Nonlocal elliptic equations (28 classes, 42h)

0. Overview and preliminaries (2 classes) (13 pages)

1. Local elliptic PDEs [*Overview on: Laplace equation, linear equations (Schauder estimates), nonlinear variational, fully nonlinear, obstacle problem*]
2. PDEs and Random walks [*from discrete to continuous, Brownian motion, expected payoff, expected hitting time, controlled diffusion, optimal stopping*]

1. Fractional Laplacian (classes) (38 pages)

1. Definition and heuristic motivation [*from the long jump random walk, Fourier symbol*]
2. Existence of solutions [*energy functional, weak solutions, fractional Sobolev inequality, integration by parts*]
3. Maximum and comparison principle [*for C^2 solutions, then weak solutions*]
4. Fundamental solution in R^n and Poisson kernel in B_1 [*corollary: smoothness of solutions*]
5. s-mean value property, Harnack inequality [*state: all functions are s-harmonic up to a small error*]
6. Extension problem and 1D solutions (Almgren frequency formula; exercise unique continuation principle)
7. Interior regularity [*interior estimate, L^infty bound*]
8. Boundary regularity [*barriers, C^s up to the boundary in C^2 domains, Hopf Lemma*]

2. Linear integro-differential equations (5 classes) (29 pages)

1. Lévy processes [*definition, Lévy-Khintchine thm, expected payoff/time, classes of kernels*]
2. Weak solutions [*existence of weak sol., Poincare inequality, integration by parts*]
3. Interior regularity [*simple proof of Liouville theorem, C^{2s+alpha} ~~for homogeneous kernels,~~* for $\mathcal{L}_0$ *corollary regular kernels, counterexamples non-regular kernels*] (Schauder estimates $K(x,y)$, case $f\in L^\infty$)
4. Boundary regularity [*remark: homogeneous and even kernels is needed here!, u/d^s in C^2 domains, state: u/d^s in C^infty*] (Prove $L(d^s)\in L^\infty(\Omega)$)

3. Nonlinear equations (6 classes) (25 pages)

1. ~~Types of~~ Fully nonlinear equations ~~[*variational: quasilinear de Giorgi, semilinear Lu=f(u), minimal surfaces; non-variational:*]~~ [*controlled diffusion with Lévy processes and fully nonlinear equations*]
2. Viscosity solutions [*definition, idea of existence*, stability of viscosity solutions]
3. Equations with bounded measurable coefficients [*M^+ and M^-, Harnack inequality via Silvestre easy proof (the two halfs Harnacks)*, Harnack implies C^α]
4. Fully nonlinear equations [*C^{1,alpha} by blow-up, state C^{2s+alpha} for convex eq*]
5. Boundary regularity for fully nonlinear equations [*state bdry regularity, counterexamples for L_0*]

4. Obstacle problems (5 classes) (21 pages)

1. Optimal stopping [*optimal stopping for Lévy processes*]
2. The classical obstacle problem [*prove C^{1,1}, recall bdry Harnack and main results for free bdry*]
3. C^{1,alpha} regularity of solutions [*weak and viscosity solutions, semiconvexity, C^{1,alpha} for L_0*]
4. Boundary Harnack [*easy proof in open sets + iterate*]
5. Regularity of the free boundary [*free bdry is C^{1,alpha}, solutions 1+s, comments on class L_0*]
6. [OPTIONAL] Fractional Laplacian [*thin obstacle problem, state Almgren frequency formula, ~~deduce~~ C^{1+s} estimate, ~~state~~ global structure and regularity for concave obstacles*]

(**5.** More general Kernels: what is similar and what is different
1. Singular Kernels (no Harnack, not known C^α for $M^\pm$)
2. Different scaling assumptions (Kassmann-Mimica))

OPEN PROBLEMS
- Regularity for fully nonlinear with singular Kernels
- Higher order boundary reg. $\leftrightarrow$ C^∞ regularity of free bdry
- Obstacle pb: complete structure of free bdry, Parabolic
- Parabolic obst. pb: $s<\frac12$, $s=\frac12$; more general Kernels

①

Nonlocal Elliptic Equations (M393C)

TTh 2pm-3:30pm
RLM 11.176
54500

Instructor: Xavier Ros-Oton (ros.oton@math.utexas.edu)

Grading: · Those students who want letter-grade will give a presentation by the end of the semester (either individually or in small groups).

