2 Linear nonlocal equations

2.1 Lévy processes

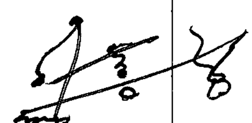
Let X_t be a stochastic process in $\mathbb{R}^n$ with:

- $X_0 = 0$
- Stationary and independent increments (no memory and $X_t - X_s \sim X_{t-s}$ in distribution)

· If we further assume that $t \mapsto X_t$ is a continuous path a.s. then X_t must be a Brownian motion (possibly non-isotropic and possibly with drift).

· However, if we relax this assumption and only assume stochastic continuity $\left(\lim_{h\downarrow 0} P(|X_{t+h} - X_t| > \varepsilon) = 0\right)$ then X_t will be a Lévy process.

Brownian motion



Lévy process (with jumps!)

Infinitesimal generators

The infinitesimal generator of a Lévy process X_t is an operator $L : C^2(\mathbb{R}^n) \to C(\mathbb{R})$ defined by

$$\boxed{Lu(x) := \lim_{t\downarrow 0} \frac{\mathbb{E}[u(x+X_t)] - u(x)}{t}}$$

It is a classical fact that this definition leads to the formula

$$\left[\;\mathbb{E}\left[u(x+X_t)\right] = u(x) + \mathbb{E}\left[\int_0^t Lu(x+X_\tau)\,d\tau\right]\;\right]$$

The Lévy-Khintchine formula states that the infinitesimal generator of any Lévy process is of the form

$$\left[\;Lu(x) = tr(A\,D^2u) + b\cdot\nabla u + \int_{\mathbb{R}^n}\left\{u(x+y) - u(x) - y\cdot\nabla u(x)\,\chi_{B_1}(y)\right\}d\nu(y)\;\right]$$

for some

- non-negative definite matrix $A \geq 0$
- $b \in \mathbb{R}^n$
- measure ν satisfying $\int_{\mathbb{R}^n} \min\{1,|y|^2\}\,d\nu(y) < \infty$

· Examples: When X_t is the Brownian motion, then $L = \Delta$, the Laplacian ($A = Id$, $b=0$, $\nu\equiv 0$).

· When X_t has no diffusion or drift part, then

$$\bullet\; Lu(x) = \int_{\mathbb{R}^n}\left\{u(x+y) - u(x) - y\cdot\nabla u(x)\,\chi_{B_1}(y)\right\}d\nu(y)$$

· Furthermore, we will always assume that ν is absolutely continuous,

$$d\nu(y) = K(y)\,dy,$$

and that the process is symmetric,

$$K(y) = K(-y).$$

In this case, we can symmetrize the integral, and

$$\left[\;Lu(x) = \frac{1}{2}\int_{\mathbb{R}^n}\left\{u(x+y) + u(x-y) - 2u(x)\right\}K(y)\,dy\;\right]$$

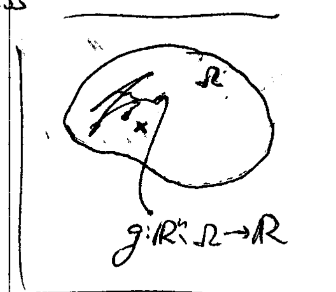
· We will mostly use this expression during the course.

· Expected payoff

· Let us "show" that the expected payoff problem leads to $\left.\begin{array}{l} Lu = 0 \text{ in } \Omega \\ u = g \text{ in } \mathbb{R}^n \setminus \Omega \end{array}\right\}$

· Namely, we have a domain $\Omega \subset \mathbb{R}^n$, and a Lévy process $X_t^x := (x + X_t)$ starting at $x \in \Omega$.

· We run the process X_t^x until it exits Ω, and goes to $\mathbb{R}^n \setminus \Omega$. When it exits Ω, we get a payoff $g(z)$, which depends on $z \in \mathbb{R}^n \setminus \Omega$.



[Question: What is the expected payoff ?]

· If we denote τ = first exit time, then we are looking for

$$u(x) = \mathbb{E}\left[g(X_\tau^x)\right].$$

~~It can be proved that, if $x \in \Omega$, then (for most events ω) the process will still be inside Ω at time $t = \delta$, with $\delta > 0$ very small.~~

At time $t = \delta$, the process will be at $x + X_\delta$, and thus we have that

$$u(x) = \mathbb{E}\left[u(x + X_\delta)\right]$$

Now, recall that

$$\mathbb{E}\left[u(x + X_t)\right] = u(x) + \mathbb{E}\left[\int_0^t Lu(x + X_s)\,ds\right]$$

and thus

$$\mathbb{E}\left[\int_0^\delta Lu(x + X_s)\,ds\right] = 0 \qquad \text{for } x \in \Omega.$$

Dividing by δ and letting $\delta \downarrow 0$, we find

$$\left[\; Lu(x) = 0 \qquad \text{for } x \in \Omega. \;\right]$$

Since $u = g$ in $\mathbb{R}^n \setminus \Omega$, then u solves

$$\boxed{\left.\begin{array}{l} Lu = 0 \text{ in } \Omega \\ u = g \text{ in } \mathbb{R}^n \setminus \Omega \end{array}\right\}}$$

④

- Similar considerations (expected time, running costs), lead to the Dirichlet problem

$$\boxed{\left.\begin{array}{l} -Lu = f \text{ in } \Omega \\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \end{array}\right\}}$$

Classes of Kernels

- From now on, we will mostly consider nonlocal operators of the form

$$\left[\, Lu(x) = \frac{1}{2}\int_{\mathbb{R}^n} \{u(x+y)+u(x-y)-2u(x)\}K(y)\,dy. \,\right]$$

with $K(y)=K(-y)$ and $\displaystyle\int_{\mathbb{R}^n} \min\{1,|y|^2\}K(y)\,dy < \infty$.

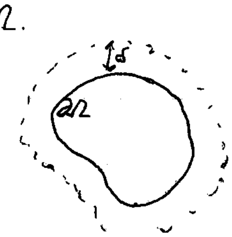
- Fractional Laplacian: The most canonical choice of $K(y)$ is the radially symmetric and homogeneous

$$\left[\, K(y) = \frac{c}{|y|^{n+2s}} \quad \text{with} \quad s\in(0,1) \,\right]$$

When c is chosen appropriately, we have $\left[-L = (-\Delta)^s\right]$

~~- Kernels with compact support: In some applications it is convenient to have Kernels with compact support in a ball B_δ ($\delta>0$). In that case, the exterior condition in the Dirichlet problem has to be posed only in $(\Omega+B_\delta)\setminus\Omega$ instead of $\mathbb{R}^n\setminus\Omega$.~~

~~From the analytical point of view, it is mainly the singularity of $K(y)$ at the origin what determines the regularity properties of L.~~



~~(Thus, we will not treat the case of Kernels with compact support.)~~

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Stable processes: These are processes that satisfy self-similarity properties, and they are also those processes appearing in a Generalized Central Limit Theorem.

The infinitesimal generators L are scale invariant, and this means that

- either $Lu = \mathrm{tr}(A D^2 u)$ (Brownian motion)
- or $Lu = \int_{\mathbb{R}^n} \{u(x+y) - u(x) - \dots\} \, d\nu(y)$, with ν homogeneous (of degree $-n-2s$, with $s \in (0,1)$).

