Statistics of the Zeta zeros: Mesoscopic and macroscopic phenomena

Brad Rodgers

Department of Mathematics
UCLA

Analysis/Number Theory Seminar
IAS Spring 2013

The Riemann Zeta function

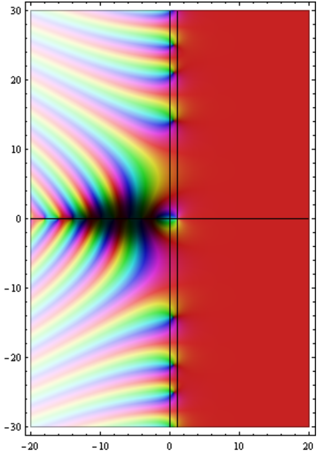

Figure: $\zeta(s)$. Hue is argument, brightness is modulus. Made with Jan Homann's ComplexGraph Mathematica code.

- Non-trivial zeros: those with real part in $(0, 1)$.
- First few: $\frac{1}{2} + i14.13$, $\frac{1}{2} + i21.02$, $\frac{1}{2} + i25.01$.
- We assume the Riemann Hypothesis in what follows: all nontrivial zeros have the form $\frac{1}{2} + i\gamma$, for $\gamma \in \mathbb{R}$.

The Riemann Zeta function

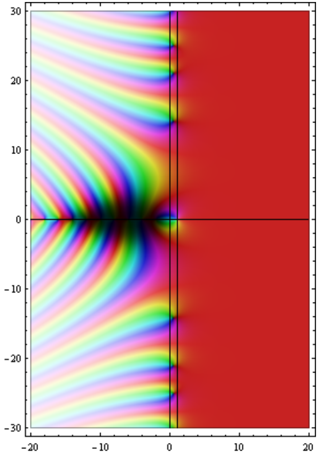

Figure: $\zeta(s)$. Hue is argument, brightness is modulus. Made with Jan Homann's ComplexGraph Mathematica code.

- Non-trivial zeros: those with real part in $(0, 1)$.
- First few: $\frac{1}{2} + i14.13$, $\frac{1}{2} + i21.02$, $\frac{1}{2} + i25.01$.
- We assume the Riemann Hypothesis in what follows: all nontrivial zeros have the form $\frac{1}{2} + i\gamma$, for $\gamma \in \mathbb{R}$.
- Around height T, zeros have density roughly $\log T/2\pi$. More precisely:

Theorem (Riemann - von Mangoldt)

$$N(T) = \#\{\gamma \in (0, T),\ \zeta(\tfrac{1}{2} + i\gamma) = 0\}$$
$$= \tfrac{T}{2\pi} \log(\tfrac{T}{2\pi}) - \tfrac{T}{2\pi} + O(\log T)$$

1 and 2-level density

- **1-level density:** For large random $s \in [T, 2T]$ and dx small,

$$\mathbb{P}\Big(\text{one } \gamma \in \big[s, s + \tfrac{2\pi\, dx}{\log T}\big]\Big) \sim dx$$

1 and 2-level density

- **1-level density:** For large random $s \in [T, 2T]$ and dx small,

$$\mathbb{P}\Big(\text{one } \gamma \in \big[s, s + \tfrac{2\pi\, dx}{\log T}\big]\Big) \sim dx$$

Since s is random, for fixed x, we can translate by $\frac{2\pi x}{\log T}$ and have the same statement

$$\mathbb{P}\big(\text{one } \gamma \in s + \tfrac{2\pi}{\log T}[x, x + dx]\big) \sim dx$$

1 and 2-level density

- **1-level density:** For large random $s \in [T, 2T]$ and dx small,

$$\mathbb{P}\Big(\text{one } \gamma \in \big[s, s + \tfrac{2\pi\, dx}{\log T}\big]\Big) \sim dx$$

Since s is random, for fixed x, we can translate by $\frac{2\pi x}{\log T}$ and have the same statement

$$\mathbb{P}\big(\text{one } \gamma \in s + \tfrac{2\pi}{\log T}[x, x + dx]\big) \sim dx$$

- **2-level density (pair correlation):** Does the presence of one zero in a location affect the likelihood of other zeros being nearby?

Conjecture:

$$\mathbb{P}\big(\text{one } \gamma \in s + \tfrac{2\pi}{\log T}[x, x + dx],\ \text{one } \gamma' \in s + \tfrac{2\pi}{\log T}[y, y + dy]\big)$$
$$\sim \Big(1 - \big(\tfrac{\sin \pi(x-y)}{\pi(x-y)}\big)^2\Big)\, dx\, dy$$

1 and 2-level density

- **1-level density:** For large random $s \in [T, 2T]$ and dx small,

$$\mathbb{P}\Big(\text{one } \gamma \in \big[s, s + \tfrac{2\pi\, dx}{\log T}\big]\Big) \sim dx$$

Since s is random, for fixed x, we can translate by $\frac{2\pi x}{\log T}$ and have the same statement

$$\mathbb{P}\big(\text{one } \gamma \in s + \tfrac{2\pi}{\log T}[x, x + dx]\big) \sim dx$$

- **2-level density (pair correlation):** Does the presence of one zero in a location affect the likelihood of other zeros being nearby?

Conjecture:

$$\mathbb{P}\big(\text{one } \gamma \in s + \tfrac{2\pi}{\log T}[x, x + dx],\ \text{one } \gamma' \in s + \tfrac{2\pi}{\log T}[y, y + dy]\big)$$
$$\sim \Big(1 - \big(\tfrac{\sin \pi(x-y)}{\pi(x-y)}\big)^2\Big)\, dx\, dy$$

- $1 - \big(\frac{\sin \pi(x-y)}{\pi(x-y)}\big)^2 \approx 0$ when $x \approx y$, so very low likelihood of two zeros being much nearer than average.
- Compare probability $dx\, dy$ for poisson process.