· If you want credit/no credit (with no letter grade) then you don't need to give any presentation or do any extra work.

Questions?

0. Overview & Preliminaries (Local equations)

0.1. Local PDEs

· Let us start with second order elliptic equations, with a very short overview on:

- Laplace equation → $-\Delta u = f$
- Linear equations → $\sum_{i,j} a_{ij}(x)\partial_{ij} u + b_i(x)\partial_i u = f(x)$

(equations with variable coefficients)

- Nonlinear variational equations:

minimizers of the functional $I[u] := \int_\Omega G(\nabla u)\,dx$; (with $u = g$ on $\partial\Omega$ for example)

(G strictly convex)

solve the PDE $\mathrm{div}\big((\nabla G)(\nabla u)\big) = 0$ in Ω

(equation in divergence form)

- Fully nonlinear equations: $F(D^2u) = 0$ or $F(D^2u, \nabla u, x) = 0$

(equations in non-divergence form)

- Obstacle problem: $\min(-\Delta u, u - \varphi) = 0$

(free boundary problem)

• During the semester we will see the <u>nonlocal</u> analogues of those second order PDEs, and will prove the main known results on those nonlocal elliptic equations.

• Before that, we will next state the main known results for 2nd order elliptic PDEs:

<u>Laplace operator</u> $\Delta u = \sum_{i=1}^{n} \partial_{ii} u$ is the most simple and canonical example of elliptic operator.

· It is <u>linear</u>, <u>translation invariant</u> (i.e., no x-dependence), and invariant under rotations. (<u>rotationally invariant</u>).

<u>Harmonic functions</u> are solutions to $\Delta u = 0$ in $\Omega \subset \mathbb{R}^n$

· Solutions to $\Delta u = 0$ are C^∞ inside Ω (and analytic...)

· This follows from the mean value property: if u is harmonic, then

$$u(x) = \fint_{\partial B_r(x)} u \qquad \left(\text{for any } r>0 \text{ such that } B_r(x) \subset \Omega\right)$$

<u>Regularity for the Laplacian</u> if $\Delta u = f(x)$ in B_1 then

$$f \in C^{0,\alpha}(B_1) \Rightarrow u \in C^{2,\alpha}(B_{1/2}) \qquad \alpha \in (0,1)$$

· Solutions are 2 derivatives more regular than the right hand side $f(x)$

$\cancel{\left(f \in L^p(B_1) \Rightarrow u \in W^{2,p}(B_{1/2}) \qquad 1<p<\infty\right)}$

· However, the result is false when $\alpha = 0$ ~~(or p=∞)~~! (but it is almost true)

In this case, u is <u>almost</u> 2 derivatives better than $f(x)$:

$$f \in L^\infty(B_1) \Rightarrow u \in C^{1,1-\varepsilon}(B_{1/2}) \qquad (\text{but } \underline{\text{not}}\ C^{1,1} \text{ in general}).$$

<u>Example:</u> $u(x) = (x^2-y^2)\log(x^2+y^2)$ in $\mathbb{R}^2$ solves

$$\Delta u = 2x \cdot \frac{2x}{x^2+y^2} + 2y \cdot \frac{2y}{x^2+y^2} + (x^2-y^2)(\dots) = \frac{8(x^2-y^2)}{x^2+y^2} = f(x,y) \in L^\infty$$

However, $\partial_{xx} u = 2\log(x^2+y^2) + \dots$ UNBOUNDED $\rightsquigarrow \boxed{u \notin C^{1,1}}$

- Thus u is 2 derivatives better than the right hand side, except when α is an integer.

Remark. We will often denote C^{α} the space $C^{0,\alpha}$ or $C^{1,\alpha-1}$, etc, for any $\alpha>0$, non-integer.
For example, $u\in C^{3/2}$ means $u\in C^{1,1/2}$, or $u\in C^{2-\varepsilon}$ means $u\in C^{1,1-\varepsilon}$.