- In our case (ν absolutely cont, and even)

$$Lu(x) = \frac{1}{2} \int_{\mathbb{R}^n} \{u(x+y) + u(x-y) - 2u(x)\} K(y)\, dy,$$

then

$$\left[K(y) \text{ homogeneous of degree } -n-2s \text{ for some } s \in (0,1) \right]$$

- Equivalently,

$$\left[K(y) = \frac{a(y/|y|)}{|y|^{n+2s}} \quad \text{for some } a \in L^1(S^{n-1}) \text{ and } s \in (0,1) \right] \quad \boxed{a \geq 0}$$

- When the process X_t is radially symmetric then $K(y) = c|y|^{-n-2s}$, ~~as before~~ and $-L = (-\Delta)^s$, as before.

(• In the limit case in which the function a consists of a sum of delta functions, then

$$-L = (-\partial_{x_1 x_1})^s + \dots + (-\partial_{x_n x_n})^s \quad)$$

Stable-like: A very typical assumption in the study of integro-differential equations is

$$\left[\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}} \quad \text{for some } 0 < \lambda \leq \Lambda < \infty \text{ and } s \in (0,1) \right]$$

- The operator L in that case is not scale invariant, but the class of kernels is scale invariant.

[• During the course we will work almost always with operators which are stable or stable-like.]

2.2. Weak solutions, comparison principle, Fourier symbol

- For operators

$$Lu(x) = \frac{1}{2}\int_{\mathbb{R}^n} \{u(x+y)+u(x-y)-2u(x)\} K(y)\,dy,$$

with

$$0 < \frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}} \qquad s\in(0,1)$$

we have the following:

- Integration by parts:

$$\left[\int_{\mathbb{R}^n} u\, Lv\, dx = -\iint_{\mathbb{R}^n\,\mathbb{R}^n} (u(x)-u(x+y))(v(x)-v(x+y))K(y)\,dy\,dx\right]$$

(Proof: Analogous to the case $-L=(-\Delta)^s$.)

- Weak solutions: u is a weak solution....

$$\left[\stackrel{\text{Def.}}{\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)}} (u(x)-u(x+y))(v(x)-v(x+y))K(y)\,dy\,dx = \int_\Omega f v \quad \text{for all } v\ldots\right]$$

- Existence of weak solutions: For any $\Omega\subset\mathbb{R}^n$ bounded, and $f\in L^2(\Omega)$, there exists a (unique) weak solution of

$$\left.\begin{array}{l} -Lu = f \text{ in } \Omega \\ u = 0 \text{ in } \mathbb{R}^n\setminus\Omega \end{array}\right\}$$

(Proof: Analogous, just use that $\iint_{\mathbb{R}\mathbb{R}} (u(x)-u(x+y))^2 K(y)\,dy\,dx$ is comparable to $K(y)=\frac{c}{|y|^{n+2s}}$.)

- Fourier symbol of L: In Fourier side, we have

$$\left[\mathcal{F}[-Lu](\xi) = \mathcal{A}(\xi)\mathcal{F}[u](\xi)\right]$$

with

$$\left[\mathcal{A}(\xi) = \int_{\mathbb{R}^n} (1-\cos(y\cdot\xi))K(y)\,dy.\right]$$

- Notice that

$$0 < \lambda|\xi|^{2s} \leq \mathcal{A}(\xi) \leq \Lambda|\xi|^{2s}$$

and that $\mathcal{A}(\xi)$ is <u>homogeneous</u> whenever $K(y)$ is <u>homogeneous</u>.

- <u>Comparison principle:</u> If u_1 and u_2 are weak solutions of $\left.\begin{array}{l}-Lu_1=f_1 \text{ in } \Omega\\ u_1=g_1 \text{ in } \mathbb{R}^n\setminus\Omega\end{array}\right\}$ $\left.\begin{array}{l}-Lu_2=f_2 \text{ in } \Omega\\ u_2=g_2 \text{ in } \mathbb{R}^n\setminus\Omega\end{array}\right\}$

then

$$\left.\begin{array}{l} f_1\geq f_2\\ g_1\geq g_2\end{array}\right\} \Longrightarrow u_1\geq u_2 \text{ in } \Omega.$$

- <u>Boundedness of weak solutions:</u> $\left[\|u\|_{L^\infty(\Omega)}\leq \|g\|_{L^\infty(\mathbb{R}^n\setminus\Omega)}+C\|f\|_{L^\infty(\Omega)}\right]$ [Exercise]

- All the previous properties are analogous to those we proved for the fractional Laplacian. However, there are several other properties that we proved for $(-\Delta)^s$ which relied on particular properties of $(-\Delta)^s$.
- For general operators

$$Lu(x)=\frac{1}{2}\int_{\mathbb{R}^n}\{u(x+y)+u(x-y)-2u(x)\}K(y)\,dy$$

we have:

- <u>No</u> explicit Poisson Kernel or mean value property (or fundamental solution)
  Thus, we need different proof of <u>interior regularity</u> and of <u>Harnack inequality</u>.

- <u>No</u> extension problem in $\mathbb{R}^{n+1}_+$.
  Thus, we need to construct barriers again in order to get <u>boundary regularity</u>.

- In the next sections we will develop the <u>interior</u> and <u>boundary</u> regularity theory for operators L with kernels K.

- It is important to notice that the operators we study in this Chapter are linear and translation invariant (no x-dependence).
- In the case of (local) 2nd order equations, this would mean $L=\sum_{ij}a_{ij}\partial_{ij}$ (constant coefficients), and thus after an affine change of variables we get the Laplacian.
- For nonlocal equations the situation is very different. The class of linear operators with "constant coefficients" is very rich, and presents interesting features that do not appear for $(-\Delta)^s$.

(1.)

2.3. Interior regularity

- The strategy to establish interior regularity estimates will be very different from the one for $(-\Delta)^s$. We will establish the estimate

$$\boxed{\|u\|_{C^{2s+\alpha}(B_{1/2})} \leq C\left(\|u\|_{C^{\alpha}(\mathbb{R}^n)} + \|f\|_{C^{\alpha}(B_1)}\right)}$$

for solutions of $\boxed{Lu = f \text{ in } B_1}$ with $\alpha\in(0,1)$ and $2s+\alpha$ not integer.

- Notice that the estimate is different from that of $(-\Delta)^s$: in the RHS we have $\|u\|_{C^{\alpha}(\mathbb{R}^n)}$ instead of $\|u\|_{L^\infty(\mathbb{R}^n)}$. This is necessary for kernels $0<\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}$.

- We will come back to this issue later. [When $[K(y)]_{C^{\alpha}(\mathbb{R}^n\setminus B_\rho)} \leq \frac{C}{\rho^{n+2s+\alpha}}$ for all $\rho>0$, then $\|u\|_{L^\infty(\mathbb{R}^n)}$ instead of $\|u\|_{C^{\alpha}(\mathbb{R}^n)}$.]

- The strategy to establish such estimate is by a contradiction and blow-up argument, combined with a Liouville theorem for L.

Liouville Theorem

Theorem. Let L be ..., with
$$0<\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}.$$
Let u be a (bounded) weak solution of $\boxed{Lu=0 \text{ in } \mathbb{R}^n}$. Then, u is constant.

Proof. Let $p(t,x)$ be the heat kernel associated to the operator L, which satisfies

① $\left.\begin{array}{l}\partial_t p - Lu = 0 \quad \text{in } \mathbb{R}^n\times(0,\infty)\\ p(0,\cdot)=\delta_0 \quad \text{for } t=0\end{array}\right\}$

② $\int_{\mathbb{R}^n} p(1,x)\,dx = 1$

③ $p(t,x)\geq 0$

④ $\hat{p} = e^{-tA(\xi)}$ in Fourier side

⑤ $\|\nabla p\|_{L^\infty(\mathbb{R}^n)} \leq C$ for $t=1$.

