MAT 636 ODES AND FOURIER SERIES
MIDTERM GROUP PROJECT

Instructions: Below are four problems. You should choose three of them and turn in solutions for these three problems only. The process of coming up with solutions is expected to be collaborative, however the task of writing up solutions should be divided among the group, with each member writing a solution to one of the problems. (By necessity, one group had only two members, and this group should choose only two problems and turn in the solutions for these.)

All resources can be used for this exam, including the course textbook by Teschl, your notes, other textbooks, internet resources, calculators and mathematics software, etc. You can even talk to people in other groups, though I ask that you do this only as a last resort.

What you hand in for each problem should consist of two parts:

Part 1 should consist of a solution to the problem at hand. Because you have two weeks to do the exam, I ask that your solutions be written in full sentences on a separate sheet of paper (both LaTeX and handwriting are OK), and that you quote any theorems you use and give a reference to the source.[1] While I ask you to write in complete sentences, grammatical and spelling mistakes will *not* be a factor in grading, so please do not spend too much time worrying about these things if your English is not great.

Part 2 should consist of a description of how your solution was arrived at. It is perfectly acceptable, and will *not* result in a lower grade for you to say "I couldn't figure out the solution, and X showed me the solution she had found." Honesty is preferred! In this case, you should ask X how she found her solution, and write down the process in your own words. It is also perfectly acceptable to say that you typed some keywords into a google search, and found a solution in Newton's *Principia* or some other source (e.g. "The main idea of the solution is to use l'Hospital's rule repeatedly..."). This will reassure Nicolas and I that you have engaged with the material rather than just copying it. This part of your written work might be something like a few paragraphs long.

Several problems have multiple parts, and partial credit will be assigned even if not all parts are completed. So even if you haven't arrived at a solution, you can talk about approaches you have attempted, and this may result in partial credit.

Please do not hesitate to ask myself or Nicolas if you have any questions about the format of the exam or the exam itself.

[1] e.g. One might write, "Theorem 6 of Artin's *Galois Theory* says the following:

Theorem 1. If F, B, E are three fields such that $F \subset B \subset E$, then $(E/F) = (B/F)(E/B)$.

From this we see that..."

Though hopefully you will not require Theorem 6 of Artin's book on Galois theory for this particular exam...

Problem 1. a) Find a function $y : [1, \infty) \to \mathbb{R}$ such that

$$\frac{dy}{dt} = y/t + e^{-y/t} \quad \text{for} \quad t \in [1, \infty), \qquad \text{with } y(1) = 0.$$

b) Find a solution to the differential equation

$$P' = rP \cdot \left(1 - \frac{P}{K}\right) \qquad \text{with } P(0) = P_0$$

defined on the interval $[0, \infty)$. Here r and K and P_0 are positive constants.

c) Explain why the differential equation in part b) might be used to model the growth of a population whose size is limited by a finite amount of resources. (It may be helpful to recall first why the differential equation $P' = rP$ models the growth of a population with no constraints on its growth.) [This part c) of the problem is not mathematical, and you may as well know: *anything* written down will receive full credit. But it is very worthwhile to think about in order to understand the context of the ODE in part b).]

Problem 2. We model the situation demonstrated (with cheesy background music!) in the first 45 seconds of the video

`http://www.youtube.com/watch?v=32FMEo_igEQ`

in which two pendulums are hung from a common string. One pendulum is pulled back and begins swinging. After a few seconds the other pendulum begins to swing and eventually takes over the motion, while the first becomes essentially stationary. This process is repeated the other way around, etc.

A simple model of this situation is given by

$$\ddot{x} + \omega^2 x = \epsilon y$$
$$\ddot{y} + \omega^2 y = \epsilon x,$$

where ω and ϵ are positive constants, with ϵ very small so that in particular $\epsilon < \omega^2$. Here x is the displacement of the first pendulum and y is the displacement of the second.

a) Solve this differential equation for $t \geq 0$ with the initial conditions $x(0) = A,\ y(0) = \dot{x}(0) = \dot{y}(0) = 0$.

b) Draw a graph of the solution for $x(t)$ and directly under it in a separate graph the solution for $y(t)$.

c) How long will it take after time $t = 0$ for the system to return to a state where y hardly moves from 0 during an entire time interval of length $2\pi/\omega$?

Problem 3. **a)** Suppose $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right) \in \mathcal{M}_{2\times 2}(\mathbb{R})$ and suppose the system of differential equations

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}y \\ \dot{y} &= A_{21}x + A_{22}y, \end{aligned}$$

has a non-constant periodic solution $(x, y) = (\phi_1(t), \phi_2(t))$.[2] Show that in this case, all solutions are periodic.

b) Suppose $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ is invertible, with n odd. Show that there exists a solution to the system of equations $\dot{x} = Ax$ that is not periodic.

[2]This means there exists some $\omega \neq 0$ such that $(\phi_1(t+\omega), \phi_2(t+\omega)) = (\phi_1(t), \phi_2(t))$ for all t.

Problem 4. [Two hints for this problem: You will need to make use of the solution to problem **1b)**. And, Lemma 1.2 in Teschl, regarding upper solutions and lower solutions of ODEs will also be helpful.]

a) Consider the system of ODEs defined on $[0, \infty)$ by

$$\dot{x} = x(1 - x/3) - xy$$
$$\dot{y} = y(1 - y/2) - xy,$$

with initial conditions

$$x(0) = x_0 > 0$$
$$y(0) = y_0 > 0.$$

Prove that for any solutions $x(t), y(t)$ we have

$$x(t) > 0$$
$$y(t) > 0,$$

for all $t \geq 0$. (Note that you are not being asked to demonstrate the existence of any solutions, just that if there does exist a solution, it will satisfy these positivity relations.)

[Further hint: suppose, say $x(t_0) = 0$, and let $Y = \sup_{t\in[0,t_0]} y(t)$. Then $\dot{x} \geq x(1-(1/3+Y)x)$, so that $x(t)$ is a super solution. What does Lemma 1.2 imply about $x(t)$? Why is this enough to derive a contradiction?]

b) For any solution $x(t), y(t)$ of this system of ODEs, prove that for any $\epsilon > 0$, there exists T such that for all $t \geq T$,

$$x(t) \leq 3 + \epsilon$$
$$y(t) \leq 2 + \epsilon.$$

[One more hint: from a), we have for instance $\dot{x} \leq x(1 - x/3)$.]

c) The system of differential equations in **a)** is known as the *Lotka-Volterra two species competition model.* Say a few words about why it is not an unreasonable model for population $x(t)$ and $y(t)$ of two different species competing for a common finite resource necessary for survival. [As in **1c)**, you should think about this for your own sake. Any answer you give to part **c)** will receive full credit.]