varepsilon (2^{-k})^{2s} \cdot 2^{k\alpha} & \text{in } B_1 \end{array}\right\}.$$

Since $\alpha \leq 2s$, (α is small), then
$$\left.\begin{array}{ll} M^+ v \geq -\varepsilon & \text{in } B_1 \\ M^- v \leq \varepsilon & \text{in } B_1 \end{array}\right\}.$$

- In order to use the Lemma, we need $\left[|v(x)| \leq 2|2x|^\alpha - 1 \text{ for } |x| \geq 1.\right]$ Let us prove this.
Take $|x| \geq 1$, and let $j \geq 0$ such that $2^j \leq |x| \leq 2^{j+1}$. Then, by inductive hypothesis,

$$v(x) = 2 \cdot 2^{k\alpha}\left(u(2^{-k}x) - m\right) \leq 2 \cdot 2^{k\alpha}\left(a_{k-j-1} - m\right) \leq 2 \cdot 2^{k\alpha}\left(a_{k-j-1} - b_{k-j-1} + b_k - m\right)$$

$$= 2 \cdot 2^{k\alpha}\left(2^{-(k-j-1)\alpha} - \frac{1}{2} \cdot 2^{-k\alpha}\right) = 2 \cdot 2^{(j+1)\alpha} - 1 \leq 2|2x|^\alpha - 1.$$

- Hence, by the previous Lemma we have $\inf_{B_{1/2}} v \geq \theta - 1$, or equivalently,

$$\left(u - m \geq \frac{1}{2}(\theta-1) 2^{-k\alpha} \text{ in } B_{2^{-k-1}}\right] \longrightarrow \left[b_k + \frac{\theta}{2} 2^{-k\alpha} \leq u \leq a_k \text{ in } B_{2^{-k-1}}\right]$$

- Taking $\alpha>0$ so that $\left[\theta = 1 - 2^{-\alpha}\right]$ we have $a_{k+1} = a_k$ and $b_{k+1} = b_k + \left(2^{-k\alpha} - 2^{(-k-1)\alpha}\right) = b_k + 2^{-k\alpha}(1 - 2^{-\alpha})$.

Then $a_{k+1} - b_{k+1} = 2^{-(k+1)\alpha}$, as desired.

- By the half Harnack inequality, (recall $w \geq 0$ in $\mathbb{R}^n$), we get:

$$\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^{n+2s}}\,dx \leq C\left(\inf_{B_{1/2}} w + \varepsilon\right).$$

In particular

$$\int_{B_{1/2}} w \leq C\left(\inf_{B_{1/2}} w + \varepsilon\right).$$

Now,

$$\int_{B_{1/2}} w = \int_{B_{1/2}} (u+1) \geq \int_{B_{1/2}} 1 \geq |B_{1/2}|,$$

so

$$\theta = c|B_{1/2}| - \varepsilon \leq \inf_{B_{1/2}} w = \inf_{B_{1/2}} u + 1$$

This means that

$$\inf_{B_{1/2}} u \geq \theta - 1$$

- We next deduce the C^α estimate from the previous Lemma.

<u>Theorem</u>. Assume that u solves
$$\left.\begin{array}{l} M^+ u \geq -C_0 \text{ in } B_1 \\ M^- u \leq C_0 \text{ in } B_1 \end{array}\right\}.$$

Then,
$$\boxed{\|u\|_{C^\alpha(B_{1/2})} \leq C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\right)}$$

<u>Proof.</u> We need to show

$$|u(x_0) - u(x)| \leq C|x - x_0|^\alpha \quad \text{for all } x_0, x \in B_{1/2}.$$

We will show it for $x_0 = 0$ (it is the same at all points). For this, notice that dividing by $2\|u\|_{L^\infty(\mathbb{R}^n)}$, we may assume $\operatorname{osc}_{\mathbb{R}^n} u = 1$. Moreover, considering the function $\tilde{u}(x) = u(rx)$, with r small, we have
$$\left.\begin{array}{l} M^+ \tilde{u} \geq -C_0 r^{2s} \text{ in } B_1 \\ M^- \tilde{u} \leq C_0 r^{2s} \text{ in } B_1 \end{array}\right\}.$$
Taking r small so that $C_0 r^{2s} \leq \varepsilon$, we may assume that
$$\left.\begin{array}{l} M^+ u \geq -\varepsilon \text{ in } B_1 \\ M^- u \leq \varepsilon \text{ in } B_1 \end{array}\right\}$$

(10.)