②

Properties 1-2-3-4 are general facts that are true for any generator of a Lévy process. (In fact, $p(t,x)$ is the probability distribution of X_t.)

Property ⑤ follows directly from the decay of the Fourier transform $\hat{p}$ in ④, recall $\mathcal{A}(\xi) \geq \lambda|\xi|^{2s} > 0$.

• Now, let $R \geq 1$, and $\bar{u}(x) := u(Rx)$. Notice that $\bar{u}$ is a weak solution of $L\bar{u} = 0$ in $\mathbb{R}^n$, and $\|\bar{u}\|_{L^\infty(\mathbb{R}^n)} = \|u\|_{L^\infty(\mathbb{R}^n)} = 1$.

• Thus, we have $\left[\bar{u} = \bar{u} * p(1,\cdot)\right]$ (Exercise: Why?)

$$\left(\bar{u} - p(1,\cdot) * \bar{u} = \left[p(t,\cdot) * \bar{u}\right]_0^1 = \int_0^1 \partial_t p * \bar{u} = \int_0^1 Lp * \bar{u} = \int_0^1 p * \underset{0}{\underset{\shortparallel}{L\bar{u}}}\right)$$

• Now, given $x, x' \in \mathbb{R}^n$ we have

$$|\bar{u}(x) - \bar{u}(x')| = \left|\int_{\mathbb{R}^n} \left(p(x-y) - p(x'-y)\right) u(y)\,dy\right| \leq CM^n \cdot |x-x'| + C\int_{|y| \geq M} p_1$$

• Here we used $|p(x-y) - p(x'-y)| \leq C|x-x'|$ in $\mathbb{R}^n$, $p_1 \geq 0$ in $\mathbb{R}^n$, and $\|\bar{u}\|_{L^\infty(\mathbb{R}^n)} = 1$.

• Setting $M = |x-x'|^{-\frac{1}{2n}}$, we get

$$|\bar{u}(x) - \bar{u}(x')| \leq C|x-x'|^{1/2} + C\int_{B^c_{|x-x'|^{-1/2n}}} p_1 := \omega(|x-x'|)$$

• Notice that since $p_1 \in L^1(\mathbb{R}^n)$, then $\omega(|x-x'|) \to 0$ as $|x-x'| \to 0$.

• Recalling that $\bar{u}(x) = u(Rx)$, this means that

$$|u(Rx) - u(Rx')| \leq \omega(|x-x'|), \quad \text{and} \quad |u(x) - u(x')| \leq \omega\left(\frac{|x-x'|}{R}\right) \quad \text{for all } R > 1.$$

Letting $R \to \infty$ we find $u(x) = u(x')$, for all $x, x' \in \mathbb{R}^n$. //

Let $V(t,x) = \bar{u} * p(t,\cdot)$.

Then, $V(t,x)$ satisfies
$$\begin{cases} V_t = LV & \text{for } t>0 \\ V(0) = \bar{u} \end{cases}$$

However, V solves in the weak sense $LV=0$, and thus $V_t \equiv 0$ for $t>0$. This means that V is constant in t, and thus $V = \bar{u}$. In particular, $\bar{u} = \bar{u} * p(1,\cdot)$.

Lemma. Let L_k be a sequence of operators with kernels $K_k(y)$ satisfying

$$(\circledast) \qquad \frac{\lambda}{|y|^{n+2s}} \le K_k(y) \le \frac{\Lambda}{|y|^{n+2s}} \qquad (0<\lambda\le\Lambda<\infty) \quad (s\in(0,1))$$

Assume that u_k satisfy $L_k u_k = f_k$ in Ω, and $\|u_k\|_{L^\infty(\mathbb{R}^n)} \le C$, and $f_k \to f$ uniformly in Ω, and $u_k \to u$ locally uniformly in $\mathbb{R}^n$.

Then, there exists an operator L with kernel $K(y)$ satisfying $\circledast$ and such that

$$Lu = f \text{ in } \Omega.$$

Proof. Let $\eta \in C_c^\infty(\Omega)$. Since $L_k u_k = f_k$ in Ω then

$$-\int_{\mathbb{R}^n} u_k L_k \eta = \int_\Omega f_k \eta .$$

Notice that $\int_\Omega f_k\eta \to \int_\Omega f\eta$. We want to show that $-\int_{\mathbb{R}^n} u_k L_k\eta \to -\int_{\mathbb{R}^n} uL\eta$ for some L.

Let us consider the sequence of measures $\left[d\mu_k(y) := c_k \min\{1,|y|^2\} K_k(y)dy\right]$ where c_k is chosen so that μ_k are probability measures in $\mathbb{R}^n$. Then, thanks to the bounds $\circledast$ we have that for every $\varepsilon>0$ there is $R>1$ large such that $\mu_k(B_R) > 1-\varepsilon$ (the mass does not "escape to infinity").

Thus, it follows (from Prokhorov's theorem) that, up to a subsequence, the sequence μ_k converges weakly to some probability measure μ in $\mathbb{R}^n$. Moreover, thanks to $\circledast$ then we will have

$$d\mu(y) = c\min(1,|y|^2)K(y) \quad \text{for some } K(y) \text{ satisfying } \circledast.$$

In other words, this means that

$$\left[\min(1,|y|^2)K_k(y) \text{ converges weakly to } \min(1,|y|^2)K(y)\right] \quad (\text{up to a subsequence})$$

Therefore, for all $x\in\mathbb{R}^n$ we have

$$L_k\eta(x) = \int_{\mathbb{R}^n} \big(\eta(x+y)+\eta(x-y)-2\eta(x)\big)K_k(y)dy = \int_{\mathbb{R}^n} \frac{\eta(x+y)+\eta(x-y)-2\eta(x)}{\min(1,|y|^2)}\cdot \min(1,|y|^2)K_k(y)dy \longrightarrow \int_{\mathbb{R}^n} \frac{\eta(x+y)+\eta(x-y)-2\eta(x)}{\min(1,|y|^2)}\min(1,|y|^2)K(y)dy$$

$$= L\eta(x).$$

Thus, $L_k\eta(x) \to L\eta(x)$ pointwise for all $x \in \mathbb{R}^n$.

Now, since $u_k(x) \to u(x)$ pointwise in $\mathbb{R}^n$, and

$$|u_k(x) L_k\eta(x)| \leq \|u_k\|_{L^\infty(\mathbb{R}^n)} \cdot \int_{\mathbb{R}^n} C \, |\eta(x+y)+\eta(x-y)-2\eta(x)| \cdot \frac{\Lambda}{|y|^{n+2s}}\, dy$$

$$\leq C \cdot \frac{C_\Omega}{1+|x|^{n+2s}}. \quad \text{(since } \eta \text{ has compact support in } \mathbb{R}^n\text{)}$$

by dominated convergence theorem we have

$$\int_{\mathbb{R}^n} u_k(x) L_k\eta(x)\,dx \longrightarrow \int_{\mathbb{R}^n} u(x) L\eta(x)\,dx.$$

Thus, u solves
$$-\int_{\mathbb{R}^n} u L\eta = \int_\Omega f\eta \quad \text{for all } \eta \in C_c^\infty(\Omega),$$

that is,
$$Lu = f \text{ in } \Omega$$ //

<u>Exercise.</u> Show that if η has compact support in $\mathbb{R}^n$ and it is C^2, then
$$|L\eta(x)| \leq \frac{C}{1+|x|^{n+2s}},$$

Interior regularity

- We want to prove the following estimate

$$\|u\|_{C^{2s+\alpha}(B_{1/2})} \leq C\left(\|f\|_{C^{\alpha}(B_1)} + \|u\|_{C^{\alpha}(\mathbb{R}^n)}\right)$$

for kernels
$$0 < \frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}.$$

- To do it, we will prove first the following result.

