Hopf algebras and combinatorics

A defense for the PhD proposal of

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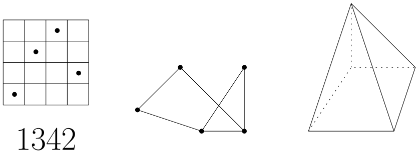
Slides can be found in
`http://user.math.uzh.ch/penaguiao/`

What is algebraic combinatorics - combinatorics

What are combinatorial objects?



Associated with a notion of not only a way to count, but also of **size** and how to **merged** and **split** them.

What is algebraic combinatorics - algebra

Goal: associate to combinatorial objects some algebraic strucutre (Groups, rings, algebras, monoidal categories).
Special algebraic structure: Hopf algebras $(H, \mu, \Delta, \iota, \varepsilon, S)$

- Start with an algebra (H, μ, ι).
- Add a coproduct Δ with a counit ε.
- Add an antipode $S : H \to H$, that plays a role of " inverse function ".

Example of a Hopf algebra: the usual algebra $\mathbb{K}[x]$ with the coproduct

$$\Delta : x \mapsto x \otimes 1 + 1 \otimes x$$

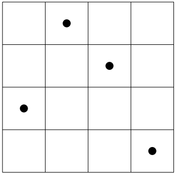
$$S : x^n \mapsto (-x)^n \, .$$

Main goals in Hopf algebras in combinatorics

What types of problems do we want to solve in algebraic combinatorics? Simple formulas for the antipode, structure theorems (freeness, cofreeness) and chromatic invariants.

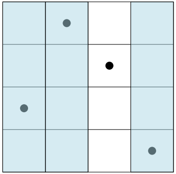
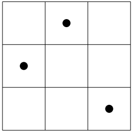
Permutations and patterns

A permutation π of size n is an arrangement on an $n \times n$ table:

$\pi =$  $= 2431$

The set of permutations of size n : $\mathcal{S}_n$. The set of all permutations : $\mathcal{S}$.

Select a set I of columns of the square configuration of π and define the **restriction** $\pi|_I$. This is a permutation.

$\pi|_{\{1,2,4\}}$  $=$  $= 231$

Pattern functions

We can count occurrences!
For permutations π, σ, we define the pattern number:

$$\mathbf{p}_\pi(\sigma) = \#\{\text{occurrences of } \pi \text{ as a restriction of } \sigma\}\,.$$

In this way we have

$$\mathbf{p}_{12}(4132) = 2,\ \mathbf{p}_{312}(4132) = 2,\ \mathbf{p}_{12}(12345) = 10$$

$$\text{and}\ \ \mathbf{p}_{312}(3675421) = 0$$

The same objects can be defined on partitions and graphs and many more! Pattern functions come from **combinatorial presheaves.**

Pattern Hopf algebras

Fact: the pattern functions $\{\mathbf{p}_\pi\}_{\pi \in \mathcal{S}}$ form a linearly independent set.

$$\mathcal{A}(\texttt{Per}) := \operatorname{span}\{\mathbf{p}_\pi\}_{\pi \in \mathcal{S}} \, .$$

Theorem (Vargas, 2014)

The space $\mathcal{A}(\texttt{Per})$ is a Hopf algebra that is freely generated by the so called ***Lyndon permutations****.*

Theorem (P., 2019)

Let $\texttt{h}$ be an associative presheaf. The space $\mathcal{A}(\texttt{h})$ is a Hopf algebra. This is a free algebra when $\texttt{h}$ is a commutative presheaf or is the presheaf on marked permutations.

Meanwhile in Twitter

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A lot of pattern algebras are free! Amazing! Are all such algebras free?

Stable Factorizations

$$\alpha^* = \iota_1^* \star \iota_2^* \star \iota_3^*$$

$$\iota_1^* = \rho_5^* \star \rho_2^* \star \rho_8^*$$

$$\iota_2^* = \rho_3^* \star \rho_4^* \star \rho_1^*$$

$$\beta^* = \rho_1^* \star \rho_2^* \star \rho_3^* \star \rho_4^* \star \rho_5^* \star \rho_6^* \star \rho_7^* \star \rho_8^*$$

$$\iota_3^* = \rho_6^* \star \rho_7^*$$

$$\gamma^* = \rho_3^* \star \rho_5^* \star \rho_4^* \star \rho_1^* \star \rho_2^* \star \rho_6^* \star \rho_7^* \star \rho_8^*$$

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Feasible regions

Fix $\mathbb{S}_j = \bigcup_{j \geq k} \mathcal{S}_k$ of permutations. What are the possible values of $\left(\mathbf{p}_\pi(\sigma)\,|\sigma|^{-|\pi|}\right)_{\pi \in \mathbb{S}_j} \in \mathbb{R}^{\mathbb{S}_j}$ when $|\sigma|$ is big? These are the so called **feasible values** and form the **feasible region**.