A histogram of the pair correlation conjecture

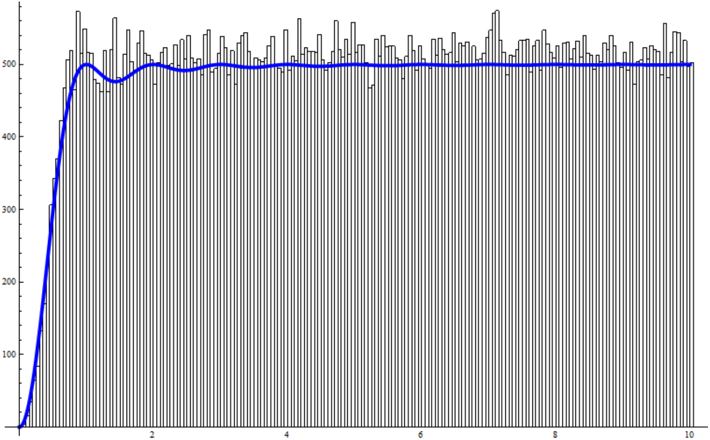

Figure: A histogram of $\frac{\log T}{2\pi}(\gamma - \gamma')$ for the first 10000 zeros, in intervals of size .05, compared to the appropriately scaled prediction $1 - (\frac{\sin \pi x}{\pi x})^2$.

k-level density

- **k-level density:** Conjecture:

$$\mathbb{P}\Big(\text{one } \gamma_1 \in s + \tfrac{2\pi}{\log T}[x_1, x_1 + dx_1],\ \text{one } \gamma_2 \in s + \tfrac{2\pi}{\log T}[x_2, x_2 + dx_2], \ldots,\ \text{one } \gamma_k \in s + \tfrac{2\pi}{\log T}[x_k, x_k + dx_k]\Big)$$

$$\sim \det \begin{pmatrix} 1 & S(x_1 - x_2) & \cdots & S(x_1 - x_k) \\ S(x_2 - x_1) & 1 & \cdots & S(x_2 - x_k) \\ \vdots & \vdots & \ddots & \vdots \\ S(x_k - x_1) & S(x_k - x_2) & \cdots & 1 \end{pmatrix} dx_1\, dx_2 \cdots dx_k$$

where $S(x) = \frac{\sin \pi x}{\pi x}$.

k-level density

- **k-level density:** Conjecture:

$$\mathbb{P}\Big(\text{one } \gamma_1 \in s + \tfrac{2\pi}{\log T}[x_1, x_1 + dx_1],\ \text{one } \gamma_2 \in s + \tfrac{2\pi}{\log T}[x_2, x_2 + dx_2], \\ \dots,\ \text{one } \gamma_k \in s + \tfrac{2\pi}{\log T}[x_k, x_k + dx_k]\Big)$$

$$\sim \det \begin{pmatrix} 1 & S(x_1 - x_2) & \cdots & S(x_1 - x_k) \\ S(x_2 - x_1) & 1 & \cdots & S(x_2 - x_k) \\ \vdots & \vdots & \ddots & \vdots \\ S(x_k - x_1) & S(x_k - x_2) & \cdots & 1 \end{pmatrix} dx_1\, dx_2 \cdots dx_k$$

where $S(x) = \frac{\sin \pi x}{\pi x}$.

- This is the same probability as

$$\mathbb{P}\Big(\tfrac{\log T}{2\pi}(\gamma_1 - s) \in [x_1, x_1 + dx_1], \dots, \tfrac{\log T}{2\pi}(\gamma_k - s) \in [x_k, x_k + dx_k]\Big)$$

A more formal statement and a comparison with the unitary group

More formally:

Conjecture (GUE)

For fixed k and fixed η (Schwartz, say)

$$\frac{1}{T}\int_T^{2T} \sum_{\substack{\gamma_1,\ldots,\gamma_k \\ \text{distinct}}} \eta\Big(\tfrac{\log T}{2\pi}(\gamma_1 - s), \ldots, \tfrac{\log T}{2\pi}(\gamma_k - s)\Big)\, ds \sim \int_{\mathbb{R}^k} \eta(x) \det_{k\times k}\big(S(x_i - x_j)\big)\, d^k x$$

This is known to be the case for unitary matrices. Let $\mathcal{U}(N)$ be the Haar-probability space of $N\times N$ random unitary matrices g, and label g's eigenvalues $\{e^{i2\pi\theta_1}, \ldots, e^{i2\pi\theta_N}\}$ with $\theta_j \in [-1/2, 1/2)$ for all j.

Theorem (Dyson-Weyl)

For fixed k and η,

$$\int_{\mathcal{U}(N)} \sum_{\substack{i_1,\ldots,i_k \\ \text{distinct}}} \eta(N\theta_{i_1}, \ldots, N\theta_{i_k})\, dg \sim \int_{\mathbb{R}^k} \eta(x) \det_{k\times k}\big(S(x_i - x_j)\big)\, d^k x$$

GUE restated

- The GUE conjecture implies: Given any fixed interval J, the random variable
$$\#_J\Big(\{\tfrac{\log T}{2\pi}(\gamma - s)\}\Big) \quad s \in [T, 2T]$$
and the random variable
$$\#_J\Big(\{N\theta_j\}\Big) \quad g \in \mathcal{U}(N)$$
tend in distribution as $T, N \to \infty$ to the same random variable.

GUE restated

- The GUE conjecture implies: Given any fixed interval J, the random variable
$$\#_J\Big(\{\tfrac{\log T}{2\pi}(\gamma - s)\}\Big) \quad s \in [T, 2T]$$
and the random variable
$$\#_J\Big(\{N\theta_j\}\Big) \quad g \in \mathcal{U}(N)$$
tend in distribution as $T, N \to \infty$ to the same random variable.