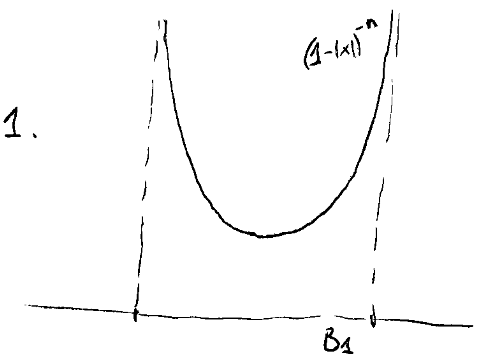
Remark. We have $\left.\begin{array}{l} M^+u \geq -C_0 \text{ in } B_1 \\ M^-u \leq C_0 \text{ in } B_1 \end{array}\right\} \to \boxed{\|u\|_{C^\alpha(B_{1/2})} \leq C\left(\|u\|_{L^1_\omega(\mathbb{R}^n)} + C_0\right)}$ (use half Harnack to bound $\|u\|_{L^\infty(B_{7/8})}$, then truncate $u\chi_{B_{7/8}}$, solves an eq. in $B_{3/4}$, so it is C^α in $B_{1/2}$).

Harnack inequality

- We have proved the half-Harnack inequality for supersolutions, and showed the C^α estimate for equations with bounded measurable coefficients.
- We now show the other half Harnack (for subsolutions), and this will yield the "full" Harnack.

Theorem Assume that u satisfies $\boxed{M^+u \geq -C_0 \text{ in } B_1}$ (subsolution) in the viscosity sense. Then

$$\boxed{\sup_{B_{1/2}} u \leq C\left(\int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\,dx + C_0\right)}$$



Proof. Dividing u by a constant, we may assume

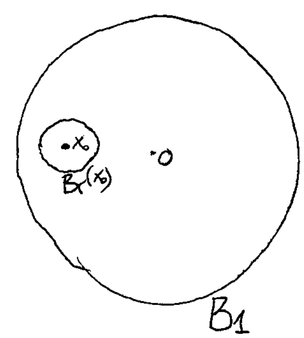
$$M^+u \geq -1 \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\,dx \leq 1.$$

Let us consider the minimum value of t such that

$$\left[u(x) \leq t(1-|x|)^{-n} \quad \text{in } B_1\right]$$

- Then, since u is continuous in $\overline{B}_1$, there must be $x_0 \in B_1$ such that $\left[u(x_0) = t(1-|x_0|)^{-n}\right]$
- We want to show that t cannot be very large. For this, let $\left[d = (1-|x_0|), \text{ and } r = d/2\right]$, and let us estimate $|B_1 \cap \{u < u(x_0)/2\}|$ and $|B_1 \cap \{u > u(x_0)/2\}|$.
- Let $A = B_1 \cap \{u > u(x_0)/2\}$. Since $\int_{B_1} |u| \leq 2$ then

$$|A| \leq 2\left|\frac{2}{u(x_0)}\right| = 4t^{-1}d^n.$$



- In particular, since $|B_r| = Cd^n$, then

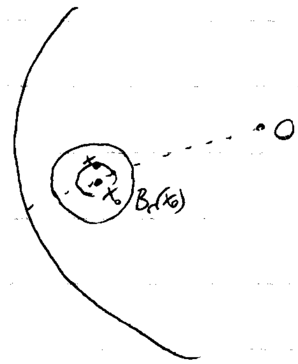
$$\left[|A \cap B_r(x_0)| \leq Ct^{-1}|B_r|\right]$$

- If t is large, then A can cover only a small portion of $B_r(x_0)$.

