Feasible regions and permutation patterns

Permutation Patterns virtual workshop 2021

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San Francisco

15th June, 2021

Slides can be found at
http://user.math.uzh.ch/penaguiao/
This talk is based on joint work with Jacopo Borga.

The feasible region

$$\widetilde{\text{occ}}(\pi, \sigma) = \#\{\text{classical occurrences of } \pi \text{ in } \sigma\} / \binom{|\sigma|}{|\pi|}.$$

$$clP_{\mathcal{A}} := \left\{ \vec{v} \in [0,1]^{\mathcal{A}} \middle| |\sigma^m| \to \infty \text{ and } \widetilde{\text{occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{A} \right\}$$

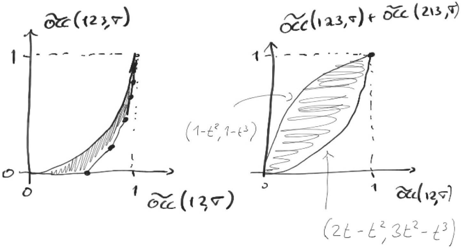

Figure: To each well-behaved sequence of permutations it corresponds a point in the feasible region.

The feasible region

Theorem (Glebov et.al. 2014, Vargas 2014)

The dimension of the feasible region is bounded below by the number of indecomposible permutations, and bounded above by the number of ***Lyndon*** *permutations.*

Conjecture

The dimension of the feasible region is precisely the number of Lyndon permutations.

Consecutive patterns

$$\widetilde{\text{c-occ}}(\pi, \sigma) = \#\{\text{consecutive occurrences of } \pi \text{ in } \sigma\}/|\sigma|.$$

$$P_k := \left\{\vec{v} \in [0,1]^{\mathcal{S}_k} \middle| |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k\right\}$$

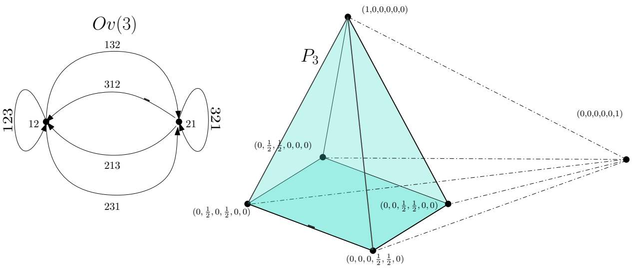

Figure: The feasible region P_3 lives in the 6-dimensional space, but is a 4-dimensional polytope.

Consecutive occurrences feasible regions

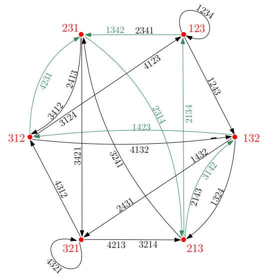

Figure: The overlap graph for $k = 4$ controls the feasible region P_4.

Theorem

The feasible region is the cycle polytope of the overlap graph. It has dimension $k! - (k-1)!$, and the vertices are indexed by simple cycles of this graph.

Restricted feasible regions

Main ingredient: a permutation class $\mathrm{Av}(B)$.

$$P_k^B := \{\vec{v} \big| \sigma^m \in \mathrm{Av}(B), |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k\}.$$

If we let our sequence of permutations vary on a permutation class, we get a smaller, restricted feasible region.
We study the geometry of this region.

Restricted feasible regions - geometry

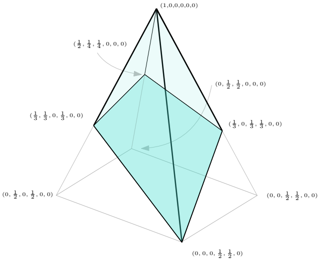

Figure: The restricted feasible region for $B = \{321\}$ and $k = 3$ lives in a 5-dimensional vector space (because there are 5 permutations in $\mathrm{Av}_3(321)$) and is a 3-dimensional polytope.

We can find a full description of this reagion for $B = \{\tau\}$, where τ is a monotone permutation, or when $|\tau| = 3$.

Restricted feasible regions - geometry

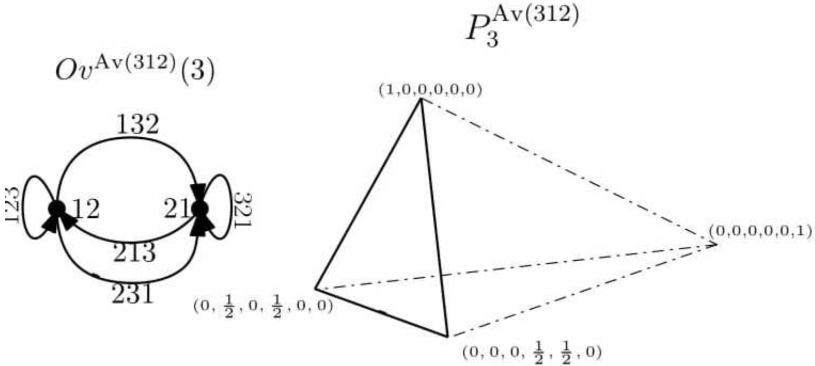

Figure: The restricted feasible region for $B = \{312\}$ and $k = 3$ lives in a 5-dimensional vector space (because there are 5 permutations in $\mathrm{Av}_3(312)$) and is a 3-dimensional polytope.

Restricted feasible regions - general results

Theorem (BP, 2021)

Whenever $\mathrm{Av}(B)$ is closed for the operation $\oplus$ or $\ominus$, we have that P_k^B is a closed, convex set with dimension:

$$\dim P_k^B = |\,\mathrm{Av}_k(B)| - |\,\mathrm{Av}_{k-1}(B)|\,.$$

Is it a polytope? We don't know!
Particular case of notice: if B is a singleton.

Other questions on feasible regions

- Can we find triangulations of these polytopes? What are the volumes of these polytopes?
- Other particular cases of restricted feasible regions - it seems to work whenever $\mathrm{Av}(\tau)$ has a structure of recursive tree.
- Is the restricted feasible region always a polytope?
- Dimension conjecture for classical patterns.

Biblio

- Borga, J. and Penaguiao, R. (2020). The feasible regions for consecutive patterns of pattern-avoiding permutations. *arXiv:2010.06273*.
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- Vargas, Y. (2014). Hopf algebra of permutation pattern functions. In Discrete Mathematics and Theoretical Computer Science (pp. 839-850). *Discrete Mathematics and Theoretical Computer Science*.
- Kenyon, R., Kral, D., Radin, C., & Winkler, P. (2015). Permutations with fixed pattern densities. *arXiv:1506.02340*.

Thank you