- The GUE conjecture is *equivalent* to: For any fixed test function η, piecewise continuous and with compact support, the random variable
$$\sum_\gamma \eta\Big(\tfrac{\log T}{2\pi}(\gamma - s)\Big)$$
and the random variable
$$\sum_j \eta(N\theta_j)$$
tend in distribution as $T, N \to \infty$ to the same random variable.

Rigorous evidence for GUE

For certain band-limited test functions, the GUE conjecture is known (on RH) to be true.

Theorem (Mongtomery, Hejhal, Rudnick-Sarnak)

For fixed k and η with $\mathrm{supp}\ \hat{\eta} \in \{y : |y_1| + \cdots + |y_k| < 2\}$

$$\frac{1}{T}\int_T^{2T} \sum_{\substack{\gamma_1,\ldots,\gamma_k \\ \text{distinct}}} \eta\big(\tfrac{\log T}{2\pi}(\gamma_1 - s), \ldots, \tfrac{\log T}{2\pi}(\gamma_k - s)\big)\, ds \sim \int_{\mathbb{R}^k} \eta(x) \det_{k\times k}\big(S(x_i - x_j)\big)\, d^k x$$

The statistics we have been talking about concern only zeros at height T separated by $O(1/\log T)$. We call such statistics "microscopic."

The statistics we have been talking about concern only zeros at height T separated by $O(1/\log T)$. We call such statistics "microscopic."

If we limit our knowledge to what I have so far talked about, we suffer two restrictions:

(1) We can't say anything rigorous about the distribution of zeros when we 'count' with test functions that are too oscillatory (too narrowly concentrated, that is, by the uncertainty principle) at the microscopic level.

(2) We can't say anything about the distribution of zeros when counted by test functions that are not essentially supported at the microscopic level. We can't say anything, for instance, about the effect the position of a zero will have on the statistics of a zero a distance of 1 away.

The statistics we have been talking about concern only zeros at height T separated by $O(1/\log T)$. We call such statistics "microscopic."

If we limit our knowledge to what I have so far talked about, we suffer two restrictions:

(1) We can't say anything rigorous about the distribution of zeros when we 'count' with test functions that are too oscillatory (too narrowly concentrated, that is, by the uncertainty principle) at the microscopic level.

(2) We can't say anything about the distribution of zeros when counted by test functions that are not essentially supported at the microscopic level. We can't say anything, for instance, about the effect the position of a zero will have on the statistics of a zero a distance of 1 away.

Philosophy: (1) is a serious obstruction to our knowledge of zeta statistics, (2) is not. Any question that can be asked about zeta zeros, provided answering it does not require counting with functions that are "too oscillatory" in the microscopic regime, can be rigorous answered.

Mesoscopic collections of zeros

Theorem (Fujii)

Let $n(T)$ be a function $\to \infty$ as $T \to \infty$ but so that $n(T) = o(\log T)$, and let s be random and uniformly distributed on $[T, 2T]$. Let $J_T = [-n(T)/2, n(T)/2]$, and define

$$\begin{aligned}\Delta_T &= \#_{J_T}(\{\tfrac{\log T}{2\pi}(\gamma - s)\}) \\ &= N(s + \tfrac{2\pi}{\log T} \cdot \tfrac{n(T)}{2}) - N(s - \tfrac{2\pi}{\log T} \cdot \tfrac{n(T)}{2})\end{aligned}$$

Mesoscopic collections of zeros

Theorem (Fujii)

Let $n(T)$ be a function $\to \infty$ as $T \to \infty$ but so that $n(T) = o(\log T)$, and let s be random and uniformly distributed on $[T, 2T]$. Let $J_T = [-n(T)/2, n(T)/2]$, and define

$$\begin{aligned}\Delta_T &= \#_{J_T}(\{\tfrac{\log T}{2\pi}(\gamma - s)\}) \\ &= N(s + \tfrac{2\pi}{\log T} \cdot \tfrac{n(T)}{2}) - N(s - \tfrac{2\pi}{\log T} \cdot \tfrac{n(T)}{2})\end{aligned}$$

we have

$$\mathbb{E}\,\Delta_T = n(T) + o(1)$$

$$\mathrm{Var}\,\Delta_T := \mathbb{E}(\Delta - \mathbb{E}\Delta)^2 \sim \frac{1}{\pi^2}\log n(T)$$

and in distribution

$$\frac{\Delta_T - \mathbb{E}\,\Delta_T}{\sqrt{\mathrm{Var}\,\Delta_T}} \Rightarrow N(0,1)$$

as $T \to \infty$.

Mesoscopic collections of zeros

Theorem (Fujii)

Let $n(T)$ be a function $\to \infty$ as $T \to \infty$ but so that $n(T) = o(\log T)$, and let s be random and uniformly distributed on $[T, 2T]$. Let $J_T = [-n(T)/2, n(T)/2]$, and define

$$\begin{aligned} \Delta_T &= \#_{J_T}(\{\tfrac{\log T}{2\pi}(\gamma - s)\}) \\ &= N(s + \tfrac{2\pi}{\log T} \cdot \tfrac{n(T)}{2}) - N(s - \tfrac{2\pi}{\log T} \cdot \tfrac{n(T)}{2}) \end{aligned}$$

we have

$$\mathbb{E}\,\Delta_T = n(T) + o(1)$$

$$\mathrm{Var}\,\Delta_T := \mathbb{E}(\Delta - \mathbb{E}\Delta)^2 \sim \frac{1}{\pi^2} \log n(T)$$

and in distribution

$$\frac{\Delta_T - \mathbb{E}\,\Delta_T}{\sqrt{\mathrm{Var}\,\Delta_T}} \Rightarrow N(0,1)$$

as $T \to \infty$.

That $n(T) = o(\log T)$ is important! Collections of zeros in this range are known as 'mesoscopic.'