• Let us now estimate $|\{u<u(x_0)/2\}\cap B_r(x_0)|$ by using the half Harnack inequality for supersolutio

More precisely, we estimate

$$|\{u<u(x_0)/2\}\cap B_{\theta r}(x_0)|$$



for $\theta>0$ small.

• For $x\in B_{\theta r}(x_0)$ we have $|x|\leq|x_0|+\theta r = 1-d+\theta d/2$, so

$$u(x)\leq t\cdot(1-|x|)^{-n}\leq t(d-\theta d/2)^{-n}$$
$$\leq t\cdot d^{-n}(1-\theta/2)^{-n} = u(x_0)\cdot(1-\theta/2)^{-n}$$

• Let

$$\left[v(x)=(1-\theta/2)^{-n}u(x_0)-u(x)\right] \qquad \left[v\geq 0 \text{ in } B_{\theta r}(x_0)\right]$$

and notice that $M^-v\leq 1$ (supersolution) since $M^+u\geq -1$.

• We want to use the half Harnack, and for this we need to consider

$$\boxed{w=v^+}$$

• We need to estimate M^-w in $B_{\theta r/2}(x_0)$, using that

$$v^-(z)=\left((1-\theta/2)^{-n}u(x_0)-u(z)\right)^-\leq\begin{cases}0 & \text{in } B_{\theta r}(x_0)\\ |u(z)| & \text{in } \mathbb{R}^n\setminus B_{\theta r}(x_0)\end{cases}$$

• We get

$$M^-w\leq M^-v+\Lambda\int_{\mathbb{R}^n\setminus B_{\theta r}(x_0)}\frac{|u(z)|}{|x-z|^{n+2s}}\,dz\leq 1+C(\theta r)^{-n-2s}\int_{\mathbb{R}^n}\frac{|u(z)|}{1+|z|^{n+2s}}\,dz\leq C(\theta r)^{-n-2s}.$$

in $B_{\theta r/2}(x_0)$.

By the half Harnack for supersolutions (rescaled to the ball $B_{\theta r/2}(x_0)$) we get

$$r^{-n}\int_{B_{\theta r/4}(x_0)}w\leq C\Big(\inf_{B_{\theta r/4}(x_0)}w+(\theta r)^{2s}\cdot C(\theta r)^{-n-2s}\Big)\leq C\Big(\inf_{B_{\theta r/4}(x_0)}w+(\theta r)^{-n}\Big).$$

(12.)

• Now, since $w(x) = \left((1-\theta/2)^{-n} u(x_0) - u(x)\right)^+$ then

$$\left|\{u < u(x_0)/2\} \cap B_{\theta r/4}\right| \leq \ldots$$

Corollary If $u \geq 0$ in $\mathbb{R}^n$ solves $\left.\begin{array}{ll} M^+ u \geq 0 & \text{in } B_1 \\ M^- u \leq 0 & \text{in } B_1 \end{array}\right\}$

then

$$\left[\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u\right]$$

Proof.

Since u is a supersolution, then

$$\int_{\mathbb{R}^n} \frac{u(x)}{1+|x|^{n+2s}}\, dx \leq C \inf_{B_{1/2}} u$$

since u is a subsolution,

$$\sup_{B_{1/2}} u \leq C \int_{\mathbb{R}^n} \frac{u(x)}{1+|x|^{n+2s}}\, dx. \quad //$$

$C^{1,\alpha}$ estimates for fully nonlinear equations

- Let

$$\left[Iu := \inf_{\beta\in B}\sup_{\gamma\in\Gamma}\left(L_{\beta\gamma}u\right)\right]$$

or

$$\left(Iu := \sup_{\gamma\in\Gamma}\left(L_\gamma u\right)\right)$$

be a fully nonlinear nonlocal operator, with $L_{\beta\gamma}$ all having Kernels

$$\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}} \qquad 0<\lambda\leq\Lambda, \quad s\in(0,1).$$

- Recall that

$$\left[M^-(u-v) \leq Iu - Iv \leq M^+(u-v)\right].$$

- Now, we want to show that if u solves $\left[Iu = 0 \text{ in } B_1\right]$ then

$$\left[\|u\|_{C^{1+\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}\right]$$

for some small $\alpha>0$.