Theorem (R. Glebov, C. Hoppen, *et al*, 2017)

The feasible region has dimension at least

$$|\{ \textit{ indecomposable permutations of size} \leq j \ \}| \, .$$

Conjecture (J. Borga, P.)

The dimension of the feasible region for classical occurrences is enumerated by the Lyndon permutations of size at most j.

Consecutive patterns

We now consider only occurrences that form **an interval.** For instance, taking $\sigma = 2413$, there are two distinct consecutive restructions of σ of size three, namely 231 and 312.

Theorem (J. Borga, P., 2019)

*The feasible region for consecutive patterns is a polytope. Specifically, it is the cycle polytope of **overlap graphs** on permutations.*

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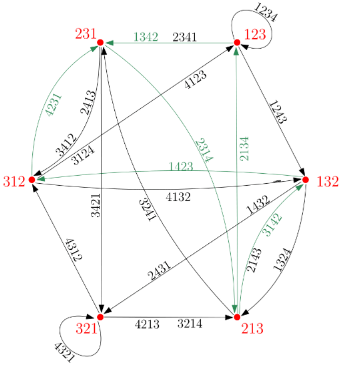
Consecutive patterns are polytopes controlled by these amazing #OverlapGraphs. @JacopoBorga



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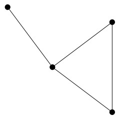
What is a graph

A graph is a pair of sets (V, E). For instance:



A partition is a list of positive integers $(\lambda_1 \leq \cdots \leq \lambda_k)$. Fixed a partition $\lambda = (\lambda_1 \leq \cdots \leq \lambda_k)$, let K_λ^c be a graph: $K_\lambda^c = (V_1 \cup \cdots \cup V_k, E)$ with maximal edges in such a way that each V_i is *independent*.

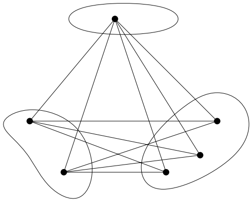


Figure: The graph $K^c_{(1,2,3)}$.

Another Hopf algebra structure on graphs

Product = disjoint union of graphs.

Figure: Product structure on the graph Hopf algebra.

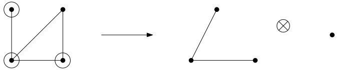


Figure: Coproduct structure on the graph Hopf algebra.

This defines maps

$$\mu : \mathbf{G} \otimes \mathbf{G} \to \mathbf{G}$$

$$\Delta : \mathbf{G} \to \mathbf{G} \otimes \mathbf{G}$$

Chromatic numbers

Given a graph $H = (V, E)$, a **stable coloring** $f : V \to \mathbb{N}$ is such that two neighboring vertices have always distinct colors.

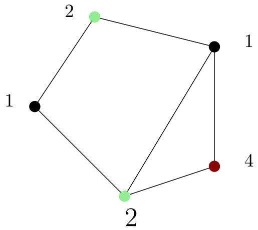


Figure: A stable coloring of a graph.

Chromatic number = lowest number of colors that still allows for a stable colouring. NP complete problem (one of the original ones).

$$\chi_H(n) = \#\{ \text{ graph-colorings with } n \text{ colors } \}.$$

Deletion - contraction relations

Given a graph H, we can define the **deletion** and a **contraction** of a set of edges.

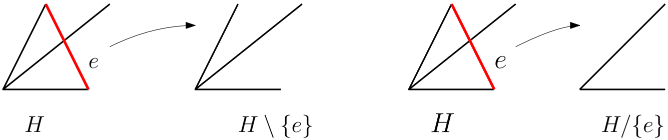


Figure: Deletion and contraction of edges.

$$\chi_H(n) = \chi_{H \setminus e}(n) - \chi_{H/e}(n)\,.$$

Reciprocity results

Given a graph, H, what is the meaning of $\chi_H(-n)$?