Mesoscopic collections of eigenvalues

Theorem (Costin-Lebowitz)

Let $n(M)$ be a function $\to \infty$ as $M \to \infty$, but so that $n(M) = o(M)$. Let $I_M = [-n(M)/2, n(M)/2]$. Consider the counting function

$$\Delta_M = \#_{I_M}(\{M\theta_i\}).$$

Then

$$\mathbb{E}_{\mathcal{U}(M)}\Delta_M = n(M)$$

$$\mathrm{Var}_{\mathcal{U}(M)}\Delta_M \sim \frac{1}{\pi^2}\log n(M)$$

and in distribution

$$\frac{\Delta_M - \mathbb{E}\Delta_M}{\sqrt{\mathrm{Var}\Delta_M}} \Rightarrow N(0,1)$$

Mesoscopic collections of eigenvalues

Theorem (Costin-Lebowitz)

Let $n(M)$ be a function $\to \infty$ as $M \to \infty$, but so that $n(M) = o(M)$. Let $I_M = [-n(M)/2, n(M)/2]$. Consider the counting function

$$\Delta_M = \#_{I_M}(\{M\theta_i\}).$$

Then

$$\mathbb{E}_{\mathcal{U}(M)}\Delta_M = n(M)$$

$$\mathrm{Var}_{\mathcal{U}(M)}\Delta_M \sim \frac{1}{\pi^2}\log n(M)$$

and in distribution

$$\frac{\Delta_M - \mathbb{E}\Delta_M}{\sqrt{\mathrm{Var}\Delta_M}} \Rightarrow N(0,1)$$

Here $n(M) = o(M)$ is a natural boundary.

Mesoscopic collections of eigenvalues

Theorem (Costin-Lebowitz)

Let $n(M)$ be a function $\to \infty$ as $M \to \infty$, but so that $n(M) = o(M)$. Let $I_M = [-n(M)/2, n(M)/2]$. Consider the counting function

$$\Delta_M = \#_{I_M}(\{M\theta_i\}).$$

Then

$$\mathbb{E}_{\mathcal{U}(M)}\Delta_M = n(M)$$

$$\mathrm{Var}_{\mathcal{U}(M)}\Delta_M \sim \frac{1}{\pi^2}\log n(M)$$

and in distribution

$$\frac{\Delta_M - \mathbb{E}\Delta_M}{\sqrt{\mathrm{Var}\Delta_M}} \Rightarrow N(0,1)$$

Here $n(M) = o(M)$ is a natural boundary.

Heuristic conjecture of Berry (1989): *The zeros look like eigenvalues not only microscopically, but also mesoscopically.*

Macroscopic collections of zeros

Theorem (Backlund)

$$N(T) = \frac{1}{\pi} \arg \Gamma(\tfrac{1}{4} + i\tfrac{T}{2}) - \frac{T}{2\pi} \log \pi + 1 + S(T)$$

$$S(T) := \frac{1}{\pi} \arg \zeta(\tfrac{1}{2} + iT)$$

$S(T)$ is small and oscillatory, and may be thought of as an error term.

Macroscopic collections of zeros

Theorem (Backlund)

$$N(T) = \frac{1}{\pi}\arg\Gamma(\tfrac{1}{4} + i\tfrac{T}{2}) - \frac{T}{2\pi}\log\pi + 1 + S(T)$$

$$S(T) := \frac{1}{\pi}\arg\zeta(\tfrac{1}{2} + iT)$$

$S(T)$ is small and oscillatory, and may be thought of as an error term.

Theorem (Fujii)

Let

$$\overline{\Delta}_T = S(s + \tfrac{2\pi}{\log T}\tfrac{n(T)}{2}) - S(s - \tfrac{2\pi}{\log T}\tfrac{n(T)}{2})$$

and $n(T) \to \infty$ *we have*

$$\mathbb{E}\,\overline{\Delta_T} = o(1)$$

$$\mathrm{Var}\,\overline{\Delta}_T \sim \begin{cases} \frac{1}{\pi^2}\log n(T) & \text{if } n(T) = o(\log T) \\ \frac{1}{\pi^2}\log\log T & \text{if } \log T \lesssim n(T) = o(T). \end{cases}$$

We still have $\overline{\Delta_T}/\mathrm{Var}\,\overline{\Delta_T} \Rightarrow N(0,1)$.

Macroscopic collections of zeros

Theorem (Backlund)

$$N(T) = \frac{1}{\pi}\arg\Gamma(\tfrac{1}{4} + i\tfrac{T}{2}) - \frac{T}{2\pi}\log\pi + 1 + S(T)$$

$$S(T) := \frac{1}{\pi}\arg\zeta(\tfrac{1}{2} + iT)$$

$S(T)$ is small and oscillatory, and may be thought of as an error term.

Theorem (Fujii)

Let

$$\overline{\Delta}_T = S(s + \tfrac{2\pi}{\log T}\tfrac{n(T)}{2}) - S(s - \tfrac{2\pi}{\log T}\tfrac{n(T)}{2})$$

and $n(T) \to \infty$ we have

$$\mathbb{E}\,\overline{\Delta_T} = o(1)$$

$$\mathrm{Var}\,\overline{\Delta}_T \sim \begin{cases} \frac{1}{\pi^2}\log n(T) & \text{if } n(T) = o(\log T) \\ \frac{1}{\pi^2}\log\log T & \text{if } \log T \lesssim n(T) = o(T). \end{cases}$$

We still have $\overline{\Delta_T}/\mathrm{Var}\,\overline{\Delta_T} \Rightarrow N(0,1)$.

This phase change does not correspond to phenomena in random matrix theory. What causes it?