(Can this be true? We need either $1+\alpha > 2s$ or further assumptions on the Kernels!)

- The easiest approach to show the estimate would be the following:

$$Iu = 0 \longrightarrow \left.\begin{matrix} M^+u \geq 0 \\ M^-u\leq 0\end{matrix}\right\} \longrightarrow \|u\|_{C^\alpha} \leq C\|u\|_{L^\infty} \longrightarrow \frac{u(x+h)-u(x)}{|h|^\alpha} \in L^\infty, \quad \left.\begin{matrix} M^+\left(\frac{u(x+h)-u(x)}{|h|^\alpha}\right) \geq 0 \\ M^-\left(\frac{u(x+h)-u(x)}{|h|^\alpha}\right) \leq 0\end{matrix}\right\}$$

$$\longrightarrow \|u\|_{C^{2\cdot\alpha}} \cong \left\|\frac{u(x+h)-u(x)}{|h|^\alpha}\right\|_{C^\alpha} \leq C\left\|\frac{u(x+h)-u(x)}{|h|^\alpha}\right\|_{L^\infty} \simeq C\|u\|_{C^\alpha} \leq C\|u\|_{L^\infty} \longrightarrow \frac{u(x+h)-u(x)}{|h|^{2\cdot\alpha}} \in L^\infty \longrightarrow \cdots$$

$$\longrightarrow \cdots \rightarrow \frac{u(x+h)-u(x)}{|h|} \in L^\infty \longrightarrow \frac{u(x+h)-u(x)}{|h|} \in C^\alpha \rightarrow u\in C^{1+\alpha}$$

- However, we were cheating here!

(2.)

- At each step one should truncate the functions, since

$$u\in L^\infty(\mathbb{R}^n) \longrightarrow u\in C^\alpha(B_{1/2}) \longrightarrow \frac{u(x+h)-u(x)}{|h|^\alpha}\in L^\infty(B_{1/2}) \text{ but \underline{not} } L^\infty(\mathbb{R}^n)\,!$$

In the next step we would need $\frac{u(x+h)-u(x)}{|h|^\alpha}\in L^\infty(\mathbb{R}^n)$ in the iteration.

- By just truncating the function at each step, the above proof works provided that one assumes that

$$\left[\; |\nabla K(y)| \le \frac{C}{|y|^{n+2s+1}} \;\right]$$

Thus, kernels must be C^1 outside the origin for this argument to work.

- To avoid that, the proof must be done by <u>blow-up and compactness</u>, as we did in case of linear equations.
- The proof would be essentially the same, but the <u>Liouville theorem</u> we need is the following.

<u>Theorem.</u> There is <u>a small $\alpha>0$</u> such that: Assume that $U\in C(\mathbb{R}^n)$ is a solution of $\left\{\begin{array}{l} M^+U\ge 0 \text{ in } \mathbb{R}^n \\ M^-U\le 0 \text{ in } \mathbb{R}^n\end{array}\right.$ and that it satisfies

$$|U(x)|\le C\left(1+|x|^\alpha\right) \quad \text{for all } x\in\mathbb{R}^n.$$

Then, U is <u>constant.</u>

<u>Proof.</u> Let $\bar\alpha>0$ be the one for which we have the estimate

$$\left.\begin{array}{l} M^+w\ge 0 \text{ in } B_1 \\ M^-w\le 0 \text{ in } B_1\end{array}\right\} \longrightarrow [w]_{C^{\bar\alpha}(B_{1/2})} \le C\left(\|w\|_{L^1_\omega(\mathbb{R}^n)}\right)$$

Let <u>$\alpha>0$</u> be any number <u>$0<\alpha<\bar\alpha$</u>.