Proposition. Let $s\in(0,1)$, $\alpha\in(0,1)$ such that $2s+\alpha$ is not integer, and $\nu = \lfloor 2s+\alpha \rfloor$ (integer part).

Let $u\in C_c^\infty(\mathbb{R}^n)$ be a function satisfying $Lu=f$ in B_1,
Then, for any $\delta>0$ there exists C such that
$$[u]_{C^{2s+\alpha}(B_{1/2})} \leq \delta [u]_{C^{2s+\alpha}(\mathbb{R}^n)} + C\left(\|u\|_{C^{\nu}(B_1)} + [f]_{C^{\alpha}(B_1)}\right)$$

- $\boxed{\text{We assume for simplicity that } K \text{ is homogeneous}}$

Proof. Assume by contradiction that there is a sequence $w_k\in C_c^\infty(\mathbb{R}^n)$ such that

$$[w_k]_{C^{2s+\alpha}(B_{1/2})} > \delta [w_k]_{C^{2s+\alpha}(\mathbb{R}^n)} + k\left(\|w_k\|_{C^{\nu}(B_1)} + [f_k]_{C^{\alpha}(B_1)}\right).$$

- Let $x_k, y_k\in B_{1/2}$ such that
$$\frac{1}{2}[w_k]_{C^{2s+\alpha}(B_{1/2})} < \frac{|D^\nu w_k(x_k) - D^\nu w_k(y_k)|}{|x_k-y_k|^{2s+\alpha-\nu}}$$

and define
$$\rho_k = |x_k - y_k|$$

- Notice that
$$\frac{1}{2}[w_k]_{C^{2s+\alpha}(B_{1/2})} < \frac{2\|w_k\|_{C^{\nu}(B_{1/2})}}{\rho_k^{2s+\alpha}} \leq \frac{2}{k}\cdot\frac{[w_k]_{C^{2s+\alpha}(B_{1/2})}}{\rho_k^{2s+\alpha}},$$

and thus $\rho_k \to 0$.

(4.)

- Now, define the blow-up sequence

$$v_k(x) := \frac{w_k(x_k+\rho_k x) - p_k(x)}{\rho_k^{2s+\alpha}\cdot [w_k]_{C^{2s+\alpha}(\mathbb{R}^n)}}$$

where $p_k(x)$ is the Taylor polynomial of order ν, so that

$$v_k(0) = \ldots = |D^\nu v_k(0)| = 0.$$

- Notice that

$$[v_k]_{C^{2s+\alpha}(\mathbb{R}^n)} \leq 1.$$

- Now, for all $x \in B_{1/2\rho_k}$ and $h \in B_1$ we have (note $\nu-1 < 2s\ldots$)

$$|L(v_k(x+h) - v(x))| = \frac{|f_k(x_k+\rho_k x+\rho_k h) - f_k(x_k+\rho_k x)|}{\rho_k^{\alpha}\cdot [w_k]_{C^{2s+\alpha}(\mathbb{R}^n)}} \leq$$

(here we are using the scale invariance of L!)

$$\leq \frac{[f_k]_{C^{\alpha}(B_1)}\cdot |h|^{\alpha}}{[w_k]_{C^{\alpha+2s}(\mathbb{R}^n)}} \leq \frac{C}{k} \to 0 \quad (\text{as } k\to\infty)$$

- On the other hand, let $z_k = \frac{y_k - x_k}{\rho_k} \in \partial B_1$. Then,

$$D^\nu v_k(z_k) = D^\nu v_k(z_k) - D^\nu v_k(0) = \frac{D^\nu w_k(y_k) - D^\nu w_k(x_k)}{\rho_k^{2s+\alpha-\nu}\cdot [w_k]_{C^{2s+\alpha}(\mathbb{R}^n)}} > \frac{\frac{1}{2}[w_k]_{C^{2s+\alpha}(B_{1/2})}}{[w_k]_{C^{2s+\alpha}(\mathbb{R}^n)}} > \frac{\delta}{2}.$$

(By definition of x_k, ρ_k)

- Thus,

$$|D^\nu v_k(z_k)| > \frac{\delta}{2}.$$

- Up to a subsequence, we have $z_k \to z \in \partial B_1$.

- Summarizing, we have: $\quad |L(v_k(x+h)-v_k(x))| \to 0$ (uniformly in compact sets of $\mathbb{R}^n$)

$$|D^\nu v_k(z_k)| > \frac{\delta}{2}, \qquad z_k \in \partial B_1$$

$$v_k(0) = \dots = |D^\nu v_k(0)| = 0 \quad \text{and} \quad [v_k]_{C^{2s+\alpha}(\mathbb{R}^n)} \leq 1.$$

- Up to a subsequence, we will have that $v_k \to v$ uniformly in compact sets, and in $C^\nu_{loc}(\mathbb{R}^n)$, to some v satisfying:

$$|D^\nu v(z)| > \frac{\delta}{2}$$

$$v(0) = \dots = |D^\nu v(0)| = 0 \quad \text{and} \quad [v]_{C^{2s+\alpha}(\mathbb{R}^n)} \leq 1.$$

- Define

$$U(x) := \begin{cases} v(x+h) - v(x) & \text{if } \underline{\nu = 0} \\ v(x+h) + v(x-h) - 2v(x) & \text{if } \underline{\nu = 1} \\ \dots & \text{if } \underline{\nu = 2} \end{cases} \qquad U_k(x) := \begin{cases} v_k(x+h) - v_k(x) & \text{if } \underline{\nu = 0} \\ \dots \\ \dots \end{cases}$$

Then, since $[v]_{C^{2s+\alpha}(\mathbb{R}^n)} \leq 1$ we have that

$$|U| \leq C|h|^{2s+\alpha} \quad \text{in } \mathbb{R}^n$$

in particular, U is <u>bounded</u>, and $\underline{U \in C^{2s+\alpha}(\mathbb{R}^n)}$.

- Moreover, $|U_k| \leq C|h|^{2s+\alpha}$ in $\mathbb{R}^n$ (with C independent of k), and $L U_k = 0$ in $\mathbb{R}^n$.

Since $U_k \to U$ uniformly in compact sets, and in $C^{2s+\alpha'}_{loc}(\mathbb{R}^n)$, for any $\alpha' < \alpha$, then

$$LU(x) = \underbrace{LU_k}_{\downarrow\, 0} + \underbrace{L(U - U_k)}_{\downarrow\, 0} \qquad \text{for every } x \in \mathbb{R}^n.$$

Here, we used that $|L(U-U_k)(0)| \leq C\|U - U_k\|_{C^{2s+\alpha'}(B_1)} + C\|U_k - U\|_{L^1_{\omega}(\mathbb{R}^n)} \to 0$ (dominated convergence)

Thus, $\boxed{LU = 0 \text{ in } \mathbb{R}^n}$. Since U is bounded, $\underline{U \equiv ct}$.

This, ~~~~ combined with $[v]_{C^{2s+\alpha}(\mathbb{R}^n)} \leq 1$ and $v(0) = \dots = |D^\nu v(0)| = 0$ yields $\underline{v \equiv 0 \text{ in } \mathbb{R}^n}$.