Regularity for linear equations with variable coefficients: Schauder estimates

If
$$\sum_{i,j} a_{ij}(x)\partial_{ij}u = f(x)$$
$$\left(\text{or} \quad tr\left(A(x)D^2u\right) = f(x), \quad \text{with } A(x)=\begin{pmatrix} a_{11}(x) & \cdots & \vdots \\ \vdots & & \vdots \\ \cdots & \cdots & a_{nn}(x)\end{pmatrix}\right)$$

and the operator is uniformly elliptic, $\lambda\,\mathrm{Id} \le A(x) \le \Lambda\,\mathrm{Id}$, $0<\lambda\le\Lambda<\infty$
(the matrix is positive definite, uniformly in x)

then we have the following:

$$\left.\begin{array}{c} f\in C^{\alpha}(B_1) \\ \text{and} \\ a_{ij}\in C^{\alpha}(B_1)\end{array}\right\} \Longrightarrow u\in C^{2+\alpha}(B_{1/2}) \quad \text{for } 0<\alpha<1.$$
(Schauder estimates)

Remark. Any uniformly elliptic equation which is translation invariant is of the form
$$\sum_{i,j} a_{ij}\partial_{ij}u = f(x)$$

- After an affine change of variables $x\mapsto \bar{x}$ the equation becomes
$$\Delta_{\bar{x}} u = \bar{f}(\bar{x}) \quad \text{in the new variables } \bar{x}.$$

- Thus, the study of linear and translation invariant elliptic PDEs reduces to that of the Laplacian.

Regularity for nonlinear variational equations

· Important question: if u is a minimizer of the functional $\int_\Omega G(\nabla u)dx$
(with G smooth and strictly convex),
is it true that $u \in C^\infty$?

- When the functional is $\int_\Omega |\nabla u|^2 dx$ then we get harmonic functions, thus $u \in C^\infty$.

But for more general functionals we get the nonlinear equation

$$(*) \qquad \sum_{i=1}^n \partial_{x_i}\left(G_i(\nabla u)\right) = 0 \quad \text{in } \Omega \quad \text{(equation in divergence form)}$$

- The regularity of u is a much more difficult question in this case, this was Hilbert 19th problem (1900), and was solved by de Giorgi and Nash in 1956.

- To prove the regularity of u, one differentiates $(*)$ with respect to x_k, to get

$$\sum_{i=1}^n \partial_{x_i}\left(G_{ij}(\nabla u)\partial_{jk} u\right) = 0$$

Now, we just look at this as an equation for $v = \partial_k u$,

$$(**) \qquad \sum_{i=1}^n \partial_{x_i}\left(a_{ij}(x)\partial_j v\right) = 0$$

where $a_{ij}(x) = G_{ij}(\nabla u)$. {A priori we don't know any regularity of $a_{ij}(x)$ in x!
But if $u \in C^{1,\alpha}$ then $a_{ij}(x) \in C^{0,\alpha}$, and thus by Schauder estimates we get $v \in C^{2,\alpha}$, so that $u \in C^{3,\alpha}$.
Iterating in this way, we see that $u \in C^{1,\alpha} \Rightarrow u \in C^\infty$.

- The important contribution of de Giorgi and Nash was to prove that u is $C^{1,\alpha}$, by looking at the equation $(**)$, with no regularity assumption on x.

This is called an equation with bounded measurable coefficients.

(The regularity for this equation can not be deduced from that of the Laplacian.)

Regularity for fully nonlinear equations

$F(D^2u) = 0$ in Ω,
(with F uniformly elliptic and smooth.)

Examples: $F(D^2u) = \sup_{\gamma\in\Gamma}\left(\sum_{ij} a_{ij}^{(\gamma)}\partial_{ij}u\right)$, with $\left(a_{ij}^{(\gamma)}\right)$ uniformly elliptic

$$F(D^2u) = \sup\left(\partial_{11}u + \partial_{22}u,\ 3\partial_{11}u + 2\partial_{22}u\right)$$

Other examples: $\det(D^2u) = 1$, with u strictly convex.