Macroscopic pair correlation 1

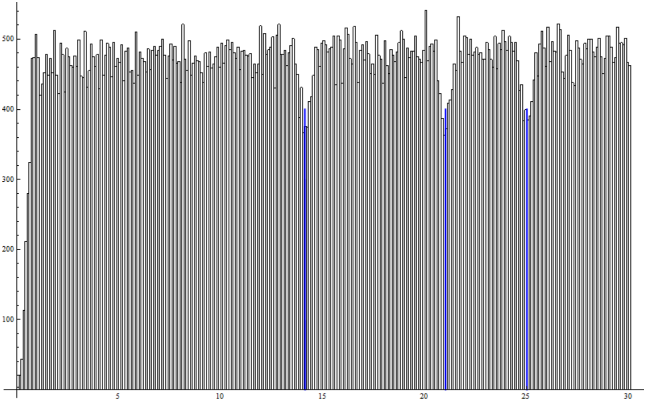

Figure: A histogram of $\gamma - \gamma'$ for the first 5000 zeros, in intervals of size .1.

Macroscopic pair correlation 2

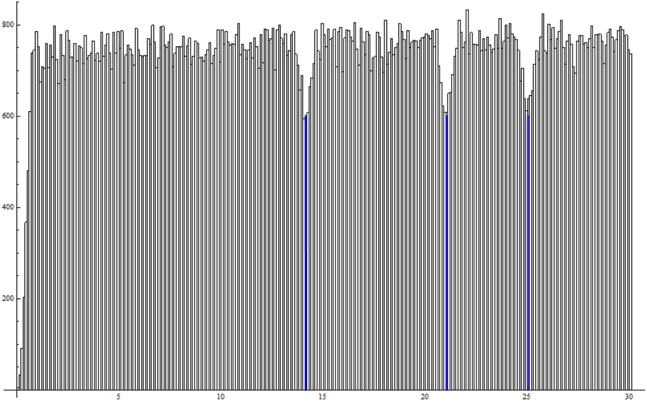

Figure: A histogram of $\gamma - \gamma'$ for the first 7500 zeros, in intervals of size .1.

Macroscopic pair correlation 3

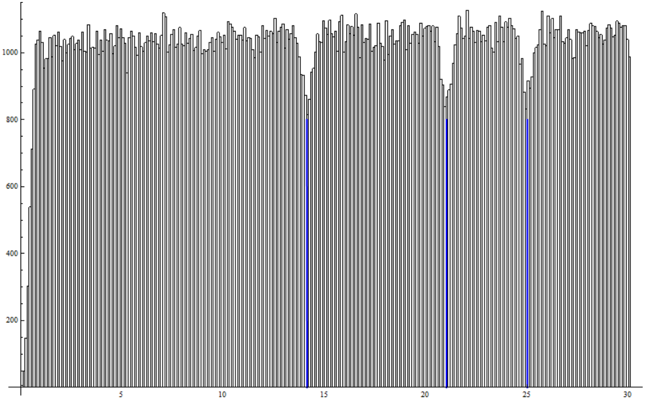

Figure: A histogram of $\gamma - \gamma'$ for the first 10000 zeros, in intervals of size .1.

The Bogomolny - Keating prediction

Figure: A histogram of $\gamma - \gamma'$ for the first 10000 zeros, in intervals of size .1, compared with the prediction of Bogomolny and Keating.

Montgomery's strong pair correlation

Theorem (Montgomery)

For fixed $\epsilon > 0$ *and* $w(u) = 4/(4+u^2)$*,*

$$\frac{1}{T\frac{\log T}{2\pi}} \sum_{0\le\gamma,\gamma'\le T} e\Big(\alpha\tfrac{\log T}{2\pi}(\gamma-\gamma')\Big) w(\gamma-\gamma')$$
$$= 1 - (1-|\alpha|)_+ + o(1) + (1+o(1))T^{-2\alpha}\log T$$
$$= (1+o(1))\int_{\mathbb{R}} e(\alpha x) w\big(\tfrac{2\pi x}{\log T}\big)\left[\delta(x) + 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right] dx$$

uniformly for $|\alpha| \le 1-\epsilon$*.*

For fixed M, this is conjectured to be true uniformly for $\alpha \le M$.

Theorem (Montgomery)

For fixed $\epsilon > 0$ and $w(u) = 4/(4+u^2)$,

$$\frac{1}{T\frac{\log T}{2\pi}} \sum_{0\le\gamma,\gamma'\le T} e\Big(\alpha \tfrac{\log T}{2\pi}(\gamma-\gamma')\Big) w(\gamma-\gamma')$$

$$= 1 - (1-|\alpha|)_+ + o(1) + (1+o(1))T^{-2\alpha}\log T$$

$$= (1+o(1)) \int_{\mathbb{R}} e(\alpha x) w\big(\tfrac{2\pi x}{\log T}\big)\Big[\delta(x) + 1 - \Big(\frac{\sin \pi x}{\pi x}\Big)^2\Big]\,dx$$

uniformly for $|\alpha| \le 1-\epsilon$.

For fixed M, this is conjectured to be true uniformly for $\alpha \le M$.

This has not only microscopic content, but first order macroscopic content! We can see, to first order, the histogram $\gamma - \gamma'$ up to a microscopically blurred resolution.

What information does this give us?

- The role of the cutoff w is only to restrict our attention to $\gamma - \gamma'$ in a prescribed bounded region

What information does this give us?

- The role of the cutoff w is only to restrict our attention to $\gamma - \gamma'$ in a prescribed bounded region - Morally w can be replaced by $\mathbf{1}_{[-30,30]}$, or anything similar.

What information does this give us?

- The role of the cutoff w is only to restrict our attention to $\gamma - \gamma'$ in a prescribed bounded region - Morally w can be replaced by $\mathbf{1}_{[-30,30]}$, or anything similar.
- By integrating the expression in α against $\hat{g}(\alpha)$, for some $\hat{g}$ supported in $(-1, 1)$, we obtain an asymptotic count of $\gamma - \gamma'$ around 0 as counted by a microscopic (but band-limited) test function: $g(\frac{\log T}{2\pi}(\gamma - \gamma'))$.

What information does this give us?