And this contradicts $|D^\nu v(z)| > \delta/2$. //

Exercise. Show that if $v \in C(\mathbb{R})$ and $v(x+h)-v(x)$ is constant (in x) for every h, then $v(x)=a\cdot x+b$.
Deduce that if $v \in C^1$ and $v(x+h)+v(x-h)-2v(x)$ is constant (in x) for every h, then $v(x)$ is a 2nd order polynomial

Remark. We only used the homogeneity of the kernel $K(y)$ in the scale invariance of L. If K satisfies

$$\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}$$

then a rescaling $v_r(x)=v(rx)$ would satisfy

$$L_r v_r(x) = r^{2s}(Lv)(rx),$$

with

$$L_r w(x) := \int (\dots)\underbrace{\left(K(ry)\cdot r^{n+2s}\right)}_{K_r(y)} dy$$

with

$$\frac{\lambda}{|y|^{n+2s}} \leq K_r(y) \leq \frac{\Lambda}{|y|^{n+2s}}.$$

- In other words, if v solves an equation $Lv=f$, then the rescaled function v_r solves a similar equation $L_r v_r = f_r$ for an operator L_r of the same form. Using this, the previous proof can be adapted to general operators L with (non-homogeneous) kernels

$$0 < \frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}.$$

Theorem. Let u be any weak solution of $Lu=f$ in B_1. Then,

(a)
$$\|u\|_{C^{\alpha+2s}(B_{1/2})} \leq C\left(\|f\|_{C^{\alpha}(B_1)} + \|u\|_{C^{\alpha}(\mathbb{R}^n)}\right),$$

provided that $\alpha+2s$ is not integer.

(b) If $[K]_{C^{\alpha}(\mathbb{R}^n\setminus B_\rho)} \leq \frac{C}{\rho^{n+2s+\alpha}}$ then $\|u\|_{C^{2s+\alpha}(B_{1/2})} \leq C\left(\|f\|_{C^{\alpha}(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)$

Proof. Assume first that $u\in C_c^\infty(\mathbb{R}^n)$. Then, we know that for any $\delta>0$ we have

$$[u]_{C^{2s+\alpha}(B_{1/2})} \leq \delta[u]_{C^{2s+\alpha}(\mathbb{R}^n)} + C_\delta\left([f]_{C^{\alpha}(B_1)} + \|u\|_{C^{\nu}(B_1)}\right)$$

where $\nu=\lfloor 2s+\alpha\rfloor$.

Let $\eta\in C_c^\infty(B_2)$ such that $\eta\equiv 1$ in $B_{3/2}$, and apply the previous estimate to $u\eta$.

$$[u]_{C^{2s+\alpha}(B_{1/2})} \leq \delta[u\eta]_{C^{2s+\alpha}(B_2)} + C_\delta\left([L(u\eta)]_{C^{\alpha}(B_1)} + \|u\eta\|_{C^{\nu}(B_1)}\right).$$

Now, since

$$L(u\eta) = Lu + L(u-u\eta) = f + L(u-u\eta) \quad \text{in } B_1,$$

and $(u-u\eta)\equiv 0$ in $B_{3/2}$, we have

$$|L(u-u\eta)(x) - L(u-u\eta)(x')| = \left|\int_{\mathbb{R}^n}(u-u\eta)(x+y)K(y)\,dy - \int_{\mathbb{R}^n}(u-u\eta)(x'+y)K(y)\,dy\right|$$

$$\leq \int_{\mathbb{R}^n}\left|(u-u\eta)(x+y)-(u-u\eta)(x'+y)\right|K(y)\,dy \qquad \left(x,x'\in B_1 \rightsquigarrow (u-u\eta)(x+y)=0 \text{ for } y\in B_{1/2}\right)$$

$$\leq \int_{\mathbb{R}^n\setminus B_{1/2}} \overset{[u]_{C^\alpha(\mathbb{R}^n)}}{C}|x-x'|^\alpha K(y)\,dy \leq C|x-x'|^\alpha \int_{\mathbb{R}^n\setminus B_{1/2}}\frac{dy}{|y|^{n+2s}} \leq C|x-x'|^\alpha [u]_{C^\alpha(\mathbb{R}^n)}.$$

- Thus, $[L(u\eta)]_{C^\alpha(B_1)} \leq [f]_{C^\alpha(B_1)} + C[u]_{C^\alpha(\mathbb{R}^n)}$.

- If $[K]_{C^{\alpha}(\mathbb{R}^n\setminus B_\rho)} \leq \frac{C}{\rho^{n+2s+\alpha}}$ then

$(u-u\eta\equiv 0 \text{ in } B_{3/2})$

$$|L(u-u\eta)(x) - L(u-u\eta)(x')| = \left|\int_{\mathbb{R}^n}(u-u\eta)(z)K(x-z)\,dz - \int_{\mathbb{R}^n}(u-u\eta)(z)K(x'-z)\,dz\right| \leq C\int_{\mathbb{R}^n\setminus B_{3/2}}\|u\|_{L^\infty(\mathbb{R}^n)}|K(x-z)-K(x'-z)|\,dz$$

$$\leq C\|u\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus B_{3/2}}\frac{|x-x'|^\alpha}{|z|^{n+2s+\alpha}}\,dz \leq C\|u\|_{L^\infty}|x-$$

· Therefore, we have

$$[u]_{C^{2s+\alpha}(B_{1/2})} \leq \delta [u]_{C^{2s+\alpha}(B_2)} + C_\delta\left([f]_{C^\alpha(B_1)} + [u]_{C^\alpha(\mathbb{R}^n)} + \|u\|_{C^\nu(B_1)} \right).$$

· Now, by interpolation inequalities, we have

$$\|u\|_{C^\nu(B_1)} \leq \delta [u]_{C^{2s+\alpha}(B_1)} + C_\delta \|u\|_{L^\infty(B_1)},$$

and hence

$$[u]_{C^{2s+\alpha}(B_{1/2})} \leq C\delta [u]_{C^{2s+\alpha}(B_2)} + C_\delta\left([f]_{C^\alpha(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^n)} \right). \qquad \circledast$$

· We now want to deduce the desired estimate from this one.
For this, let us introduce the weighted norm

$$[w]^*_{\beta;\Omega} := \sup_{B_{2r}(z)\subset\Omega} \left(r^\beta [w]_{C^\beta(B_r(z))} \right) \qquad (\beta \text{ not integer}).$$

$$\|w\|^*_{\beta;\Omega} := [w]^*_{\beta;\Omega} + \sum_{k=0}^{\lfloor\beta\rfloor} \sup_{x\in\Omega} \left(d^k(x) \cdot |D^k w(x)| \right)$$

Rescaling and translating the estimate $\circledast$ to any ball $B_r(z) \subset B_{4r}(z) \subset B_1$ we get

$$r^{2s+\alpha} [u]_{C^{2s+\alpha}(B_{r/2}(z))} \leq C\delta [u]_{C^{2s+\alpha}(B_{2r}(z))} \cdot r^{2s+\alpha} + C_\delta\left(r^\alpha [f]_{C^\alpha(B_r(z))} + r^\alpha [u]_{C^\alpha(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right)$$

$$\leq C\delta [u]^*_{2s+\alpha;B_1} + C_\delta\left([f]_{C^\alpha(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^n)} \right).$$

(In the last inequality we used $r \leq 1$.)