- Uniform ellipticity: The linearized operator is uniformly elliptic if $F(M)$ satisfies

$$\lambda\,\mathrm{Id} \leq \frac{\partial F}{\partial M_{ij}} \leq \Lambda\,\mathrm{Id}$$

~~where $F_{ij}(M)$ is the derivative of $F(M)$ with respect to the component m_{ij}.~~

- Equivalently, $F(D^2u)$ can be written as

$$F(D^2u) = \inf_{\beta\in B}\sup_{\gamma\in\Gamma}\left(\sum_{ij} a_{ij}^{(\beta,\gamma)}\partial_{ij}u + c_{\beta,\gamma}\right)$$

- Regularity: If we differentiate the equation $F(D^2u)=0$ with respect to x_k, we get

$$(*)\quad F_{ij}(D^2u)\,\partial_{ijk}u = 0$$

If we denote $v = \partial_k u$, and $a_{ij}(x) = F_{ij}(D^2u(x))$, this equation is

$$(**)\quad a_{ij}(x)\,\partial_{ij}v = 0$$

Now, if $u\in C^{2,\alpha}$ then $a_{ij}(x)$ are $C^{0,\alpha}$, and thus by Schauder estimates and $(**)$, $v\in C^{2,\alpha}$, so that $u\in C^{3,\alpha}$. Iterating this (whenever F is C^∞), we get $u\in C^\infty$.

- The important and difficult problem is: is u always $C^{2,\alpha}$?

  The answer is in general, no.

  - For any uniformly elliptic $F(D^2u)=0$, we have $u \in C^{1,\alpha}$ (Krylov-Safonov 1979)
  - If F is in addition convex, then $u \in C^{2,\alpha}$ (Evans and Krylov 1982)
  - In dimension $n=2$, we always have $u \in C^{2,\alpha}$ (Nirenberg 1953)
  - In dimensions $n \geq 5$, there are solutions u that are not $C^{1,1}$ (Nadirashvili – Vladut 2008-2012)

Weak and viscosity solutions

- Usually, the existence of solution for a PDE is proved in a generalized class of functions, and then by regularity theory we end up showing that such solution is in fact more regular.
- During the course we will focus on the regularity theory, rather than on the existence of solutions.
- Still, it is important to have in mind the following:

  - Weak solution. u is a weak solution of $\displaystyle\sum_{i=1}^{n} \partial_{x_i}\big(G_i(\nabla u)\big) = 0$ in Ω

  if $u \in H^1(\Omega)$ and $\displaystyle\int_\Omega G_i(\nabla u)\cdot \partial_{x_i} w \, dx = 0$ for all $w \in C_c^\infty(\Omega)$.

  - In case of the Laplace equation $\Delta u = 0$, this reduces to

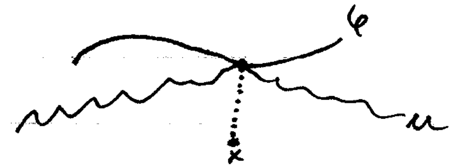
$$\int_\Omega \nabla u \cdot \nabla w \, dx = 0 \quad \text{for all } w \in C_c^\infty(\Omega).$$

  - Classical solution. $u \in C^2(\Omega)$ and solves the equation pointwise.

- <u>Viscosity solutions</u> $u \in C(\Omega)$ satisfies $F(D^2u) \geq 0$ in Ω in the viscosity sense if for every function $\varphi \in C^2(\Omega)$ such that $\varphi(x) = u(x)$ and $\varphi \geq u$ in Ω, then $F(D^2\varphi(x)) \geq 0$.

The definition of $F(D^2u) \leq 0$ in the viscosity sense is analogous.

u solves $F(D^2u) = 0$ when it satisfies both $F(D^2u) \geq 0$ and $F(D^2u) \leq 0$ in the viscosity sense.



[see Lecture notes on Viscosity solutions at the webpage of Luis Silvestre.]

Variational equations (divergence-form) $\rightsquigarrow$ weak solutions

Fully nonlinear eq. (non-divergence form) $\rightsquigarrow$ viscosity solutions

(Laplace eq. $\rightsquigarrow$ any of the two)

<u>Obstacle problem</u>

- We will talk about it later during the course.
- To prove existence of solutions, either weak or viscosity solutions.
- It is similar to a fully nonlinear eq.

$$\left[\min(-\Delta u,\ u - \varphi) = 0\right]$$

- Optimal regularity: $u \in C^{1,1}$.