Let $\left(w(x)=R^{-\alpha}\,U(Rx)\right)$ with <u>$R>1$</u>. Then, we have

$$|w(x)|\le CR^{-\alpha}\left(1+|Rx|^\alpha\right)\le C\left(1+|x|^\alpha\right),\quad \text{with } C \text{ independent of } R>1.$$

(3)

- In particular, we have

$$\|w\|_{L^1_w(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \frac{C(1+|x|^\alpha)}{1+|x|^{n+2s}}\,dx \leq C \qquad (\text{since } \underline{\alpha<2s}).$$

- Thus, we have $\quad [w]_{C^{\bar\alpha}(B_{1/2})} \leq C\|w\|_{L^1_w(\mathbb{R}^n)} \leq C \quad$ (for every $R>1$).

- Now, this means

$$[v]_{C^{\bar\alpha}(B_{R/2})} = R^{\alpha-\bar\alpha}[w]_{C^{\bar\alpha}(B_{1/2})} \leq CR^{\alpha-\bar\alpha} \longrightarrow 0$$

as $R\to\infty$.

- This yields $[v]_{C^{\bar\alpha}(\mathbb{R}^n)} = 0$, so v is constant.//

- Using this "Liouville theorem", we get the following.

<u>Theorem.</u> **Let** $\underline{\alpha>0}$ be given by the previous result.

Let I be a fully nonlinear operator, $\quad \left[Iu = \inf_{\beta\in B}\sup_{\gamma\in\Gamma}(L_{\beta\gamma}u)\right]$

with $L_{\beta\gamma}$ being nonlocal operators with kernels

$$\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}.$$

If $s\leq 1/2$ we assume <u>in addition</u> that $\quad \left[\,[K]_{C^{1+\alpha-2s}(B_r)} \leq \frac{C}{r^{n+1+\alpha}}\right]$

Then, any solution u of $\underline{Iu=0 \text{ in } B_1}$ satisfies

$$\|u\|_{C^{1+\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}$$

<u>Proof.</u> Exercise. (using the previous Liouville theorem, the proof is very similar to the one we did for linear equations).

Boundary regularity for fully nonlinear nonlocal equations

- Recall: for the fractional Laplacian we showed that solutions "look like" d^s, where $d(x) := \mathrm{dist}(x, \Omega^c)$.

$$\left.\begin{array}{l}(-\Delta)^s u = f \text{ in } \Omega\\ u = 0 \text{ in } \Omega^c\\ f \in L^\infty(\Omega)\end{array}\right\} \to \left[\, |u| \leq C d^s \,\right]$$

- Moreover, we saw that the same happens for all operators with kernels

$$\left[\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}} \quad \text{with } \underline{K(y) \text{ homogeneous}}\right]$$

- For such operators L we saw that

$$\left.\begin{array}{l}Lu = f \text{ in } \Omega\\ u = 0 \text{ in } \Omega^c\end{array}\right\} \longrightarrow \left[u(x) = c_z d^s(x) + o(|x-z|^{s+\alpha})\right] \ \underline{\forall \alpha \in (0,s)}$$

whenever $\underline{f \in L^\infty}$ and $\underline{\Omega \text{ is } C^2}$.

- This was equivalent to $\underline{u/d^s \in C^\alpha(\overline{\Omega})}$.

- Furthermore, when $f \in C^\infty(\overline{\Omega})$, Ω is C^∞, and the $K(y)$ is C^∞ outside the origin, then $u/d^s \in C^\infty(\overline{\Omega})$.