· Taking the supremum over all balls $B_r(z) \subset B_{4r}(z)$, we get

$$[u]^*_{2s+\alpha;B_1} \leq C\delta [u]^*_{2s+\alpha;B_1} + C_\delta\left([f]_{C^\alpha(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^n)} \right).$$

· Taking $\delta > 0$ small enough so that $C\delta \leq \frac{1}{2}$, we get

$$[u]^*_{2s+\alpha;B_1} \leq C\left([f]_{C^\alpha(B_1)} + \|u\|_{C^\alpha(B_1)} \right).$$

· Since $[u]_{C^{2s+\alpha}(B_{1/2})} \leq [u]^*_{2s+\alpha;B_1}$ then we get the estimate for $u \in C_c^\infty(\mathbb{R}^n)$.

• Finally, if u is any weak solution of $Lu=f$ in B_1, then we consider

$$u_\varepsilon = u * \eta_\varepsilon, \qquad \eta_\varepsilon \in C_c^\infty(B_\varepsilon), \quad \text{smooth mollifier,} \quad \left(\begin{array}{l}\eta_\varepsilon(x)=\eta\left(\frac{x}{\varepsilon}\right), \ \int_{B_1}\eta=1\\ \eta\in C_c^\infty(B_1)\end{array}\right)$$

~~[illegible]~~ since L is translation invariant, then

$$Lu_\varepsilon = f*\eta_\varepsilon := f_\varepsilon \quad \text{in } B_{1-\varepsilon}$$

in the weak sense, (and since $u_\varepsilon \in C^\infty(\mathbb{R}^n)$, also ~~[illegible]~~ pointwise.)

• Thus, we have (for all $\varepsilon\leq 1/2$)

$$\|u_\varepsilon\|_{C^{2s+\alpha}(B_{1/4})} \leq C\left(\|f_\varepsilon\|_{C^\alpha(B_{1/2})} + \|u_\varepsilon\|_{C^\alpha(\mathbb{R}^n)}\right)$$
$$\leq C\left(\|f\|_{C^\alpha(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^n)}\right).$$

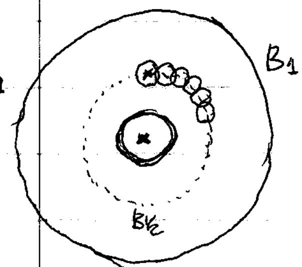
• Letting $\varepsilon\downarrow 0$ we find that $u\in C^{2s+\alpha}(B_{1/4})$ and

$$\|u\|_{C^{2s+\alpha}(B_{1/4})} \leq C\left(\|f\|_{C^\alpha(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^n)}\right).$$

• By a <u>covering argument</u>, we deduce

$$\|u\|_{C^{2s+\alpha}(B_{1/2})} \leq C\left(\|f\|_{C^\alpha(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^n)}\right). \quad //$$

(Cover $B_{1/2}$ with a finite number of balls $B_{r/4}(z)\subset B_r(z)\subset B_1$ and use the estimate in each $B_r(z)$.)



<u>Theorem.</u> Let L be an operator with kernel $\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}$. ~~[illegible]~~, If $Lu=f$ in B_1, then

(a) If $s\neq\frac{1}{2}$ $\quad \|u\|_{C^{2s}(B_{1/2})} \leq C\left(\|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)$

(b) If $s=\frac{1}{2}$, $\quad \|u\|_{C^{2s-\varepsilon}(B_{1/2})} \leq C_\varepsilon\left(\|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)$ for all $\varepsilon>0$.

1

24. Boundary regularity

- Let us now turn our attention to the boundary regularity of solutions.

- For the fractional Laplacian we proved that $u \in C^s(\overline{\Omega})$ $\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$ by constructing appropriate barriers and using interior estimates.

- For general operators L, the situation is very similar; if we can construct appropriate barriers, then the regularity up to the boundary follows by the exact same method.

- It turns out that, if we want a fine description of the boundary regularity of solutions, we need the kernels $K(y)$ to be homogeneous.
- This was not important in the interior regularity, but turns out to be very important in the boundary regularity.

The main difference comes from the following:

1D solution

Proposition. Let $u(x) = (x\cdot e)_+^s$ in $\mathbb{R}^n$, with $e \in S^{n-1}$. Let L be an operator with kernel
$$\frac{\lambda}{|y|^{n+2s}} \le K(y) \le \frac{\Lambda}{|y|^{n+2s}}$$
and such that $K(y)$ is homogeneous.

Then, $\begin{cases} Lu = 0 & \text{in } \{x\cdot e > 0\} \\ u = 0 & \text{in } \{x\cdot e \le 0\} \end{cases}$

Proof. Since $K(y)$ is homogeneous, we can write it as
$$\boxed{K(y) = \frac{a(y/|y|)}{|y|^{n+2s}}}, \quad \text{with } a \in L^\infty(S^{n-1}), \quad 0 < \lambda \le a(\theta) \le \Lambda, \ \forall \theta \in S^{n-1}$$

(2.)

- Now, using polar coordinates,

$$\mathcal{L}u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \{u(x+y)+u(x-y)-2u(x)\}\frac{a(y/|y|)}{|y|^{n+2s}}\,dy = \frac{1}{4}\int_{S^{n-1}}\int_{-\infty}^{\infty}\{u(x+r\theta)+u(x-r\theta)-2u(x)\}\frac{a(\theta)}{|r|^{1+2s}}\,dr\,d\theta =$$

$$= \frac{1}{4}\int_{S^{n-1}}\left(\int_{-\infty}^{\infty}\{u(x+r\theta)+u(x-r\theta)-2u(x)\}\frac{dr}{|r|^{1+2s}}\right)a(\theta)\,d\theta$$

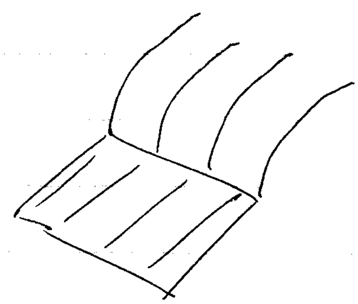
- Now, since $u(x) = (x\cdot e)_+^s$, then

$$u(x+r\theta)+u(x-r\theta)-2u(x) = (x\cdot e + r|\theta\cdot e|)_+^s + (x\cdot e - r|\theta\cdot e|)_+^s - 2(x\cdot e)_+^s,$$

so with the change of variables $\boxed{t = r|\theta\cdot e|}$, $(dt = dr\,|\theta\cdot e|)$

$$\mathcal{L}u(x) = \frac{1}{4}\int_{S^{n-1}}\underbrace{\left(\int_{-\infty}^{\infty}\{(x\cdot e+t)_+^s+(x\cdot e-t)_+^s-2(x\cdot e)_+^s\}\frac{dt}{|t|^{1+2s}}\right)}_{(-\Delta)^s(t_+^s)=0 \text{ in dimension } 1}|\theta\cdot e|^{2s}a(\theta)\,d\theta = 0$$

- Thus, $\mathcal{L}u(x) = 0$ if $x\cdot e > 0$, and we are done//



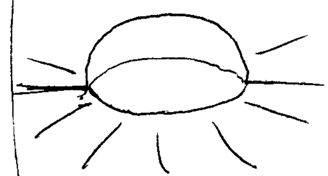
- This means that for homogeneous Kernels the 1D function $(x\cdot e)_+^s$ is a solution, exactly as in case $(-\Delta)^s$.
- Thus, with the exact same proof we find:

Proposition. Assume that the Kernel of $\mathcal{L}$ is homogeneous, and let u solve $\left.\begin{array}{l}\mathcal{L}u = f \text{ in } \Omega\\ u = 0 \text{ in } \mathbb{R}^n\setminus\Omega\end{array}\right\}$.