Theorem (R. Stanley, 1973)

$$\chi_H(-1) = (-1)^n \#\{\ \textit{acyclic orientations of the graph } H\ \}.$$

Interpretations of $\chi_H(-k)$ for $k > 1$ also exist that relate the chromatic polynomial and acyclic orientations.

The antipode

Theorem (B. Humpert and J. Martin, 2012)

Let $H = (V, E)$ be a graph, then

$$S(H) = \sum_{F \subseteq E \text{ flat}} (-1)^{n-\mathrm{rank}(F)} a(H/_F) H_{(V,F)} .$$

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{\chi} & K[x] \\ \downarrow S & & \downarrow S \\ \mathbf{G} & \xrightarrow{\chi} & K[x] \end{array}$$

$$\chi_H(-n) = S(\chi_H(n)) = \chi_{S(H)}(n) .$$

The ring of symmetric functions

A symmetric function on the variables $x_1, \dots$ (infinitely many variables) is a **formal sum** of monomials on $x_1, \dots$ with bounded degree.

Examples: 1 , $x_1 + x_2 + \dots,$ $x_1^2 + x_2^2 + \dots$

$$x_1 x_2^2 + x_1^2 x_2 + x_1 x_3^2 + x_1^2 x_3 + x_2 x_3^2 + x_2^2 x_3 + x_1 x_4^2 \dots$$

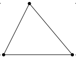
This forms a Hopf algebra.

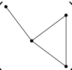
The CSF

The ultimate chromatic invariant.

$$\Psi_{\mathbf{G}}(H) = \sum_{f \text{ stable coloring}} x_{f(1)} \dots x_{f(n)} = \sum_{f \text{ stable coloring}} x_f \in Sym\,.$$

Examples of CSF:

$\Psi_{\mathbf{G}} \Big($  $\Big) = 6(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + \dots)\,,$

$\Psi_{\mathbf{G}} \Big($  $\Big) = 4(x_1^2x_2x_3 + \dots) + 24(x_1x_2x_3x_4 + \dots)\,.$

The tree conjecture

Conjecture (Tree conjecture)

*Given two **non-isomorphic** trees T_1, T_2, their chromatic symmetric functions are distinct, i.e. $\Psi_{\mathbf{G}}(T_1) \neq \Psi_{\mathbf{G}}(T_2)$.*

Partial results: Proper caterpillars, “ labelled ” case.

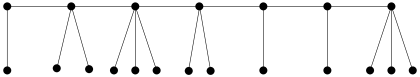


Figure: A tree that is a proper caterpillar.

Modular relations on graphs

Are there any linear relations that Ψ_G satisfy?

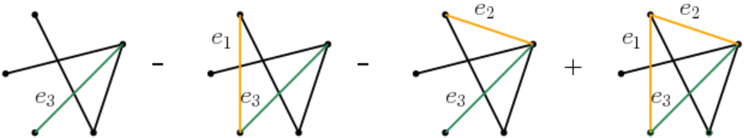


Figure: Modular relations that annihilate Ψ_G, [GP13] and [OM14].

Figure: Two unicyclic graphs with the same CSF.

The kernel problem

Theorem (P., 2018)

The kernel of $\Psi_{\mathbf{G}} : \mathbf{G} \to Sym$ *is generated by modular relations.*

The tree conjecture is established if we find a graph invariant ζ that satisfies the following two properties

- The invariant ζ distinguishes trees, *i.e.* for two non-isomorphic trees T_1, T_2 we have that $\zeta(T_1) \neq \zeta(T_2)$.
- The invariant ζ satisfies the modular relations.

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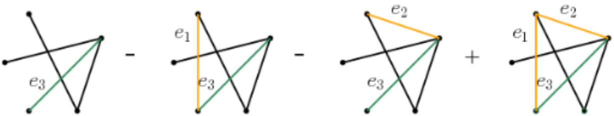
The modular relations are enough to find a rewriting between two graphs with the same CSF! Noice!



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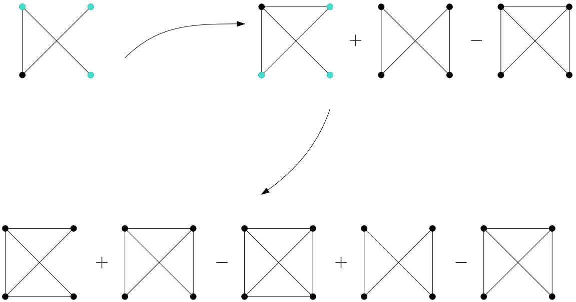
The kernel solution

Proof: Start with an expression $\sum_i \alpha_i G_i \in \ker \Psi_{\mathbf{G}}$. Always increase the number of edges on a graph!