M 393C: Nonlocal elliptic equations

(Unique number: 54500)

INSTRUCTOR **Xavier Ros-Oton**
Lectures: TTh 2pm - 3:30pm in RLM 11.176
Office Location: RLM 11.108
Email: ros.oton@math.utexas.edu

TEXTBOOK No required textbook, we will use some articles.

GRADING INFORMATION Evaluation will be by student presentation.

COURSE TOPICS

0. Overview and Preliminaries
0.1 (Local) elliptic PDEs
0.2 PDEs and random walks

1. Fractional Laplacian
1.1 Basics (maximum principle, Fourier symbol, etc.)
1.2 Mean value properties, Harnack inequality
1.3 Extension problem
1.4 Regularity

2. Linear nonlocal equations
2.1 Lévy processes
2.2 Weak and viscosity solutions
2.3 The Dirichlet problem
2.4 Regularity

3. Nonlinear equations
3.1 Equations with bounded measurable coefficients
3.2 Regularity results for fully nonlinear equations

4. Obstacle problem
4.1 The classical obstacle problem
4.2 Semiconvexity and $C^{1,\alpha}$ regularity of solutions
4.3 Boundary Harnack
4.3 Optimal regularity of solutions and regularity of free boundaries

GENERAL INFORMATION

- The University of Texas at Austin provides upon request appropriate academic accommodations for qualified students with disabilities. For more information, contact Services for Students with Disabilities: http://ddce.utexas.edu/disability/, 471-6259, 471-6441 TTY.

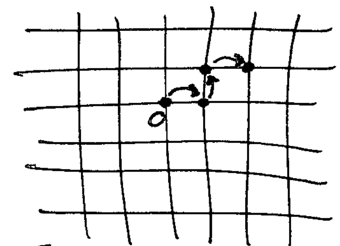
(0.)

0.2 PDEs and random walks

From discrete to continuous time random walks

- Let us consider a particle starting at the origin in $\mathbb{R}^2$, and such that after a time step $\tau>0$ it jumps a distance $h>0$ to the right/left/up/down.

- We denote $u(x,t)$ the probability that the particle is at position $x\in h\mathbb{Z}^2$ at time $t\in \tau\mathbb{N}$.



[Discrete random walk]

- Then, we have: $\quad u(0,0)=1,\quad u(x,0)=0$ for $x\neq 0$

$$\left[\begin{aligned} u(x,t+\tau) = {} & \tfrac{1}{4}u(x+he_1,t)+\tfrac{1}{4}u(x-he_1,t)+ \\ & +\tfrac{1}{4}u(x+he_2,t)+\tfrac{1}{4}u(x-he_2,t)\end{aligned}\right] \circledast$$

where $e_1=(1,0)$ and $e_2=(0,1)$.

- We now let h and τ go to zero, and obtain a stochastic process:

subtract $u(x,t)$ in to get

$$u(x,t+\tau)-u(x,t) = \frac{1}{4}\Big(u(x+he_1,t)+u(x-he_1,t)-2u(x,t)\Big)+\frac{1}{4}\Big(u(x+he_2,t)+u(x-he_2,t)-2u(x,t)\Big)$$

divide by h^2 and choose $\boxed{\tau=2h^2}$

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \frac{u(x+he_1,t)+u(x-he_1,t)-2u(x,t)}{2h^2}+\frac{\cdots}{2h^2}$$

Now take h and $\tau\to 0$, to get

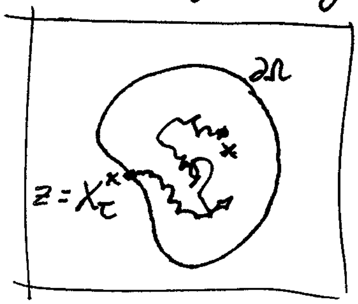
$$\partial_t u = \partial_{x_1x_1}u+\partial_{x_2x_2}u \qquad \text{i.e.} \qquad \left[\partial_t u-\Delta u=0 \text{ in } \mathbb{R}^2\right]$$

- Moreover, $u(x,0)$ is the Dirac delta function at 0.