Then, $$\|u\|_{C^s(\overline{\Omega})} \le C\|f\|_{L^\infty(\Omega)}.$$

- We also have the following:

Lemma. Let L be such that $K(y)$ is homogeneous. Then, $\boxed{u(x) = (1-|x|^2)_+^s}$ solves
$$\begin{cases} -Lu = c_L & \text{in } B_1 \\ u = 0 & \text{in } \mathbb{R}^n \setminus B_1 \end{cases}$$
for a certain constant $c_L > 0$ (which depends on L).



- In particular, we have a Hopf Lemma, exactly as in $(-\Delta)^s$.

~~Non-homogeneous kernels~~

Higher order boundary regularity

- For second order elliptic PDEs, like $\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$, we know that if $f \in C^{\alpha}(\overline{\Omega})$ and $\partial\Omega$ is $C^{2,\alpha}$, then $u \in C^{2+\alpha}(\overline{\Omega})$.
- For integro-differential equations like $Lu = f$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, we have seen that for homogeneous kernels $K(y) = \frac{a(y/|y|)}{|y|^{n+2s}}$, solutions satisfy $u \in C^s(\overline{\Omega})$, and that it is in general optimal.
- Inside Ω, we have $u \in C^{2s+\alpha}$ whenever $f \in C^{\alpha}$, but on the boundary we only get $u \in C^s(\overline{\Omega})$, because u behaves like d^s,
$$u(x) \simeq d^s(x) \text{ near } \partial\Omega.$$

- Question: Can we find a fine expansion for $u(x)$ near $\partial\Omega$?
  - If $f \in C^{\alpha}(\overline{\Omega})$ and $\partial\Omega$ is $C^{2,\alpha}$, we should expect an expansion of order $2s+\alpha$.
  - If $f \in L^{\infty}(\Omega)$ and $\partial\Omega$ is C^2, we should expect an expansion of order $2s-\varepsilon$ (or $2s$)
  - If $f \in C^{\infty}(\overline{\Omega})$ and $\partial\Omega$ is C^{∞}, we should expect an expansion of order ∞.

④

It turns out that: If $K(y)=\frac{a(y/|y|)}{|y|^{n+2s}}$, $\frac{\lambda}{|y|^{n+2s}}\le K(y)\le\frac{\Lambda}{|y|^{n+2s}}$,

and u solves $\left.\begin{array}{l} Lu=f \text{ in } \Omega \\ u=0 \text{ in } \mathbb{R}^n\setminus\Omega \end{array}\right\}$

Then:

(a) If $\underline{\Omega \text{ is } C^2}$ and $\underline{f\in L^\infty(\Omega)}$, then for every $z\in\partial\Omega$ we have

$$\left[u(x)=c_z d^s(x)+O\left(|x-z|^{2s-\epsilon}\right)\right] \quad \forall\epsilon>0$$

(b) If $\underline{\Omega \text{ is } C^{2,\alpha}}$, $\underline{f\in C^\alpha(\overline{\Omega})}$, and $\underline{a\in C^2(S^{n-1})}$, then for every $z\in\partial\Omega$ we have

$$\begin{cases} u(x)=a_z d^s(z)+\vec{b}_z\cdot(x-z)\,d^s(x)+O\left(|x-z|^{2s+\alpha}\right) & \text{if } 2s+\alpha>1+s \\ u(x)=a_z d^s(z)+O\left(|x-z|^{2s+\alpha}\right) & \text{if } 2s+\alpha\le 1+s \end{cases}$$

(c) If $\underline{\Omega \text{ is } C^\infty}$, $\underline{f\in C^\infty(\overline{\Omega})}$, and $\underline{a\in C^\infty(S^{n-1})}$, then for every $z\in\partial\Omega$ we have

$$u(x)=a_z d^s(x)+p_1^z(x)d^s(x)+p_2^z(x)d^s(x)+\dots$$

where $p_k^z(x)$ is a polynomial of degree k.

• By using such expansions, it can be shown the following:

$\underline{\text{Theorem}}$. Let $K(y)=\frac{a(y/|y|)}{|y|^{n+2s}}$ as before, and $\left.\begin{array}{l} Lu=\varphi \text{ in } \Omega \\ u=0 \text{ in } \mathbb{R}^n\setminus\Omega \end{array}\right\}$. Then,

(a) $u/d^s\in C^{s-\epsilon}(\overline{\Omega})$ $\forall\epsilon>0$

(b) $u/d^s\in C^{s+\alpha}(\overline{\Omega})$

(c) $u/d^s\in C^\infty(\overline{\Omega})$

Remark. Notice that $u/d^s \in C^{s-\varepsilon}(\overline{\Omega})$ implies that for every $z\in\partial\Omega$ there exists the limit $\lim_{x\to z} \frac{u(x)}{d^s(x)} = c_z$. Then, we have

$$\left| \frac{u(x)}{d^s(x)} - c_z \right| \le C|x-z|^{s-\varepsilon} \quad \left(\text{by definition of } C^{s-\varepsilon}(\overline{\Omega})\right).$$

Thus, multiplying by $d^s(x)$ we get

$$|u(x) - c_z d^s(x)| \le C d^s(x)\,|x-z|^{s-\varepsilon} \le C|x-z|^{2s-\varepsilon}.$$

Thus, the expansion in (a) is equivalent to $u/d^s \in C^{s-\varepsilon}(\overline{\Omega})$.

Let us give a sketch of the proof of case (a):

Theorem. Assume $K(y) = \frac{a(y/|y|)}{|y|^{n+2s}}$ and $\left.\begin{array}{l} Lu = f \text{ in } \Omega \\ u = 0 \text{ in } \mathbb{R}^n\setminus\Omega \end{array}\right\}$.

Then,

$$\|u/d^s\|_{C^{s-\varepsilon}(\overline{\Omega})} \le C\|f\|_{L^\infty(\Omega)} \qquad \forall \varepsilon>0.$$

Proof (sketch). First of all, one needs to show that d^s satisfies

$$|L(d^s)| \le C_\Omega \quad \text{in } \Omega.$$

This is not easy to prove, requires some fine computations (even for the fractional Laplacian).

- We want to prove that, for each $z\in\partial\Omega$, there is Q_z such that
$$|u(x) - Q_z d^s(x)| \le C|x-z|^{2s-\varepsilon}. \qquad (\circledast)$$

Once this is established, one can prove that it yields $u/d^s \in C^{s-\varepsilon}(\overline{\Omega})$.

- The proof of $\circledast$ is by a blow-up argument. Assume that for some $z\in\partial\Omega$ the expansion $\circledast$ does not hold for any $Q\in\mathbb{R}$. Then,

$$\sup_{r>0} r^{\varepsilon-2s}\|u - Qd^s\|_{L^\infty(B_r(z))} = \infty \quad \text{for all } Q\in\mathbb{R}.$$

- Then, one can show that this yields

$$\sup_{r>0} r^{\varepsilon-2s} \|u - Q(r) d^s\|_{L^\infty(B_r(z))} = \infty \qquad \text{for } Q(r) = \frac{\int_{B_r(z)} u d^s}{\int_{B_r(z)} d^{2s}}.$$

- Notice that this choice of $Q(r)$ is the one that minimizes the L^2 distance between u and Qd^s in $B_r(z)$.

- We now define the blow-up sequence

$$v_r(x) := \frac{u(z+rx) - Q(r) d^s(z+rx)}{\|u - Q(r) d^s\|_{L^\infty(B_r(z))}}$$

which satisfies
$$\|v_r\|_{L^\infty(B_1)} = 1$$

and (by definition of $Q(r)$)
$$\int_{B_1} v_r(x) d^s(z+rx)\,dx = 0$$

(That is, v_r is orthogonal to a rescaling of d^s.)