- The role of the cutoff w is only to restrict our attention to $\gamma - \gamma'$ in a prescribed bounded region - Morally w can be replaced by $\mathbf{1}_{[-30,30]}$, or anything similar.
- By integrating the expression in α against $\hat{g}(\alpha)$, for some $\hat{g}$ supported in $(-1,1)$, we obtain an asymptotic count of $\gamma - \gamma'$ around 0 as counted by a microscopic (but band-limited) test function: $g(\frac{\log T}{2\pi}(\gamma - \gamma'))$.
- By integrating against $\hat{g}(\alpha)e(-\alpha\frac{\log T}{2\pi}r)$, we obtain an asymptotic count of $\gamma - \gamma$ microscopically near r: $g(\frac{\log T}{2\pi}(\gamma - \gamma' - r)$. This asymptotic is uniform in r.

What information does this give us?

- The role of the cutoff w is only to restrict our attention to $\gamma - \gamma'$ in a prescribed bounded region - Morally w can be replaced by $\mathbf{1}_{[-30,30]}$, or anything similar.
- By integrating the expression in α against $\hat{g}(\alpha)$, for some $\hat{g}$ supported in $(-1,1)$, we obtain an asymptotic count of $\gamma - \gamma'$ around 0 as counted by a microscopic (but band-limited) test function: $g(\frac{\log T}{2\pi}(\gamma - \gamma'))$.
- By integrating against $\hat{g}(\alpha)e(-\alpha\frac{\log T}{2\pi}r)$, we obtain an asymptotic count of $\gamma - \gamma$ microscopically near r: $g(\frac{\log T}{2\pi}(\gamma - \gamma' - r))$. This asymptotic is uniform in r.
- By adding these microscopic counts at different r, we can obtain asymptotics of meso- and macroscopic counts, even against test functions not band limited.

Macroscopic pair correlation: An exact formulation

Theorem (R.)

For fixed $\epsilon > 0$ and fixed ω with a smooth and compactly supported Fourier transform,

$$\frac{1}{T}\sum_{0<\gamma\neq\gamma'\leq T} \omega(\gamma-\gamma')e\big(\alpha\tfrac{\log T}{2\pi}(\gamma-\gamma')\big)$$
$$= O_\delta\Big(\frac{1}{T^\delta}\Big) + \int_{\mathbb{R}} \omega(u)e\big(\alpha\tfrac{\log T}{2\pi}u\big)\left[\frac{1}{T}\int_0^T \Big(\frac{\log(t/2\pi)}{2\pi}\Big)^2 + Q_t(u)\,dt\right]du$$

for any $\delta < \epsilon/2$, uniformly for $|\alpha| < 1-\epsilon$.

where

$$Q_t(u) := \frac{1}{4\pi^2}\bigg(\Big(\frac{\zeta'}{\zeta}\Big)'(1+iu) - B(iu) + \Big(\frac{\zeta'}{\zeta}\Big)'(1-iu) - B(-iu)$$
$$+ \Big(\frac{t}{2\pi}\Big)^{-iu}\zeta(1-iu)\zeta(1+iu)A(iu) + \Big(\frac{t}{2\pi}\Big)^{iu}\zeta(1+iu)\zeta(1-iu)A(-iu)\bigg),$$

defined by continuity at $u = 0$, and

$$A(s) := \prod_p \frac{(1-\frac{1}{p^{1+s}})(1-\frac{2}{p}+\frac{1}{p^{1+s}})}{(1-\frac{1}{p})^2} = \prod_p \bigg(1 - \frac{(1-p^{-s})^2}{(p-1)^2}\bigg) = 1 + O(s^2),$$

and

$$B(s) := \sum_p \frac{\log^2 p}{(p^{1+s}-1)^2}.$$

What produces the troughs?

- $$\left(\frac{\zeta'}{\zeta}\right)'(1+iu) = \frac{d^2}{d^2\, s}\log\zeta\Big|_{1+iu}$$

What produces the troughs?

- $$\left(\frac{\zeta'}{\zeta}\right)'(1+iu) = \frac{d^2}{d^2\,s}\log\zeta\Big|_{1+iu}$$

- $$\left(\frac{\zeta'}{\zeta}\right)'(1+iu) = H(u) - \sum_{\gamma}\phi(u-\gamma)$$

  for $H(u)$ regular and not too large when $u \neq 0$, and

  $$\phi(x) := 2\frac{\frac{1}{4}-x^2}{(\frac{1}{4}+x^2)^2}$$

What produces the troughs?

- $$\left(\frac{\zeta'}{\zeta}\right)'(1+iu) = \frac{d^2}{d^2\, s}\log\zeta\Big|_{1+iu}$$

- $$\left(\frac{\zeta'}{\zeta}\right)'(1+iu) = H(u) - \sum_{\gamma}\phi(u-\gamma)$$

  for $H(u)$ regular and not too large when $u \neq 0$, and

  $$\phi(x) := 2\frac{\frac{1}{4}-x^2}{(\frac{1}{4}+x^2)^2}$$

- It ends up that

  $$\left(\frac{\zeta'}{\zeta}\right)'(1+iu) - B(iu) = \sum_{k=1}^{\infty} c_k \left(\frac{\zeta'}{\zeta}\right)'(ks)$$

  for $c_k = \sum_{d|k}\mu(d)d$.

Macroscopic pair correlation: A reformulation for $|\alpha| < 1$

Theorem

For fixed $\epsilon > 0$ and fixed ω with a smooth and compactly supported Fourier transform,

$$\frac{1}{T} \sum_{0<\gamma\neq\gamma'\leq T} \omega(\gamma-\gamma') e\big(\alpha \tfrac{\log T}{2\pi}(\gamma-\gamma')\big)$$

$$= O_\delta\Big(\frac{1}{T^\delta}\Big) + \int_{\mathbb{R}} \omega(u) e\big(\alpha \tfrac{\log T}{2\pi} u\big) \Big[\frac{1}{T}\int_0^T \Big(\frac{\log(t/2\pi)}{2\pi}\Big)^2 + \tilde{Q}_t(u)\, dt\Big]\, du$$

for any $\delta < \epsilon/2$, uniformly for $|\alpha| < 1 - \epsilon$.

where

$$\tilde{Q}_t(u) := \frac{1}{4\pi^2}\Bigg(\sum \frac{\Lambda^2(n)}{n^{1+iu}} + \sum \frac{\Lambda^2(n)}{n^{1-iu}} + \frac{e(-\frac{\log(t/2\pi)}{2\pi}u) + e(\frac{\log(t/2\pi)}{2\pi}u)}{u^2}\Bigg),$$

defined by continuity at $u = 0$.