- <u>Brownian motion</u>: stochastic process X_t, $t \geq 0$

  - continuous paths
  - no memory
  - stationary increments ($X_t - X_s$ is equal in distribution to X_{t-s})
  - $X_0 = 0$
  - rotationally symmetric (in distribution)

- <u>Expected payoff</u>. Given a domain $\Omega \subset \mathbb{R}^n$, and a Brownian motion X_t^x starting at $x \in \Omega$, when we first hit the boundary $\partial\Omega$ we get a <u>payoff</u> $g(z)$, which depends on the point $z \in \partial\Omega$.



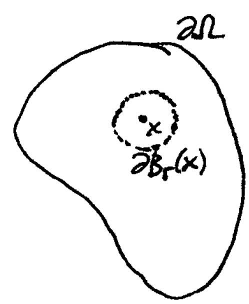
<u>Question</u>: What is the <u>expected payoff</u> we will get?

To answer this question, we define $[\tau = \text{first hitting time of } X_t^x]$

$$\boxed{u(x) = \mathbb{E}\left[g\left(X_\tau^x\right)\right]} \quad \text{(value function)}$$

- To <u>find $u(x)$</u>, we try to relate it with values of $u(y)$ for $y \neq x$. This will yield <u>a PDE for u</u>, and by solving it we find $u(x)$.

- Indeed, let us consider a ball $B_r(x) \subset \Omega$, with $r > 0$. For any such ball, we know that the process X_t^x will hit any point on $\partial B_r(x)$ with the same probability (the process is rotationally symmetric!).



- Since the process has no memory, this means that

$$u(x) = \int_{y \in \partial B_r(x)} u(y)\,dy$$

- This means that u satisfies the <u>mean value property</u>, and thus u is harmonic, $\Delta u = 0$ in Ω.

- Since we also know that $u = g$ on $\partial\Omega$, then u is the unique solution to

$$\boxed{\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}}$$

- Once we know u, we can find $u(x)$ for any x we want.

- Expected hitting time

We are given a domain $\Omega \subset \mathbb{R}^n$, and a Brownian motion X_t^x, $t \geq 0$.

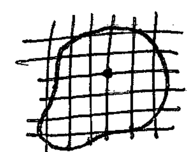
<u>Question:</u> What is the expected hitting time? (first time X_t^x will hit the boundary $\partial\Omega$)

- With an argument similar to the previous one, it is not difficult to see that

$$u(x) = \mathbb{E}[\tau] \quad \text{(where } \tau \text{ is the first exit time)}$$

satisfies

$$\boxed{\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}}$$



$\partial\Omega$

$u(x, t+\tau) = \tau + \frac{1}{4} u(x+he_1, t) + \cdots$

- Running costs

If, instead of measuring time, we have a running cost (a price we pay for passing through points in Ω), then

$u(x) =$ expected total cost

satisfies

$$\boxed{\begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}}$$

The same problem appears when considering expected hitting time in a <u>non-homogeneous</u> medium.

- More general diffusions

If, instead of considering the Brownian motion, we consider more general diffusions, (non rotationally symmetric) then we are led to linear operators of the form

$$Lu = \sum_{i,j} a_{ij} \partial_{ij} u$$

or even with x-dependence

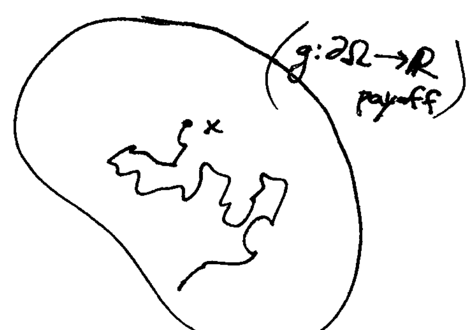
$$Lu = \sum_{i,j} a_{ij}(x) \partial_{ij} u$$

- Controlled diffusions

Say now that we have a set of different diffusions, each one corresponding to an operator

$$L_\gamma u = \sum_{i,j} a^{ij}_\gamma \partial_{ij} u, \qquad \text{with } \gamma \in \Gamma$$

We can control the parameter $\gamma \in \Gamma$ in order to maximize the expected payoff. This means that at every moment we can choose the diffusion parameter γ that is more convenient for us (in order to maximize the expected payoff.