Consider the path of length three:



For which graphs aren't we able to add edges? Graphs of the form K_λ^c. Fact: $\{\Psi_{\mathbf{G}}(K_\lambda^c)\}_{\lambda \vdash n}$ is a linearly independent set.

Polytopes - Simplices

Let's go to the n-dimentional space $\mathbb{R}^n$, that has a canonical basis $\{\vec{e}_i | i = 1, \ldots, n\}$.

$$\mathrm{Sim}_I = \mathrm{conv}\{\vec{e}_i | i \in I\} .$$

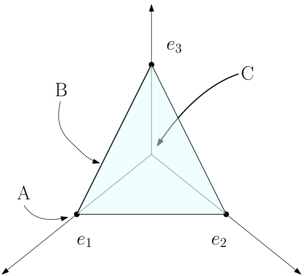
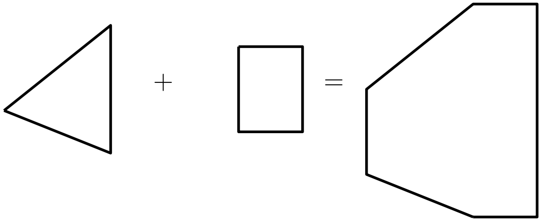


Figure: The polytopes $A = \mathrm{Sim}_{\{1\}}, B = \mathrm{Sim}_{\{1,3\}}, C = \mathrm{Sim}_{\{1,2,3\}}$ are simplices in $\mathbb{R}^3$.

Polytopes - Minkowski operations

Given two convex sets A, B and a non-negative real number λ, their *Minkowski sum* is given by $A + B = \{a + b | a \in A, b \in B\}$ and the *Minkowski dilation* is $\lambda A = \{\lambda a | a \in A\}$.



For convex sets A, B, the set $A - B$ is the convex set C such that $B + C = A$. May not exists, but it is unique.

Polytopes - Generalized permutahedra

A generalized permutahedron is a polytope that results as

$$\left(\sum_{a_J>0} a_J \,\mathrm{Sim}_J\right) - \left(\sum_{a_J<0} |a_J|\,\mathrm{Sim}_J\right).$$

A hypergraphic polytope is a polytope of the form $\sum_{a_J>0} a_J \,\mathrm{Sim}_J$.

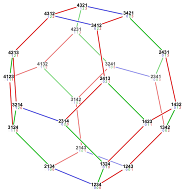


Figure: The permutahedra for $n = 4$.

Faces and linear optimization

Given a polytope $\mathfrak{q}$, a face of $\mathfrak{q}$ is a set of the form

$$\mathfrak{q}_f := \min_{x \in \mathfrak{q}} f(x)\,,$$

for some linear function f.

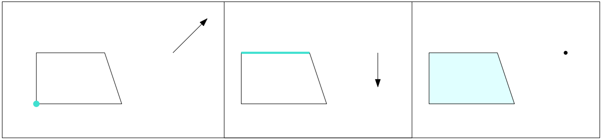


Figure: For some linear functionals the corresponding face.

A chromatic invariant on generalized permutahedra

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots \mapsto (\alpha_1, \alpha_2, \dots)\,.$$

Define $\Psi_{\mathbf{GP}}(\mathfrak{q}) = \sum_{\alpha \text{ vector}} x^{(\alpha)} \mathbb{1}[\mathfrak{q}_\alpha \text{ is a point}]\,.$

This is a chromatic invariant as well! But is not in Sym anymore.

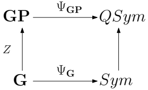


Figure: Commutative diagram of chromatic invariants.

Theorem (P. , 2018)

Explicit generators of the kernel of $\Psi_{\mathbf{GP}}$ can be given, when restricted to $\mathbf{HGP}$.

Future work

- Find all modular relations for the chromatic invariant on $\mathbf{GP}$.
- Freeness conjecture on pattern algebras, antipode formulas for pattern algebras.
- More exciting feasible regions!

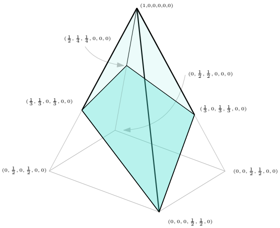


Figure: The feasible region for consecutive occurrences of permutations avoiding 321.

The end