Another application of this philosophy: An analogue of the Strong Szegő Theorem

Theorem (R., Bourgade-Kuan)

Let $n(T) \to \infty$, but $n(T) = o(\log T)$. For a fixed η define

$$\Delta_{\eta,T} = \sum_{\gamma} \eta\Big(\tfrac{\log T}{2\pi n(T)}(\gamma - s)\Big),$$

For all η with compact support and bounded variation when $\int |x||\hat{\eta}(x)|^2\, dx$ diverges, and nearly all such η when the integral converges, we have

$$\mathbb{E}\,\Delta_{\eta,T} = n(T)\int_{\mathbb{R}} \eta(\xi)d\xi + o(1),$$

$$\mathrm{Var}\,\Delta_{\eta,T} \sim \int_{-n(T)}^{n(T)} |x||\hat{\eta}(x)|^2 dx$$

and in distribution

$$\frac{\Delta_{\eta,T} - \mathbb{E}\Delta_{\eta,T}}{\sqrt{\mathrm{Var}\Delta_{\eta,T}}} \Rightarrow N(0,1)$$

as $T \to \infty$.

Ideas in proof: Explicit formulas

Ex: $\Lambda(n) \approx 1 - \sum_\gamma n^{-1/2+i\gamma} + \text{lower order}$

Ideas in proof: Explicit formulas

Ex: $\Lambda(n) \approx 1 - \sum_\gamma n^{-1/2+i\gamma} + \text{lower order}$

- Must use an explicit formula which is exact, or extremely close to being exact
- Must use an explicit formula which takes into account the functional equation

Ideas in proof: Explicit formulas

Ex: $\Lambda(n) \approx 1 - \sum_\gamma n^{-1/2+i\gamma} + \text{lower order}$

- Must use an explicit formula which is exact, or extremely close to being exact
- Must use an explicit formula which takes into account the functional equation

Recall $N(T) = \frac{1}{\pi}\arg\Gamma(\frac{1}{4} + i\frac{T}{2}) - \frac{T}{2\pi}\log\pi + 1 + S(T)$.

Ideas in proof: Explicit formulas

Ex: $\Lambda(n) \approx 1 - \sum_\gamma n^{-1/2+i\gamma} + \text{lower order}$

- Must use an explicit formula which is exact, or extremely close to being exact
- Must use an explicit formula which takes into account the functional equation

Recall $N(T) = \frac{1}{\pi}\arg\Gamma(\frac{1}{4} + i\frac{T}{2}) - \frac{T}{2\pi}\log\pi + 1 + S(T)$.

$$\Rightarrow: dN(\xi) = \sum_\gamma \delta_\gamma(\xi)d\xi = \frac{\Omega(\xi)}{2\pi} + dS(\xi)$$

where $\Omega(\xi)/2\pi$ is regular and $\approx \log(\xi)/2\pi$

Ideas in proof: Explicit formulas

Ex: $\Lambda(n) \approx 1 - \sum_\gamma n^{-1/2+i\gamma} + \text{lower order}$

- Must use an explicit formula which is exact, or extremely close to being exact
- Must use an explicit formula which takes into account the functional equation

Recall $N(T) = \frac{1}{\pi}\arg\Gamma(\frac{1}{4} + i\frac{T}{2}) - \frac{T}{2\pi}\log\pi + 1 + S(T)$.

$$\Rightarrow: dN(\xi) = \sum_\gamma \delta_\gamma(\xi)d\xi = \frac{\Omega(\xi)}{2\pi} + dS(\xi)$$

where $\Omega(\xi)/2\pi$ is regular and $\approx \log(\xi)/2\pi$

Pair correlation $\Leftrightarrow$ Knowing about $dN(\xi_1 + t)dN(\xi_2 + t)$ on average
$\Leftrightarrow$ Knowing about $dS(\xi_1 + t)dS(\xi_2 + t)$ on average

Ideas in proof: Explicit formulas

Theorem (Riemann-Guinand-Weil)

For nice g

$$\int_{\mathbb{R}} \hat{g}\left(\frac{\xi}{2\pi}\right) dS(\xi) = \int_{-\infty}^{\infty} [g(x) + g(-x)] e^{-x/2} d\left(e^x - \psi(e^x)\right)$$

Here $\psi(x) = \sum_{n \leq x} \Lambda(n)$.

Ideas in proof: Explicit formulas

Theorem (Riemann-Guinand-Weil)

For nice g

$$\int_{\mathbb{R}} \hat{g}\left(\frac{\xi}{2\pi}\right) dS(\xi) = \int_{-\infty}^{\infty} [g(x) + g(-x)] e^{-x/2} d\left(e^x - \psi(e^x)\right)$$

Here $\psi(x) = \sum_{n \le x} \Lambda(n)$.

This is a Fourier duality between the error term of the prime counting function, and the error term of the zero counting function.

Ideas in proof: Smooth averages

- Replace

$$\frac{1}{T}\int_T^{2T} \cdots\, ds = \int_{\mathbb{R}} \frac{\mathbf{1}_{[1,2]}(s/T)}{T} \cdots\, ds \text{ with } \int_{\mathbb{R}} \frac{\sigma(s/T)}{T} \cdots\, ds$$

for $\hat{\sigma}$ compactly supported, and σ of mass 1 (so $\hat{\sigma}(0) = 1$).