It turns out that

$$u(x) = \text{maximal possible expected value of } g \text{ the first time } X_t \text{ exits } \Omega.$$

solves the fully nonlinear equation

$$\begin{cases} \sup_{\gamma \in \Gamma} L_\gamma u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

- Notice that
$$F(M) = \sup_{\gamma\in\Gamma}\left(\sum_{i,j} a^{ij}_{\gamma} m_{ij}\right) \qquad \left(M=(m_{ij}),\ \text{symmetric}\right)$$
is a <u>convex</u> function (it is the sup of linear functions).

- Thus, the equation
$$\left[F(D^2u) = \sup_{\gamma\in\Gamma} L_\gamma u = 0\right]$$
is a fully nonlinear elliptic equation, with F convex.

- This equation is known as the Bellman equation.

- Similar considerations in the context of two-player zero sum games lead to the Isaacs equation
$$\left[\inf_{\beta\in B}\sup_{\gamma\in\Gamma} L_{\beta\gamma} u = 0\right]$$

- This is a fully nonlinear equation as well, with
$$F(M) = \inf_{\beta\in B}\sup_{\gamma\in\Gamma}\left(\sum_{i,j} a^{ij}_{\beta\gamma} m_{ij}\right).$$

In this case, F is <u>not convex</u>.

<u>Linear equations</u>

Expected payoff $\rightsquigarrow$ $\begin{cases}\Delta u = 0 & \text{in } \Omega\\ u = g & \text{on } \partial\Omega\end{cases}$

Expected time/running costs $\rightsquigarrow$ $\begin{cases}-\Delta u = f & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega\end{cases}$

- When the process is Brownian motion, then $L=\Delta$, the Laplacian.

<u>Nonlinear equations: stochastic control</u>

Controlled diffusion $\rightsquigarrow$ $\sup_{\gamma\in\Gamma} L_\gamma u = 0$

Stochastic games $\rightsquigarrow$ $\inf_{\beta\in B}\sup_{\gamma\in\Gamma} L_{\beta\gamma} u = 0$

(fully nonlin)

Optimal stopping $\longrightarrow$ $\min(-Lu,\ u-\varphi) = 0$ (obstacle problem)

- Optimal stopping

· We have a stochastic process X_t in $\mathbb{R}^n$, and we have the choice of stopping it at any time τ. When we stop, we are given a payoff $\varphi(X_\tau)$. The problem is to choose the optimal stopping time that maximizes the value of the expected payoff.

·
$$u(x) = \sup_{\substack{\text{all stopping}\\ \text{times } \tau}} \mathbb{E}\left[\varphi(X_\tau^x)\right]$$

· The space $\mathbb{R}^n$ will be divided into two zones: $\{u=\varphi\}$ (where it is better to stop) and $\{u>\varphi\}$ (where it is better not to stop).

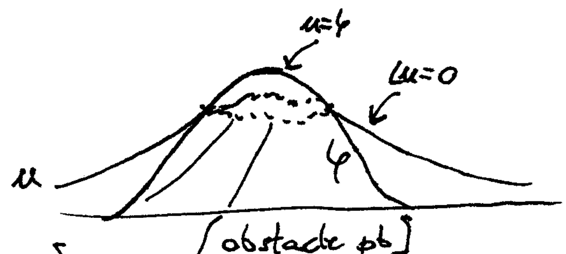
· At those points where it is better to continue, the function u will satisfy the PDE $Lu=0$, that is,

$$Lu=0 \text{ in } \{u>\varphi\}.$$

· At those points where it is better to stop, we have $u=\varphi$. Moreover, if it is better to stop it is because continuing would not improve the expected payoff, therefore $Lu \leq 0$ at those points.

· Summarizing,

$$\begin{cases} u \geq \varphi & \text{in } \mathbb{R}^n \\ Lu = 0 & \text{in } \{u>\varphi\} \\ -Lu \geq 0 & \text{in } \mathbb{R}^n \end{cases}$$



and thus

$$\boxed{\min(-Lu,\ u-\varphi) = 0 \text{ in } \mathbb{R}^n}$$

Summarizing:

Minimization of energies/functionals	⟿	divergence-form PDEs (weak solutions)
Probability/ stochastic processes	⟿	non-divergence form PDEs (viscosity solutions)

· Some equations (like $\Delta u=0$) have both interpretations.