- For fully nonlinear equations, one has the following:

Theorem. Assume Ω is $C^{2,\alpha}$, and $\left[\, Iu = \inf_{\beta \in B} \sup_{\gamma \in A} L_{\beta\gamma} u \,\right]$ with $L_{\beta\gamma}$ being nonlocal operators with kernels

$$\left[\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}, \text{ with } \underline{K(y) \text{ homogeneous}} \text{ and } [K]_{C^\alpha(\mathbb{R}^n \setminus B_r)} \leq \frac{C}{r^{n+2s+\alpha}}.\right]$$

Let u be any solution of

$$\left.\begin{array}{l}Iu = f \text{ in } \Omega\\ u = 0 \text{ in } \Omega^c\end{array}\right\} \text{ with } f \in C^\alpha(\overline{\Omega}).$$

Then,

$$\left[\, \|u/d^s\|_{C^{s+\alpha}(\overline{\Omega})} \leq C \|f\|_{C^\alpha(\overline{\Omega})} \,\right]$$

(with $\alpha > 0$ small.)

- This is an estimate of $\underline{\text{order } 2s+\alpha}$: $\left[\, u(x) = c_z d^s(x) + (b_z \cdot x) d^s(x) + O(|x-z|^{2s+\alpha}) \,\right]$

- In the limit $s \uparrow 1$ it says that "u is $C^{2+\alpha}$ on the boundary".

• To show this theorem, we define

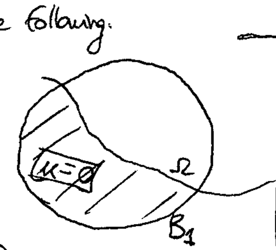
$$\boxed{M_*^+ w := \sup_{\substack{L \text{ is an operator with} \\ \frac{\lambda}{|y|^{n+2s}} \le K(y) \le \frac{\Lambda}{|y|^{n+2s}} \\ \text{and } K(y) \text{ is homogeneous}}} Lw} \qquad \boxed{M_*^- w := \inf_{\substack{\dots \\ K(y) \text{ homogeneous}}} Lw}$$

• Then, if $I u$ is as in the Theorem, we will have

$$M_*^-(u-v) \le Iu - Iv \le M_*^+(u-v).$$

• For such operators $M_*^\pm$ (with homogeneous kernels), we have the following.

<u>Theorem.</u> Assume Ω is C^2 and $\left.\begin{array}{l} M_*^+ u \ge -C_0 \text{ in } \Omega\cap B_1 \\ M_*^- u \le C_0 \text{ in } \Omega\cap B_1 \\ u = 0 \text{ in } B_1\setminus\Omega \end{array}\right\}$



Then,

$$\left[\| u/d^s \|_{C^\alpha(\overline{\Omega}\cap B_{1/2})} \le C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0 \right) \right]$$

for some small $\underline{\alpha > 0}$

• This is the analogue of the C^α interior estimate for equations with bounded measurable coefficients.

• In this case, however, "the boundary data helps", and the estimate is of <u>order $s+\alpha$ on the boundary</u>:

$$\left[u(x) = c_z d^s(x) + O(|x-z|^{s+\alpha}) \right]$$

• This result is proved by a boundary Harnack type estimate, and an iteration that improves the oscillation of u/d^s in dyadic balls:

$$b_k \le u/d^s \le a_k \quad \text{in } B_{2^{-k}}, \qquad \text{with } b_k \le b_{k+1} \le \dots \le a_{k+1} \le a_k \le \dots$$

• Then, the estimate for fully nonlinear equations is proved by a blow-up and compactness argument. (much more delicate than the interior...).

Boundary regularity for non-homogeneous Kernels

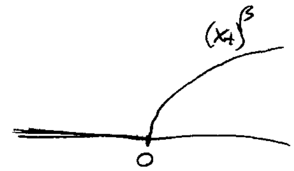
Question: What happens with the boundary regularity for non-homogeneous Kernels?

- Let us try to answer this question in dimension 1. (In $\mathbb{R}^n$ would be similar.)