Ideas in proof: Smooth averages

- Replace

$$\frac{1}{T}\int_T^{2T} \cdots \, ds = \int_{\mathbb{R}} \frac{\mathbf{1}_{[1,2]}(s/T)}{T} \cdots \, ds \text{ with } \int_{\mathbb{R}} \frac{\sigma(s/T)}{T} \cdots \, ds$$

  for $\hat{\sigma}$ compactly supported, and σ of mass 1 (so $\hat{\sigma}(0) = 1$).

We want to know about:

$$A = \int_{\mathbb{R}} \frac{\sigma(s/T)}{T} \int_{\mathbb{R}^2} e\big(\alpha \tfrac{\log T}{2\pi}(\xi_1 - s) - \alpha \tfrac{\log T}{2\pi}(\xi_2 - s)\big) r\Big(\tfrac{\xi_1 - s}{2\pi}\Big) r\Big(\tfrac{\xi_2 - s}{2\pi}\Big)\, dS(\xi_1)\, dS(\xi_2)\, ds$$

$$= \sum_{\varepsilon \in \{-1,1\}^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\sigma}\Big(\tfrac{T}{2\pi}(\varepsilon_1 x_1 + \varepsilon_2 x_2)\Big) \hat{r}(\varepsilon_1 x_1 - \alpha \log T) \hat{r}(\varepsilon_2 x_2 + \alpha \log T)$$

$$\times\, e^{-(x_1+x_2)/2} d\big(e^{x_1} - \psi(e^{x_1})\big) d\big(e^{x_2} - \psi(e^{x_2})\big)$$

This is really four integrals, over different measures:

$$d\big(e^{x_1} - \psi(e^{x_1})\big) d\big(e^{x_2} - \psi(e^{x_2})\big) = d(e^{x_1})d(e^{x_2}) - d(e^{x_1})d\psi(e^{x_2}) - d\psi(e^{x_1})d(e^{x_2}) + d\psi(e^{x_1})d\psi(e^{x_2})$$

Ideas in proof

The term $\hat{\sigma}\Big(\frac{T}{2\pi}(\varepsilon_1 x_1 + \varepsilon_2 x_2)\Big)$ forces $\varepsilon_1 x_1 + \varepsilon_2 x_2 = O(1/T)$:

$$\begin{aligned} A = O\Big(\frac{1}{T^{1-\alpha}}\Big) + \sum_{\varepsilon \in \{-1,1\}^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\sigma}\Big(\tfrac{T}{2\pi}(\varepsilon_1 x_1 + \varepsilon_2 x_2)\Big)\hat{r}(\varepsilon_1 x_1 - \alpha \log T) \\ \times \hat{r}(\varepsilon_2 x_2 + \alpha \log T) e^{-(x_1+x_2)/2} d\psi(e^{x_1}) d\psi(e^{x_2}) \\ = O\Big(\frac{1}{T^{1-\alpha}}\Big) + \sum_n \frac{\Lambda^2(n)}{n}\big[\hat{r}(-\log n - \alpha \log T)\hat{r}(\log n - \alpha \log T) \\ + \hat{r}(\log n - \alpha \log T)\hat{r}(-\log n - \alpha \log T)\big] \end{aligned}$$

- This can be untangled with some complex analysis to give the form we're after.
- Some additional work is needed to untangle $dS(\xi_1 + t)dS(\xi_2 + t)$.

A more ambitious application: Moments

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt$$

$\leftrightarrow$

$$\int_0^1 \int_{\mathcal{U}(n)} |\det(1-e^{i2\pi\theta}g)|^{2k}\,dg\,d\theta$$
$$=\int_{\mathcal{U}(n)} |\det(1-g)|^{2k}\,dg$$

Macroscopic information in k-point correlation functions, with microscopic band-limitations: Fourier support in $\{y : |y_1| + \cdots + |y_k| \leq 2\}$

$\leftrightarrow$

Using only knowledge of the k-point correlation functions

$$\int \sum_{\substack{j_1,\ldots,j_k \\ \text{distinct}}} \eta(e^{i2\pi\theta_{j_1}},\ldots,e^{i2\pi\theta_{j_k}})\,dg$$

for $\eta : \mathbb{T}^k \to \mathbb{R}$, $\mathrm{supp}\ \hat{\eta} \subset \{r \in \mathbb{Z}^k : |r_1| + \cdots + |r_k| \leq 2n\}$

A more ambitious application: Moments

But with this information, we can deduce

$$\int |\det(1-g)|^{2k}\, dg = \int \prod_{j=1}^{n} (2 - e^{i2\pi\theta_j} - e^{-i2\pi\theta_j})^k\, dg$$

for $k = 1, 2$ but no higher.

A more ambitious application: Moments

But with this information, we can deduce

$$\int |\det(1-g)|^{2k}\, dg = \int \prod_{j=1}^{n} (2 - e^{i2\pi\theta_j} - e^{-i2\pi\theta_j})^k\, dg$$

for $k = 1, 2$ but no higher.

Classical knowledge about the zeta function, having nothing to do with random matrix theory, let's us rigorously deduce the asymptotics of

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt$$

for $k = 1, 2$, but no higher.

A more ambitious application: Moments

But with this information, we can deduce

$$\int |\det(1-g)|^{2k}\, dg = \int \prod_{j=1}^{n} (2 - e^{i2\pi\theta_j} - e^{-i2\pi\theta_j})^k\, dg$$

for $k = 1, 2$ but no higher.

Classical knowledge about the zeta function, having nothing to do with random matrix theory, let's us rigorously deduce the asymptotics of

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k}\, dt$$

for $k = 1, 2$, but no higher.

Question: Is there a way to understand these computations in terms of macroscopic k-point correlation functions?

What about the conjectured asymptotics of higher moments? (Keating-Snaith conjecture)

Thanks:

Thanks!