- Now, moreover, since $\sup_{r>0} r^{\varepsilon-2s} \|u - Q(r) d^s\|_{L^\infty(B_r(z))} = \infty$,

then we can take a sequence $r_k \to 0$ such that

$$\|u - Q(r_k) d^s\|_{L^\infty(B_{r_k}(z))} \geq r_k^{2s-\varepsilon}$$

and

$$\|v_{r_k}\|_{L^\infty(B_R)} \leq CR^{2s-\varepsilon} \quad \text{for } R \geq 1.$$

- Let us denote $v_k = v_{r_k}$.

- Now, let us see what is the equation that v_k satisfies:

$$L v_k(x) = \frac{r_k^{2s}}{\|u - Q(r_k) d^s\|_{L^\infty(B_{r_k}(z))}} \cdot \left\{ Lu(z + r_k x) - Q(r_m) L(d^s)(z + r_k x) \right\}. \quad \text{in } \Omega_k := \frac{1}{r_k}(\Omega - z)$$

- Since Lu and $L(d^s)$ are bounded, and $\|u - Q(r_k) d^s\|_{L^\infty(B_{r_k}(z))} \geq r_k^{2s-\varepsilon}$ then

$$|L v_k| \leq \frac{C r_k^{2s}}{r_k^{2s-\varepsilon}} \leq C r_k^{\varepsilon} \longrightarrow 0 \quad \text{uniformly in compact sets in } \{x \cdot e > 0\}.$$

- Here, we use that Ω_k converges to a half-space $\{x \cdot e > 0\}$ as $k \to \infty$.
- Using the <u>growth condition</u>

$$\|v_k\|_{L^\infty(B_R)} \leq C R^{2s-\varepsilon} \quad \text{for } R \geq 1,$$

we can pass to the limit the equation and get $v_k \to v$ locally uniformly, and

$$\begin{cases} Lv = 0 & \text{in } \{x \cdot e > 0\} \\ v = 0 & \text{in } \{x \cdot e \leq 0\} \end{cases} \qquad \|v\|_{L^\infty(B_R)} \leq C R^{2s-\varepsilon} \quad \text{for } R > 1.$$

- Such solutions can be classified (Liouville type theorem), and

$$v(x) = A (x \cdot e)_+^s \quad \text{for some } A \in \mathbb{R}.$$

- However, we had that

$$\|v_k\|_{L^\infty(B_1)} = 1 \quad \text{and} \quad \int_{B_1} v_k(x) d^s(z + r_k x)\, dx = 0.$$

Passing to the limit we get

$$\|v\|_{L^\infty(B_1)} = 1 \quad \text{and} \quad \int_{B_1} v(x) \cdot (x \cdot e)_+^s\, dx = 0.$$

This is a contradiction with $v(x) = A(x \cdot e)_+^s$, so we are done. //

Comments / Remarks.

- It was very important to chose the right subsequence, so that we have the right growth condition at infinity!

Essentially, from the assumption $\|u - Q(r_k)d^s\|_{L^\infty(B_r)} \gg r_k^{2s-\varepsilon}$ we constructed a subsequence such that "the growth at the origin is transformed into growth at infinity for the blow-up sequence."

- To classify solutions of $\begin{cases} Lv = 0 & \text{in } \{x \cdot e > 0\} \\ v = 0 & \text{in } \{x \cdot e \leq 0\} \end{cases}$ the idea is to differentiate the equation in the directions orthogonal to e, to prove that $v(x)$ must be a 1D solution.

For 1D functions, L is just the fractional Laplacian, and everything reduces to a 1D computation.

⑨

Lemma. Let $\{w_a\}$ be a ~~sequence~~ family of functions such that $\|w_a\|_{L^\infty(\mathbb{R}^n)} \leq C$,

and
$$\sup_{a\in\mathcal{A}} \sup_{r\in(0,1)} \frac{\|w_a\|_{L^\infty(B_r)}}{r^{2s-\varepsilon}} = \infty$$

Then, there is a sequence $r_k \to 0$, $w_k = w_{a_k}$, for which $\|w_k\|_{L^\infty(B_{r_k})} \geq c r_k^{2s-\varepsilon}$ and the rescaled functions
$$v_k(x) := \frac{w_k(r_k x)}{\|w_k\|_{L^\infty(B_{r_k})}}$$
satisfy
$$|v_k(x)| \leq C\left(1+|x|^{2s-\varepsilon}\right) \text{ in } \mathbb{R}^n, \qquad \|v_k\|_{L^\infty(B_1)} = 1.$$

• We will discuss this in more detail later on during the course.

Proof. Let
$$\theta(r) := \sup_{a\in\mathcal{A}} \sup_{\rho\in(r,1)} \frac{\|w_a\|_{L^\infty(B_\rho)}}{\rho^{2s-\varepsilon}} \longrightarrow \infty \quad \text{as } r\to 0$$

Notice that $\theta(r)$ is nonincreasing in r, and $\theta(r) < \infty$ for every $r>0$.

Then, for every $k\in\mathbb{N}$ there is $r_k \geq \frac{1}{k}$ and $a_k \in \mathcal{A}$ such that
$$\frac{\|w_{a_k}\|_{L^\infty(B_{r_k})}}{r_k^{2s-\varepsilon}} \geq \frac{1}{2}\theta\left(\frac{1}{k}\right) \geq \frac{1}{2}\theta(r_k)$$

(Note that since $\|w_a\|_{L^\infty(\mathbb{R}^n)} \leq C$, then $r_k \to 0$.)

Thus, we find
$$\|v_k\|_{L^\infty(B_R)} = \frac{\|w_k\|_{L^\infty(B_{Rr_k})}}{\|w_k\|_{L^\infty(B_{r_k})}} \leq \frac{\theta(r_k R)\,(r_k R)^{2s-\varepsilon}}{\frac{1}{2}(r_k)^{2s-\varepsilon}\theta(r_k)} \leq 2R^{2s-\varepsilon} \quad \text{for } R\geq 1. \quad //$$

a

- We are now done with Chapters 1 and 2 on linear equations.
- We will start now Chapters 3 and 4 on nonlinear equations.

Divergence-form equations:

$$\min_{\substack{u\in H^1(\Omega)\\ u=g \text{ on } \partial\Omega}} \left\{ \int_\Omega L(Du)\,dx \right\}$$

$$\Downarrow$$

$$\operatorname{div}\big((\nabla L)(Du)\big)=0 \text{ in } \Omega$$

or

$$\sum_{i=1}^n \partial_{x_i}\big(L_{p_i}(Du)\big)=0 \text{ in } \Omega$$

Semilinear equations

$$\mathcal{E}(u)=\frac{1}{2}\int_\Omega |\nabla u|^2-\int_\Omega G(u)$$

$$\Downarrow$$

$$-\Delta u=g(u) \text{ in } \Omega$$

(reaction-diffusion eq)

Fully nonlinear equations

$$F(D^2u)=0 \text{ in } \Omega$$

Obstacle problems

$$\min\{-\Delta u,\ u-\varphi\}=0 \text{ in } \Omega$$

- We will study:
  - Ch 3. Fully nonlinear nonlocal equations
  - Ch 4: Obstacle problems for nonlocal operators

- The study of "divergence-form" nonlocal equations, or semilinear equations $(-\Delta)^s u=g(u)$, could be works for presentations of students.
- There are other options for presentations, just let me know what are you interested on.
(parabolic or drift-diffusion equations, obstacle problems, regularity for more general kernels, minimal surfaces, etc