- If we denote M^+ and M^- the extremal operators $\left(M^+w = \sup_{\frac{\lambda}{|y|^{1+2s}} \le K(y) \le \frac{\Lambda}{|y|^{1+2s}}} Lw \quad \text{and} \quad M^-w = \inf Lw. \right)$ then, these operators are scale invariant of order $2s$.

- Namely, if $w_r(x) = w(rx)$ then $(M^+w_r)(x) = r^{2s}(M^+w)(rx)$.

- Thus, if we consider the 1D functions $\boxed{(x_+)^\beta}$, with $0<\beta<2s$



we will have that the functions $M^+(x_+)^\beta$ and $M^-(x_+)^\beta$ will be homogeneous of degree $\beta - 2s$. In particular,

$$M^+(x_+)^\beta = c_1(\beta)\cdot(x_+)^{\beta-2s} \quad \text{in } (0,\infty)$$
$$M^-(x_+)^\beta = c_2(\beta)\cdot(x_+)^{\beta-2s} \quad \text{in } (0,\infty).$$

- For the fractional Laplacian, we showed that $(-\Delta)^s(x_+)^s = 0$ in $(0,\infty)$.

- For the operators M^+ and M^- we have the following:
  - The constants $c_1(\beta)$ and $c_2(\beta)$, are continuous in β for $\beta \in (0,2s)$.
  - We have $\lim_{\beta\to 0} c_1(\beta) < 0$ and $\lim_{\beta\to 2s} c_1(\beta) = +\infty$.
  - In particular, there exist β_1 and β_2 such that $c_1(\beta_1) = c_2(\beta_2) = 0$, i.e. $\boxed{\begin{array}{c} M^+(x_+)^{\beta_1} = 0 \text{ in } (0,\infty) \\ M^-(x_+)^{\beta_2} = 0 \text{ in } (0,\infty) \end{array}}$
  - Such β_1 and β_2 are unique, and satisfy $\boxed{0 < \beta_1 < s < \beta_2 < 2s}$

(4.)

- This is very different to what happens for homogeneous kernels:

Homogeneous kernels in $\mathbb{R}^n$

$\left.\begin{array}{l} Iu=f \text{ in } \Omega \\ u=0 \text{ in } \Omega^c \end{array}\right\} \to |u| \le Cd^s$

and if $f>0$ then $0<cd^s \le u \le Cd^s$

General kernels $\frac{\lambda}{|y|^{n+2s}} \le K(y) \le \frac{\Lambda}{|y|^{n+2s}}$ in $\mathbb{R}^n$

- solutions do not look like d^s.
- In general, we only have $|u| \le Cd^{\beta_1}$ and if $f \ge 0$ then $u \ge cd^{\beta_2}$, but $\beta_1 < s < \beta_2$.

- This is very different to the local case $F(D^2u)=0$, all solutions are $C^{1+\alpha}(\bar{\Omega})$ and in particular they look like $d(x)$.

- This finishes the Chapter on fully nonlinear equations.

- Some results that we did not cover and could be good for presentations:

  - $C^{2s+\alpha}$ estimate for convex equations $\left(Iu = \sup_{\alpha \in \Gamma} L_\alpha u\right) \to$ Yijing
    (solutions are classical for all $s \in (0,1)$)

  - Parabolic equations $\left[u_t - Iu = 0 \text{ in } B_1 \times (0,1)\right] \to$ Hernán

  - Equations nonlocal in space-time: $\left[Lu(t,x) = \int_0^\infty \int_{\mathbb{R}^n} \big(u(t-s, x+y) - u(t,x)\big) K(s,y)\, dy\, ds\right]$
    (of order $2s$ in x and β in time)
    with $\left[K(s,y) \approx \frac{1}{|y|^{n+2s+\beta} + s^{\frac{n+2s+\beta}{\beta}}}\right]$

  - Equations with kernels $K(y) \approx |y|^{-n} \cdot \chi_{B_1}$
    or $K(y) \approx |y|^{-n-2s} \,|\log|y||$
    or other types of scaling conditions